

# *Fractional Bosonic Strings*

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Based on **J.Math.Phys. 59 (2018) no.3, 033509**,  
in collaboration with A. Giusti



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# Outline

- Motivation

**String Theory!!!!**

**Fractional Calculus???**

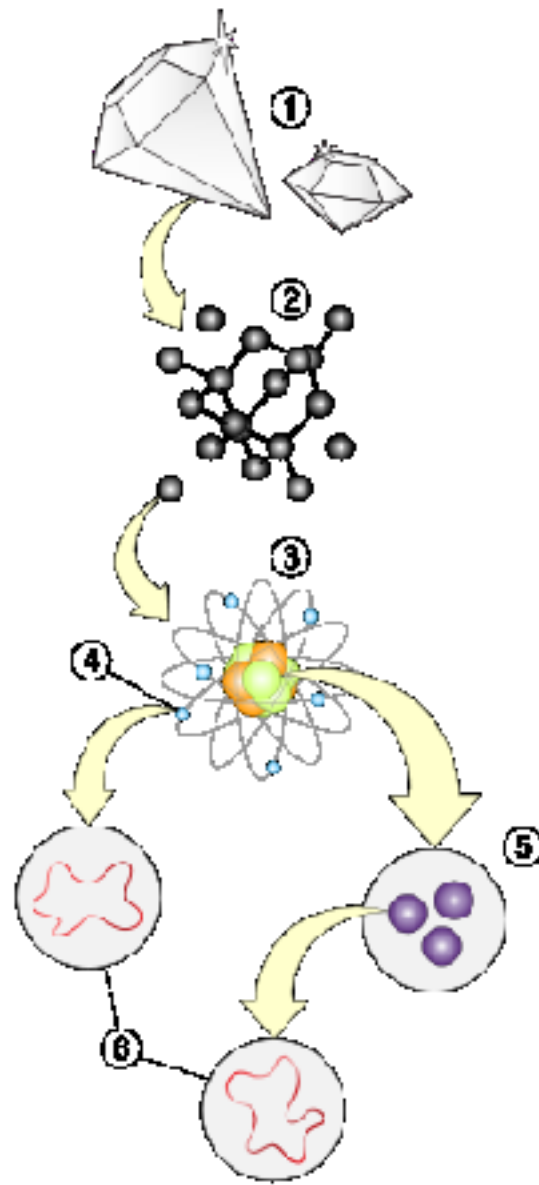
The review

- ♦ Classical Bosonic S.T.
- ♦ Basic Fractional Calculus

✱ Fractional Bosonic Strings

Summary & conclusions

# String Theory!!!



- String scale

$$\ell_p = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-33} \text{ cm}$$

$$m_p = \sqrt{\frac{\hbar c}{G\mathcal{V}}} \sim \frac{1.221 \times 10^{19}}{\sqrt{\mathcal{V}}} \text{ GeV}$$

- particles = string oscillations modes

Ex.: Graviton

Gauge Vectors

In S.T. the gauge and gravitational interactions are unified at the quantum level

# *Fractional Calculus??*



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$$\frac{d^{\pi}}{dx^{\pi}} \left[ e^{4 \ln x^2} \right]$$



It has been used in many different fields such as numerical analysis & physics

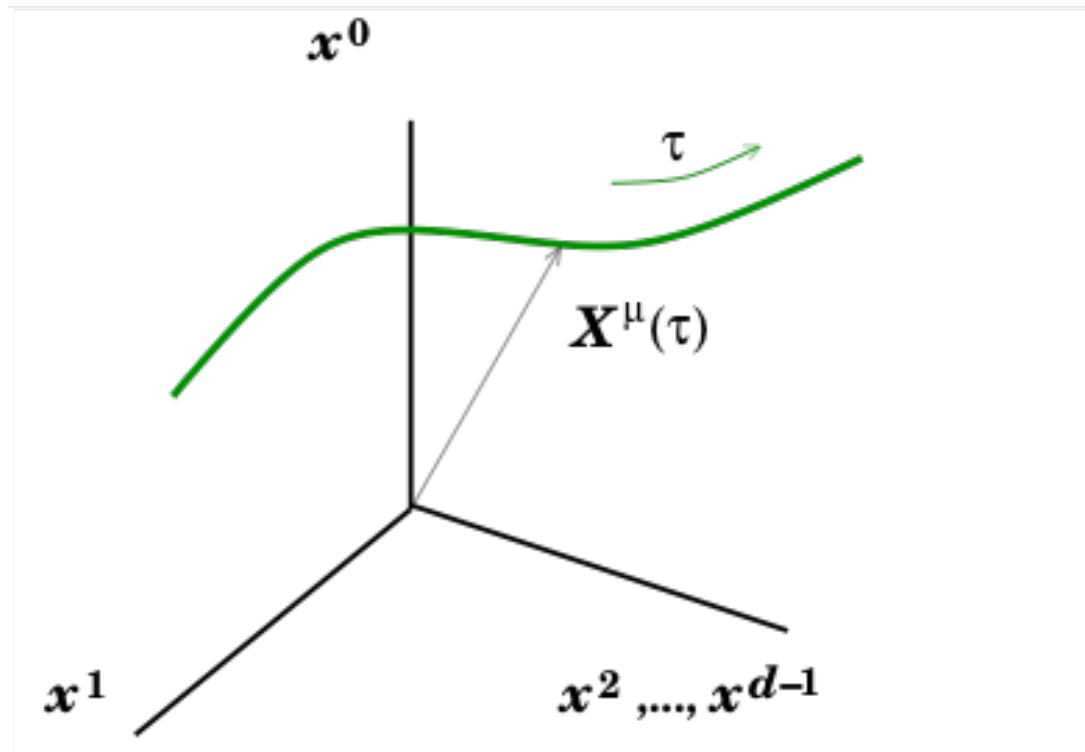
# *The Review*

# Classical Bosonic String Theory

(in a nutshell)

[Polchinski, Zwiebach, Green, Schwarz, Witten, Blumenhagen, Lüst, Theisen, Font,...]

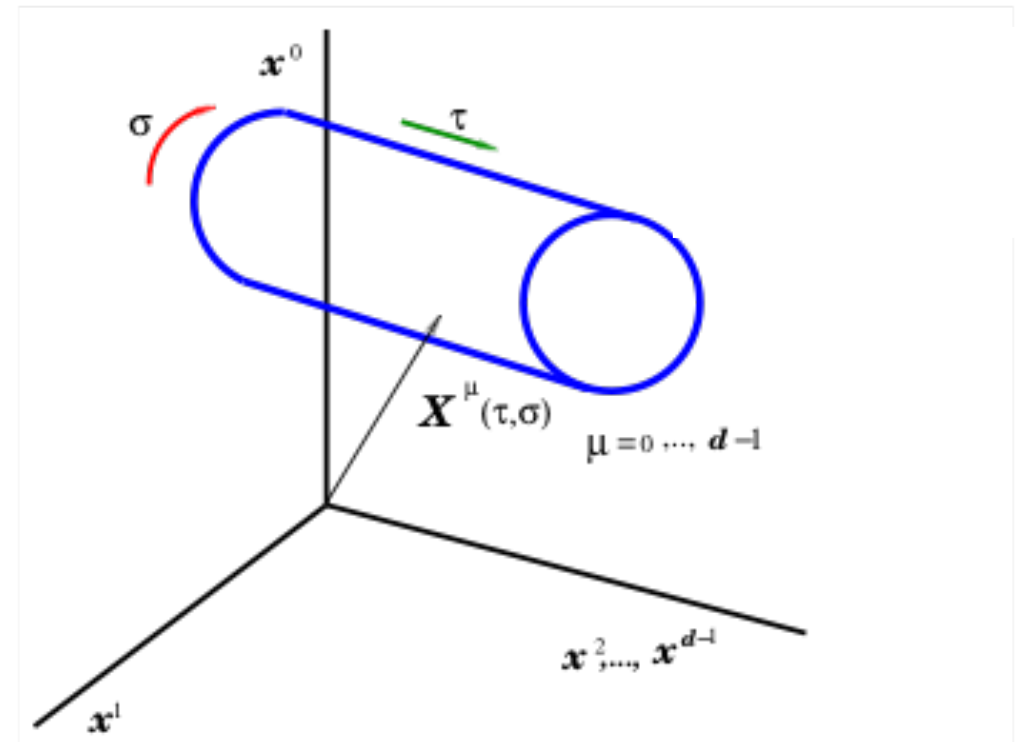
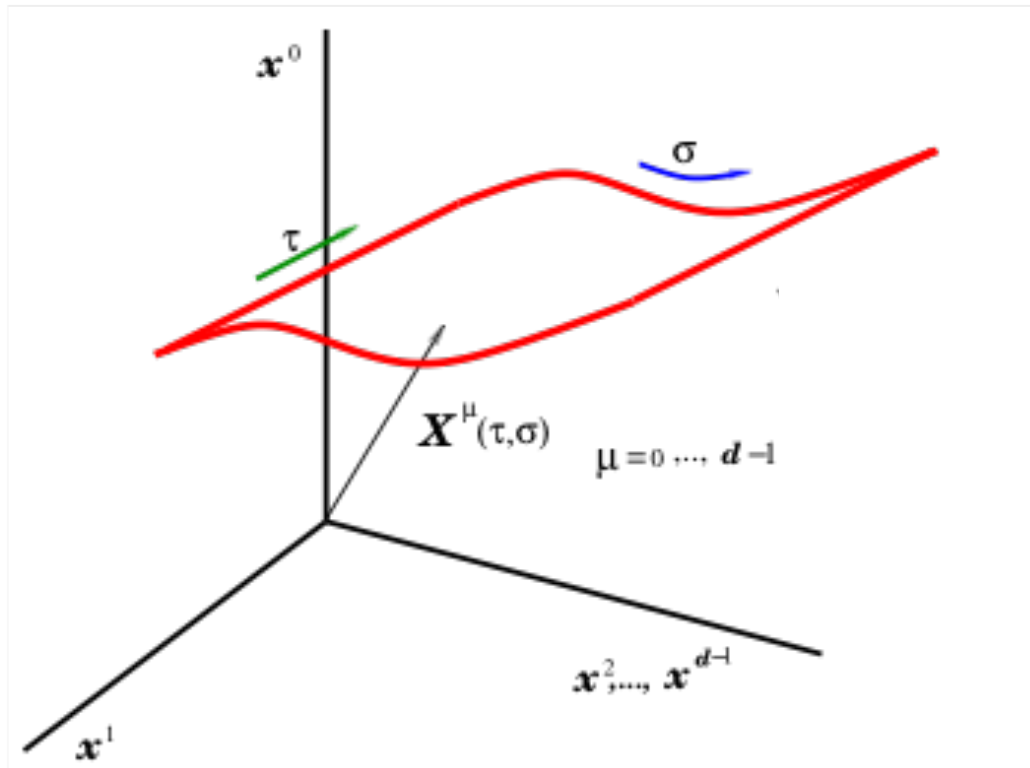
## Relativistic particle :



## Action for a relativistic particle

$$S \propto [\text{length}]$$

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} ; \quad \dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$$



Nambu-Goto 
$$S_{NG} = -\frac{T}{2} \int d\tau \int d\sigma \sqrt{-g}$$

$$g_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\mu, \nu = 0, \dots, d-1 \quad ; \quad a, b = 0, 1 \text{ ó } \tau, \sigma \quad ; \quad \sigma \in [0, 2\pi]$$

$$T = \frac{1}{2\pi\alpha'}$$

Polyakov 
$$S_p = -\frac{T}{2} \int d\tau \int d\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = \iint d\tau d\sigma \mathcal{L}_p$$



**Polyakov Lagrangian :** 
$$\mathcal{L}_p = -\frac{1}{4\pi\alpha'} \left[ \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \right]$$

## Symmetries of the theory :

- **Poincaré Invariance**
- **Invariance under reparametrisation of the WS**
- **Weyl invariance** 
$$h_{ab}(\tau, \sigma) \longrightarrow \Omega^2(\tau, \sigma) h_{ab}(\tau, \sigma) \quad ; \quad a, b = 0, 1$$

**Euler–Lagrange Eqs.** 
$$\partial_a \left( \frac{\partial \mathcal{L}_p}{\partial (\partial_a \phi)} \right) - \frac{\partial \mathcal{L}_p}{\partial \phi} = 0$$

$$\partial_a (\sqrt{-h} h^{ab} \partial_b X_\mu) = 0$$

$$\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = 0$$

**Lets fix the gauge!!!**

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Conformal gauge:

$$h_{ab} = \rho^2(\tau, \sigma) \eta_{ab} \quad ; \quad a, b = 0, 1 \quad ; \quad \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

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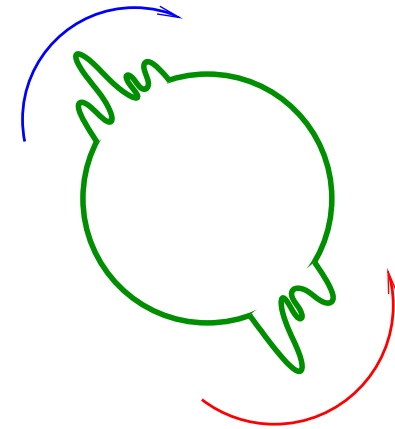
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Wave Equation!!!!!!



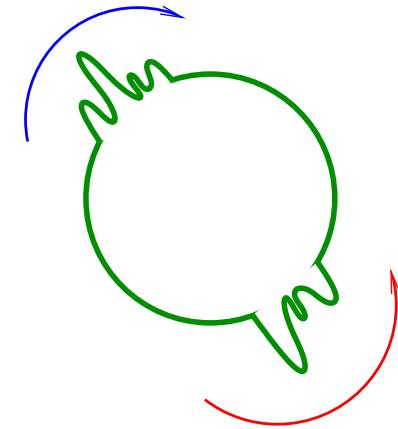
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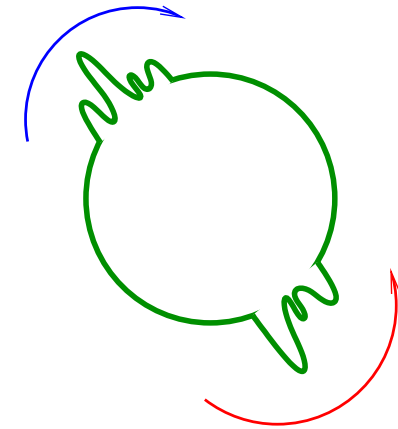
$$X_{\textcolor{red}{R}}^\mu(\tau - \sigma) = \frac{1}{2}X_0^\mu + \frac{\alpha'}{2}p^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m^\mu}{m} e^{-im(\tau - \sigma)}, \quad \alpha_{-n}^\mu = \alpha_n^{\mu*}$$

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$$X_0^\mu : \text{constant} \quad ; \quad p^\mu = \int_0^{2\pi} d\sigma \mathcal{P}^{\tau\mu} \quad \text{space-time momentum}$$

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## Hamiltonian

$$H = \int_0^{2\pi} d\sigma \left( \mathcal{P}^{\tau\mu} \dot{X}^\mu - \mathcal{L}_P \right) \quad \text{canonical momentum : } \mathcal{P}^{\tau\mu} = \frac{\partial \mathcal{L}_P}{\partial \dot{X}_\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu$$

# *The Second Review*

*The Second Review* (or not)



# *Fractional Calculus*

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Applications of derivatives & integrals of arbitrary order

[Leibniz, Euler, Caputo, Maniardi,...]

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**Riemann–Liouville Approach :**

**Let  $f(x)$  a continuous function on the real line.**

**Let  $G$  be a linear operator acting on the space of cont. funct. on the real line**



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## Applications of derivatives & integrals of arbitrary order

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**Natural question:**

Is there an  $G$  such that  $G^2 f(x) = \frac{d}{dx} f(x)$  or better  $G^a f(x) = \frac{d}{dx} f(x)$  ?

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or better

$$\frac{d^\pi}{dx^\pi} \left[ e^{4 \ln x^2} \right] = \frac{5040}{\Gamma(8-\pi)} x^{8-\pi}$$



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In all these definition for a generic function the order of the derivative has to be  $0 \leq a \leq 1$

# *Fractional Bosonic Strings*



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time-fractional Polyakov action

$$S_\alpha \equiv -\frac{1}{4\pi\alpha'\Gamma(\alpha)} \int_0^{2\pi} d\sigma \int_{-\infty}^t (t-\tau)^{\alpha-1} d\tau \left[ \sqrt{-h} h^{ab}(\tau, \sigma) \partial_a X^\mu(\tau, \sigma) \partial_b X^\nu(\tau, \sigma) \eta_{\mu\nu} \right]$$

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**Lagrangian :**  $\mathcal{L}_\alpha = -\frac{1}{4\pi\alpha' \Gamma(\alpha)} (t-\tau)^{\alpha-1} \left[ \sqrt{-h} h^{ab}(\tau, \sigma) \partial_a X^\mu(\tau, \sigma) \partial_b X^\nu(\tau, \sigma) \eta_{\mu\nu} \right]$

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Symmetries of the theory :

- Poincaré Invariance
- Invariance under reparametrisation along the string
- Weyl invariance  $h_{ab}(\tau, \sigma) \longrightarrow \Omega^2(\tau, \sigma) h_{ab}(\tau, \sigma) \quad ; \quad a, b = 0, 1$

Euler–Lagrange Eqs.

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Fractional conformal gauge:

$$h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & [f^2(\tau, \sigma)]^{\alpha-1} \end{pmatrix} \quad \text{with } f(\tau, \sigma) = 1$$

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not a free wave equation!!!!!!

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$$||\dot{X} \pm X'||^2 = 0$$

Virasoro  
constraints!!!



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$$X^\mu(z, \sigma) = X_0^\mu - \frac{\alpha' z^{2\nu} \Gamma(2 - 2\nu)}{2\nu} p^\mu - i \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \left( \frac{\alpha_m^\mu}{m} e^{i m \sigma} + \frac{\tilde{\alpha}_m^\mu}{m} e^{-i m \sigma} \right) \mathcal{E}_\nu^m(z)$$

$$\mathcal{E}_\nu^m(z) := \sqrt{\frac{\pi |m|}{2}} z^\nu \begin{cases} H_{-\nu}^{(1)}(|m|z), & m \in \mathbb{Z}^-, \\ H_{-\nu}^{(2)}(|m|z), & m \in \mathbb{N}, \end{cases}$$

with

$$z = t - \tau$$

$$\alpha = 2 - 2\nu$$

$$\mathcal{E}_\nu^m(z) := \sqrt{\frac{\pi |m|}{2}} z^\nu \left[ J_{-\nu}(|m|z) - i \operatorname{Sgn}(m) Y_{-\nu}(|m|z) \right]$$

# *Summary & Outlooks*

- We review the classical bosonic string theory
- We review the Riemann–Liouville approach
- We merge the idea of fractional calculus in the context of bosonic strings

creating fractional bosonic strings

## *To do:*

- Properly define the Virasoro operators
- Study the conformal symmetry of the theory

Gracias.