Planar Two-loop Integrals for WW+jet Production at the LHC

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CERN & The University of Edinburgh
together with Dima Chicherin, Vasily Sotnikov and Simone Zoia

Ringberg, 2024

Two-Loop Planar Integrals for Five-Point Processes with Two Massive External Legs

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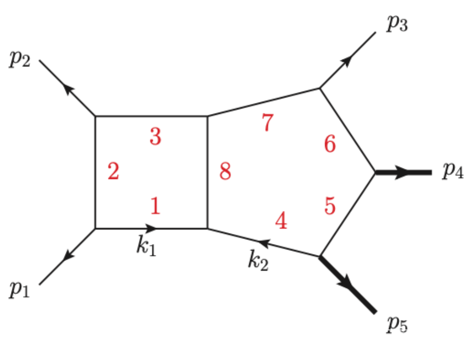
Goal, Focus and Slogans

Main goal: compute Feynman integrals to make their analytic structure transparent, and so that we can evaluate them in a stable and efficient way

Focus: Planar Feynman integrals for processes with five external particles, two of them massive, and with massless propagators



- This should be easy and boring!
  - Everything that can be made massless was made massless
  - Describing processes with 3 particles in the final state at 2nd order in perturbation theory
  - Functions that appear are the ones we've been saying we understand well for a long time!
  - Five-point one-mass @ 2 loops was not that easy…
  - … but we have better tools and it actually was simple for five-point two-mass @ 2 loops!
  - First explorations in [2401.07632, Jiang, Liu, Xu, Yang, 24]

Precision!

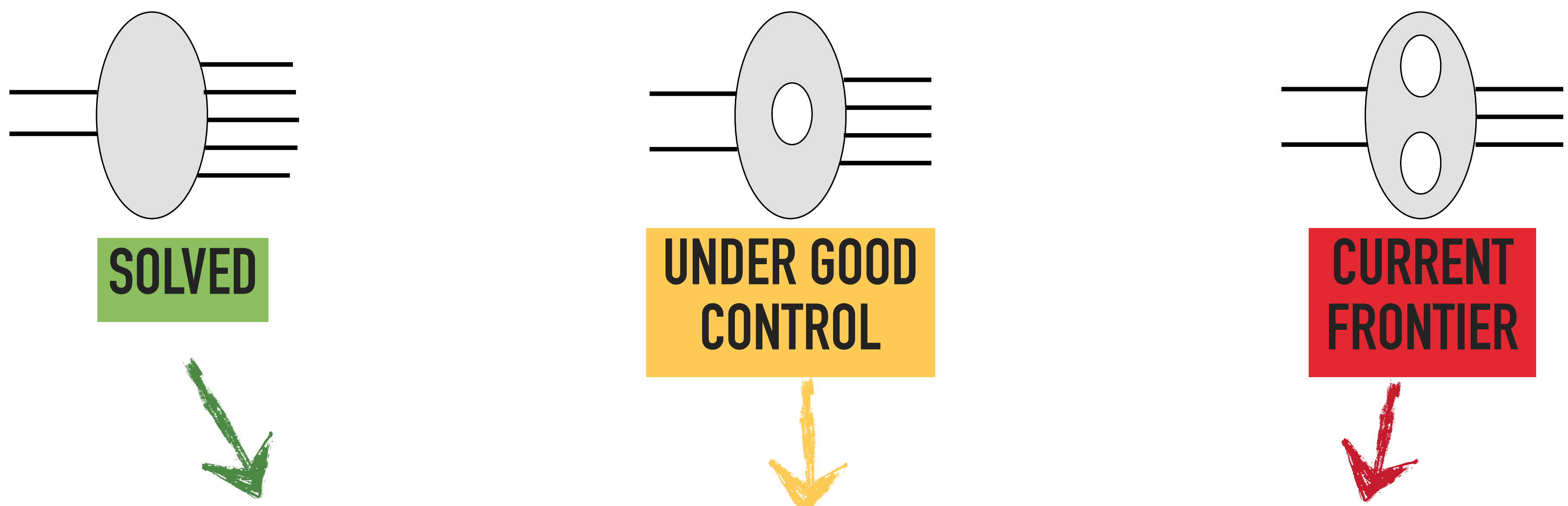
- Percent-level precision

[See Fabrizio's, Federico's, Vasily's, Andreas' talks]

$$\sigma = \sigma_{LO} \left(1 + \alpha_s \sigma_{NLO} + \alpha_s^2 \sigma_{NNLO} \right) + \mathcal{O}(\alpha_s^3)$$

$\sim \mathcal{O}(10\%)$ $\qquad \sim \mathcal{O}(1\%)$

- Amplitudes for NNLO corrections (five-point processes)



$$\sigma_{NNLO} = \sigma_{RR} + \sigma_{RV} + \sigma_{VV}$$

- Factorisation of work: amplitudes and phase-space integration

$$\sigma \sim \int d\Phi \, |\mathscr{A}|^2$$

NB: Divergences appear, work in Dimensional Regularisation, $D = 4 \to D = 4 - 2\epsilon$

Amplitudes and Feynman Integrals

- Natural factorisation

$$\mathscr{A} = \sum c_i(\vec{p};\epsilon)\ m_i(\vec{p};\epsilon)$$

Master coefficients
- process/theory specific
- rational functions

Master integrals
- kinematic dependent
- `special' functions

1. Feynman integrals as vector spaces
   - ✓ Integration-by-parts (IBP) relations and master integrals

2. How to compute (multi-scale) Feynman integrals?
   - ✓ Differential equations and pure basis

   } Enough for formal studies, e.g., $\mathcal{N} = 4$ sYM

3. How to (efficiently) evaluate Feynman integrals?
   - ✓ Numerical methods and pentagon functions

   } Non-trivial, required for pheno studies

Feynman Integrals as Vector Spaces: IBP relations

$$I(p_1,\ldots,p_E;m_1^2,\ldots,m_p^2;\nu;D) = \int\left(\prod_{j=1}^{L} e^{\gamma_E\epsilon}\frac{d^Dk_j}{i\pi^{D/2}}\right)\frac{\mathcal{N}(\{k_j\cdot k_l, k_j\cdot p_l\};D)}{\prod_{j=1}^{p}(m_j^2-q_j^2-i\varepsilon)^{\nu_j}}$$

[Tkachov; Chetyrkin, Tkachov, 81]

$$\int d^Dk_i\,\frac{\partial}{\partial k_i^\mu}\left[v^\mu\,\frac{\mathcal{N}(\{k_j\cdot k_l, k_j\cdot p_l\};D)}{\prod_{j=1}^{p}(m_j^2-q_j^2-i\varepsilon)^{\nu_j}}\right]=0$$

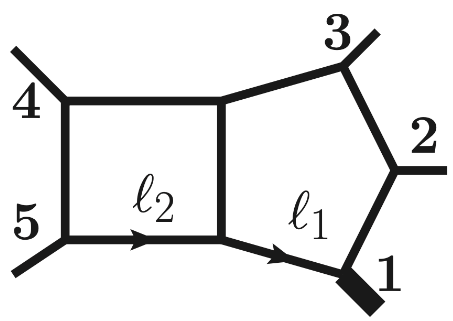
- Linear relations of integrals with different ν_j

- IBP relations can generate integrals with new propagators
  - A *family/topology* contains enough propagators for this not to happen
- Integrals in a family related by IBP relations, rational in scales and D
  - Reduce integrals to a set of master integrals
- The number of master integrals is always finite
  - Computed from critical points, Euler characteristics, …
  - Finite number of integrals needed to solve a family
- Each family defines a (finite dimensional) vector space
  - Like for any vector space, some bases are better than others

Feynman Integrals as Vector Spaces

Example 1: five-point one-mass scattering at two loops ; Planar VS Non-Planar

- ✓ Depend on 6 variables
- ✓ Penta-boxes:
  [2005.04195]



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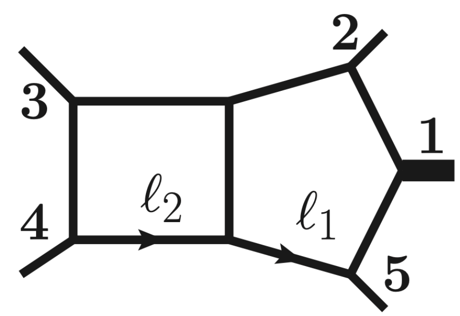

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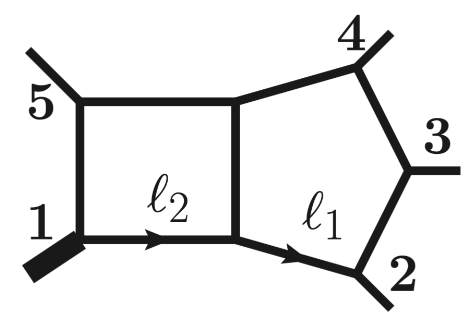

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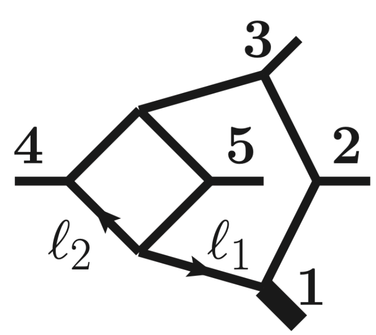
- ✓ Hexa-boxes:
  [2107.14180]



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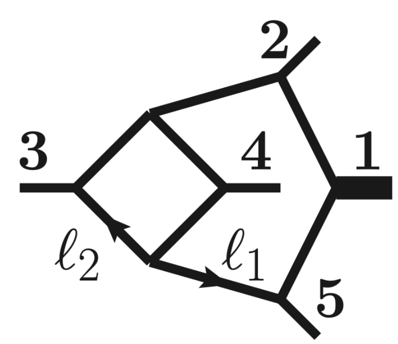

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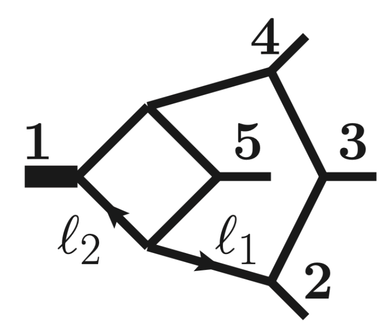

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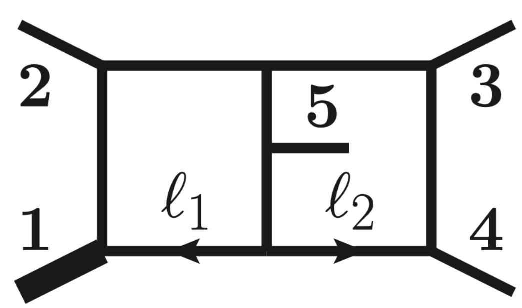
- ✓ Double pentagons:
  [2306.15431]



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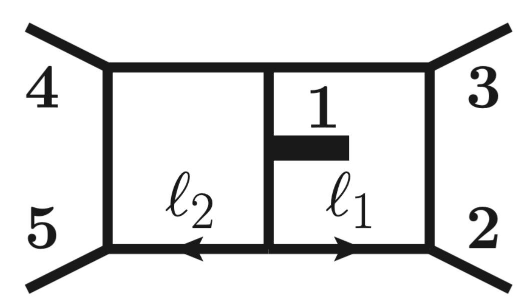

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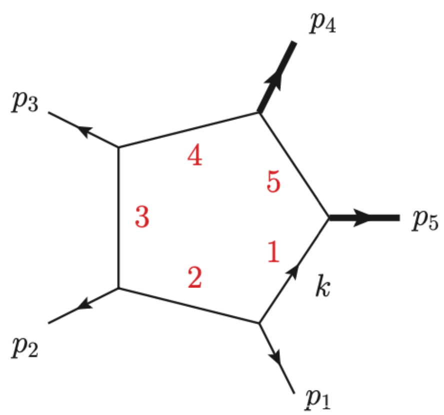
Feynman Integrals as Vector Spaces

Example 2: five-point two-mass scattering ; one VS two loops

- ✓ Depend on 7 variables



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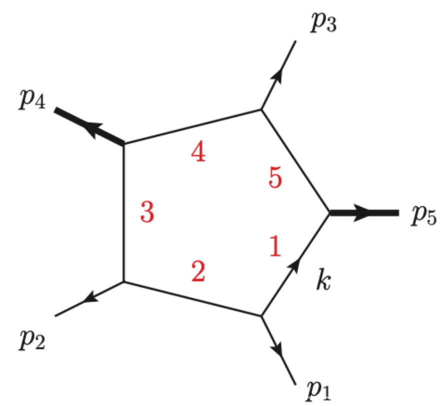

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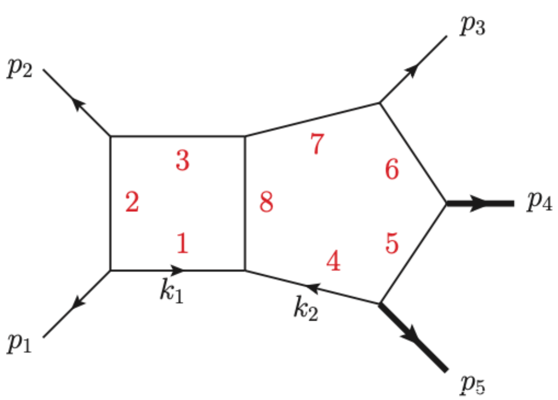

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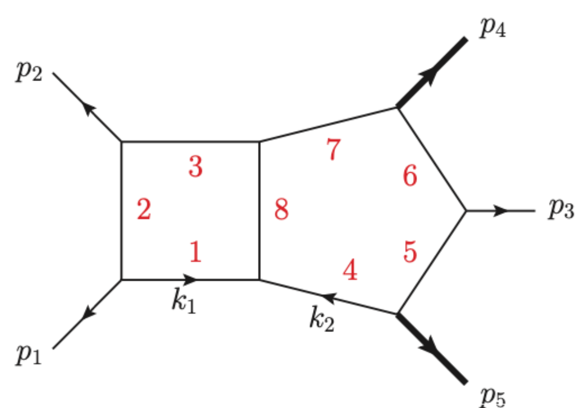

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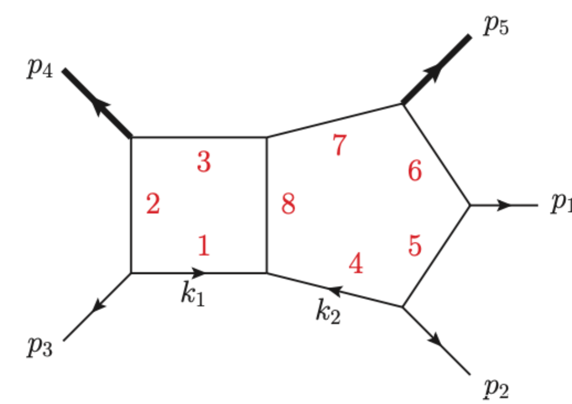

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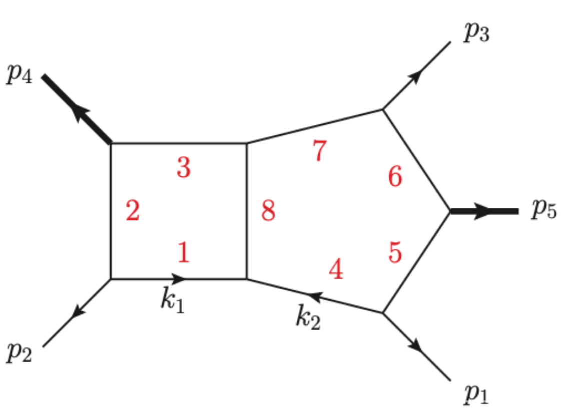

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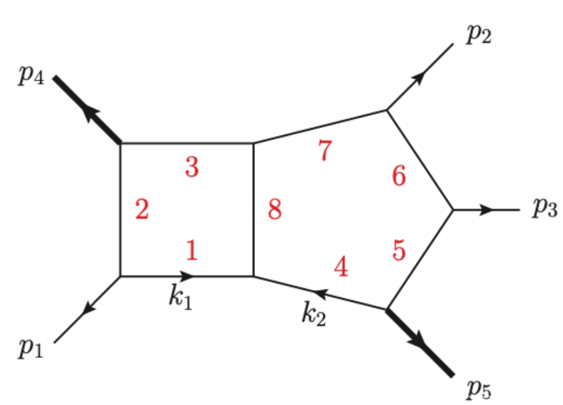

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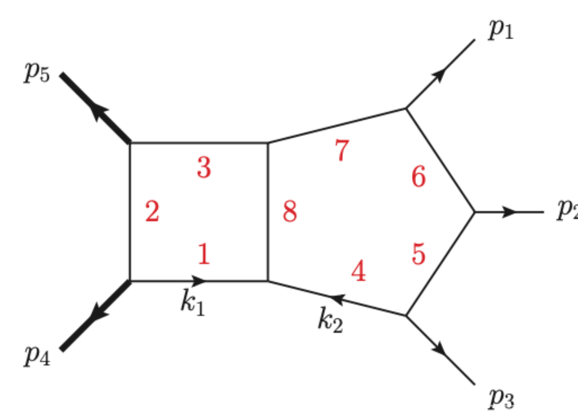

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[2401.07632, Jiang, Liu, Xu, Yang, 24]

Feynman Integrals as Vector Spaces: Summary

- All about solving IBP relations
  - IBP relations are easy to write, but hard to solve
  - Several approaches: Laporta's algorithm (most successful approach), intersection theory, recurrence relations, …

- Implemented in several public codes
  - Kira, FIRE, NeatIBP, FiniteFlow, Reduze, LiteRed …

- Bottleneck in many applications
  - Only use analytics when it cannot be avoided
  - Bypass large analytic expressions with numerical evaluations (in finite fields)

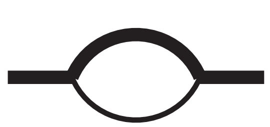
Computing Feynman Integrals: Differential Equations

- **Goal:** evaluate integrals around $D = 4$ dimensions (as expansion in ϵ)
- Many ways to compute Feynman integrals
  - ✓ Analytic/numerical integration of parametric representation
  - ✓ Transform into differential equation problem [Kotikov, 91; Bern et al, 94; Remiddi, 97; Gehrmann, Remiddi 00]
- Let $\vec{\mathcal{I}}$ be a set of master integrals ; it is closed under differentiation

$$\partial_{x_i} \vec{\mathcal{I}}(x,\epsilon) = A_{x_i}(x,\epsilon)\, \vec{\mathcal{I}}(x,\epsilon)$$

  - ✓ Derivatives change powers of propagators $\Rightarrow$ reduce to masters with IBPs
  - ✓ IBPs are rational in x and $D = 4 - 2\epsilon \;\Rightarrow A_{x_i}(x,\epsilon)$ has rational entries
  - ✓ For generic $\vec{\mathcal{I}}$, not clear we gain a lot… but some bases are better than others!

Example: one-loop bubble with one massive propagator, $\mathcal{I} = \{I(1,1), I(1,0)\}$



$$\partial_{m_1^2} \vec{\mathcal{I}} = \begin{pmatrix} -I(2,1) \\ -I(2,0) \end{pmatrix} = \begin{pmatrix} \frac{(D-3)(m_1^2-p^2)}{(p^2-m_1^2)^2} & \frac{(D-2)(m_1^2-p^2)}{2m_1^2(p^2-m_1^2)^2} \\ 0 & \frac{D-2}{2m_1^2} \end{pmatrix} \vec{\mathcal{I}}$$

Computing Feynman Integrals: Pure Bases

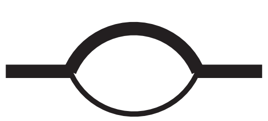
- If possible (!!), find new basis $\vec{\mathcal{J}}(x,\epsilon)$ such that [Henn, 13]

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\, A(x)\, \vec{\mathcal{J}}(x,\epsilon) \qquad A(x) = \sum_i A_i\, d\log W_i$$

  - ✓ A_i are matrices of rational numbers, all x dependence in W_i
  - ✓ only has logarithmic singularities, explicit in the differential equation
  - ✓ organises ϵ dependence, easier to solve order by order
  - ✓ solution trivial to write in terms of iterated integrals, order by order in ϵ

- All analytic information made manifest
  - ✓ W_i give logarithmic singularities/branch cuts: symbol alphabet
  - ✓ A_i tell us how singularities interact: (extended) Steinmann relations, ...

Example: Pure basis for one-loop bubble with one massive propagator ($u = p^2/m_1^2$)



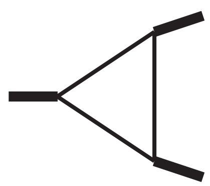
$$\partial_u \vec{\mathcal{J}}(u;\epsilon) = \epsilon \left[\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \mathrm{dlog}\,(1-u) + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mathrm{dlog}\, u \right] \vec{\mathcal{J}}(u;\epsilon)$$

Computing Feynman Integrals: Pure Bases

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\Big(\sum_i A_i\, d\log W_i(x)\Big)\,\vec{\mathcal{J}}(x,\epsilon)$$

- No general algorithm to find a pure basis (automated codes exist, with limitations)
  - ✓ leading singularities
  - ✓ cuts/on-shell differential equations
  - ✓ ideas from $\mathcal{N}=4$ sYM

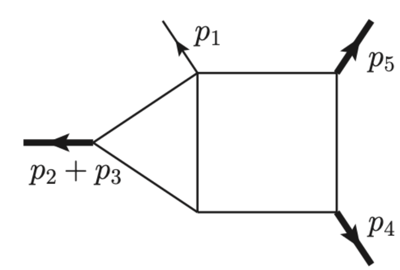
- Leading singularities: this is where square roots appear!

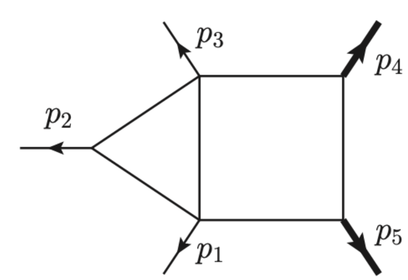


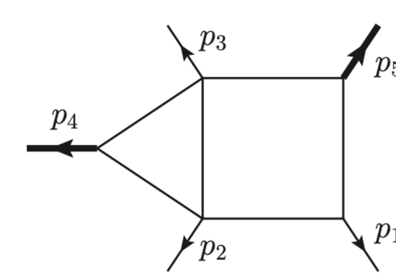
$$\sim \frac{1}{\sqrt{\Delta_3}}\mathcal{T}$$

  - ✓ Determine Δ_3 without computing the integral
  - ✓ Compute as residue of integrand

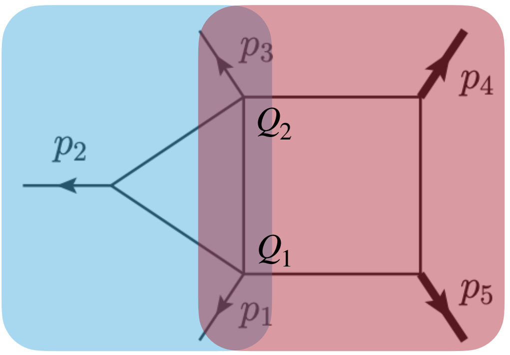
- 44 square roots for 2-loop 5-pt 2mass (10 for 2-loop 5-pt 1m)!
  - ✓ 3-point Gram Δ_3, degree 2: 7 permutations
  - ✓ 5-point Gram Δ_5, degree 4: 1 permutation
  - ✓ 4-point 3-mass root, degree 4: 18 permutations
  - ✓ New degree 4 root: 6 permutations
  - ✓ New degree 4 root: 12 permutations





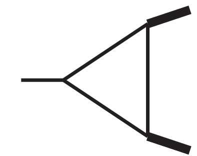


Computing Feynman Integrals: The New Roots

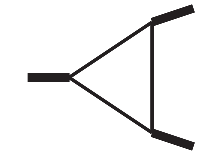


$$= \int d\ell_1 \frac{1}{D_1 D_2} \frac{1}{\sqrt{\Delta_\Box(\ell_1)}}$$

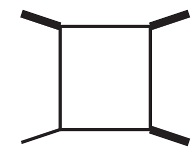
$$= \int d\ell_2 \frac{1}{D_1 D_2 D_3} \frac{1}{Q_1^2(\ell_2) - Q_2^2(\ell_2)}$$



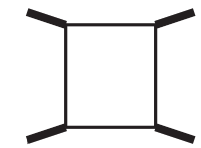
$\sim \frac{1}{Q_1^2 - Q_2^2}$



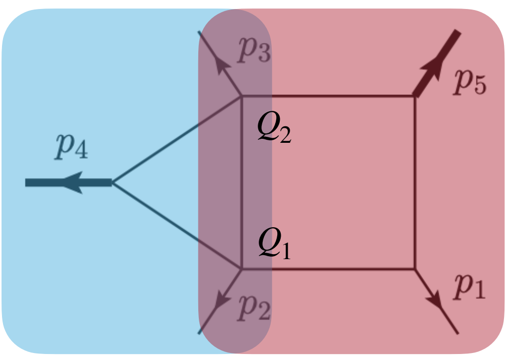
$\sim \frac{1}{\sqrt{\Delta_3}}$



$\sim \frac{1}{st - Q_1^2 Q_3^2}$



$\sim \frac{1}{\sqrt{\Delta_\Box}}$

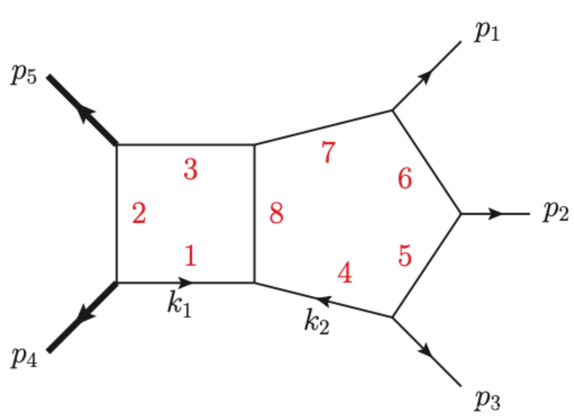


$$= \int d\ell_2 \frac{1}{D_1 D_2 D_3} \frac{1}{\sqrt{\Delta_3(\ell_2)}}$$

$$= \int d\ell_1 \frac{1}{D_1 D_2} \frac{1}{st(\ell_1) - Q_1^2(\ell_1) p_5^2}$$

- ✓ Need to work a bit harder to compute root…

- Side comment: one of the integrals comes with two roots!

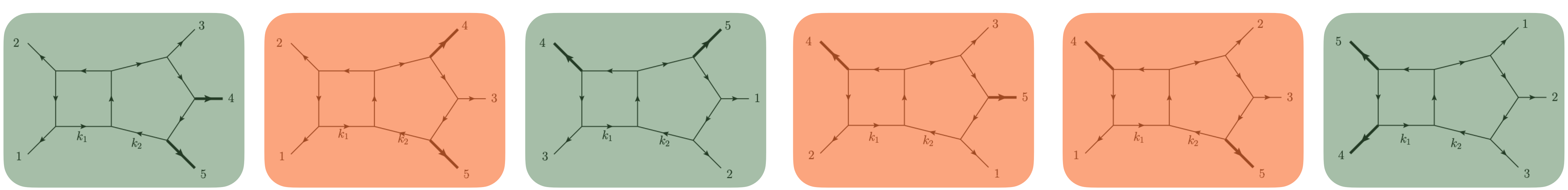


$$\sim \frac{1}{\sqrt{\Delta_3}\sqrt{\Delta_5}}$$

Computing Feynman Integrals: Alphabets and Letters

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\Big(\sum_i A_i\, d\log W_i(x)\Big)\vec{\mathcal{J}}(x,\epsilon)$$

- Getting diff. eq. relies on IBPs: difficult to do analytically…
- If the W_i are known, determine the A_i from numerical IBPs!
  - ✓ removes the IBP bottleneck, allows to attack multi-scale problems
- The W_i give singularities of Feynman integrals ⇒ Landau conditions
  - ✓ Factorisation of work: determine W_i without computing the differential equation!
  - ✓ Active area of research in Amplitudes area: coactions, solving Landau conditions, principal A-determinants, Gram determinants, Schubert problem, …
  - ✓ Two highlights: **[2311.14669, Fevola, Mizera, Telen]**, **[2401.07632, Jiang, Liu, Xu, Yang, 24]**
- Baikovletter [2401.07632] misses one of the new five-point roots…
  - ✓ Not really an issue, we know it's there



Computing Feynman Integrals: Symbol Alphabet

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon \left(\sum_i A_i \, d\log W_i(x) \right) \vec{\mathcal{J}}(x,\epsilon)$$

family	dim(fam)	family	dim(fam)
Pa	16	PBmzz	105
Pb	15	PBzmz	104
PBmmz	94	PBzzm	104
PBmzm	87	PBzzz	127

Table 1: Number of master integrals in each family

family	dim($\mathcal{A}_{\text{fam}}$)	family	dim($\mathcal{A}_{\text{fam}}$)
Pa	43	PBmzz	80
Pb	39	PBzmz	96
PBmmz	85	PBzzm	82
PBmzm	52	PBzzz	104

Table 2: Dimension of the alphabet for each family

- ✓ Overall, 570 independent letters for two-loop five-point two-mass kinematics
- ✓ Even letters (215): polynomials/rational functions in the kinematic variables
- ✓ Odd letters in one square root (236): $W = \dfrac{P(\vec{s}) + Q(\vec{s})\sqrt{\Lambda(\vec{s})}}{P(\vec{s}) - Q(\vec{s})\sqrt{\Lambda(\vec{s})}}$
  - in this case, there are 44 different $\Lambda(\vec{s})$
- ✓ Odd letters in two square roots (119): $W = \dfrac{P(\vec{s}) + Q(\vec{s})\sqrt{\Lambda_1(s)}\sqrt{\Lambda_2(s)}}{P(\vec{s}) - Q(\vec{s})\sqrt{\Lambda_1(s)}\sqrt{\Lambda_2(s)}}$
- ✓ Most letters from `Baikovletter`, others (mostly odd) we determine ourselves

[Heller, von Manteuffel, Schabinger, 20]

[Abreu, Ita, Page, Tschernow, 20]

Computing Feynman Integrals: Summary

- Differential equations compute all master integrals in one go
  - ✓ Getting the diff. eq. relies on IBPs, find ways around it
- Pure bases: singular structure manifest and simplify ϵ dependence
  - ✓ Factorised problem: determine the singularities
  - ✓ Use numerical IBPs to get analytic differential equations
- Very explicit and compact analytic representation for Feynman integrals
  - ✓ Gives important information for amplitude calculation
  - ✓ Sufficient for formal studies $\Rightarrow \mathcal{N} = 4$ sYM calculations
- How do we solve the differential equations? i.e., how to get numbers!?
  - ✓ Determine the initial conditions
  - ✓ Find efficient ways to get numerical evaluations

Evaluating Feynman Integrals: Initial Condition

$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\Big(\sum_i A_i\, d\log W_i(x)\Big)\vec{\mathcal{J}}(x,\epsilon)$$

- General solution singular at all $W_i = 0$ but Feynman integrals are not
  - ✓ Imposing this condition allows to determine the initial condition!

Used for 5pt 1m @ 2loops, [Abreu, Ita, Moriello, Page, Tschernow, Zeng, 20, 21]

- AMFlow approach: [Liu, Ma, 22]
  - ✓ Go to (non-physical) limit where all integrals become tadpoles, known to 5 loops
  - ✓ Evolve back to physical points — Used for 5pt 1m @ 2loops, [Abreu et al, 23]
  - ✓ Obtain high-precision ($\mathcal{O}(100)$ digits) numerical evaluation at random point

- In our case: Euclidean/physical-region initial conditions $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_4, s_5\}$

$$X_{\rm eu} = \left(-\frac{3}{2}, -3, -\frac{57}{8}, -\frac{23}{4}, -\frac{5}{8}, -11, -1\right) \qquad X_0 = (7, -1, 2, 5, -2, 1, 1)$$

  - ✓ 80 digits evaluations (took ~ 1 week). Sufficient for pentagon functions

Evaluating Feynman Integrals: Solving the DEs

- Trivial solution in terms of Chen iterated integrals, order by order in ϵ

$$[W_{i_1}, \ldots, W_{i_w}]_{\vec{s}_0}(\vec{s}) = \int_\gamma [W_{i_1}, \ldots, W_{i_{w-1}}]_{\vec{s}_0} \mathrm{dlog}\, W_{i_w}$$

  - γ connects $\vec{s}_0$ and $\vec{s}$; $[]_{\vec{s}_0} \equiv 1$

  - ✓ Formal solution, not trivial to evaluate…

- Numerical solution [Moriello, 19 ; Hidding, 20; Armadillo et al, 22; Liu, Ma, 22]
  - ✓ Start from known initial condition, and evolve along path
  - ✓ Generalised power-series solution with finite convergence radius

$$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{N_{i,k}} \mathbf{c}_k^{(i,j_1,j_2)} (t - t_k)^{\frac{j_1}{2}} \log{(t - t_k)}^{j_2}$$

  - ✓ High-precision, but slow…

- Write solution in terms of special functions (multiple polylogarithms, …) …

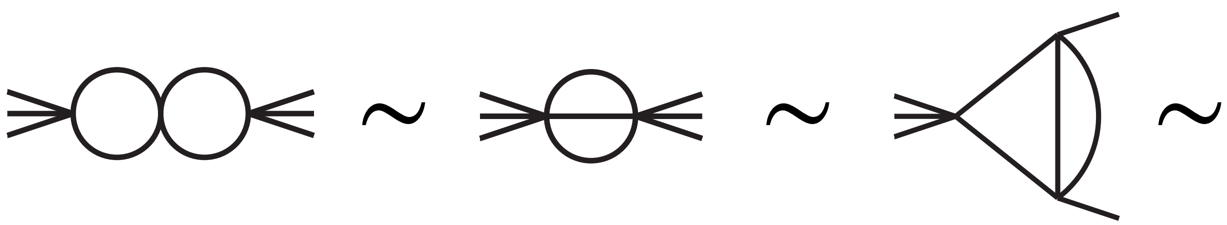
For planar 5pt 1m @ 2loops, [Canko, Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 20-22]

- Roots make it hard/impossible, and not the most convenient representation
  - ✓ Introduces spurious singularities
  - ✓ complicated branch cut structure means expression only valid in small region

Evaluating Feynman Integrals: Pentagon Functions

- Master integrals are linearly independent *before expansion in ϵ* [Gehrmann, Henn, Lo Presti, 18] [Chicherin, Sotnikov, 20]

- After expansion in ϵ, there are new relations:



$$\sim r_0 + r_1\,\epsilon\,\ln(s) + r_2\,\epsilon^2\,\ln^2(s) + \ldots$$

- Make relations explicit: build basis of special functions at each order in ϵ

- Improved algorithm for two-loop five-point one-mass processes

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 23]

1. Solve in terms of Chen iterated integrals, order by order in ϵ

$$[W_{i_1}, \ldots, W_{i_w}]_{\vec{s}_0}(\vec{s}) = \int_\gamma [W_{i_1}, \ldots, W_{i_{w-1}}]_{\vec{s}_0} \mathrm{dlog}\, W_{i_w}$$

   ✓ Simple algebra for Chen iterated integrals (with dlog kernels)!

2. Choose components of Feynman integrals as pentagon functions

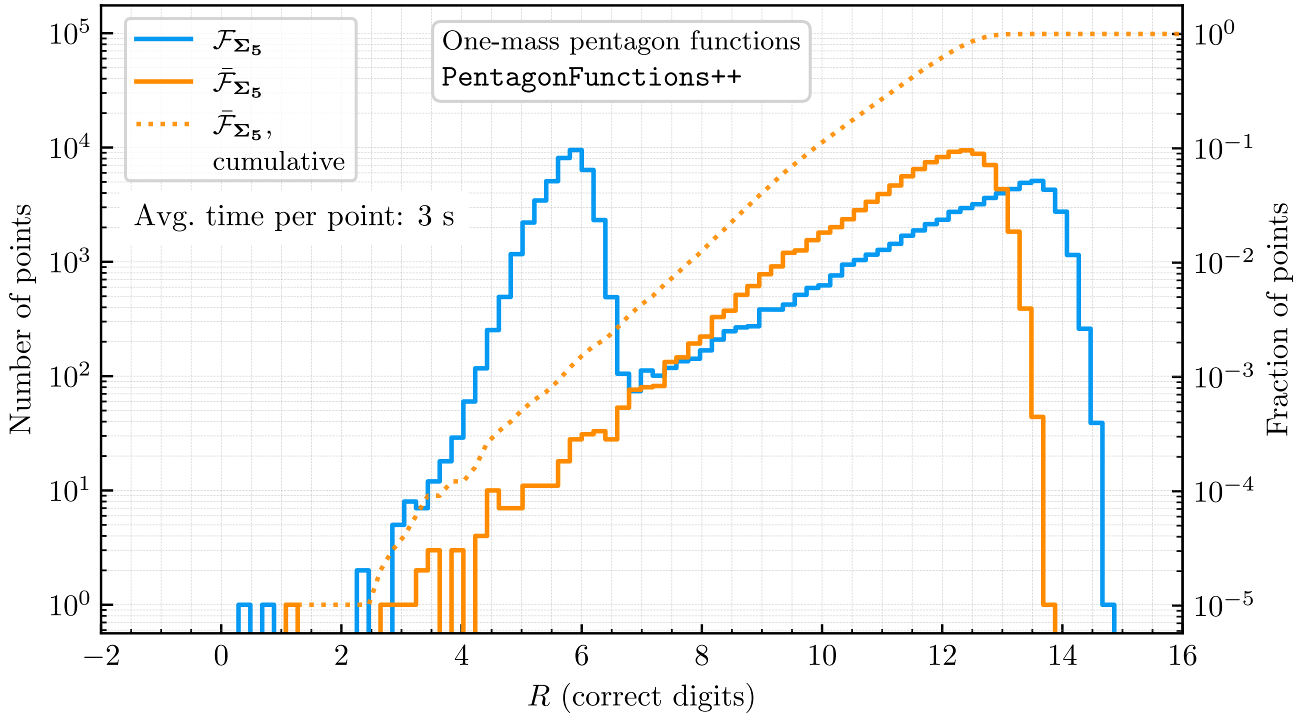
3. Use \`symbol technology' to write all integrals in terms of basis

4. Implement in C++ code

Evaluating Feynman Integrals: Pentagon Functions

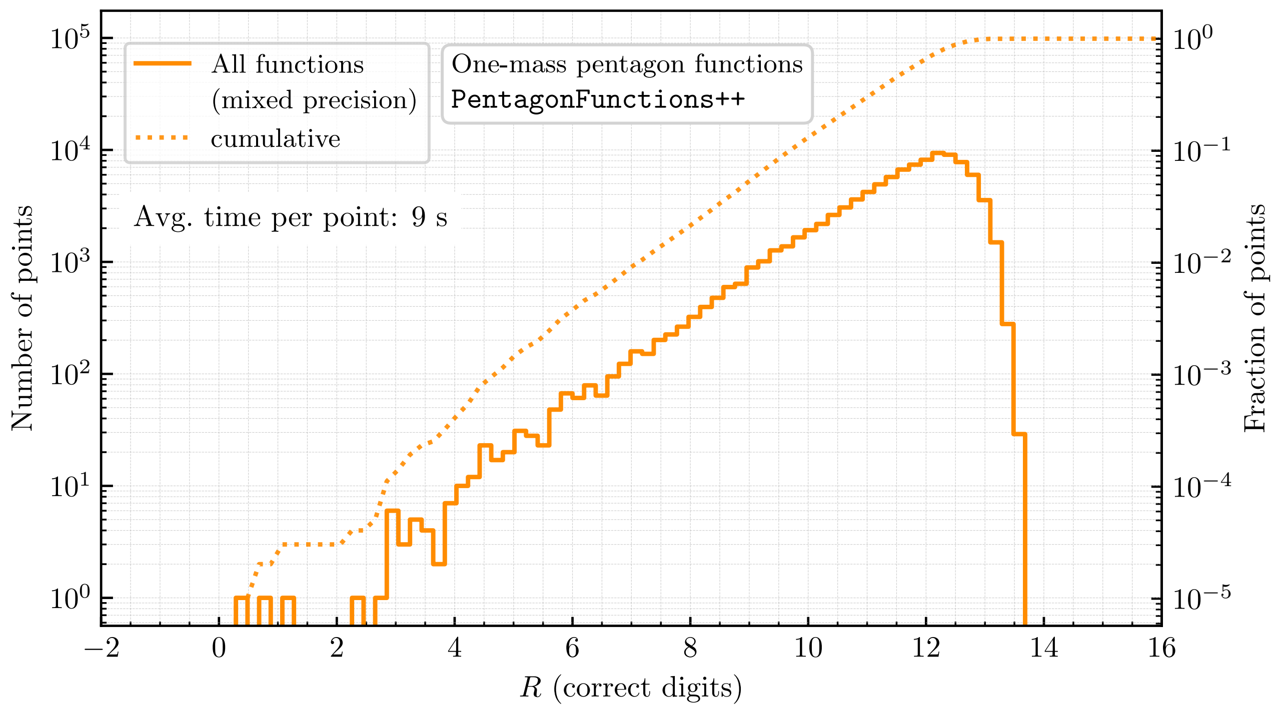
Five-point one-mass scattering at two loops

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 23]



- ✓ Standard precision, no rescue system
- ✓ Precision loss because of square root

- ✓ With rescue system
- ✓ Easy to implement if good control of analytic structure



- ✓ Ready for phenomenological applications!

Summary and Outlook

- We have mature tools that allow us to push the state of the art
  - Pheno-ready integrals available for 5pt massless and 5pt one-mass processes
  - Progress in two-loop five-point two-mass processes was much faster

    [Abreu, Chicherin, Sotnikov, Zoia, to appear]

- New results obtained with pheno in mind leading to new formal studies
  - Five-point one-mass integrals used in $\mathcal{N} = 4$ bootstrap program

    [e.g., Dixon, Gurdogan, Liu, McLeod, Wilhelm 23]

- Are pentagon functions actually a good basis?
  - We know that they are not at one loop
  - Include rational factors to make them have better behaved limits

- New challenges ahead: what if singularities are not all dlogs?
  - Elliptic integrals and beyond!
  - A lot of developments, but still missing heavy machinery

THANK YOU!