Feynman periods - on graphs, integrals, polytopes and tropical physics

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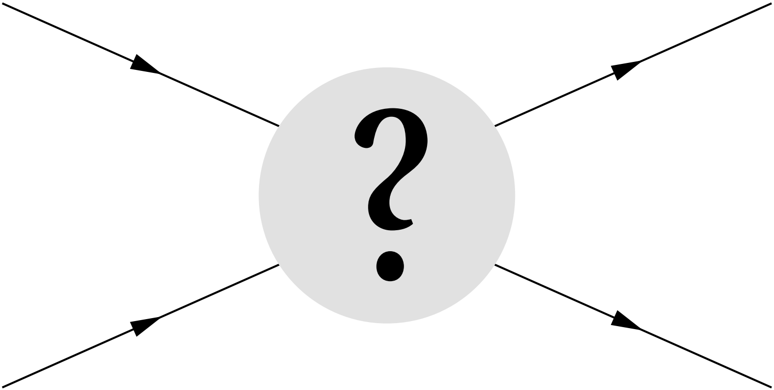
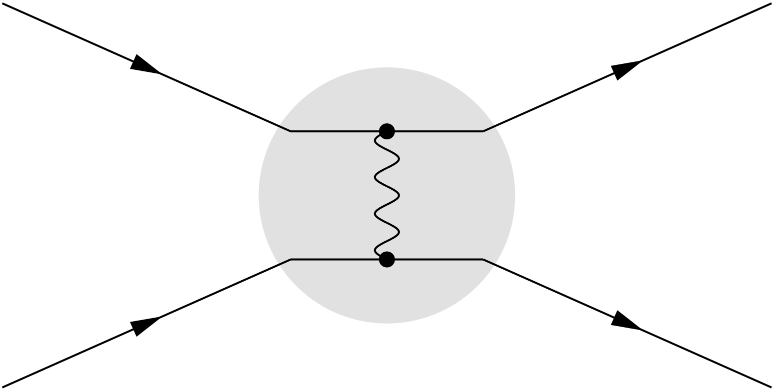
MPP Munich

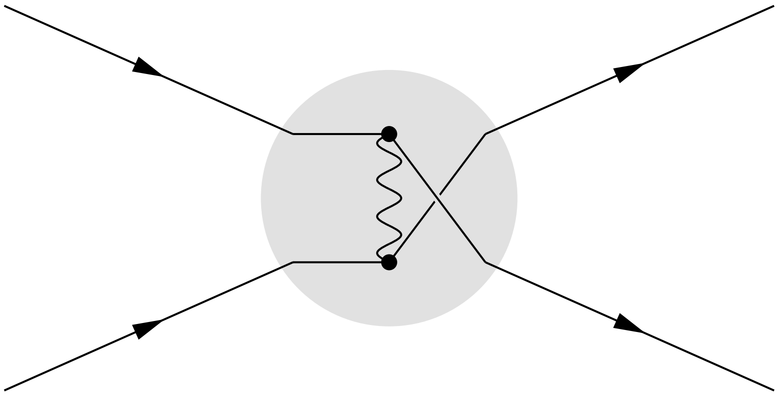
Perturbative Quantum Field Theory



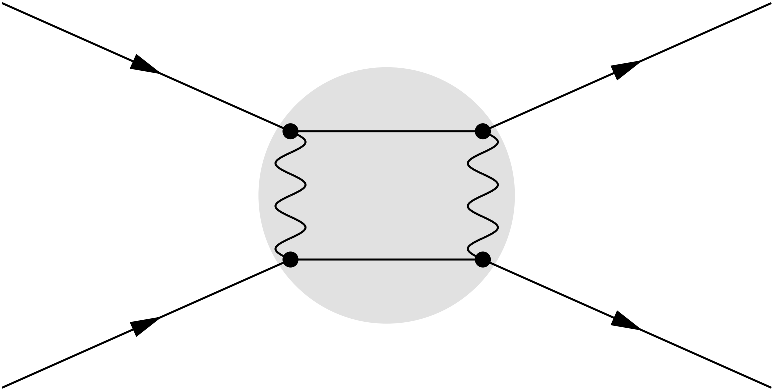
Perturbative Quantum Field Theory



Perturbative Quantum Field Theory



Perturbative Quantum Field Theory



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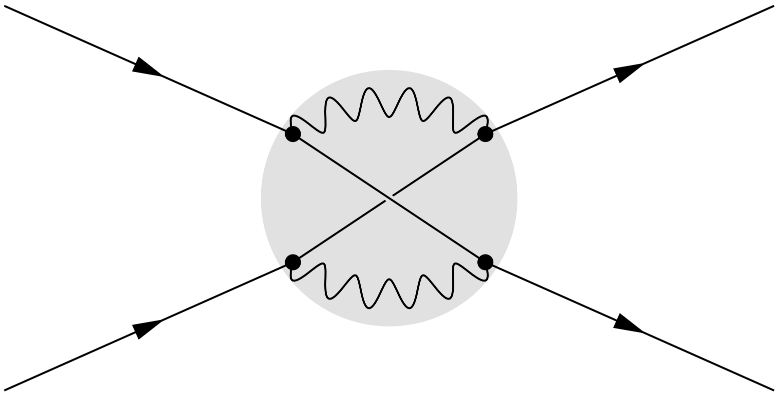

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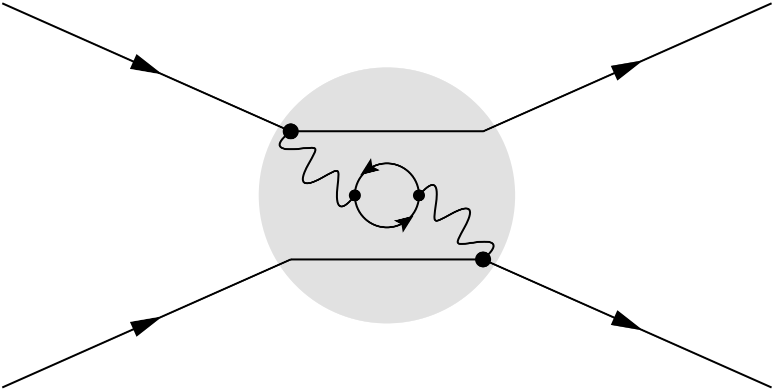

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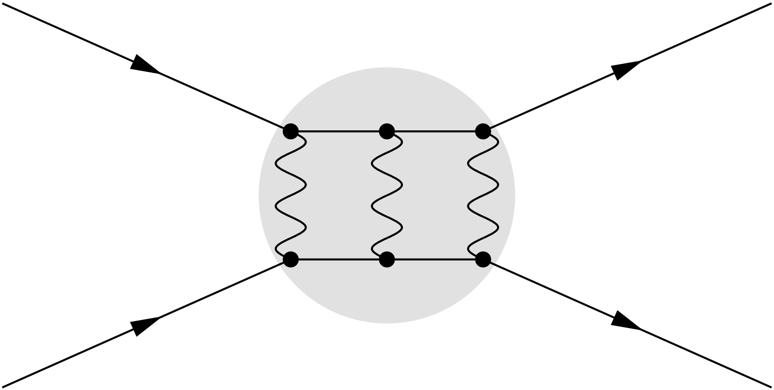

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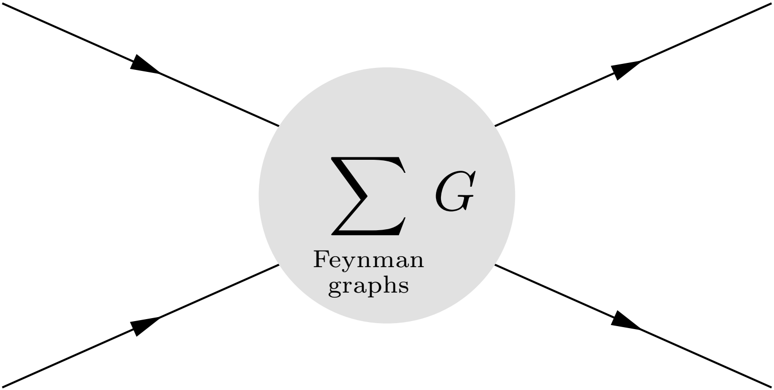

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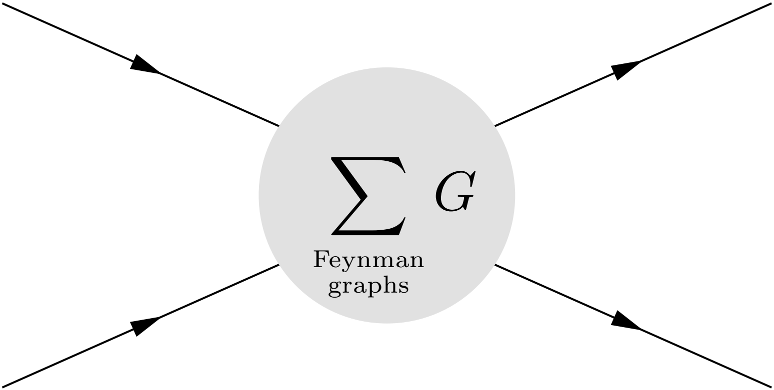
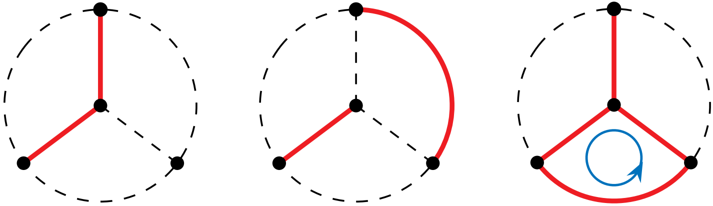


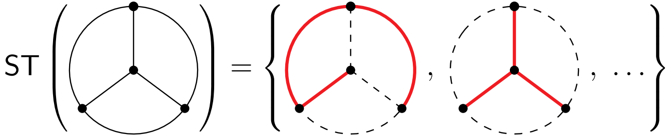
- Feynman graph $G \mapsto$ Feynman integral $\Phi(G, \{m_i^2, \vec{p}_i \cdot \vec{p}_j\})$
- compute more graphs $\sum_G \Phi(G) \Rightarrow$ higher precision



not spanning   not connected   has a loop

Definition

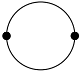
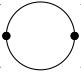
A spanning tree $T \subset G$ is a spanning, simply connected subgraph.



Definition

The graph polynomial $\mathcal{U}$ and Feynman period of G are

$$\mathcal{U} = \sum_{T \in \mathrm{ST}(G)} \prod_{e \notin T} x_e \qquad \text{and} \qquad \mathcal{P}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{\mathcal{U}^2|_{x_1=1}}$$

$G =$  $\quad \Rightarrow \quad \mathcal{U} = x_1 + x_2$ and $\mathcal{P}\Big($  $\Big) = \displaystyle\int_0^\infty \frac{\mathrm{d}x_2}{(1+x_2)^2} = 1$

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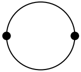
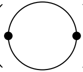
$G =$  $\Rightarrow \quad \mathcal{U} = x_1 + x_2$ and $\mathcal{P}\Big($  $\Big) = \int_0^\infty \frac{\mathrm{d}x_2}{(1+x_2)^2} = 1$

Assumptions:

1. logarithmic divergence: $\omega(G) := |E(G)| - 2 \cdot \ell(G) \overset{!}{=} 0$

$$\ell(G) = h_1(G) = |E(G)| - |V(G)| + 1 \quad \text{(loop number)}$$

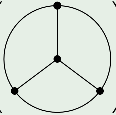
2. no subdivergences: $\quad \omega(\gamma) > 0 \quad \text{for all} \quad \emptyset \neq \gamma \subsetneq G$

All such periods contribute to the β-function of the field theory.

$\Rightarrow$ *renormalization constants, running coupling, critical exponents*

- These are periods in the sense of Kontsevich and Zagier
  $\Rightarrow$ interesting transcendental numbers, motivic Galois theory

Example

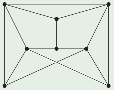
$\mathcal{P}\Bigg($  $\Bigg)$

$$= \int_{\mathbb{R}_+^5} \frac{\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}x_5\mathrm{d}x_6}{(x_1x_2x_3 + 15 \text{ more terms})^2|_{x_1=1}} = 6\zeta(3) = 6\sum_{n=1}^{\infty}\frac{1}{n^3}$$

- Sometimes expressible as multiple zeta values

$$\zeta(s_1,\ldots,s_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{n_1^{s_1}\cdots n_d^{s_d}}$$

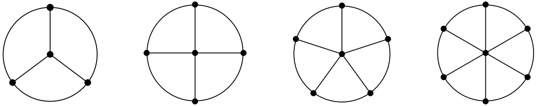
Example (Broadhurst & Schnetz)

$\mathcal{P}\Bigg($  $\Bigg)$

$$= \frac{92943}{160}\zeta(11) + 896\zeta(3)\left(\frac{27}{80}\zeta(3,5) + \frac{45}{64}\zeta(3)\zeta(5) - \frac{261}{320}\zeta(8)\right) + \frac{3381}{20}\left(\zeta(3,5,3) - \zeta(3,5)\zeta(3)\right) - \frac{1155}{4}\zeta(3)^2\zeta(5)$$

- These integrals are very hard to compute (even numerically).

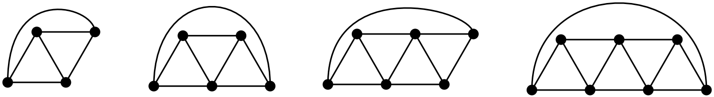
Only two infinite families of periods are known: wheels and zigzags



W_3 W_4 W_5 W_6

Theorem (Broadhurst 1985)

$$\mathcal{P}(\mathrm{W}_n) = \binom{2n-2}{n-1}\zeta(2n-3)$$



ZZ_3 ZZ_4 ZZ_5 ZZ_6

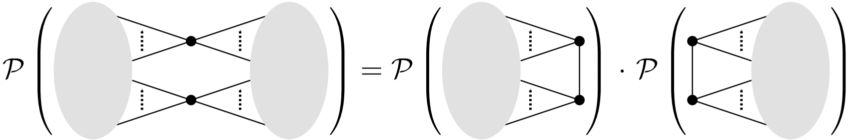
Theorem (Brown & Schnetz 2012)

$$\mathcal{P}(\mathrm{ZZ}_n) = 4\frac{(2n-2)!}{n!(n-1)!}\left(1-\frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3)$$

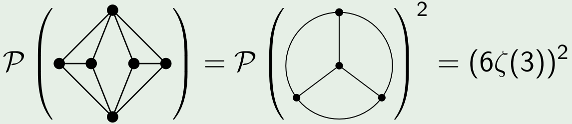
$>$ 1000 more periods are known [Broadhurst, Schnetz, Panzer]

When is $\mathcal{P}(G_1) = \mathcal{P}(G_2)$?

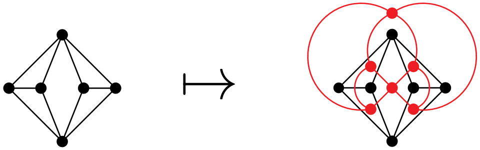
1. Product:



Example



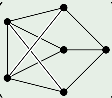
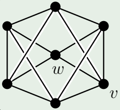
2. Planar duality: $\mathcal{P}(G) = \mathcal{P}(G^{\text{dual}})$

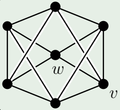


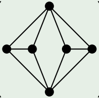
3. Completion: If G is 4-regular and v, w are vertices G, then

$$\mathcal{P}(G \setminus v) = \mathcal{P}(G \setminus w)$$

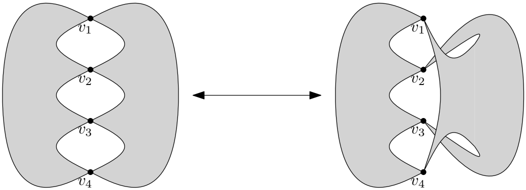
Example

$\mathcal{P}\Big($  $\Big) = \mathcal{P}\Big($ 

$\setminus v\Big) = \mathcal{P}\Big($ 

$\setminus w\Big) = \mathcal{P}\Big($  $\Big)$

4. Twist:



[drawing by Crump]

Goal:

Construct simpler graph invariants with those symmetries.

$$c_2(G)(p) = \frac{1}{p^2}\left|\left\{\vec{x} \in (\mathbb{Z}/p\mathbb{Z})^N\colon\ \mathcal{U}(\vec{x}) = 0\right\}\right|$$

$P_{7,11}$

p	2	3	5	7	11	13	17	19	23
$c_2(p)$	1	0	1	-1	1	-1	1	-1	1
Perm(p)		0	1	1	1	11	5	0	22

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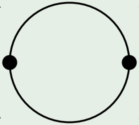
	product	duality	completion	twist	
c_2 [Schnetz]	yes	yes [Doryn]	for $p = 2$ [Yeats, Doryn]	open	few values, sees number theory
permanent [Crump]	yes	yes	yes	yes	almost faithful
$\mathcal{H}$epp	yes	yes	yes	yes	faithful (conj.), sees magnitude

Hepp bound

$$\mathcal{H}(G) := \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{\mathcal{U}^2_{\max}|_{x_1=1}} \quad \text{where} \quad \mathcal{U}_{\max} := \max_{T \in \mathrm{ST}} \prod_{e \notin T} x_e$$

Example

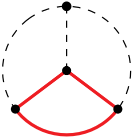
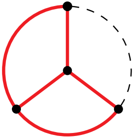
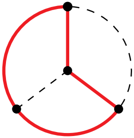
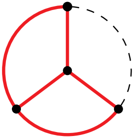
$\mathcal{H}\Bigg($  $\Bigg)$

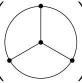
$$= \int_0^\infty \frac{\mathrm{d}x_2}{(\max\{1, x_2\})^2} = \int_0^1 \mathrm{d}x_2 + \int_1^\infty \frac{\mathrm{d}x_2}{x_2^2} = 2$$

- $\mathcal{H}(G) > \mathcal{P}(G) > \mathcal{H}(G) / |\mathrm{ST}(G)|^2$
- fulfils the four symmetries
- $\mathcal{H}(G) \in \mathbb{Q}_{>0}$
- can be computed very efficiently

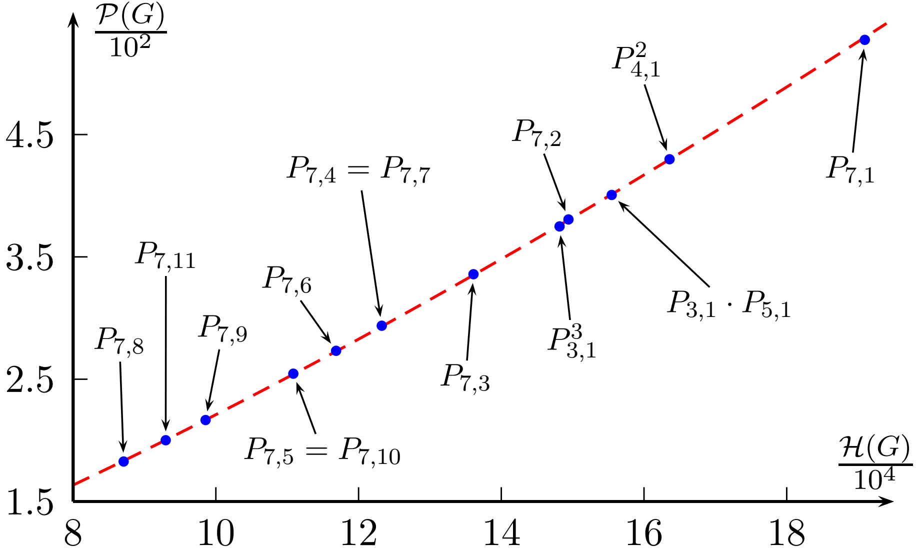
Theorem

$$\mathcal{H}(G) = \sum_{\substack{\gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_\ell = G \\ \text{each } \gamma_i \text{ is 1PI}}} \frac{|\gamma_1| \cdot |\gamma_2 \setminus \gamma_1| \cdots |G \setminus \gamma_{\ell-1}|}{\omega(\gamma_1) \cdots \omega(\gamma_{\ell-1})}$$

γ_1	$\subset$	γ_2	summand	#	$\sum$
	$\subset$		$\frac{3\cdot2\cdot1}{1\cdot1} = 6$	12	72
	$\subset$		$\frac{4\cdot1\cdot1}{2\cdot1} = 2$	6	12

$\Rightarrow \mathcal{H}\Big($  $\Big) = 84$

Hepp-Period correlation



Conjecture

$$\mathcal{H}(G_1) = \mathcal{H}(G_2) \qquad \Leftrightarrow \qquad \mathcal{P}(G_1) = \mathcal{P}(G_2)$$

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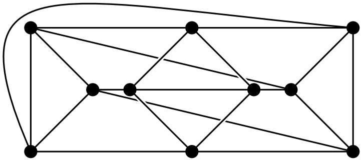
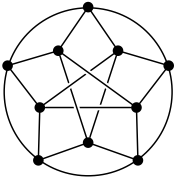
For example, we find a pair of unknown 8 loop periods with:

- $\mathcal{H}(P_{8,30}) = \frac{1724488}{3} = \mathcal{H}(P_{8,36})$
- $\mathcal{P}(P_{8,30}) \approx 505.5 \approx \mathcal{P}(P_{8,36})$

$\mathcal{P}\Bigg($  $\Bigg) \quad = \quad \mathcal{P}\Bigg($  $\Bigg) \quad ?$

Spanning tree polytope and its polar (relevant for sector decomposition):

$$\vec{\mathrm{v}}_T = \vec{T} - \vec{T}^c \in \{1, -1\}^{E_G}$$

$$\mathcal{N}_G = \operatorname{conv}\{\vec{\mathrm{v}}_T\colon\ T \in \mathrm{ST}\} \subset \mathbb{R}^{E_G}$$

$$\mathcal{N}_G^\circ = \bigcap_{T \in \mathrm{ST}} \{\vec{a}\colon\ \vec{a} \cdot \vec{\mathrm{v}}_T \leq 1\}$$

The Hepp bound is the volume of the polar polytope

$$\mathcal{H}(G) = (E_G - 1)! \cdot \operatorname{Vol}\left(\mathcal{N}_G^\circ \cap \{a_1 = 0\}\right)$$

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$$\mathcal{H}(G) = (E_G - 1)! \cdot \mathrm{Vol}\,(\mathcal{N}_G^\circ \cap \{a_1 = 0\})$$

Facets of $\mathcal{N}_G$/vertices of $\mathcal{N}_G^\circ$ are indexed by subgraphs:

$$\{\gamma \subset G \colon\ \gamma \text{ and } G/\gamma \text{ are 2-vertex connected}\}$$

Factorisation of the facets:

$$\mathcal{N}_G \cap \{\vec{\gamma} \cdot \vec{a} = \omega_\gamma\} \cong \mathcal{N}_\gamma \times \mathcal{N}_{G/\gamma}$$

Roughly, $\mathcal{N}_G$ looks like a cube, and $\mathcal{N}_G^\circ$ is a cross-polytope: very “spikey” and all volume concentrated near the centre.

Multivariate version & canonical form

Now consider arbitrary indices:

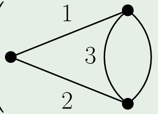
$$\mathcal{H}(G;\vec{a}) := \left(\prod_{e>1}\int_0^\infty x_e^{a_e-1}\mathrm{d}x_e\right)\frac{1}{\mathcal{U}_{\max}^{D/2}|_{x_1=1}}$$

The dimension is fixed by $\omega(G) = \sum_e a_e - (D/2)\cdot\ell(G) \overset{!}{=} 0$.

Example

The flag formula generalizes to this case, e.g.

$\mathcal{H}\Bigg($



$;\vec{a}\Bigg)$

$$= \frac{1}{a_1a_2a_3a_4}\times\left\{\frac{(a_1+a_2+a_3)a_4}{a_1+a_2+a_3-D/2} + \frac{(a_1+a_2+a_4)a_3}{a_1+a_2+a_4-D/2} + \frac{(a_3+a_4)(a_1+a_2)}{a_3+a_4-D/2}\right\}$$

Consider the Hepp bound $\mathcal{H}(G;\vec{a})$:

- it is a rational function in $\vec{a}$
- it has simple poles
- at hyperplanes $\omega(\gamma)=0$ for 1PI subgraphs γ

Factorization of residues

$$\operatorname*{Res}_{\omega(\gamma)=0} \mathcal{H}(G;\vec{a}) = \mathcal{H}(\gamma;\vec{a}_\gamma)\Big|_{\omega(\gamma)=0} \cdot \mathcal{H}(G/\gamma;\vec{a}_{G/\gamma})\Big|_{\omega(G/\gamma)=0}$$

Example: edge contraction

$$\operatorname*{Res}_{a_e=0} \mathcal{H}(G;\vec{a}) = \mathcal{H}(G/e;\vec{a}_{G/e})$$

- it is the volume of a polytope:

$$\mathcal{H}(G;\vec{a}) = (E-1)! \cdot \mathrm{Vol}\left(\left(\mathcal{N}_G + (\vec{a}-\vec{1})\right)^{\circ} \cap \{a_1=0\}\right)$$

$\Rightarrow$ canonical form

The period of a graph can be written as

$$\mathcal{P}(G) = (N-2)! \int_{y_N=0} \frac{\Omega}{[\omega(\vec{y})]^{N-1}} F(\vec{y})$$

where $F(\vec{y})$ is the projectively invariant function

$$F(\vec{y}) := \frac{1}{(N-2)!} \int_0^\infty \lambda^{N-2} \mathrm{d}\lambda \left\{ \sum_{T \in \mathrm{ST}} \exp\left(\frac{\lambda}{2} \frac{\vec{y} \cdot \vec{\mathrm{v}}_T}{\vec{y} \cdot \vec{\mathrm{v}}_{T_{\max}}} \right) \right\}^{-D/2}$$

The Hepp bound is precisely obtained by the approximation $F \leq 1$.

Lemma

Within each tree sector (constant $T_{\max}$), the function F is log-concave.

$\Rightarrow$ efficient sampling of log-concave distributions

Summary

- There is a rational version of Feynman periods.
- It captures identities and gives numeric estimates.
- Volume of a polytope with factorizing residues.
- Generalizes to matroids.

Outlook

- add kinematics
- dimensional regularization
- renormalization
- tropical field theory
- asymptotics
- numerics for Feynman integrals