Recent developments in multiscale loop scattering amplitudes

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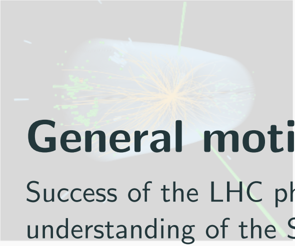
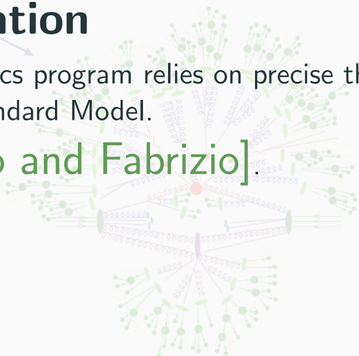
The usual picture





General motivation

Success of the LHC physics program relies on precise theoretical understanding of the Standard Model.

[talks by Federico and Fabrizio].

Fixed order partonic cross sections

Collinear factorization:

"Hard" partonic cross section

$$d\sigma_{h_1 h_2 \to X}(p_1, p_2) = \sum_{i,j} \int \mathrm{d}x_1\, \mathrm{d}x_2\, f_i(x_1, \mu) f_j(x_2, \mu)\, \boxed{\mathrm{d}\hat{\sigma}_{ij \to X}(x_1 p_1, x_2 p_2, \mu)} \qquad + \mathcal{O}(\Lambda_{\mathsf{QCD}}/Q)$$

Series truncation uncertainty

$$\mathrm{d}\hat{\sigma}_0 \left(1 + \alpha_s \sigma^{(1,0)} + \alpha_s^2 \sigma^{(2,0)} + \alpha \sigma^{(0,1)} + \alpha_s^3 \sigma^{(3,0)} + \alpha \alpha_s \sigma^{(1,1)} + \dots\right) \qquad \begin{aligned} \alpha_s(M_Z) &\sim 0.1 \\ \alpha(M_Z) &\sim 0.01 \end{aligned}$$

At least NNLO QCD and NLO EW corrections must be included to achieve percent level theory uncertainties ($\oplus$ PDFs, parton showers, resummations).

This talk: recent advances in multi-scale NNLO QCD corrections.

NNLO QCD multiplicity frontier

Ultimate goal:

fully differential NNLO cross sections for all (interesting) SM processes

NNLO QCD $2 \to 3$
current frontier

$$\sigma^{F+X}_{\text{NNLO}} = \sigma^{F+X}_{\text{NLO}} + \int_{\Phi_{F+2}} \mathrm{d}\sigma_{\text{RR}} + \int_{\Phi_{F+1}} \mathrm{d}\sigma_{\text{RV}} + \int_{\Phi_F} \mathrm{d}\sigma_{\text{VV}}$$

IR divergences

One-loop stable in IR, e.g. OpenLoops2 [Buccioni et al. '19]

Two-loop amplitudes

- Main missing ingredient, no automation in sight
- Both technical and conceptual challenges

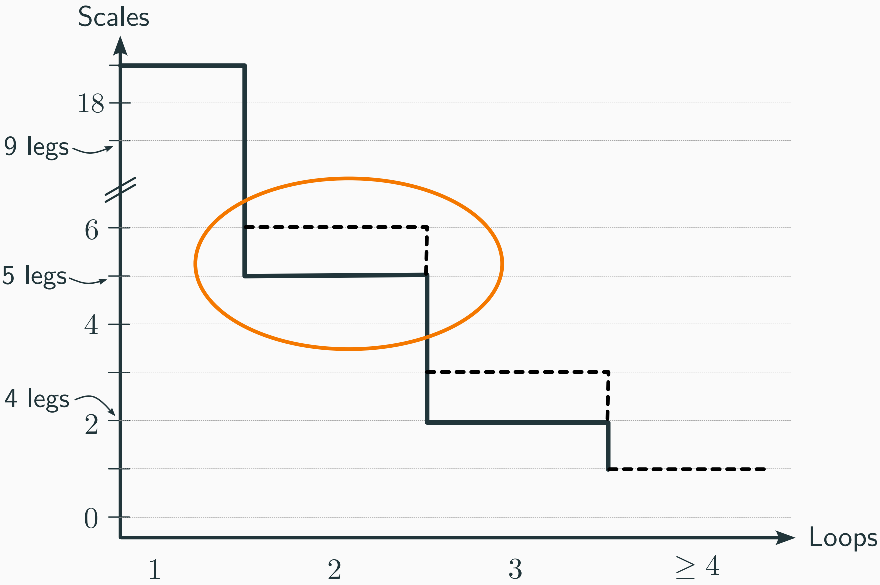
Cancellation of IR divergences at NNLO automated in principle
[Czakon '11] [Czakon, Heymes '14]
[Chen, Gehrmann, Glover, Huss, Marcoli '22]
[Grazzini, Kallweit, Wiesemann '17]
[Gehrmann, Glover, Marcoli '23]

But:

- Doing it efficiently is hard
- General purpose public codes still missing

Two-loop multi-scale amplitudes: state of the art

Loops & legs: state of the art



Two-loop five-point amplitudes: massless

Complete since the end of last year

	Comment	Complete analytic results	Public code	Cross sections
$pp \to \gamma\gamma\gamma$	l.c.*	[4, 5]	[4]	[11, 12]
$pp \to \gamma\gamma j$	l.c.*	[2, 3]	[2]	[10]
$pp \to jjj$	l.c.	[1]	[1]	[8, 9]
$pp \to \gamma\gamma\gamma$	NLO loop induced	[14]	[14]	
$pp \to \gamma\gamma j$	NLO loop induced	[6]		
$gg \to \gamma\gamma g$	NLO loop induced	[7]	[7]	[13]
$pp \to \gamma jj$	NLO loop induced	[15]		[15]
$pp \to jjj$	NLO loop induced	[16,17,18]	[17]	
$pp \to t\bar{t}H$	$m_t, m_H \to 0$ limit	[19]		

[1] [Abreu, Febres Cordero, Ita, Page, VS '21]
[2] [Agarwal, Buccioni, von Manteuffel, Tancredi '21]
[3] [Chawdry, Czakon, Mitov, Poncelet '21]
[4] [Abreu, Page, Pascual, VS '20]
[5] [Chawdry, Czakon, Mitov, Poncelet '20]
[6] [Agarwal, Buccioni, von Manteuffel, Tancredi '21]
[7] [Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]
[8] [Czakon, Mitov, Poncelet '21]
[9] [Chen, Gehrmann, Glover, Huss, Marcoli '21]
[10] [Chawdry, Czakon, Mitov, Poncelet '21]
[11] [Chawdry, Czakon, Mitov, Poncelet '19]
[12] [Kallweit, VS, Wiesemann '20]
[13] [Badger, Gehrmann, Marcoli, Moodie '21]
[14] [Abreu, de Laurentis, Ita, Klinkert, Page, VS '23]
[15] [Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia '23]
[16] [de Laurentis, Ita, Klinkert, VS '23]
[17] [de Laurentis, Ita, VS '23]
[18] [Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi '23]
[19] [Wang, Xia, Yang, Ye '24]

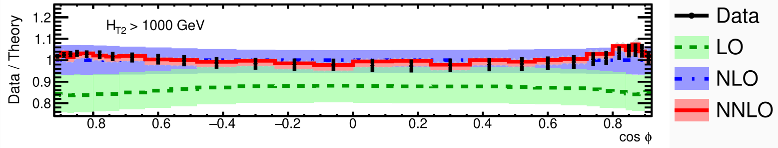
Evaluation of Feynman integrals: pentagon functions [Chicherin, VS '20]

Application in α_s measurement

Determination of the strong coupling constant from transverse energy—energy correlations in multijet events at $\sqrt{s} = 13$ TeV with the ATLAS detector

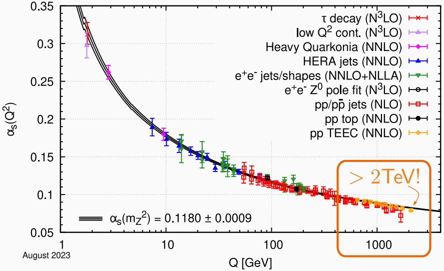
ATLAS Collaboration • Georges Aad (Marseille, CPPM) Show All(2916) (see also [Alvarez, Cantero, Czakon, Llorente, Mitov, Poncelet '23])

Jan 23, 2023



- Remarkable agreement between NNLO and data
- α_s measured at record scales
- Milestone for pQCD tool development

STRIPPER
[Czakon '11] [Czakon, Heymes '14]
Two-loop corrections
[Abreu, Febres Cordero, Ita, Page, VS '21]
[Chicherin, VS '20]



To be published in PDG 2024 review [arXiv:2312.14015]

Two-loop five-point amplitudes: one external mass

	Comment	Complete analytic results	Public code	Cross sections
$pp \to Wb\bar{b}$	l.c.*, on-shell W	[1]		
$pp \to W(l\nu)b\bar{b}$	l.c., $m_b = 0$	[2, 3]	[10]	[3, 4, 7]
$pp \to W(l\nu)t\bar{t}$	l.c., $m_t = 0$	[2, 3]	[10]	[8]
$pp \to Z(ll)b\bar{b}$	l.c.*, $m_b = 0$	[2]	[10]	[9]
$pp \to W(l\nu)jj$	l.c.	[2]	[10]	
$pp \to Z(l\bar{l})jj$	l.c.*	[2]	[10]	
$pp \to W(l\nu)\gamma j$	l.c.*	[5]		
$pp \to Hb\bar{b}$	l.c., $m_b = 0$	[6]		[Christian's talk]

[1] [Badger, Hartanto, Zoia '21]
[2] [Abreu, Febres Cordero, Ita, Klinkert, Page, VS '21]
[3] [Hartanto, Poncelet, Popescu, Zoia '22]
[4] [Hartanto, Poncelet, Popescu, Zoia '22]
[5] [Badger, Hartanto, Kryś, Zoia '22]
[6] [Badger, Hartanto, Kryś, Zoia '21]
[7] [Buonocore, Devoto, Grazzini, Kallweit, Mazzitelli, Rottoli, Savoini '22]
[8] [Buonocore, Devoto, Grazzini, Kallweit, Mazzitelli, Rottoli, Savoini '23]
[9] [Mazzitelli, VS, Wiesemann '24]
[10] [de Laurentis, Ita, Page, VS in preparation]

Evaluation of Feynman integrals: pentagon functions

[Chicherin, VS, Zoia '21] [Abreu, Chicherin, Ita, Page, VS, Tschernow, Zoia '23]

Two-loop five-point amplitudes: beyond one external mass

	Comment	Complete analytic results	Public code	Cross sections

[Image by DALL-E]

⟶ [Samuel's talk]

Two-loop five-point scattering: first results with masses in loops

No complete amplitudes or integral families known

Integrals

- Analytic study of integral families for $pp \to t\bar{t}j$ (l.c.)
  [Badger, Becchetti, Chaubey, Marzucca '23] [Badger, Becchetti, Giraudo, Zoia '24]
- Analytic study of integrals for $pp \to t\bar{t}H$ contribution with a light quark loop in l.c.
  [Febres Cordero, Figueiredo, Kraus, Page, Reina '23]
- Numerical evaluation on a few points possible with `AMFlow` approach [Liu, Ma '21,'22]

Amplitudes

- Numerical evaluation of light and heavy quark loop contributions to $q\bar{q} \to t\bar{t}H$
  [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson '24]

Warning: two-lop “mass-in-the-loop” frontier

⚠ With massive particles in loops analytic (mathematical) complexity may escalate abruptly and dramatically!

Underlying reason: integrals associated with nontrivial algebraic curves and surfaces (e.g. elliptic curves)

Example: $pp \to t\bar{t}$

- ✓ analytic results for $q\bar{q} \to t\bar{t}$ with top loops [Mandal, Mastrolia, Ronca, Bobadilla '22], evaluation “easy”
- ⚠ analytic results for $g\bar{g} \to t\bar{t}$ with top loops [Adams, Chaubey, Weinzierl '17,'18] [Badger, Chaubey, Hartanto, Marzucca '21], but unclear how to evaluate efficiently due to the presence of elliptic curves

But

- Cross sections computed with numerical methods and interpolation grids since long time ago [Czakon '08] [Bärnreuther, Czakon, Fiedler '13]
- Recent example: NLO corrections for $gg \to ZZ$ [Agarwal, Jones, Kerner, von Manteuffel '24]

Not discussed in this talk $\longrightarrow$ [Andreas's talk]

Dynamic and fixed scales

Dynamic scales

- Mandelstam invariants s_{ij}, off-shell legs p_i^2
- Monte Carlo integrals over phase space $\int \mathrm{d}\Phi_n\,(s_{ij}, p_i^2)\big|\mathcal{A}_{2\to n}(s_{ij}, p_i^2)\big|^2$
- Need fast and robust numerical evaluation of $\mathcal{A}_{2\to n}$ over phase space

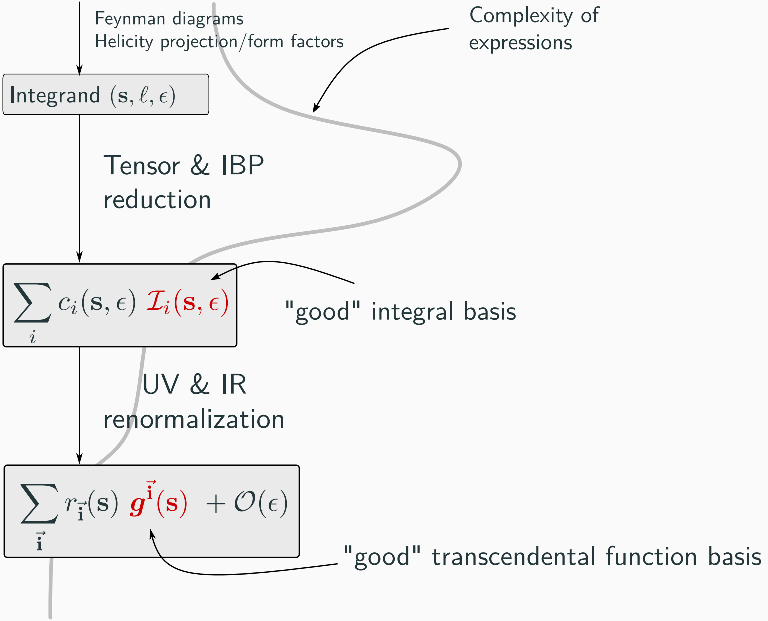
Fixed scales

- Particle (complex) masses, e.g. m_t, m_W
- Mathematical complexity can escalate very quickly
- With few dynamic scales can profit the most from numerical methods $\oplus$ interpolation grids

In the following I mainly highlight dealing with many dynamic scales.

Analytic methods: selected highlights

Analytic multi-loop amplitude calculations



What is a “good” transcendental functions basis?

Analytic properties

- No hidden identities (basis)
- Analytic cancellation of UV and IR divergences (minimize regularization artifacts)
- Control over physics properties (amplitudeology friendly)
- Compact rational coefficients

Numerical evaluation

- Over whole physical phase space
- Fast (Monte-Carlo integration over large phase space)
- Stable

Analytic methods: selected highlights

Feynman integrals

Pure integrals and canonical differential equations

Pure Feynman integrals
[Henn '13]

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\; A_i$$

Letters of symbol alphabet
Encode singularity structure and branch cuts

Rational matrix

Pure integrals and canonical differential equations

Pure Feynman integrals
[Henn '13]

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$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\ A_i$$

Letters of symbol alphabet
Encode singularity structure and branch cuts

Rational matrix

⚠ Canonical DE very challenging to obtain for multi-scale integrals

Cutting edge examples:

# scales	# massive lines	Reference
6	0	[Abreu, Chicherin, Ita, Page, VS, Tschernow, Zoia '23]
7	0	[Jiang, Liu, Xu, Lin Yang '24] [Samuel's talk]
8	0	[Henn, Peraro, Xu, Zhang '21] [Henn, Matijašić, Miczajka, Peraro, Xu, Zhang '24]
6	≤ 1	[Badger, Becchetti, Chaubey, Marzucca '22]
6	≤ 2	[Badger, Becchetti, Giraudo, Zoia '24]
7	≤ 2	[Febres Cordero, Figueiredo, Kraus, Page, Reina '23]

beyond $\mathrm{d}\log$ forms!

How to solve DE?

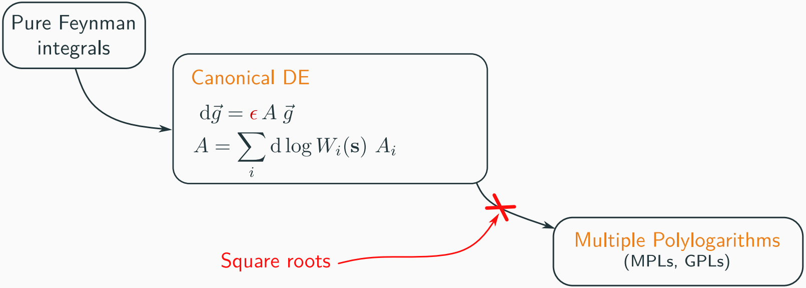
Pure Feynman integrals

Canonical DE

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\ A_i$$

How to solve DE?



⚠ Map may not exist [Duhr, Brown '20]

Sometimes still possible

[Heller, von Manteuffel, Schabinger '19][Heller '21]
[Bonetti, Panzer, Smirnov, Tancredi '20]
[Kreer, Weinzierl '21][Duhr, Smirnov, Tancredi '21]
[Papadopoulos, Tommasini, Wever '15] [Papadopoulos '14]
[Canko, Papadopoulos, Syrrakos '20]

⚠ but “good” representation even more challenging

How to solve DE?

Pure Feynman integrals

Canonical DE

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\; A_i$$

Semi-numerical DE solution
(local power-log series expansions along a path)

`DiffExp` [Moriello '19] [Hidding '20]
`AMFLow` [Liu, Ma, Wang '17] [Liu, Ma '21]
`SeaSyde` [Armadillo, Bonciani, Devoto, Rana, Vicini '22]
[Hidding, Usovitsch '22]

✓ universal and easy to set up

✓ canonical DE not required

⚠ evaluation of integrals (no function basis)

- hidden cancellations
- prohibitive run times
- stability over phase space?

Pentagon functions construction

[Abreu, Chicherin, Ita, Page, VS, Tschernow, Zoia '23]
(see also [Chicherin, VS '20] [Badger, Hartanto, Zoia '21] [Chicherin, VS, Zoia '21])

Pure Feynman integrals

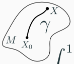
Canonical DE

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\; A_i$$

weight = length = ϵ order

Chen iterated integrals [Chen '77]



$$\int_0^1 \mathrm{d}\log W_n(t_n)\; \ldots \int_0^{t_2} \mathrm{d}\log W_1(t_1)$$

"Good" basis of functions

Use properties of iterated integrals and simple linear algebra

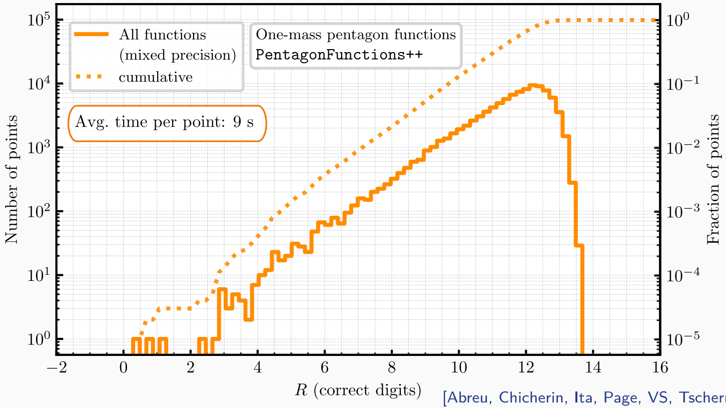
- ✓ complete
- ✓ non-redundant

Initial conditions (`AMFlow`)

Efficient numerical evaluation

Control over analytic properties $\Longrightarrow$ optimized one-fold integral representations

Example: one-mass pentagon functions



[Abreu, Chicherin, Ita, Page, VS, Tschernow, Zoia '23]

- Compelte basis of two-loop transcendental functions for NNLO corrections for Vjj, Hjj, etc.
- Timing to evaluate all 1291 functions on one CPU
- Excellent numerical performance

The only viable method for $2 \to 3$ phenomenology so far!

Case study: importance of transcendental function basis

Consider triphoton hadroproduction in NNLO QCD (l.c.)

[Chawdry, Czakon, Mitov, Poncelet '19]

using earlier incarnation of (planar) pentagon functions [Gehrmann, Henn, Lo Presti '18]

- Rationalized kinematics required due to precision loss
- Average time 17 minutes to typically get 2 digits
- Need interpolation grids (very challenging for many dynamic scales)

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[Abreu, Page, Pascual, VS '20]

using "good" function basis [Chicherin, VS '20]

- Double precision sufficient
- 1 second to typically get 11 digits
- Same efficiency for full color !
  [Abreu, de Laurentis, Ita, Klinkert, Page, VS '23]

Analytic methods: selected highlights

Rational coefficients

Rational coefficients

Feynman diagrams
Helicity projection/form factors

Integrand $(\mathbf{s}, \ell, \epsilon)$

Tensor & IBP reduction

$$\sum_i c_i(\mathbf{s}, \epsilon)\ \mathcal{I}_i(\mathbf{s}, \epsilon)$$

"good" integral basis

UV & IR renormalization

$$\sum_{\vec{\mathbf{i}}} r_{\vec{\mathbf{i}}}(\mathbf{s})\ \boldsymbol{g}^{\vec{\mathbf{i}}}(\mathbf{s})\ + \mathcal{O}(\epsilon)$$

"good" transcendental function basis

Analytics from (exact) numerics

Main lesson

Rational coefficients $r_{\vec{\mathbf{i}}}$ simple (given a good $\boldsymbol{g}^{\vec{\mathbf{i}}}$ function basis).

- Bypass intermediate expression swell by exact numerical evaluations over $\mathbb{F}_p$:
  $p <$ machine integer $\implies$ efficient
- Reconstruct analytic expressions from numeric samples [von Manteuffel, Schabinger '14] [Peraro '16]
- Important bonus: enables parallelization

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Most powerful when applied to physical quantities (reconstruct finite remainders)

Caravel [Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, Pascual, Ruf, VS '20]

`FiniteFlow` [Peraro '19]

Less powerful, but also useful for IBP reduction only (reconstruct integral reduction rules)

`LiteRed+FiniteFlow`, `Kira+FireFly`, `FIRE6`

numerous private codes (e.g. `Finred` by A. von Manteuffel)

Remarks on integration-by-parts reduction

- Problem conceptually solved by Laporta's algorithm
- In practice, IBP equation systems remain major bottleneck in loop calculations
- No major breakthroughs, but process specific optimizations make the difference

Multiscale problems @ 2 loops

- Situation majorly improved by (exact) numerical frameworks
- Solving systems over $\mathbb{F}_p$ (setting s, ϵ to integers) typically straightforward
- Eventually number of samples for reconstruction becomes the issue

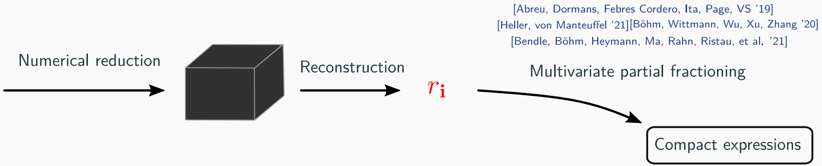
General observation

Avoiding generating identities that introduce auxiliary integrals (e.g. higher denominator powers [Gluza, Kadja, Kosower '11]) typically helpful.

Public proof-of-principle implementation: `NeatIBP` [Wu, Boehm, Ma, Xu, Zhang '23].

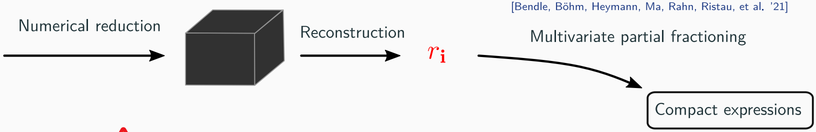
Note: lots of experimentation and ideas in the literature not discussed here! [Andreas's talk]

Analytics from numerics workflow



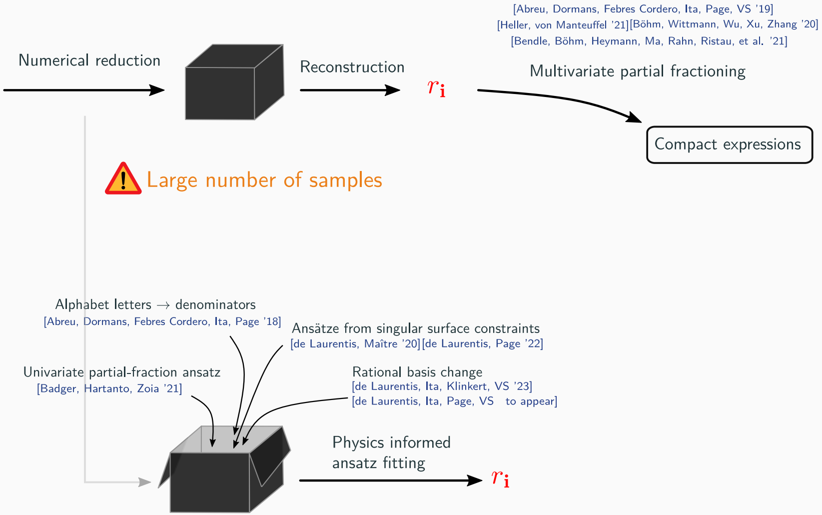
Analytics from numerics workflow

[Abreu, Dormans, Febres Cordero, Ita, Page, VS '19]
[Heller, von Manteuffel '21][Böhm, Wittmann, Wu, Xu, Zhang '20]
[Bendle, Böhm, Heymann, Ma, Rahn, Ristau, et al. '21]

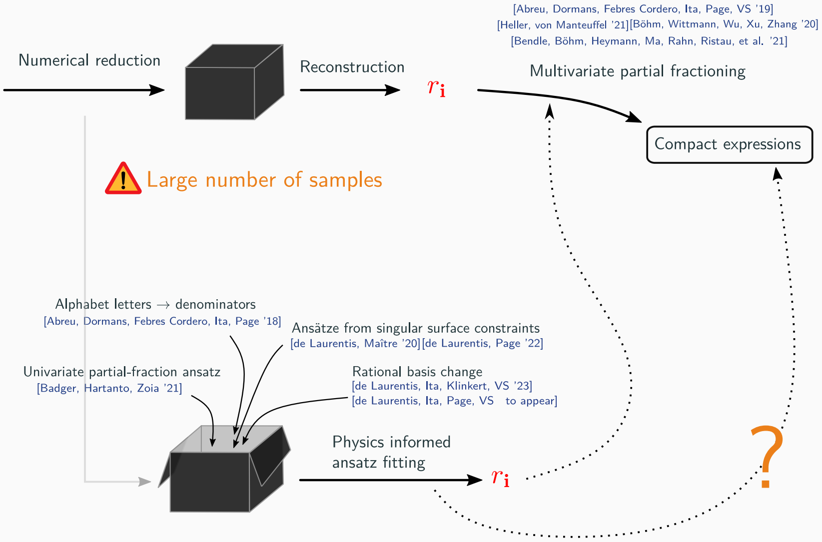


⚠ Large number of samples

Analytics from numerics workflow



Analytics from numerics workflow



Rational basis change

[Abreu, Dormans, Febres Cordero, Ita, Page '18]
[de Laurentis, Ita, Klinkert, VS '23]
[de Laurentis, Ita, Page, VS to appear]

rational matrix

$$\sum_i r_i \, g_i \longrightarrow \sum_i r_i \, M_{ij} \, g_j \longrightarrow \sum_i r'_i \, M'_{ij} \, g_j$$

$$r'_i = Q_{ij} r_j, \quad M'_{ij} = (Q^{-1})_{ik} M_{kj}$$

r_i generally has spurious denominators, which amplitudes not allowed to have

- exponent too high, e.g. $1/s_{23}^3$
- unphysical pole, e.g. $1/(s_{12} - s_{14})$

Idea: find transformations Q_{ij} to maximally cancel spurious denominators

- Can be done before full analytic reconstruction $\Rightarrow$ reduced number of samples
- Positive impact on numerical stability expected

Rational basis change examples

Example 1

Three-jet production (full color) [de Laurentis, Ita, Klinkert, VS '23].

(see also [Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi '23]).

Numerical samples generated by CARAVEL.

- Reduction from 250k to 15k samples (reconstruct the latter)
- Rational basis (after additional massaging) printed in the paper, 4 pages

Example 2

Analytic results for Vjj production from [Abreu, Febres Cordero, Ita, Klinkert, Page, VS '21]

- Reconstructed analytic form that is hard to use (large numerical cancellations, large memory footprint)
- Multivariate partial fractioning fails due to complicated Gröbner basis
- With basis change 1.2Gb $\to$ 25Mb [de Laurentis, Ita, Page, VS to appear]

Numerical methods

Mixed analytic-numerical

Light and heavy quark loop contributions to $q\bar{q} \to t\bar{t}H$

[Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson '24]

(see also similar approach to Wjj production [Hartanto, Badger, Brønnum-Hansen, Peraro '19])

Amplitude reduction

- Numerical with rationalized kinematics, highly optimized systems of IBP equations.
- Impressive performance: 2 minutes on one CPU.

Evaluation of integrals

- Basis optimized for sector decomposition (quasi-finite integrals [Andreas's talk]).
- Large numerical cancellations handled by Quasi-Monte-Carlo sampling (`pySecDec`).
- 5 minutes on a modern GPU.

Questions

- Scaling to more complex integrands (complete amplitudes)?
- Do high-dimensional interpolation grids work?
- Two-loop corrections expected small [Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Savoini '22]
  $\longrightarrow$ how many evaluations to validate approximations?

Fully numerical: IR finite integrands

Idea

- Universal IR structure of color-singlet production
  $\Longrightarrow$ locally finite integrands (before loop integration).
  $q\bar{q} \to F$ [Anastasiou, Sterman, Venkata '22]
  $gg \to F$ [Anastasiou, Karlen, Sterman, Venkata '24]
- Simultaneous Monte-Carlo integration over loops and phase-space.
- No IBP reduction, no dedicated computations of Feynman integrals.

- E.g. applicable to $pp \to VVV$, which is challenging with current analytic methods.
- Proof-of-principle computation: closed-quark contributions for $q\bar{q} \to \gamma\gamma\gamma$
  [Matilde Vicini's talk @ Loops&Legs 2024]

Questions

- Scaling to more complex integrands (complete amplitudes)?
- Minkowski (or threshold) singularities?
- Easy to adapt standard cross section frameworks?

Conclusions & Outlook

Feynman integrals: analytic vs numerical

Analytic

- ✓ Map to MPLs for $2 \to 2$, when possible.
- ✓ For $2 \to 3$ "pentagon functions" method, when possible.
- ⚠ Biggest issue: general class of functions not understood, even mathematically. Pentagon functions beyond $\mathrm{d}\log$s?

(Semi-)numerical

Solving DEs by matching local series expansions, or numerical Monte-Carlo integration of optimized bases.

- ✓ Successfully sidestep analytic complexity with few dynamic scales.
- ✓ Less sensitive to analytic complexity, masses may actually help in practice.
- ✗ No function basis $\implies$ analytic rational coefficients hard, large numerical cancellations
- ⚠ Too slow for many dynamic scales?

Conclusions & Outlook

NNLO ~~r~~evolution

- Steady progress for $2 \to 3$ processes:
  all massless complete, first result with external masses.
- Significant progress due to paradigm shift from symbolic computations to analytic reconstruction.
- Good grasp on analytic structure of Feynman integrals and associated function spaces has been essential.
- Multi-scale loop amplitudes remain major bottleneck, case by case computations.

Conclusions & Outlook

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Outlook

- 5-point with massless loops (e.g. Hjj, VVj, VVV): feasible based on current methods.
- N^3LO applications more challenging, potentially better analytic control will be needed.
- Massive loops, few dynamic scales: feasible with (semi-)numerical methods.
- Massive loops, many dynamic scales (e.g. $t\bar{t}j, t\bar{t}H, t\bar{t}W$):
  requires major breakthroughs.
- Beyond 5-point: currently unimaginable (any relevant processes?).

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