# Introduction to loop calculations and recent developments (and some future directions)

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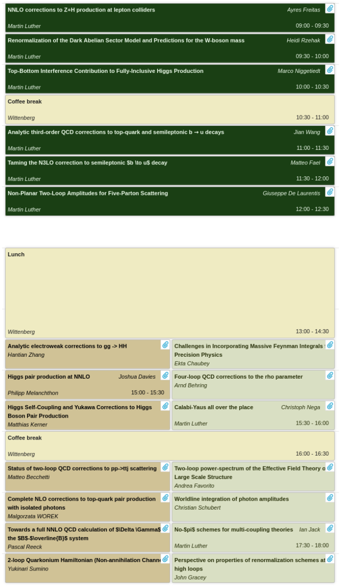
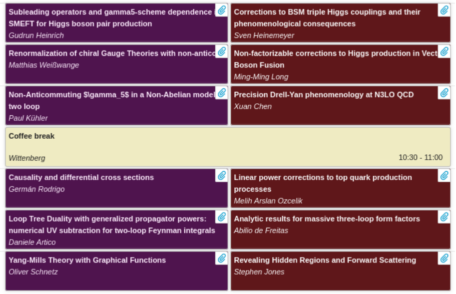
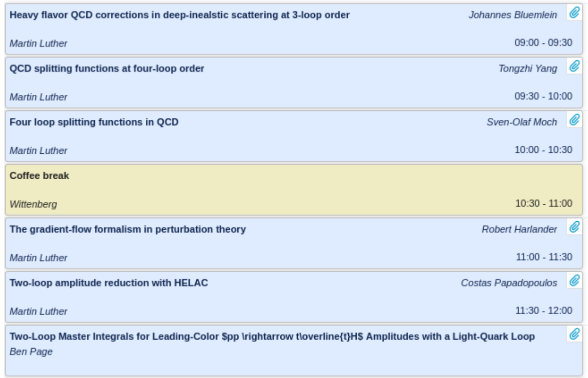
Ringberg 2024: 2nd Workshop on Tools for High Precision LHC Simulations
Schloss Ringberg

# ~~Catholic~~ *protestant* ecumenical council on loop business (2024 edition)






# Amplitudes and precision phenomenology

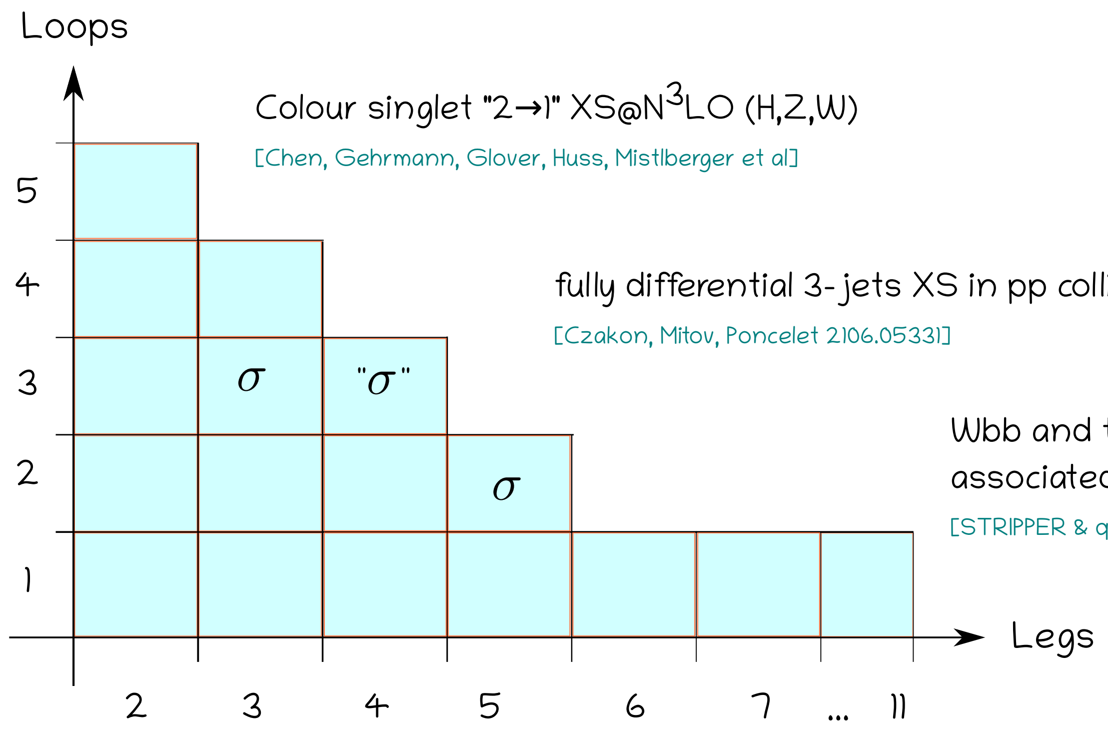
$$\sigma = \sigma_{\rm LO} + \alpha_s \sigma_{\rm NLO} + \alpha_s^2 \sigma_{\rm NNLO} + \alpha_s^3 \sigma_{\rm N^3LO} + \dots$$

10-20%   1-10%

@Fixed order:

- Multi-loop, multi-leg Amplitudes
- IR Subtraction schemes

Not only, loops. Need to include more legs for relevant LHC pheno




Colour singlet "2→1" XS@N$^3$LO (H,Z,W)

[Chen, Gehrmann, Glover, Huss, Mistlberger et al]

fully differential 3-jets XS in pp collisions

[Czakon, Mitov, Poncelet 2106.05331]

Wbb and ttX
associated production (X=H/W)

[STRIPPER & $q_T$ slicing]

From jets cross section point of view:

- In local subtraction schemes, from dijet and above, subtraction is conceptually understood (technically is a different story)
- massive radiators under control

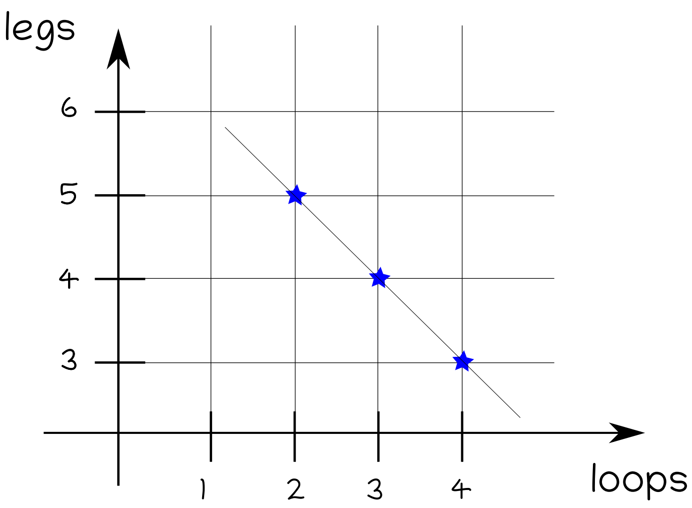
extreme: any XS with NNLO QCD in principle possilbe?

In practice not really true:
automation, efficiency, generalisation/universality.
aka: no CS/FKS at NNLO yet

# Multiloop scattering amplitudes

Complexity rapidly increases with #loops and #scales:
availability of multiscale-multiloop amplitudes are now arguably the bottleneck of NNLO predictions

Current frontier (loops > 1): loops + legs = 7




★ Done

Mostly manageable with analytical methods

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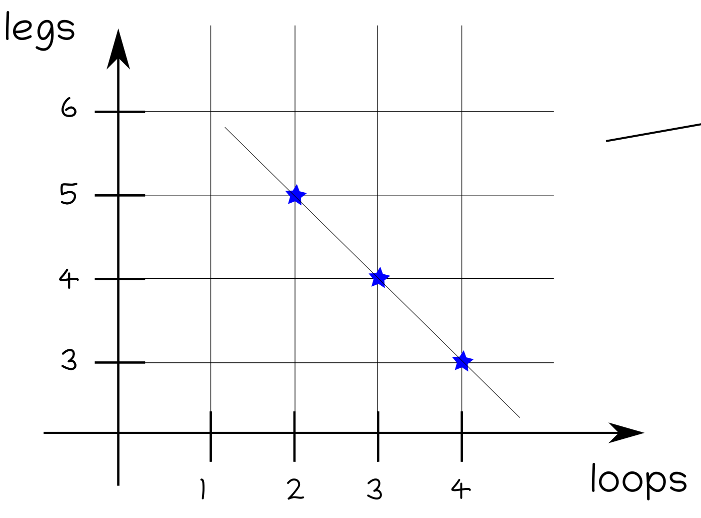
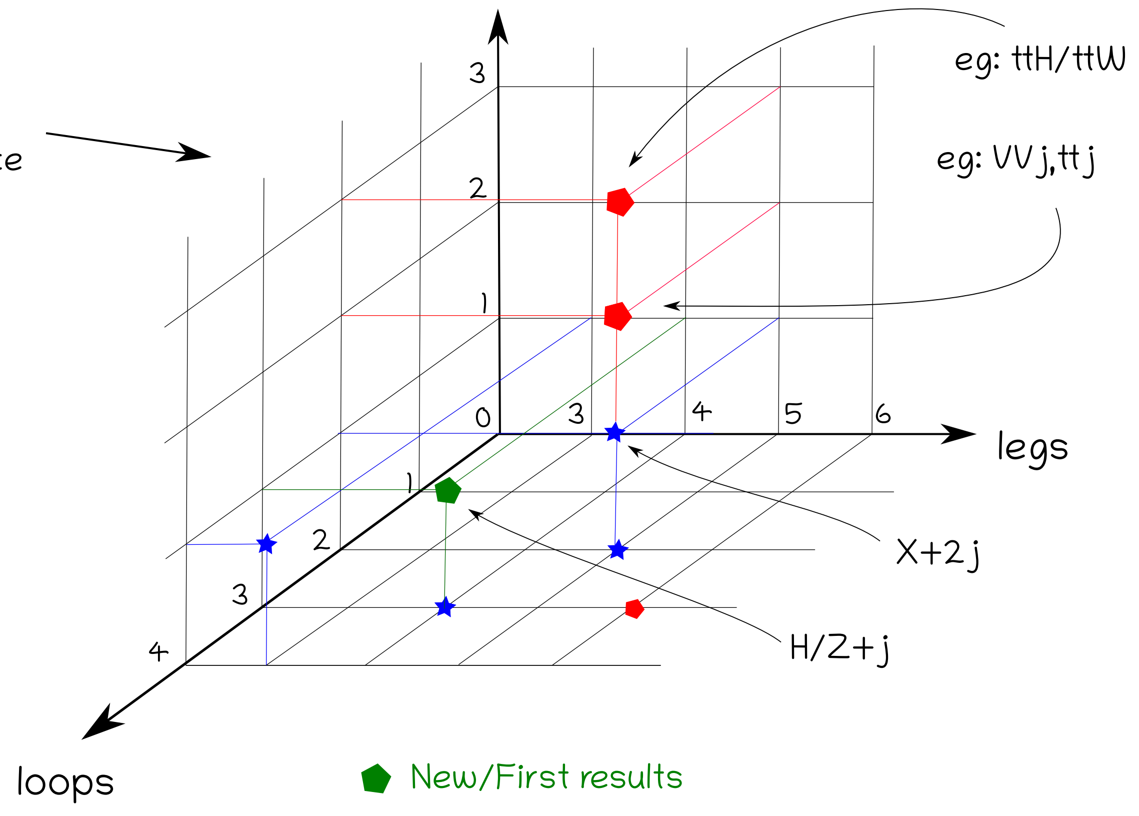



Third direction
in complexity space




★ Done

Mostly manageable with analytical methods

New/First results

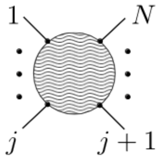
Work in progress (mostly planar integrals)

# One-loop amplitudes

Nowadays almost nobody talks about one-loop anymore (maybe fair enough)

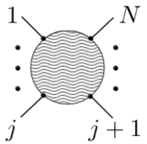
We certainly do not get much excited about one-loop...it is fully automated anyway!

Key components so far of LHC pheno truly indispensable

$$\mathcal{A}_N =$$ 

$$= \sum_{\Omega_0} \int \mathrm{d}^D q \frac{\mathcal{N}^{(\Omega)}(q)}{D_0^{(\Omega)} \dots D_{n-1}^{(\Omega)}}$$

## Unitarity-based/On-shell methods



$$= \sum_i d_i \mathcal{I}_{4,i} + \sum_i c_i \mathcal{I}_{3,i} + \sum_i b_i \mathcal{I}_{2,i} + \sum_i a_i \mathcal{I}_{1,i} + \mathcal{R}$$

Started as an analytic program [Bern, Dixon, Kosower]

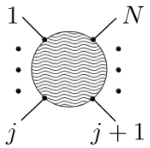
boost from OPP [Ossola, Papadopolous, Pittau]

Automation in computer codes

- Rocket [Giele, Zanderighi 0805.2152]
- Black Hat [Berger, Bern, Dixon et al 0803.4180]
- NJet [Badger, Biedermann, Uwer, Yundin 1209.0100]

shows that at large multiplicity N, algorithm scales as $N^9$

## Off-shell methods/Tensor-reduction



$$= \sum_{\Omega} \sum_{r=0}^{n} \mathcal{N}^{(\Omega)}_{\mu_1 \dots \mu_r} \int \mathrm{d}^D q \frac{q^{\mu_1} \cdots q^{\mu_r}}{D_0^{(\Omega)} \dots D_{N-1}^{(\Omega)}} + R_2$$

estimatation: exponential scaling $\sim 2^N N^4$

- HELAC-NLO [Czakon, Papadopoulos, Worek, 0905.0883; Bevilacqua et al 1110.1499]
- Recola [Actis et al 1211.6316; Denner, Lang, Uccirati 1711.07388]
- Madloop [Hirschi, Frederix, Frixione et al 1103.0621]
- OpenLoops [Cascioli, Maierhofer, Pozzorini 1111.5206; FB, Lang et al 1907.13071]

routinely adopted these days

# Pillars of multiloop calculations

Ability to express
the loop integrand

Reduction
to MIs (IBPs)

Formulated in very simple terms
but very hard to solve (often bottleneck)

Laporta Algorithm + public codes
e.g. Reduze, Kira, Lite-Red/Fire
(successful for 2→2 @2-loop)

Finite-Fields reconstruction techniques
[Manteuffel, Schabinger 1406.4513, Peraro 1905.08019]

+ insight/ideas ~ amplitudes-IBPs interplay
and algebraic-geometry inspired
[Gluza, Kajda, Kosower 1009.0472, Ita 1510.05626 ]

broken the multileg frontier +
2→2 @3-loop and 2→1 @4-loop

$\mathcal{A}$

Calculation of integrals
via differential equations

ε-factorised canonical form [Henn 1304.1806]

not necessarily a must, but it massively helps
(practically and often conceptually)

algorithmic approaches + 1-scale problems

general (enough) for multiscale ones

if "right basis" of integrals, dlog form,
integration "trivial"

basis of functions have well
understood properties

These are not the only way to
evaluate multiloop amplitudes:
we'll see alternative ones later

# IBPs and algebraic complexity

Relate any integral in a given problem as a linear combination of a limited set of integrals (MIs) → via IBP identities

problems of algebraic nature

However: huge number of these identities (complexity grows severely with loops and legs/scales)

Moreover: for multiscale problems the rational functions appearing in IBPs can be prohibitively complicated (ratios of huge polynomials)

Questions about:

- Very generally: efficient generation and solution of linear system of identities
- How to minimise the number of equations still having the same info? Amplitudes-guided principles? Algebraic geometry?
- possible to sidestep manipulating huge symbolic expressions?

[all answers to these questions + details + references in Andrea's and Vasily's talks]

# IBPs and algebraic complexity

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Even then: for multiscale problems IBPs can stil be horrific (example of massless 5pt amplitudes)

Exploit multivariate/polynomial nature of the problem: partial fraction decomposition

Drastic reduction of algebraic complexity. IBPs tractable in a fully symbolic fashion

Proposals/approaches for MVPFD:
[Pak 1111.0868], [Abreu et al, 1904.00945],
[Boehm, Wittmann, Wu, Xu, Zhang, 2008.13194]
[Heller, von Manteuffel, 2101.08283]

Examples:

| | | common den. | | MVPFD |
|---|---|---|---|---|
| PB: | INT[TA,8,255,8,5,{1,1,1,1,1,1,1,1,-5,0,0}] | 162 mb | → | 3.9 mb |
| HB: | INT[TB,8,255,8,5,{1,1,1,1,1,1,1,1,-4,0,-1}] | 513 mb | → | 9.9 mb |
| DP: | INT[TB,8,510,8,5,{0,1,1,1,1,1,1,1,1,-5,0}] | 2.9 gb | → | 24 mb |

The largest simplifications for most complicated integrals: factor ~ 100 in reduction size!

New proposals to directly reconstruct in partial-fraction decomposed form
[Chawdhry 2312.03672]

# Multiloop integrals via differential equations

Take derivatives wrt invariants and solve differential equations, ideal scenario:

Differential equations in canonical form, $I_j(s)$ a vector of UT integrals

$$\mathrm{d}I_i(\vec{s}) = \epsilon\, \mathrm{d}A_{ij}(\vec{s}) I_j(\vec{s}) \qquad \mathrm{d}A_{ij}(\vec{s}) = \sum_{n=1} a^n_{ij} \mathrm{d}\log(W_n)$$

$W_n$

Letters of the alphabet

algebraic functions of the invariants:
rat. functions + square roots

Provided boundary conditions are known

Often fixed required analyticity conditions or compute/evaluate integrals in a specific limit point

Differential equations can be solved systematically: HPLs, GPLs, Iterated integrals

ideal for symbolic manipulations and numerical implementation

fast numerical evaluation, also in "arbitrarily" high numerical accuracy

$$I(\vec{s}) = \mathrm{P}\exp\left[\epsilon \int^{\vec{s}} A(\vec{x})\mathrm{d}\vec{x}\right] I(\vec{s}_0)$$

$$I^{(\omega)}(\vec{s}) = \int_\gamma \mathrm{d}\log W_{i_1} \ldots \mathrm{d}\log W_{i_n}$$

$\omega$ integrations

Typical issues:

- finding a well-suited canonical basis can be very hard
- write a solution general enough in any kinematic region
- fixing boundary conditions tricky at times/technical challenges

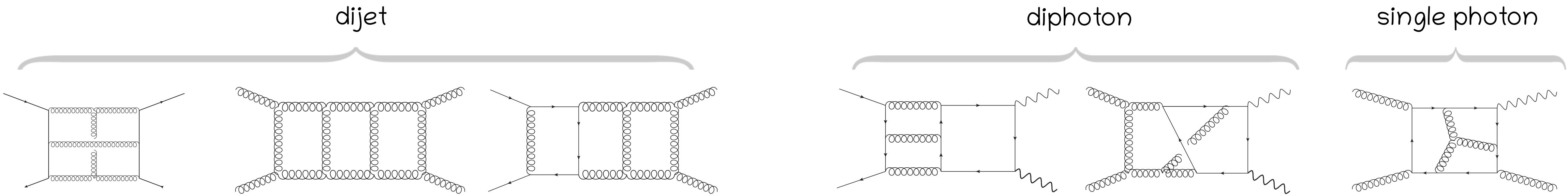
Scenario not always like this (we'll see later)

# (selection of) Recent results from multiloop-multileg in QCD

# Scattering amplitudes: 2→2 @ 3-loops in massless QCD

All 3-loop 2→2 amplitudes with external massless partons are now available

Master Integrals [Henn, Mistlberger, Wasser '20] + calculation of the amplitudes [Bargiela, Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi '21,'22]



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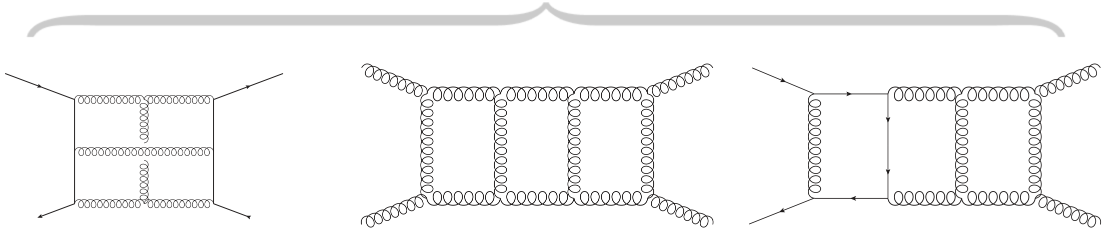
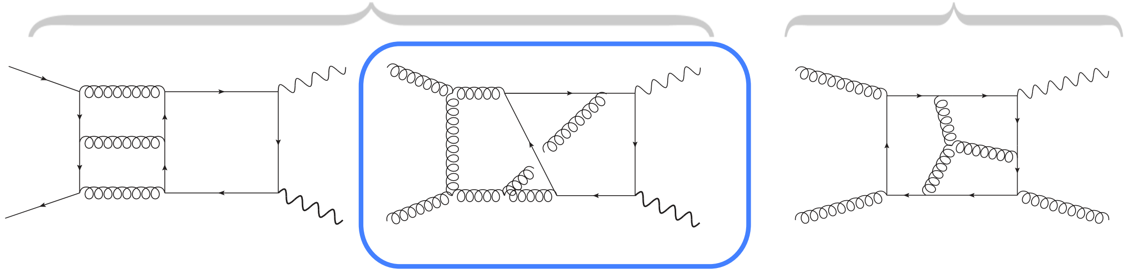
dijet



building blocks of Regge factorization: 3-loop Regge trajectory

quadruple colour insertion for the first time (in pert. expansion)

diphoton | single photon



$N^3LO$ pheno application in sight

Basis of functions:

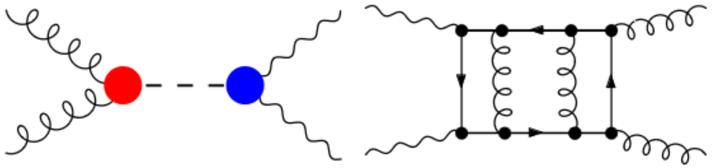
- canonical form "algorithmically" + dLog integrand [Henn et al 2002.09492]
- HPLs up to transcendental weight 6

Reduction to MIs:

- FinRed [von Manteuffel]: syzygy + finite-fields reconstruction

Application of a 3-loop QCD amplitude

Signal-background interference in Higgs-mediated diphoton production [Bargiela, FB, Caola, Devoto, von Manteuffel, Tancredi '23]



Destructive interference effects ~ 1.8% reduction of signal XS

# H/V+jet @3-loops

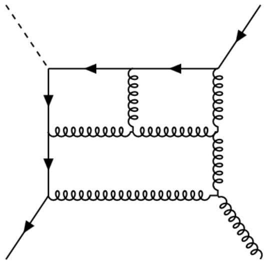
Arguably one of the most important class of processes:

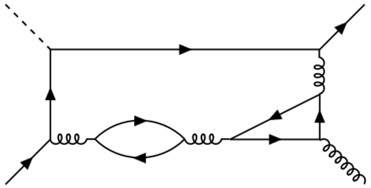
a resonant colour singlet recoiling against a hard jet

three-loop QCD corrections V+jet and H+jet

First results: planar (LC) contribution to Z+jet amplitude

[Gehrmann, Jakubčik, Mella, Syrrakos, Tancredi 2307.15405]





First results: planar (LC) contribution to Z+jet amplitude

Canonical bases not a bottleneck (still non-trivial)

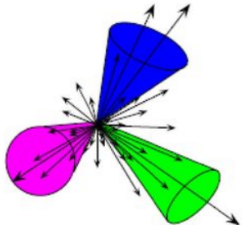
solutions in terms of 2d-HPLs [Gehrmann, Remiddi hep-ph/9912329]

IBP reduction manageable with public code (Kira)

Higgs?

Z (H) decaying to three jets

crucial for QCD studies at future lepton colliders



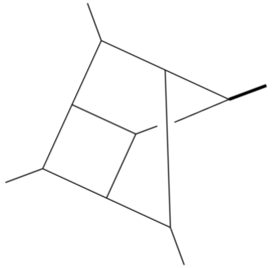
Going beyond planar sectors (Higgs at LC and beyond)

- find a candidate canonical basis is very hard
- alphabet richer and more complicated

  great progress towards non-planar LC MIs in H+j

  [C. Mella talk at Loops&Legs 2024]

- IBP reduction of amplitudes to MIs not feasible with standard public codes



needs experimenting and good ideas!

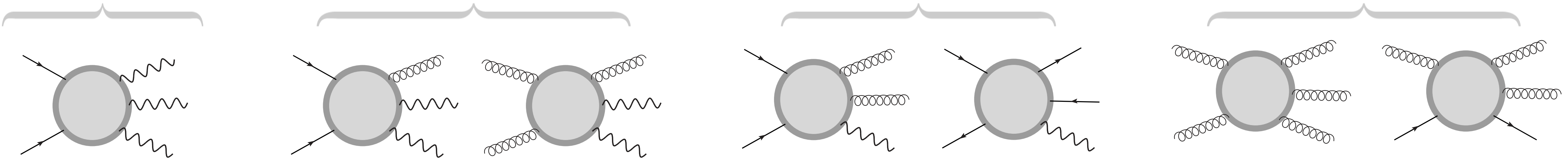
# Scattering amplitudes: 2→3 massless QCD

Contributors: [Abreu, Agarwal, Badger, Brønnum-Hansen FB, Chawdhry, Chicherin, Czakon, Cordero Febres, De Laurentiis, Gehrmann, Hartanto, Henn, Ita, Klinkert, von Manteuffel, Marcoli, Mitov, Moodie, Page, Pascual, Peraro, Poncelet, Sotnikov, Tancredi, Zoia]

All 2→3 massless amplitudes available in full colour (massless QCD)

Big boost from availability & fast evaluation of "Pentagon Functions" [Chicherin, Sotnikov '20] + new methods to cope with algebraic complexity

three photons | diphoton + jet | dijet + photon | three jets

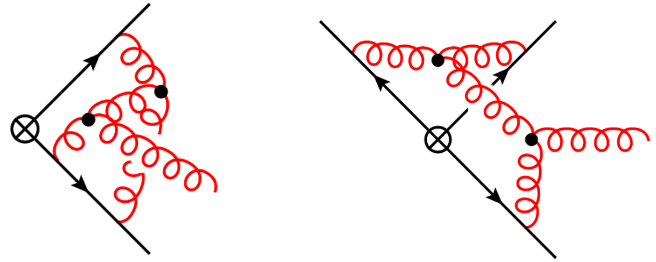


Phenomelogical predictions for all particles signatures

dijet + photon pheno
full colour 2-loop amplitudes

Interesting observations/studies from full colour QCD results

## Violation of collinear factorisation



Dipole | Tripole

2L one-gluon soft-current.
Tripole only present at SLC in > 4-jets amplitudes.
Responsible for breaking of collinear-factorisation

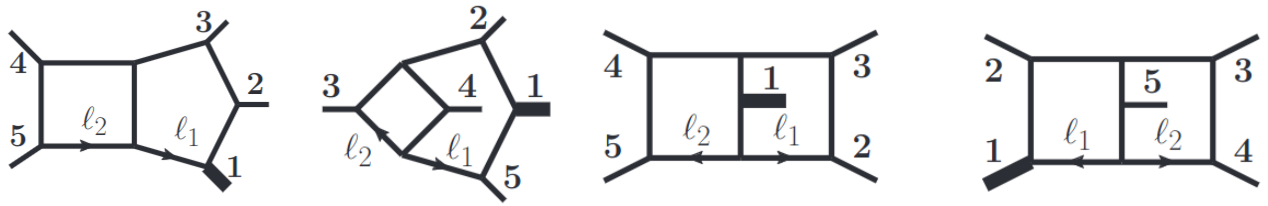
[Dixon, Hermann, Yan, Zhu 1912.09370]

## Challenges for the (near) future:

- any of these amplitudes with a jet: RVV @ $N^3LO$
- controlling two-loop amplitudes in unresolved regions: high-numerical stability required
- quadruple precision? Rather expansions $N^kLP$

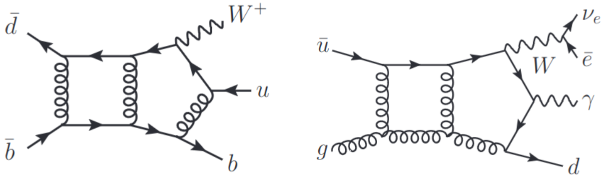
# Scattering amplitudes: 5 pt with one external mass

Great progress on one-mass 5pt scattering amplitudes



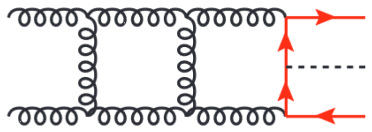
Impressive progress on 5pt with one mass amplitudes

W + bb/jγ



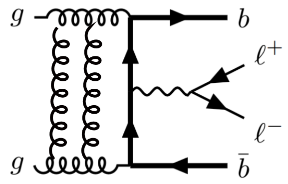
[Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 2110.07541 ]
[Badger, Bayu Hartanto, Kryś, Zoia 2201.04075]

H + bb



[Badger, Bayu Hartanto, Kryś, Zoia 2107.14733]

Z + bb



[Mazzitelli, Sotnikov, Wiesemann 2404.08598]

All results derived in leading-colour, i.e. large $N_c$ approximation

Where are H+jets amplitudes?

Boosters:

computation of all relevant functions (one-mass pentagon functions)

application of finite-fields reconstruction methods for IBPs (Caravel, FiniteFlow)

Recently: complete set of all one-mass PF (planar+non-planar)

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2306.15431]

Such progress will also be crucial for FCC-ee studies

- $e^-e^+ \to 4$ jets @ NNLO QCD
  1-mass 5pt in decay kinematics Z→qqbgg/4q
- $e^-e^+ \to 3$ jets @ $N^3$LO QCD
  VVR contribution

numerical/technical challenges will arise in unresolved kinematic regions
true in general, also for LHC applications

# Scattering amplitudes: more masses

Starting to see preliminary results for 2→3 with external two and more external masses

VVj and friends: see talk by Samuel

tt associated production

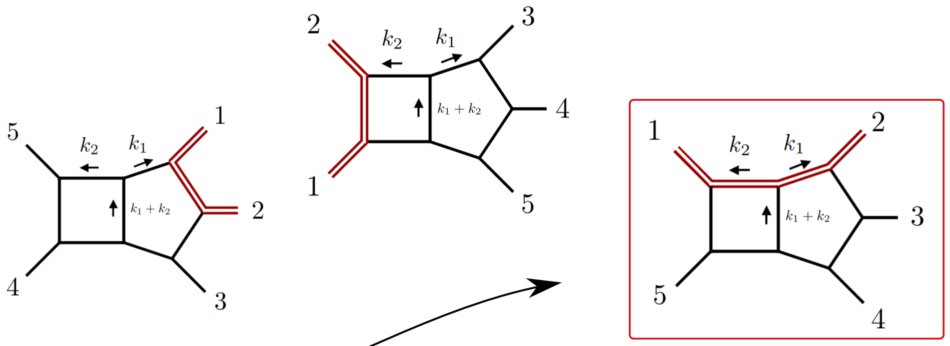
ttj

completed evaluation of MIs contributing to Leading Colour ttj amplitude

[Badger, Becchetti, Giraudo, Zoia 2404.12325]

for most integral families, deqs admit a dLog form representation

presence of elliptic sectors → non-logarithmic differential forms



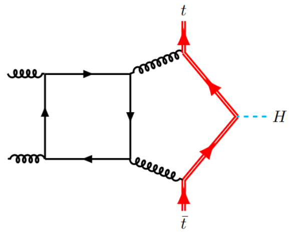

solution via generalized series expansion

(Froebenius method)

ttH/ttW

two-loop MIs for ttH production with a Light-Quark Loop

[F. Febres Cordero, G. Figueiredo, M. Kraus, B. Page, L. Reina 2312.08131]

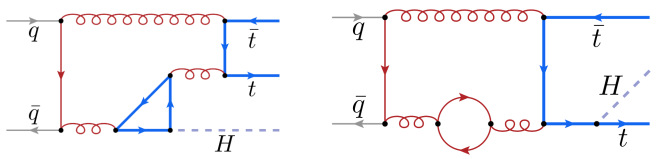



two-loop MIs for ttH production with a Light-Quark Loop

solution via canonical deqs

Two-loop amplitudes for ttH production, the Nf-part

[Bakul Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson 2402.03301]




MIs computed numerically: pySecDec

(selection of )

# Recent results from multiloop QCDxEW and NNLO-EW

# QCD: a very sociable theory

Transfer of knowledge from various subfields of high-energy (precision) physics

QCD Meets EW

5–9 Feb 2024
CERN
Europe/Zurich timezone

QCD Meets Gravity 2023

11 Dec 2023, 08:30 → 15 Dec 2023, 19:00 Europe/Zurich
4/3-006 - TH Conference Room (CERN)

Galaxies meet QCD

21–23 Feb 2024
ETH Zurich
UTC timezone

The focus of the workshop is on computational techniques (mostly analytical, but possibly numerical) for cosmological large-scale structure. We anticipate lively discussions on how to leverage some of the expertis from the QCD, amplitudes, and related communities for computing observables (especially higher n-point functions and higher loops) in LSS.

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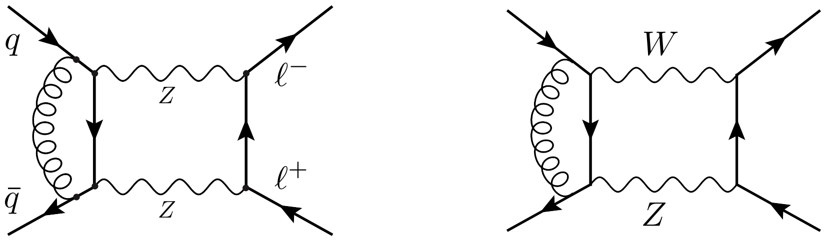
# Mixed QCDxEW corrections

Natural and most well motivated starting point: Drell-Yan

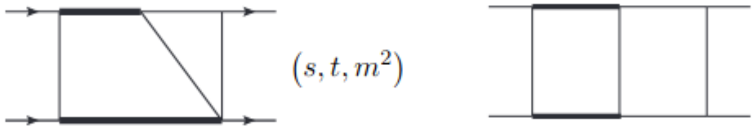



Analytical approach: [Heller, von Manteuffel, Schabinger, Spiesberger 1907.00491,2012.05918]

Result in terms of GPLs → fit polylog ansatz via symbol calculus.
Fast and flexible evaluation 0.8 s/psp. Needs care for thresholds

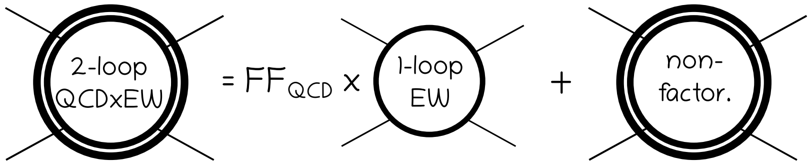
Semi-numerical approach [Armadillo, Bonciani, Devoto, Rana, Vicini 2201.01754,2405.00612]

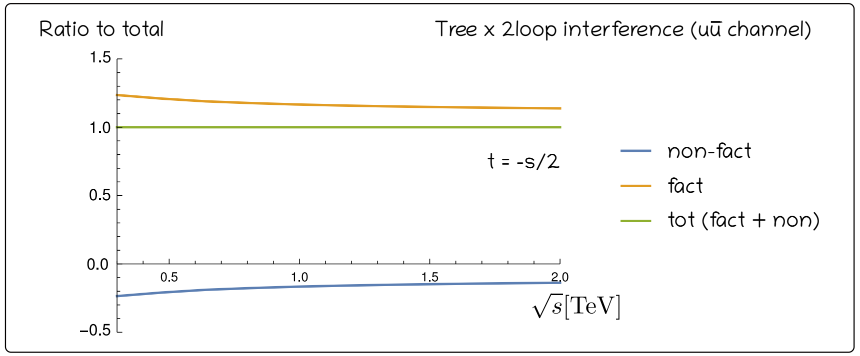
Compute MIs via series expansions + grid for MC evaluation
QCDxEW amplititudes for CCDY (fresh off the press)




non-rationalizable squared roots

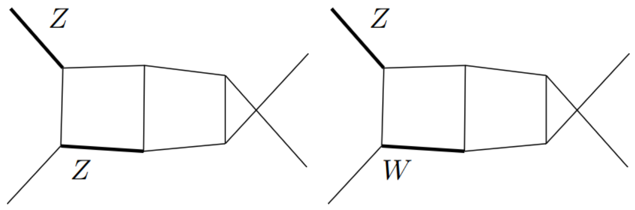
Interesting observation:




V+jet

[Bargiela, Caola, Chawdhry, Liu 2312.14145]




$l^+l^-$ recoiling against a hard jet

Dominant contribution from resonant Z boson

on-shell Z+jet

bosonic contributions only for now

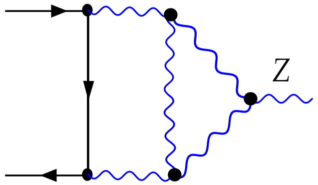
two-loop MIs via AMFlow + 2-d grid

# Electroweak physics at the Z-pole: NNLO EW

EW corrections are essential for precision measurements at lepton colliders: perturbative approach

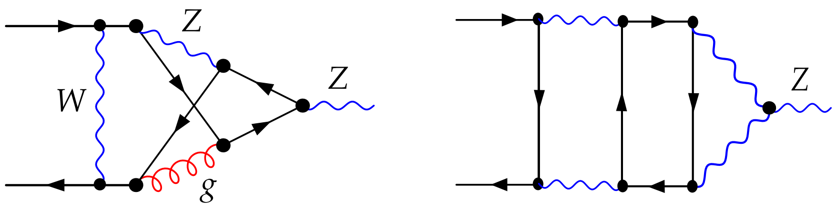
Delicate relation EWPOs ↔ measured quantities: absolute control on theory

✓ complete 2-loop EW corrections to Zff form-factor + detailed study of impact on EWPOs [Dubovyk, Freitas, Gluza, Riemann, Usovitsch 1906.08815]




From the technical point of view:
2-loop integrals using numerical techniques
(mostly sector-decomposition and Mellin-Barnes)

✗ 3-loop EW and QCDxEW form factors needed for target precision

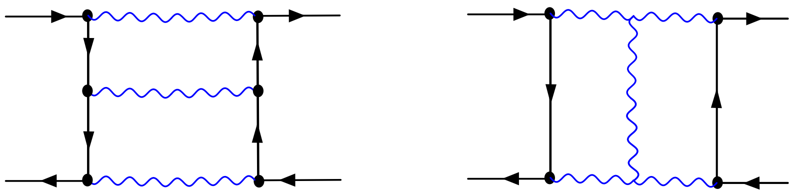



Numerical approaches seem the most solid route right now

Nowadays: underlying mathematical structure emerging: high potential

Theory input required for full line-shape description

✗ Complete 2-loop EW corrections to $e^-e^+ \to l^-l^+$



NNLO-EW corrections in Drell-Yan (LHC and FCC)

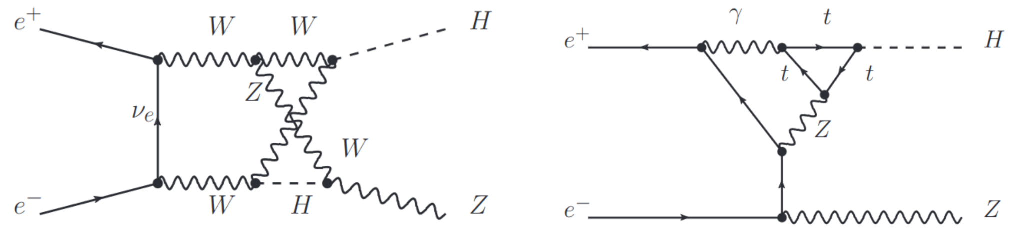
> Take care!
>
> consistent renormalization in presence of unstable particles, aka. complex-mass scheme @NNLO-EW
>
> careful and detailed study

# Higgs physics at FCC-ee and NNLO EW

Measurement of ZH cross-section with expected precision of 0.4%

NNLO electroweak corrections of commensurate size (although calculations are monstruos)

Complete two-loop amplitudes calculation a' la AMFlow: [Chen et al 2209.14953]



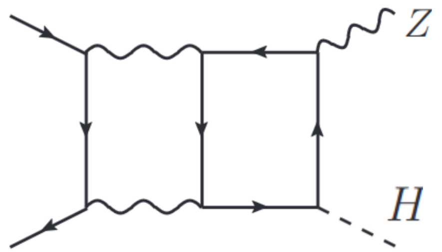

Computed for fixed ratios of invariants/masses

Still hard: full-fledged NNLO EW implementation for real-events simulation

numerical reliability, efficiency, large-scal usage, etc

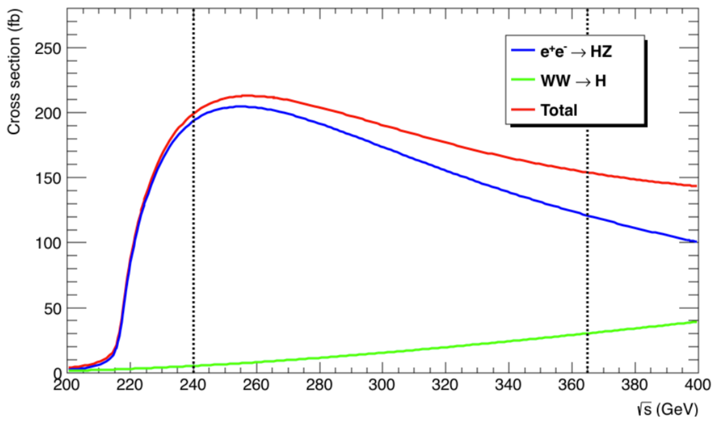
Numerical approaches to
Higher-order EW corrections to ZH

NNLO EW with nf-enhanced contributions [Freitas, Song 2209.07612]

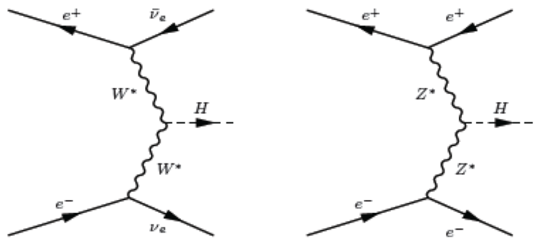



Impact at cross section level for s ~ (240 GeV)$^2$
increase NLO-EW prediction by 0.7%

Bosonic contributions could be significantly harder




Dreaming big (and crazy): H in VBF : e-e+ → Hνν




Two-loop electroweak corrections to 5-point amplitude with one off-shell leg

possibly beyond current technology:

# (some recent, and not so recent...) Methods

# Numerical loop integration

Integrate over energy component of loop momentum → Loop-Tree duality inspired methods

Integrate the rest via Monte Carlo ~ like a phase-space integration

Idea: consider the whole amplitude as a "loop MC" integrand

Potential scope for generalisation/automation

Loop integrand is singular: UV and IR

UV is easy (local UV counterterm)

IR local counterterm way more involved

locally finite 2-loop amplitudes

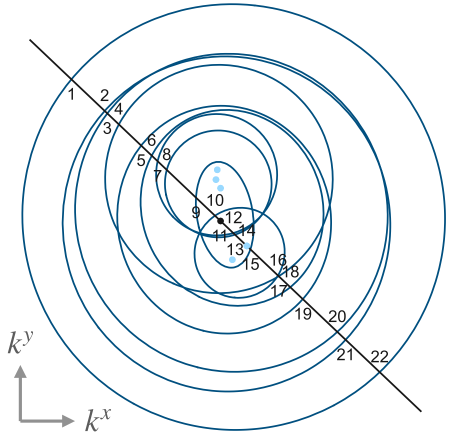
proof of concept application to EW gauge production in qq and gg

[Anastasiou, Sterman 2212.12162 ]

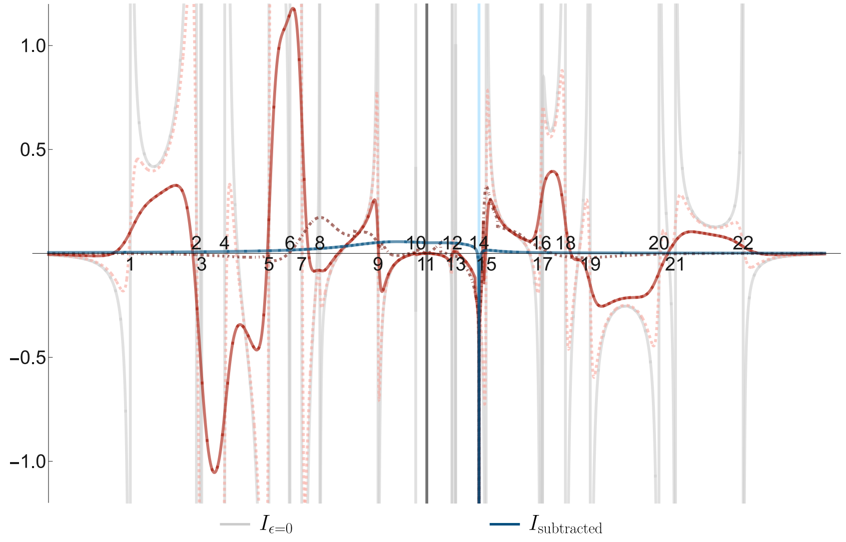
[Anastasiou, Karlen,Sterman, Venkata 24-03.13712]

Threshold singularities of loop integrand (can be very nasty)

local subtraction of thresholds [Kermanschah 2110.06869, Capatti 2211.09653]



threshold singularities of a pentagon



threshold subtraction contour deformation

# Evaluation of MIs via generalised series expansion

tipycal situation:

non-necessary or unavailability to cast deqs in canonical form

connection matrix A "too complicated" or equations are coupled

$$\mathrm{d}I_i(\vec{x},\epsilon) = A_{ij}(\vec{x},\epsilon)I_j(\vec{s},\epsilon)$$

Ansatz for a general solution around a (non)-regular point

$$f(x) \sim a_{nm}x^n \log(x)^m$$

once an initial condition is known, trasport solution
to new disc of convergence: cuts & poles in the complex plane

(selection of) Examples:

Application to elliptic sectors in H+j production
with full top/bottom mass dependence

[Moriello 1907.13234]

[Bonciani, Del Duca, Frellesvig, Hidding, Hirschi, Moriello, Salvatori, Somogyi, Tramontano 2206.10490]

diphoton at NNLO QCD with fullo top-mass dependence

[Becchetti, Bonciani, Cieri, Coro, Ripani 2308.11412]

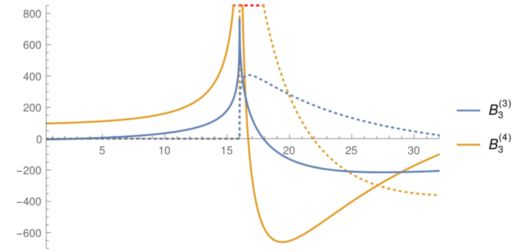
Application of Frobenius method to solve differential equations

high potential for algorithmic implementation

DiffExp

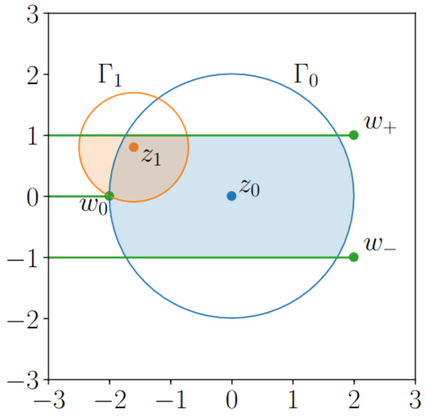
[Hidding 2006.05510]

well-established and
highly optimized




Seasyde

[Armadillo, Bonciani, Devoto, Rana, Vicini 2205.03345]




transporting differential equation
in the complex plane

ideal for (N)NLOEW corrections
with complex masses (resonances)

application to mixed QCDxEW
corrections

not really usable point-by-point (pheno application)
need to rely on grid implementation

# Auxiliary mass flow

Key idea behind: introduce an auxiliary imaginary mass η and fix the boundary condtion at "η~-I*∞"

[Liu, Ma, Wang 1711.09572, Liu, Ma 2107.01864]

$$I(\vec{\nu},\vec{s},\epsilon) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$I_{\mathrm{aux}}(\vec{\nu},\vec{s},\epsilon,\eta) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \eta)^{\nu_1} \cdots (\mathcal{D}_K - \eta)^{\nu_K}}$$

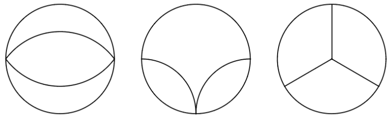
Physical result recovered as:

$$I(\vec{\nu},\vec{s},\epsilon) = \lim_{\eta \to \mathrm{i}0^-} I_{\mathrm{aux}}(\vec{\nu},\vec{s},\epsilon,\eta)$$

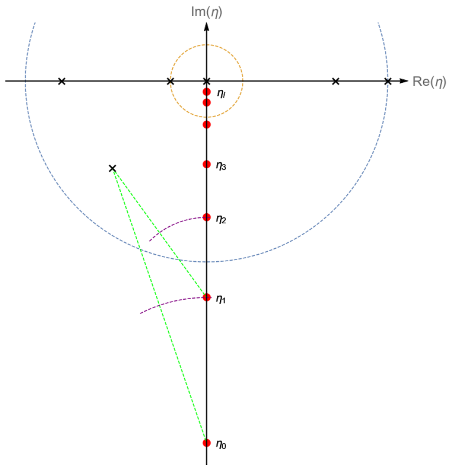
Calculation of the auxiliary integral via:

$$\frac{\partial}{\partial \eta} \vec{\mathcal{I}}_{\mathrm{aux}}(\vec{s},\epsilon,\eta) = A(\epsilon,\eta) \vec{\mathcal{I}}_{\mathrm{aux}}(\vec{s},\epsilon,\eta)$$

Boundary conditions trivialize at η~-I*∞



massive vacuum integrals



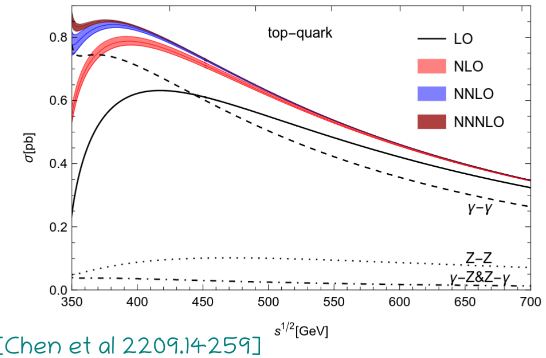

Solution of the deq via series expansion in η

Successive steps to transport solution and expansion around regular points

Dictated by poles/cuts structure in Re[η] and radii of convergence in η

Impressive results and extremely handy tool

$t\bar{t}$ in $e^-e^+$: total rate at $N^3LO$ (above threshold)




[Chen et al 2209.14259]

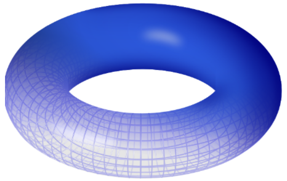
can deliver a value for an integral in a specif point with arbitrarily many digits

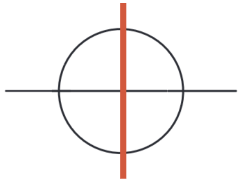
more and more often used to fix BC for other general series solvers

Also able to deal with phase-space integrals and linear propagators

# Elliptic amplitudes

Cases where deqs are doubly coupled or the maximal cut describes by an elliptic curve





$$\mathrm{MaxCut}(I_{1,1,1,0,0}) \sim \int \frac{\mathrm{d}z_4}{\sqrt{P_4(z_4)}}$$

$$P_4(z_4) = (z_4 - a_1)(z_4 - a_2)(z_4 - a_3)(z_4 - a_4)$$

$$a_1 = (m_2 - m_3)^2\,,\ a_2 = (m_2 + m_3)^2\,,\ a_3 = \left(m_1 - \sqrt{p^2}\right)^2\,,\ a_4 = \left(m_1 + \sqrt{p^2}\right)^2$$

First issue: canonical differential equation beyond MPLs cases?

New approaches and ideas

[Pögel, Wang, Weinzierl 2211.04292] [Görges, Nega, Tancredi, Wagner 2305.14090]

$$\mathrm{d}I_i(x,\epsilon) = \mathrm{GM}_{ij}(\vec{x},\epsilon) I_j(\vec{x},\epsilon) \longrightarrow \mathrm{d}J_m(x,\epsilon) = \epsilon\, \mathrm{GM}^{\epsilon}_{mn}(\vec{x}) J_n(\vec{x},\epsilon)$$

$$\longrightarrow \mathrm{GM}^{\epsilon}_{mn}(\vec{x}) \sim \omega(\vec{x})$$

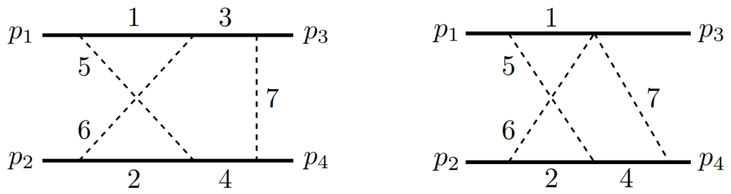
examples of 3-equal mass sunrise, x=(s,m$^2$)

$$\omega(\vec{x}) \sim \mathrm{K}(k^2) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

NNLO QED correction to Bhabha, Møller scattering:

First complete analytic results for a scattering amplitude with elliptic integrals

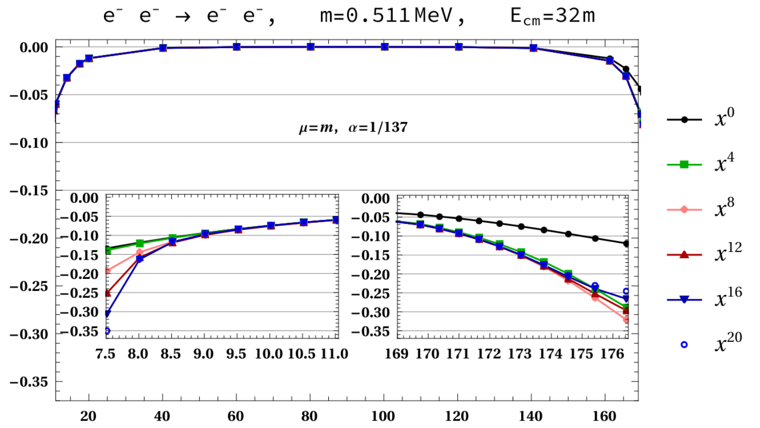
[Delto, Duhr, Tancredi, Zhu 2211.04292]




start from ep-factorized deq to systematically obtain a small mass-expansion (generalised series)

coefficients are HPLs

Compact expressions for the whole/final amplitude



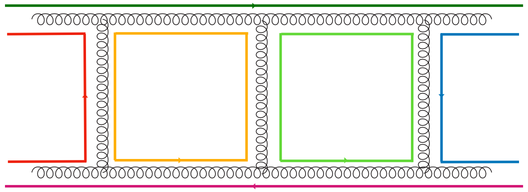
# (couple words on)
# Approximations
# (that we can hear about in this workshop)

# (non-)leading colour and (non-)planar diagrams

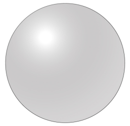
Consider a U(N) gauge theory (also SU(N) is fine), in the t'Hooft limit N→ ∞ at λ = g 2 *N fixed [t'Hooft '73]

diagram ~ $\lambda^{loops} N^{X}$

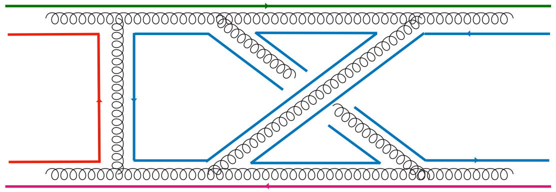
sphere X = 2



X = 2



otherwise X = 2 - holes



X = 1

at Leading power in N
the diagram is planar (topologically)

Tipically: planar topologies much easier to handle with

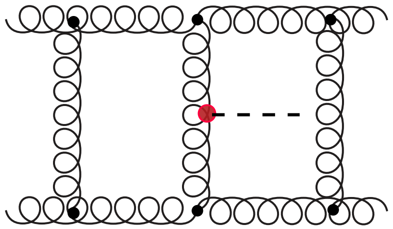
Non-planar: more complicated functions/cuts and algebraic complexity much worse

To give an idea: IBP identity for the single worst
planar integral in 5-pt massless scattering ~ 50mb

To give an idea: IBP identity for the single worst
non-planar integral in 5-pt massless scattering ~ 3gb

as they come out
of an IBP reducer

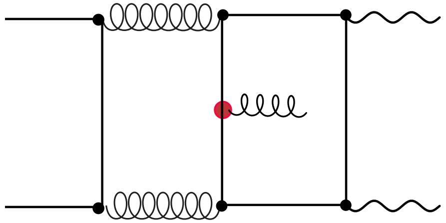
Sometimes: planar != LC



moreover, Hgg is an effective coupling
result: higher numerator rank than full QCD amplitudes
algebraic complexity significantly increases

Parametrically: LC, i.e. $N^{2}$, not justified

In QCD, the expansion parameter: $N_f/N_c$ (once leading $N_c$ factored out)



$$\sim N_c \left(n_u Q_u^2 + n_d Q_d^2\right) \qquad \frac{\left(n_u Q_u^2 + n_d Q_d^2\right)}{N_c Q_d^2} = \frac{11}{3}$$

ratio to LC
for dd channel

# Masses, when they matter

Most often perturbative QCD calculations performed including only massless fermionic contributions or, in HEFT (infinite top-mass limit)

Justified when such contributions are suppressed by the large mass ~ $(1/m_t)^p$

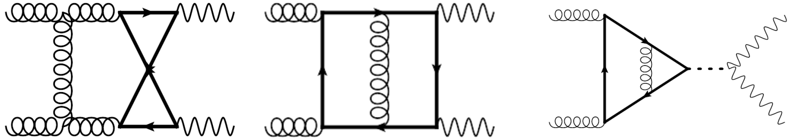
Cases where the presence of internal masses does matter, typical example EW corrections, Higgs Y coupling to internal masses + large pT, etc

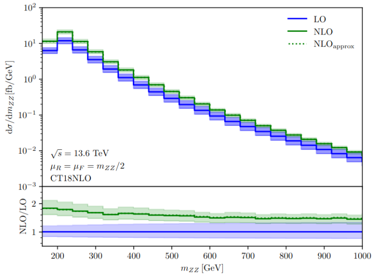
Presence of internal masses makes everything significantly more complicated, even for low multiplicity

see talk by Andreas

ZZ production in gg fusion

[Agarwal, Jones, von Manteuffel 2011.15113 ]
[Agarwal, Jones, Kerner, von Manteuffel 2105.04436]



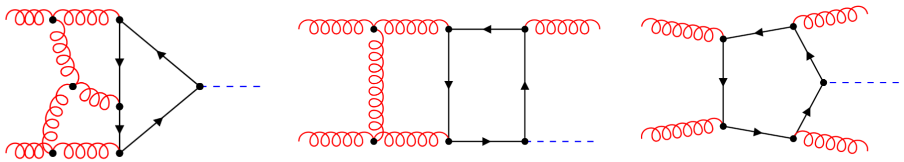



see talk by Marco

1/mt effects and top-bottom interference in gg→H

[Czakon, Harlander, Klappert, Niggetiedt 2105.04436]

[Czakon, Eschment, Niggetiedt, Poncelet, Schellenberger 2312.09896 ]



crucial in addressing one of the leading theory uncertainties on the gluon-fusion cross section

# Summary and outlook

- Great progress on loop and more generally amplitudes calculation
- General advancements and deeper understanding of algebraic/transcendendatal properties of amplitudes
- Broken the 2→3 phenomenology barrier:

  fully differential predictions for classes of processes (massless particles + one mass + ttX)

  for some processes non-availability of loop amplitudes current bottleneck

- Where/when new results become available, ideally "fully exact" results should be included where possible

  lift approximations (be them justified/process dependent) when relevant: masses, LC vs SLC

- Analytic method may not be suited with massive contributions ~ NNLO-EW and mixed QCDxEW → numerical/seminumerical methods will most likely be the way to go
- Great parallel progress on understanding geometry of amplitudes beyond MPLs (elliptics and beyond)
- If amplitudes not available:
  estimates/approximation only possibility (well motivated approximations)

  More refined calculation will shed light.
  More generally: when complete results are avalaible understanding leading contributions important for the future