Collaborative Research Center TRR 257

Particle Physics Phenomenology after the Higgs Discovery

Research Training Group
Physics of the Heaviest
Particles at the LHC

Subleading effects in soft-gluon emission at one-loop in massless QCD

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Czakon, Eschment, Schellenberger, JHEP 12 (2023) 126

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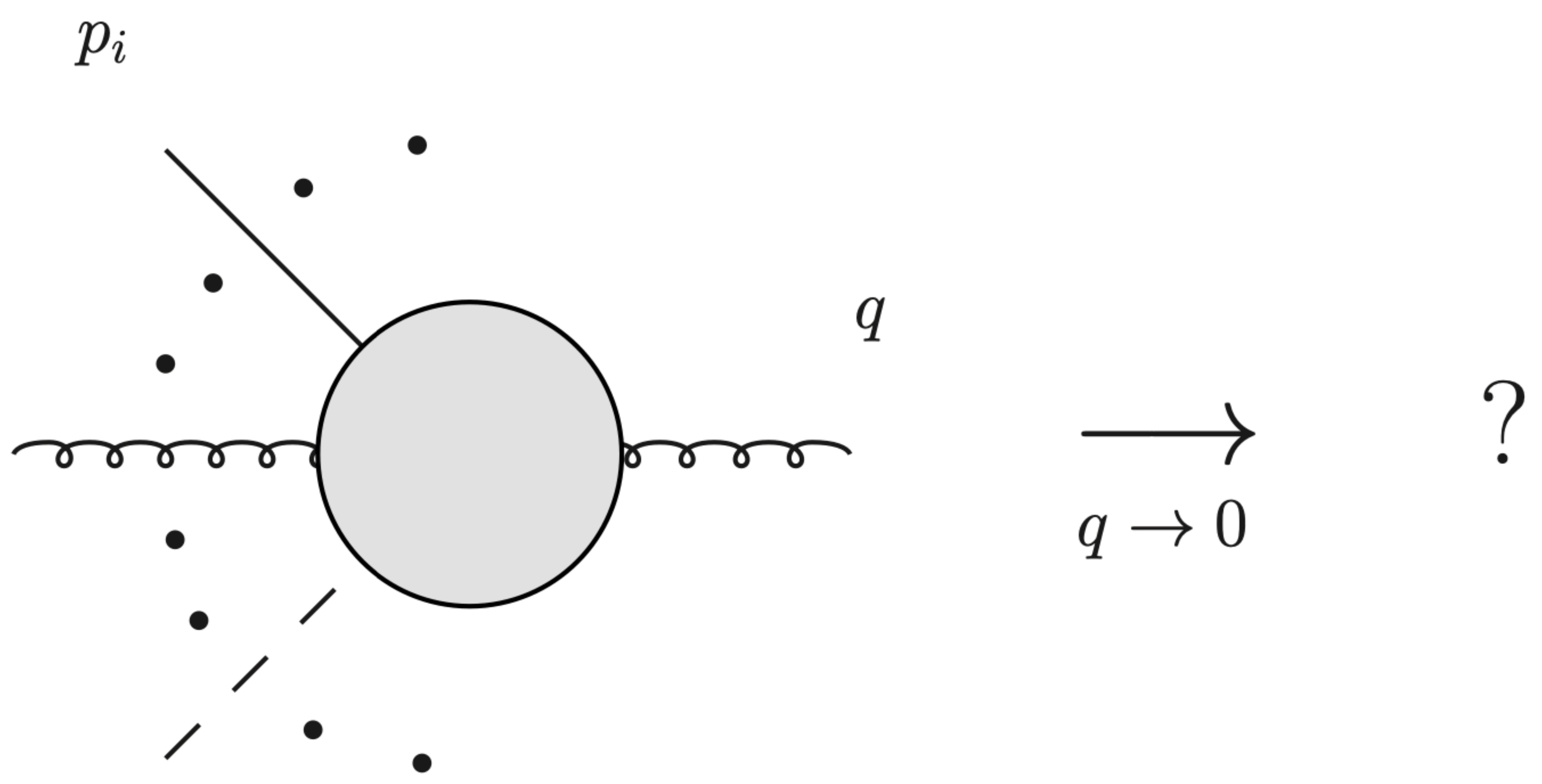
Motivation

- Classic problem:



- At leading power: eikonal approximation
  you need to calculate a soft current,
  but at least the structure is understood !

- Structure at next-to-leading power understood at tree-level QED by Low (1958), Burnett and Kroll (1968)

- Necessity to include virtual collinear enhancements at higher orders noticed by del Duca (1990)

- Extension to tree-level QCD described in 1404.5551, 1406.6987, 1406.6574

- Why bother (if you don't like pure theory) ?
  - needed to obtain cross sections approximations at subleading power in different kinematic variables
  - can be used to improve numerical stability in cross section calculations

Structure of the Result

- Several attempts to understand one-loop QCD amplitudes (more results for photon emission):
  - based on SCET: 1412.3108, 1912.01585, 2112.00018
  - based on Feynman-diagram analysis: 1503.05156, 1610.06842

- Complete characterisation in Czakon, Eschment, Schellenberger, JHEP 12 (2023) 126

$$\begin{aligned}\left|M_g^{(1)}(\{p_i+\delta_i\},q)\right\rangle &= \mathbf{S}^{(0)}(\{p_i\},\{\delta_i\},q)\left|M^{(1)}(\{p_i\})\right\rangle \\ &+ \mathbf{S}^{(1)}(\{p_i\},\{\delta_i\},q)\left|M^{(0)}(\{p_i\})\right\rangle + \int_0^1 \mathrm{d}x \sum_i \mathbf{J}_i^{(1)}(x,p_i,q)\left|H_{g,i}^{(0)}(x,\{p_i\},q)\right\rangle \\ &+ \sum_{i\neq j}\sum_{\substack{\tilde{a}_i\neq a_i\\ \tilde{a}_j\neq a_j}} \tilde{\mathbf{S}}^{(1)}_{a_i a_j\leftarrow\tilde{a}_i\tilde{a}_j,\,ij}(p_i,p_j,q)\left|M^{(0)}(\{p_i\})\Big|_{\substack{a_i\to\tilde{a}_i\\ a_j\to\tilde{a}_j}}\right\rangle + \int_0^1 \mathrm{d}x \sum_{\substack{i\\ a_i=g}} \tilde{\mathbf{J}}_i^{(1)}(x,p_i,q)\left|H_{\bar{q},i}^{(0)}(x,\{p_i\},q)\right\rangle + \mathcal{O}(\lambda)\end{aligned}$$

$$\mathbf{P}_g(\sigma,c)\,\mathbf{S}^{(0)}(\{p_i\},\{\delta_i\},q) = -\sum_i \mathbf{T}_i^c \otimes \mathbf{S}_i^{(0)}(p_i,\delta_i,q,\sigma)\left|M^{(0)}(\{p_i\})\right\rangle ,$$

$$\mathbf{S}_i^{(0)} = \frac{p_i\cdot\epsilon^*}{p_i\cdot q} + \frac{1}{p_i\cdot q}\left[\left(\epsilon^* - \frac{p_i\cdot\epsilon^*}{p_i\cdot q}q\right)\cdot\delta_i + p_i\cdot\epsilon^*\sum_j \delta_j\cdot\partial_j + \frac{1}{2}F_{\mu\nu}\left(J_i^{\mu\nu} - \mathbf{K}_i^{\mu\nu}\right)\right]$$

Kinematics and Squares

- Importance of proper definition of kinematics (see also 2401.01820) - an expansion requires an expansion parameter !

$$\boxed{0 \to a_1(p_1+\delta_1,\sigma_1,c_1)+\cdots+a_n(p_n+\delta_n,\sigma_n,c_n)+g(q,\sigma_{n+1},c_{n+1})\,, \qquad a_i \in \{q,\bar{q},g\}\,.}$$

$$\sum p_i = 0\,, \qquad \sum \delta_i + q = 0 \qquad\qquad p_i^2 = (p_i+\delta_i)^2 = m_i^2\,, \qquad q^2 = 0$$

$$p_i^\mu = \mathcal{O}(1) = \mathcal{O}(\lambda^0) \gg \lambda\,, \qquad \delta_i^\mu = \mathcal{O}(\lambda)\,, \qquad q^\mu = \mathcal{O}(\lambda) \qquad\qquad p_i\cdot\delta_i = \mathcal{O}(\lambda^2)$$

- Subleading behaviour for squared amplitudes 1706.04018 - here including the massive case

$$\left\langle M_g^{(0)}(\{k_l\},q)\middle|M_g^{(0)}(\{k_l\},q)\right\rangle = -\sum_{i\neq j}\left(\frac{k_i\cdot k_j}{(k_i\cdot q)(k_j\cdot q)}-\frac{m_i^2}{2(k_i\cdot q)^2}-\frac{m_j^2}{2(k_j\cdot q)^2}\right)\left\langle M^{(0)}(\{k_l+\delta_{il}\Delta_i+\delta_{jl}\Delta_j\})\middle|\mathbf{T}_i\cdot\mathbf{T}_j\middle|M^{(0)}(\{k_l+\delta_{il}\Delta_i+\delta_{jl}\Delta_j\})\right\rangle$$

$$k_i \equiv p_i+\delta_i \qquad \Delta_i \equiv \frac{1}{N_{ij}}\left[\left(1-\frac{m_i^2(p_j\cdot q)}{(p_j\cdot p_i)(p_i\cdot q)}\right)q+\frac{p_j\cdot q}{p_j\cdot p_i}p_i-\frac{p_i\cdot q}{p_i\cdot p_j}p_j\right] \qquad \Delta_j \equiv \frac{1}{N_{ij}}\left[\left(1-\frac{m_j^2(p_i\cdot q)}{(p_i\cdot p_j)(p_j\cdot q)}\right)q-\frac{p_j\cdot q}{p_i\cdot p_j}p_i+\frac{p_i\cdot q}{p_i\cdot p_j}p_j\right]$$

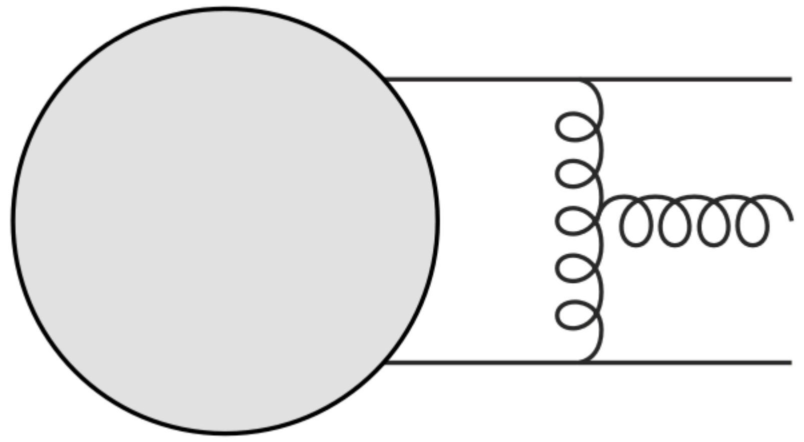
$$N_{ij} \equiv 2-\frac{m_i^2(p_j\cdot q)}{(p_j\cdot p_i)(p_i\cdot q)}-\frac{m_j^2(p_i\cdot q)}{(p_i\cdot p_j)(p_j\cdot q)}$$

- Kinematics satisfies momentum conservation and on-shellness

$$\sum_l k_l+\delta_{il}\Delta_i+\delta_{jl}\Delta_j = 0\,, \qquad (k_l+\delta_{il}\Delta_i+\delta_{jl}\Delta_j)^2 = m_l^2+\mathcal{O}(\lambda^2)$$

Flavour-Diagonal Soft Operators

- Extension of soft current to subleading behaviour



$$\mathbf{S}^{(1)}(\{p_i\},\{\delta_i\},q)\left|M^{(0)}(\{p_i\})\right\rangle$$

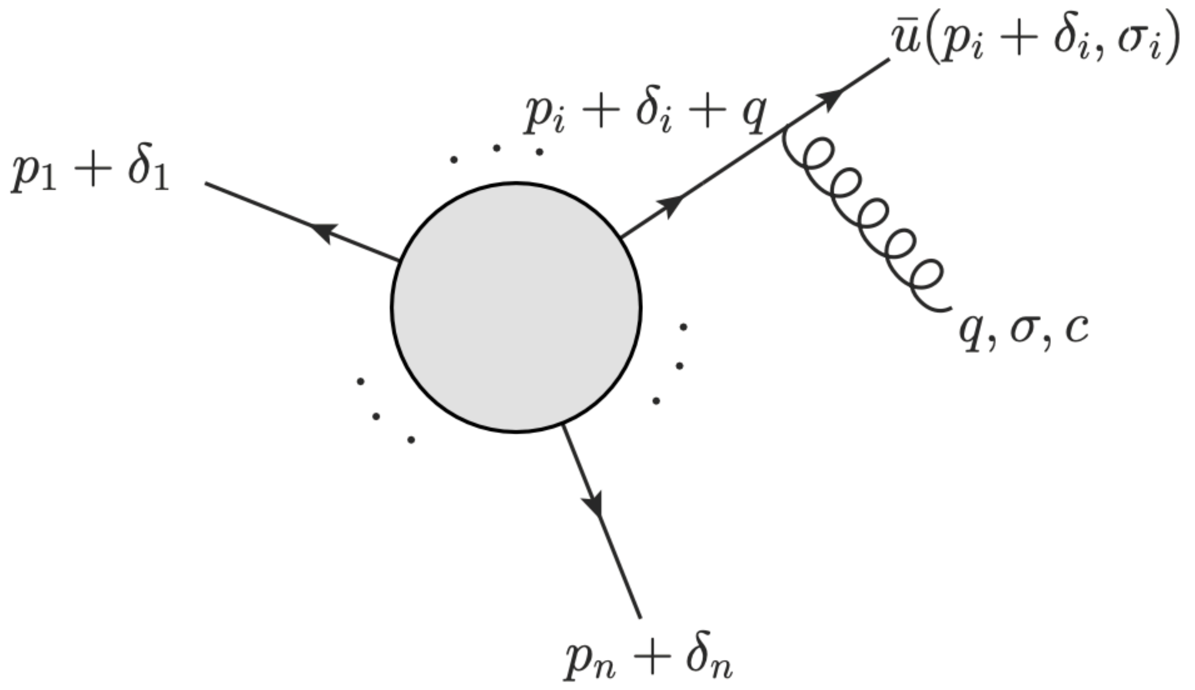
$$\mathbf{P}_g(\sigma,c)\,\mathbf{S}^{(1)}(\{p_i\},\{\delta_i\},q)+\mathcal{O}(\lambda)=$$

$$\frac{2\,r_{\text{Soft}}}{\epsilon^2}\sum_{i\neq j} i f^{abc}\mathbf{T}_i^a\mathbf{T}_j^b\otimes\left(-\frac{\mu^2 s_{ij}^{(\delta)}}{s_{iq}^{(\delta)}s_{jq}^{(\delta)}}\right)^{\epsilon}\left[\mathbf{S}_i^{(0)}(p_i,\delta_i,q,\sigma)+\frac{\epsilon}{1-2\epsilon}\frac{1}{p_i\cdot p_j}\left(\frac{p_i^\mu p_j^\nu-p_j^\mu p_i^\nu}{p_i\cdot q}+\frac{p_j^\mu p_j^\nu}{p_j\cdot q}\right)F_{\mu\rho}(q,\sigma)\left(J_i-\mathbf{K}_i\right)_\nu^{\ \rho}\right]$$

$$s_{ij}^{(\delta)}\equiv 2\,(p_i+\delta_i)\cdot(p_j+\delta_j)+i0^+\,,\qquad s_{iq}^{(\delta)}\equiv 2\,(p_i+\delta_i)\cdot q+i0^+\,,\qquad s_{jq}^{(\delta)}\equiv 2\,(p_j+\delta_j)\cdot q+i0^+\qquad r_{\text{Soft}}\equiv\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}=1+\mathcal{O}(\epsilon)$$

- Contains the tree-level soft current

$$\mathbf{S}_i^{(0)}=\frac{p_i\cdot\epsilon^*}{p_i\cdot q}+\frac{1}{p_i\cdot q}\left[\left(\epsilon^*-\frac{p_i\cdot\epsilon^*}{p_i\cdot q}q\right)\cdot\delta_i+p_i\cdot\epsilon^*\sum_j\delta_j\cdot\partial_j+\frac{1}{2}F_{\mu\nu}\left(J_i^{\mu\nu}-\mathbf{K}_i^{\mu\nu}\right)\right]$$



- Constraints on differential operators - gauge invariance not Ward identity !

$$J^{\mu\nu}(p)\equiv i\left(p^\mu\partial_p^\nu-p^\nu\partial_p^\mu\right),\qquad \partial_p^\mu\equiv\frac{\partial}{\partial p_\mu}\qquad\qquad \sum_{\sigma'}K^{\mu\nu}_{q,\sigma\sigma'}(p)\,\bar{u}(p,\sigma')\equiv J^{\mu\nu}(p)\,\bar{u}(p,\sigma)-\frac{1}{2}\bar{u}(p,\sigma)\,\sigma^{\mu\nu},\qquad \sigma^{\mu\nu}\equiv\frac{i}{2}[\gamma^\mu,\gamma^\nu],$$

Flavour-Off-Diagonal Soft Operators

- Soft quarks introduce splitting functions

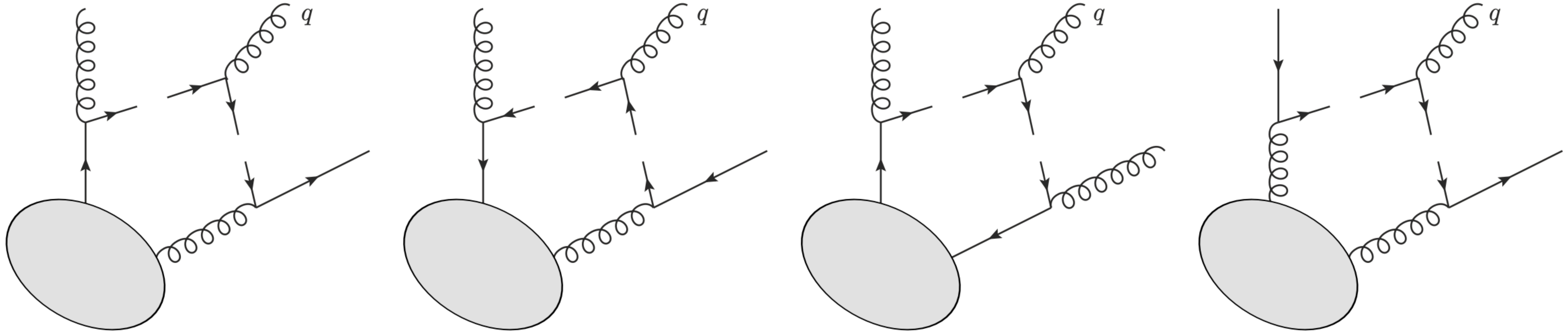
$$\sum_{i\neq j}\sum_{\substack{\tilde{a}_i\neq a_i\\ \tilde{a}_j\neq a_j}} \tilde{\mathbf{S}}^{(1)}_{a_ia_j\leftarrow\tilde{a}_i\tilde{a}_j,ij}(p_i,p_j,q)\left|M^{(0)}(\{p_i\})\right|^{a_i\to\tilde{a}_i}_{a_j\to\tilde{a}_j}\Big\rangle$$

$$\tilde{\mathbf{S}}^{(1)}_{a_ia_j\leftarrow\tilde{a}_i\tilde{a}_j,ij}(p_i,p_j,q)\left|\ldots,c_i',\ldots,c_j',\ldots;\ldots,\sigma_i',\ldots,\sigma_j',\ldots\right\rangle$$

$$= -\frac{r_{\mathrm{Soft}}}{\epsilon(1-2\epsilon)}\left(-\frac{\mu^2 s_{ij}}{s_{iq}s_{jq}}\right)^{\epsilon}\sum_{\sigma c}\sum_{\sigma_i c_i}\sum_{\sigma_j c_j}\sum_{\sigma_i'' c_i''}\sum_{\sigma_j'' c_j'} \begin{cases} T^c_{c_j''c_i''}\,\bar{v}(p_j,\sigma_j'')\,\not{\epsilon}^*(q,p_i,\sigma)\,u(p_i,\sigma_i'') & \text{for } a_i=q \text{ or } \tilde{a}_i=\bar{q} \\ T^c_{c_i''c_j''}\,\bar{v}(p_i,\sigma_i'')\,\not{\epsilon}^*(q,p_i,\sigma)\,u(p_j,\sigma_j'') & \text{for } a_i=\bar{q} \text{ or } \tilde{a}_i=q \end{cases}$$

$$\left\langle c_i,c_j'';\sigma_i,\sigma_j''\right|\mathbf{Split}^{(0)}_{a_i\tilde{a}_j\leftarrow\tilde{a}_i}(p_i,p_j,p_i)\left|c_i';\sigma_i'\right\rangle\left\langle c_j,c_i'';\sigma_j,\sigma_i''\right|\mathbf{Split}^{(0)}_{a_j\tilde{a}_i\leftarrow\tilde{a}_j}(p_j,p_i,p_j)\left|c_j';\sigma_j'\right\rangle\ \left|\ldots,c_i,\ldots,c_j,\ldots,c;\ldots,\sigma_i,\ldots,\sigma_j,\ldots,\sigma\right\rangle$$

$$\epsilon^*_\mu(q,p_i,\sigma)\equiv\epsilon^*_\mu(q,\sigma)-\frac{p_i\cdot\epsilon^*(q,\sigma)}{p_i\cdot q}q_\mu = iF_{\mu\nu}(q,\sigma)\frac{p_i^\nu}{p_i\cdot q}$$



Virtual Collinear Enhancements

$$\int_0^1 \mathrm{d}x \sum_i \mathbf{J}_i^{(1)}(x,p_i,q)\left|H_{g,i}^{(0)}(x,\{p_i\},q)\right\rangle$$

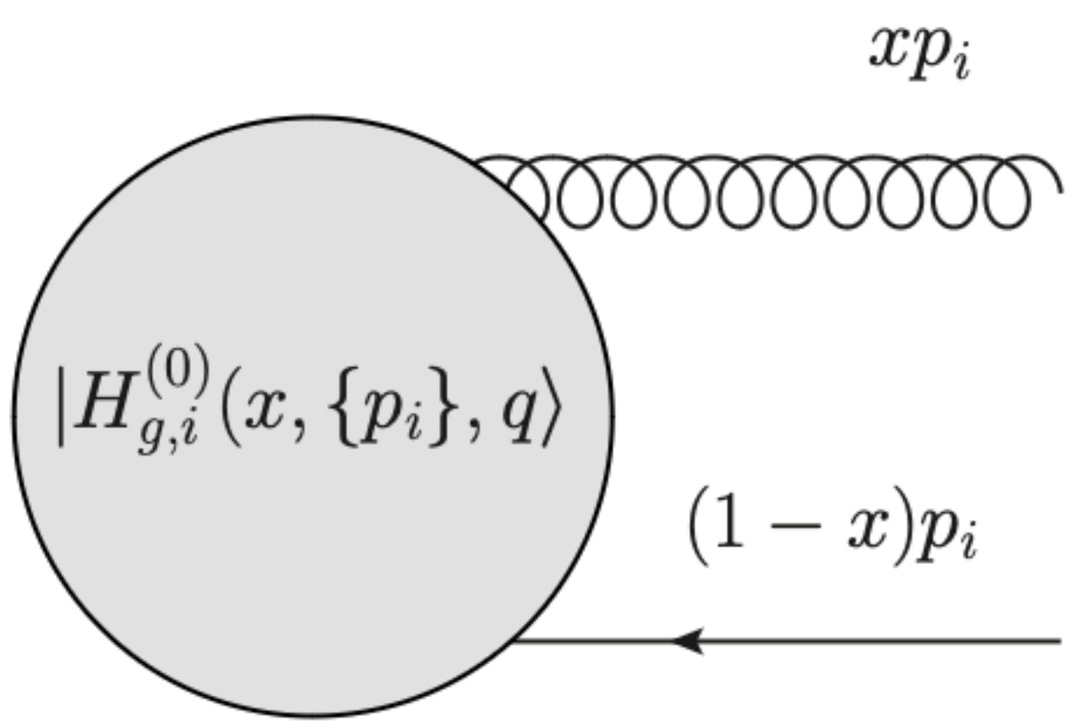
- Gauge-invariant collinear amplitudes satisfy Ward identity

  gauge invariance is not equivalent to Ward identity !

$$\boxed{a_i = g} \qquad \mathbf{P}_i(\sigma_i,c_i)\mathbf{P}_{n+1}(\sigma_{n+1},c_{n+1})\left|H_{g,i}^{(0)}(x,\{p_i\},q)\right\rangle \equiv (1-x)^{-\dim(a_i)}\mathbf{P}_i(\sigma_i,c_i)\mathbf{P}_{n+1}(\sigma_{n+1},c_{n+1})\left|\Delta M_g^{(0)}(x,\{p_i\},q)\right\rangle$$

$$-\frac{1}{x}\frac{q\cdot\epsilon^*(p_i,\sigma_{n+1})}{q\cdot p_i}\mathbf{P}_i(\sigma_i,c_i)\mathbf{T}_i^{c_{n+1}}\left|M^{(0)}(\{p_i\})\right\rangle - \frac{1}{1-x}\frac{q\cdot\epsilon^*(p_i,\sigma_i)}{q\cdot p_i}\mathbf{P}_i(\sigma_{n+1},c_{n+1})\mathbf{T}_i^{c_i}\left|M^{(0)}(\{p_i\})\right\rangle$$

- One would hope that collinear-enhanced contributions are given by collinear asymptotics



$$\left|\Delta M_{g,i}^{(0)}(x,\{p_i\},q)\right\rangle \equiv \lim_{l_\perp\to 0}\left[\left|M_g^{(0)}(\{k_i\}_{i=1}^n,k_g)\right\rangle - \mathbf{Split}_{i,n+1\leftarrow i}^{(0)}(k_i,k_g,p_i)\left|M^{(0)}(\{p_i\})\right\rangle\right]$$

$$k_g \equiv xp_i + l_\perp - \frac{l_\perp^2}{2x}\frac{q}{p_i\cdot q}\,, \qquad \text{with} \qquad l_\perp\cdot p_i = l_\perp\cdot q = 0\,,$$

$$k_i \equiv (1-x)p_i - l_\perp - \frac{l_\perp^2}{2(1-x)}\frac{q}{p_i\cdot q}\,, \qquad \text{and} \qquad k_j \equiv p_j + \mathcal{O}(l_\perp^2)\,, \qquad j\neq i$$

Collinear Amplitudes

- Collinear asymptotics from modified diagrams ?

$$\int_0^1 \mathrm{d}x \sum_i \mathbf{J}_i^{(1)}(x,p_i,q)\left|H^{(0)}_{g,i}(x,\{p_i\},q)\right\rangle$$

$$\mathbf{P}_i(\sigma_i,c_i)\mathbf{P}_g(\sigma,c)\left|\Delta M^{(0)}_{g,i}(x,\{p_i\},q)\right\rangle =$$
$$\Big[\mathbf{P}_i(\sigma_i,c_i)\mathbf{P}_g(\sigma,c)\left|M_g^{(0)}(\{p_1,\ldots,(1-x)p_i,\ldots,p_n\},xp_i)\right\rangle\Big]_{\substack{\text{non-singular}\\ \text{diagrams}}} - \delta_{\sigma_i,-s_i\sigma}\sum_{c_i'} T^c_{a_i,c_ic_i'}\left[\begin{cases} \dfrac{\bar u\big((1-x)p_i,\sigma_i\big)\not\epsilon^*(p_i,\sigma)\not q}{2\,p_i\cdot q}\dfrac{\partial}{\partial \bar u_i} & \text{if } a_i=q \\ \dfrac{\not q\not\epsilon^*(p_i,\sigma)v\big((1-x)p_i,\sigma_i\big)}{2\,p_i\cdot q}\dfrac{\partial}{\partial v_i} & \text{if } a_i=\bar q \\ \dfrac{(2x-1)q}{p_i\cdot q}\cdot\dfrac{\partial}{\partial \epsilon_i^*} & \text{if } a_i=g \end{cases}\right]\mathbf{P}_i(\sigma_i,c_i')\left|M^{(0)}(\{p_i\})\right\rangle$$

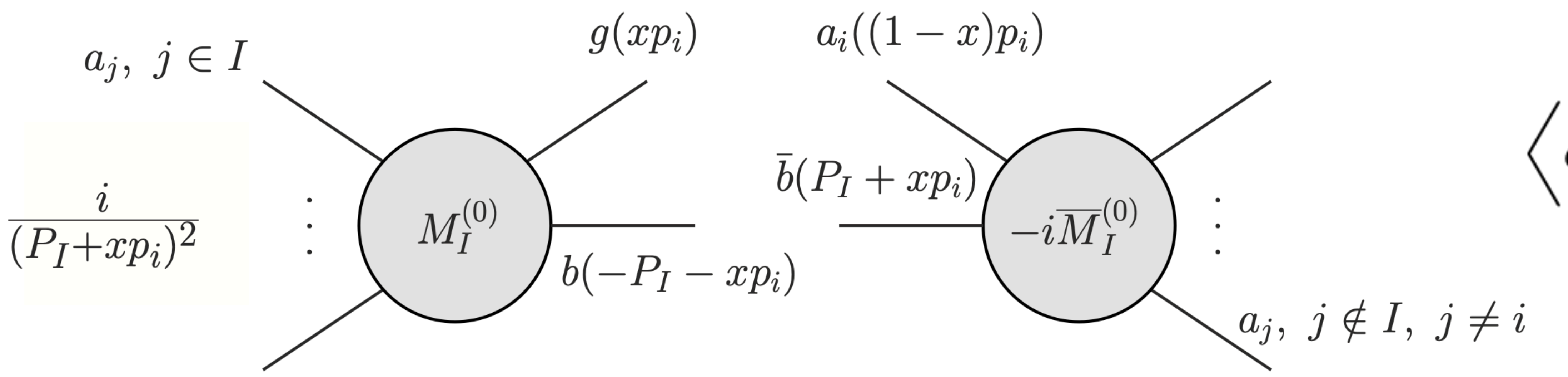
- Much better: tree-level amplitude is rational in x $\Longrightarrow$ use partial fractioning to uncover structure

LP-soft g | NLP-soft g | LP-soft i | "Regular" | "Linear"

$$\left|H^{(0)}_{g,i}(x,\{p_i\},q)\right\rangle = \left(\frac{1}{x}+\dim(a_i)\right)\left|S^{(0)}_{g,i}(\{p_i\},q)\right\rangle + \left|C^{(0)}_{g,i}(\{p_i\},q)\right\rangle + \frac{x}{1-x}\left|\bar S^{(0)}_{g,i}(\{p_i\},q)\right\rangle + \sum_I\left(\frac{1}{x_I-x}-\frac{1}{x_I}\right)\left|R^{(0)}_{g,i,I}(\{p_i\})\right\rangle + x\left|L^{(0)}_{g,i}(\{p_i\},q)\right\rangle$$

from symmetry only if i is gluon

$$\left|L^{(0)}_{g,i}(\{p_i\},q)\right\rangle = \left|\bar S^{(0)}_{g,i}(\{p_i\},q)\right\rangle - \left|S^{(0)}_{g,i}(\{p_i\},q)\right\rangle + \left|\bar C^{(0)}_{g,i}(\{p_i\},q)\right\rangle - \left|C^{(0)}_{g,i}(\{p_i\},q)\right\rangle + \frac{1}{2}\sum_I\left(\frac{1}{x_I}+\frac{1}{1-x_I}\right)\left(\left|R^{(0)}_{g,i,I}(\{p_i\})\right\rangle - \left|\bar R^{(0)}_{g,i,I}(\{p_i\})\right\rangle\right)$$



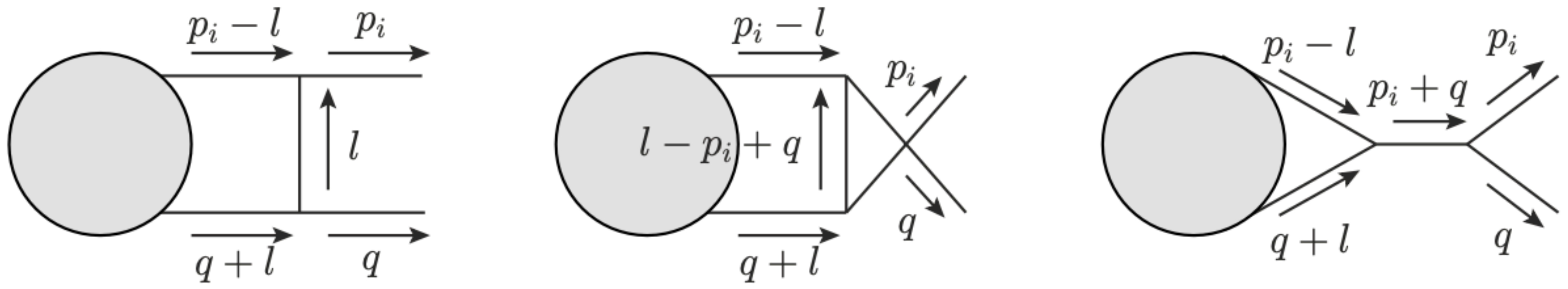
$$\left\langle c_1,\ldots,c_{n+1};\sigma_1,\ldots,\sigma_{n+1}\middle|R^{(0)}_{g,i,I}(\{p_i\})\right\rangle =$$
$$(1-x_I)^{-\dim(a_i)}\frac{1}{2p_i\cdot P_I}\sum_{\sigma c} M_I^{(0)}(\{p_i\},\{\sigma_i\},\{c_i\},\sigma,c)\,\overline{M}_I^{(0)}(\{p_i\},\{\sigma_i\},\{c_i\},\sigma,c)$$

Flavour-Diagonal Jet Operators

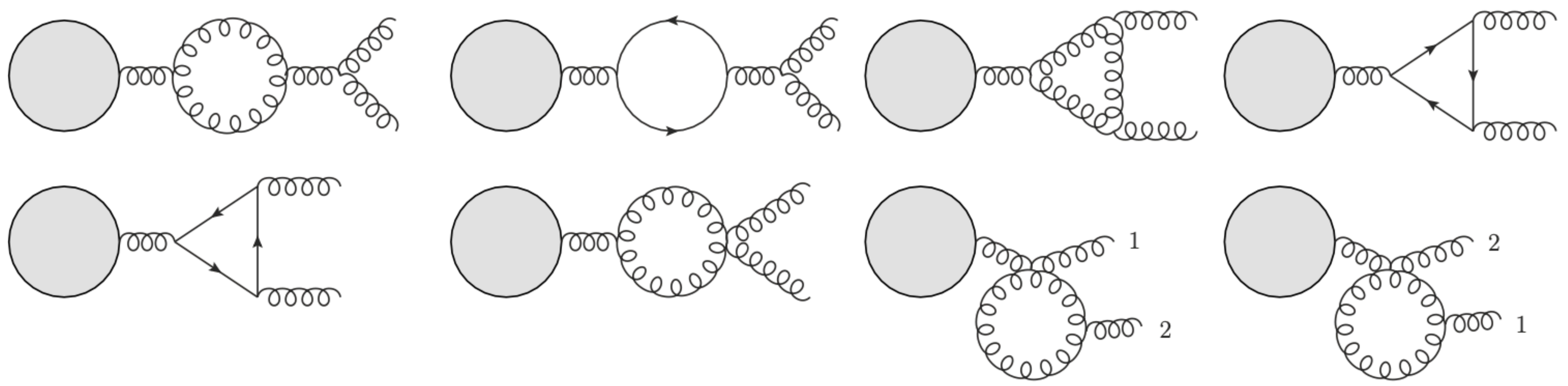
$$\int_0^1 \mathrm{d}x \sum_i \mathbf{J}_i^{(1)}(x,p_i,q)\left|H_{g,i}^{(0)}(x,\{p_i\},q)\right\rangle$$

- “jet”-operators should be determined in physical gauge !

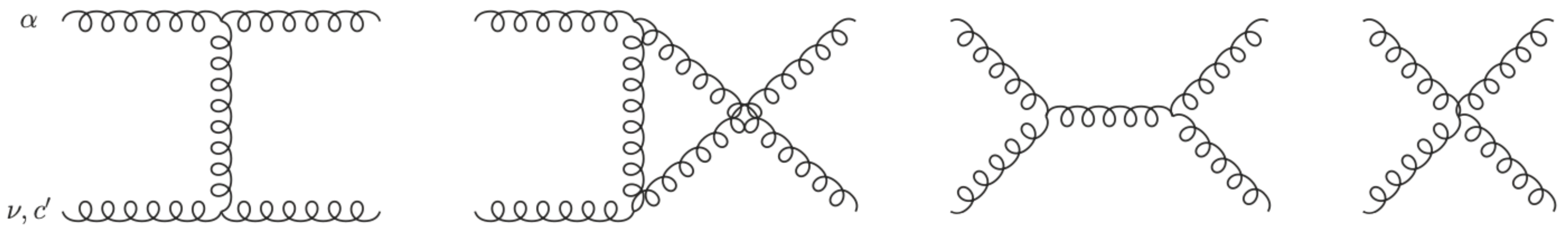
generic topologies



one-line contributions



two-line contributions



- Combine and exploit properties of the “hard”-functions

$$\mathbf{P}_g(\sigma,c)\,\mathbf{J}_i^{(1)}(x,p_i,q)$$
$$= \frac{\Gamma(1+\epsilon)}{1-\epsilon}\left(-\frac{\mu^2}{s_{iq}}\right)^{\epsilon}(x(1-x))^{-\epsilon}\epsilon^*(q,p_i,\sigma)\cdot\epsilon(p_i,-\sigma)\sum_{c'}\mathbf{P}_g(-\sigma,c')\times\left[\left(\mathbf{T}_i^c\mathbf{T}_i^{c'}+\frac{1}{x}if^{cdc'}\mathbf{T}_i^d\right)\otimes\left(-2+x(1+\mathbf{\Sigma}_{g,i})\right)\right]$$

Flavour-Off-Diagonal Jet Operators

$$\int_0^1 \mathrm{d}x \sum_{\substack{i \\ a_i=g}} \tilde{\mathbf{J}}_i^{(1)}(x,p_i,q)\left|H_{\bar{q},i}^{(0)}(x,\{p_i\},q)\right\rangle$$

- Only contribute in the case of $g \to q\bar{q}$ splitting
- Correspond to crossing of the diagrams

$$\mathbf{P}_g(\sigma,c)\,\mathbf{J}_i^{(1)}(x,p_i,q)$$
$$= \frac{\Gamma(1+\epsilon)}{1-\epsilon}\left(-\frac{\mu^2}{s_{iq}}\right)^{\epsilon}(x(1-x))^{-\epsilon}\epsilon^*(q,p_i,\sigma)\cdot\epsilon(p_i,-\sigma)\sum_{c'}\mathbf{P}_g(-\sigma,c')\times\left[\left(\mathbf{T}_i^c\mathbf{T}_i^{c'}+\frac{1}{x}if^{cdc'}\mathbf{T}_i^d\right)\otimes\left(-2+x(1+\boldsymbol{\Sigma}_{g,i})\right)\right]$$

$$\Downarrow$$

$$\tilde{\mathbf{J}}_i^{(1)}(x,p_i,q)\,|\ldots,c_i',\ldots,c';\ldots,\sigma_i',\ldots,\sigma'\rangle$$
$$= \frac{\Gamma(1+\epsilon)}{1-\epsilon}\left(-\frac{\mu^2}{s_{iq}}\right)^{\epsilon}(x(1-x))^{-\epsilon}\sum_{cc_i}\left(T_q^cT_q^{c_i}+xif^{cdc_i}T_q^d\right)_{c'c_i'}\delta_{-\sigma'\sigma_i'}\sum_{\sigma\sigma_i}\delta_{\sigma\sigma_i}\epsilon^*(q,p_i,\sigma)\cdot\epsilon^*(p_i,\sigma_i)$$
$$\times\left(-2x+1+\mathrm{sgn}(\sigma_i\sigma')\right)|\ldots,c_i,\ldots,c;\ldots,\sigma_i,\ldots,\sigma\rangle\,.$$

Collinear Convolutions

- Collinear convolutions evaluated in a “process-independent” form

$$\int_0^1 \mathrm{d}x \sum_i \mathbf{J}_i^{(1)}(x,p_i,q) \left| H_{g,i}^{(0)}(x,\{p_i\},q) \right\rangle$$

$$\int_0^1 \mathrm{d}x \sum_{\substack{i \\ a_i=g}} \tilde{\mathbf{J}}_i^{(1)}(x,p_i,q) \left| H_{\bar{q},i}^{(0)}(x,\{p_i\},q) \right\rangle$$

$$\begin{aligned}
&\mathbf{P}_g(\sigma,c) \int_0^1 \mathrm{d}x\, \mathbf{J}_i^{(1)}(x,p_i,q) \left| H_{g,i}^{(0)}(x,\{p_i\},q) \right\rangle \\
&= \frac{r_\Gamma}{\epsilon(1-\epsilon)(1-2\epsilon)} \left(-\frac{\mu^2}{s_{iq}}\right)^{\epsilon} \epsilon^*(q,p_i,\sigma)\cdot\epsilon(p_i,-\sigma) \sum_{c'} \mathbf{P}_g(-\sigma,c') \\
&\quad \Bigg\{ \mathbf{T}_i^{c'} \mathbf{T}_i^{c} \Bigg[-\frac{1-2\epsilon}{1+\epsilon} (1-3\epsilon+(1+\epsilon)\mathbf{\Sigma}_{g,i}) \left| S_{g,i}^{(0)} \right\rangle + (1-3\epsilon-(1-\epsilon)\mathbf{\Sigma}_{g,i}) \left| \bar{S}_{g,i}^{(0)} \right\rangle \\
&\quad + (2-3\epsilon+\epsilon\mathbf{\Sigma}_{g,i}) \left(\left| C_{g,i}^{(0)} \right\rangle + \dim(a_i) \left| S_{g,i}^{(0)} \right\rangle \right) - \frac{\epsilon}{2} (3-\mathbf{\Sigma}_{g,i}) \left| L_{g,i}^{(0)} \right\rangle \\
&\quad + \sum_I \frac{\epsilon}{2x_I^2(1-x_I)} \left(2x_I - 2x_I\,\mathbf{\Sigma}_{g,i} - (2-x_I-x_I\,\mathbf{\Sigma}_{g,i})\,{}_2F_1(1,1-\epsilon,3-2\epsilon,1/x_I)\right) \left| R_{g,i,I}^{(0)} \right\rangle \Bigg] \\
&\quad + \mathbf{T}_i^{c} \mathbf{T}_i^{c'} \Bigg[\frac{1-\epsilon}{1+\epsilon} (3-3\epsilon+(1+\epsilon)\mathbf{\Sigma}_{g,i}) \left| S_{g,i}^{(0)} \right\rangle + \frac{\epsilon}{2} (3-\mathbf{\Sigma}_{g,i}) \left| \bar{S}_{g,i}^{(0)} \right\rangle \\
&\quad - \frac{1}{2}(4-3\epsilon+\epsilon\,\mathbf{\Sigma}_{g,i}) \left(\left| C_{g,i}^{(0)} \right\rangle + \dim(a_i) \left| S_{g,i}^{(0)} \right\rangle \right) + \frac{\epsilon}{2(3-2\epsilon)} (5-3\epsilon-(1-\epsilon)\mathbf{\Sigma}_{g,i}) \left| L_{g,i}^{(0)} \right\rangle \\
&\quad + \sum_I \frac{\epsilon}{2x_I^2} \left(x_I + x_I\,\mathbf{\Sigma}_{g,i} + (2-x_I-x_I\,\mathbf{\Sigma}_{g,i})\,{}_2F_1(1,1-\epsilon,3-2\epsilon,1/x_I)\right) \left| R_{g,i,I}^{(0)} \right\rangle \Bigg] \Bigg\},
\end{aligned}$$

$$\begin{aligned}
&\mathbf{P}_i(\sigma_i,c_i)\mathbf{P}_g(\sigma,c) \int_0^1 \mathrm{d}x\, \tilde{\mathbf{J}}_i^{(1)}(x,p_i,q) \left| H_{\bar{q},i}^{(0)}(x,\{p_i\},q) \right\rangle \\
&= \frac{r_\Gamma}{(1-\epsilon)(1-2\epsilon)} \left(-\frac{\mu^2}{s_{iq}}\right)^{\epsilon} \epsilon^*(q,p_i,\sigma)\cdot\epsilon^*(p_i,\sigma_i) \sum_{\sigma' c'} \sum_{c_i'} \mathbf{P}_i(-\sigma',c_i')\mathbf{P}_{n+1}(\sigma',c') \\
&\quad \Bigg\{ (T_q^{c_i} T_q^{c})_{c'c_i'} \Bigg[2\sigma_i\sigma' \left| S_{\bar{q},i}^{(0)} \right\rangle + \left(\frac{1-(2-\epsilon)\sigma_i\sigma'}{\epsilon} + \frac{1}{2(3-2\epsilon)} \right) \left| \bar{S}_{\bar{q},i}^{(0)} \right\rangle + \left(\sigma_i\sigma' - \frac{1}{2(3-2\epsilon)} \right) \left| C_{\bar{q},i}^{(0)} \right\rangle \\
&\quad + \sum_I \frac{1}{x_I} \left(2x_I^2 - (1+2x_I)\sigma_i\sigma' + \frac{1}{2(3-2\epsilon)} + x_I(1-2x_I+2\sigma_i\sigma')\,{}_2F_1(1,1-\epsilon,2-2\epsilon,1/x_I) \right) \left| R_{\bar{q},i,I}^{(0)} \right\rangle \Bigg] \\
&\quad + (T_q^{c} T_q^{c_i})_{c'c_i'} \Bigg[\left(2\sigma_i\sigma' - \frac{1+2\sigma_i\sigma'}{\epsilon} \right) \left| S_{\bar{q},i}^{(0)} \right\rangle + \left(\sigma_i\sigma' - \frac{1}{2(3-2\epsilon)} \right) \left| \bar{S}_{\bar{q},i}^{(0)} \right\rangle + \left(\sigma_i\sigma' + \frac{1}{2(3-2\epsilon)} \right) \left| C_{\bar{q},i}^{(0)} \right\rangle \\
&\quad + \sum_I \frac{1}{x_I} \Bigg(2x_I - 2x_I^2 - (1-2x_I)\sigma_i\sigma' - \frac{1}{2(3-2\epsilon)} \\
&\quad + (1-x_I)(1-2x_I+2\sigma_i\sigma')\,{}_2F_1(1,1-\epsilon,2-2\epsilon,1/x_I) \Bigg) \left| R_{\bar{q},i,I}^{(0)} \right\rangle \Bigg] \Bigg\}.
\end{aligned}$$

Collinear Limit at Tree-Level

- Bonus result - first time in the literature !

$$k_{n+1} \equiv x p_i + l_\perp - \frac{l_\perp^2}{2x}\frac{q}{p_i \cdot q},$$

$$k_i \equiv (1-x)p_i - l_\perp - \frac{l_\perp^2}{2(1-x)}\frac{q}{p_i \cdot q},$$

$a_i = a_{n+1} = g$

$$\mathbf{P}_i(\sigma_i, c_i)\mathbf{P}_{n+1}(\sigma_{n+1}, c_{n+1}) \left| M^{(0)}(\{k_i\}_{i=1}^{n+1}) \right\rangle =$$

$$\mathbf{P}_i(\sigma_i, c_i)\mathbf{P}_{n+1}(\sigma_{n+1}, c_{n+1}) \Bigg[\mathbf{Split}^{(0)}_{i,n+1 \leftarrow i}(k_i, k_{n+1}, p_i) \left| M^{(0)}(\{p_i\}) \right\rangle$$

$$+ \left(\frac{1-x^2}{x} + \frac{1-(1-x)^2}{1-x}\mathbf{E}_{i,n+1} \right) \left| S^{(0)}_{g,i}(\{p_i\}, q) \right\rangle + ((1-x) + x\mathbf{E}_{i,n+1}) \left| C^{(0)}_{g,i}(\{p_i\}, q) \right\rangle$$

$$+ \frac{1}{2}\sum_I \frac{x(1-x)}{x_I(1-x_I)} \left(\frac{1}{x_I - x} + \frac{1}{x_I - (1-x)}\mathbf{E}_{i,n+1} \right) \left| R^{(0)}_{g,i,I}(\{p_i\}) \right\rangle \Bigg]$$

$$+ \left[\frac{1}{x}\frac{q \cdot \epsilon^*(p_i, \sigma_{n+1})}{q \cdot p_i}\mathbf{P}_i(\sigma_i, c_i)\mathbf{T}_i^{c_{n+1}} + \frac{1}{1-x}\frac{q \cdot \epsilon^*(p_i, \sigma_i)}{q \cdot p_i}\mathbf{P}_i(\sigma_{n+1}, c_{n+1})\mathbf{T}_i^{c_i} \right] \left| M^{(0)}(\{p_i\}) \right\rangle$$

$a_i \in \{q, \bar{q}\},\ a_{n+1} = g$

$$\mathbf{P}_{n+1}(\sigma_{n+1}, c_{n+1}) \left| M^{(0)}(\{k_i\}_{i=1}^{n+1}) \right\rangle =$$

$$\mathbf{P}_{n+1}(\sigma_{n+1}, c_{n+1}) \Bigg[\mathbf{Split}^{(0)}_{i,n+1 \leftarrow i}(k_i, k_{n+1}, p_i) \left| M^{(0)}(\{p_i\}) \right\rangle$$

$$+ \sqrt{1-x}\Bigg(\left(\frac{1}{x} + \frac{1}{2} \right) \left| S^{(0)}_{g,i}(\{p_i\}, q) \right\rangle + \left| C^{(0)}_{g,i}(\{p_i\}, q) \right\rangle + \frac{x}{1-x} \left| \bar{S}^{(0)}_{g,i}(\{p_i\}, q) \right\rangle$$

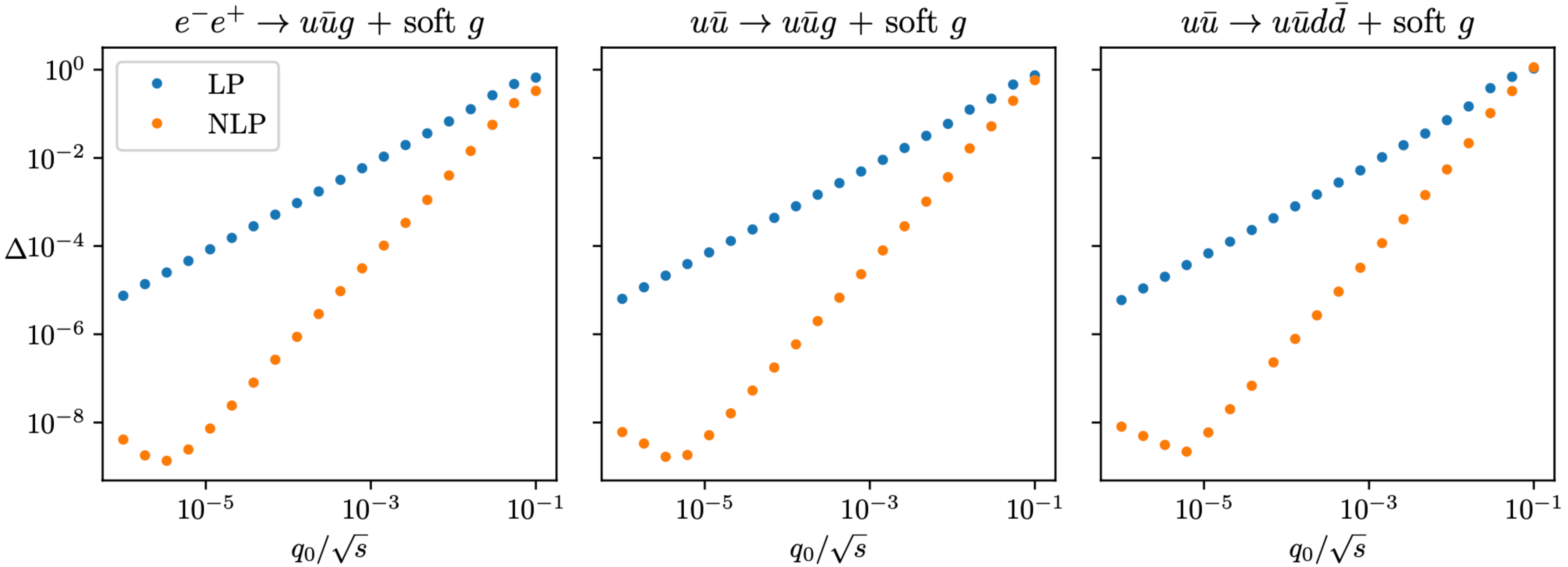
$$+ \sum_I \left(\frac{1}{x_I - x} - \frac{1}{x_I} \right) \left| R^{(0)}_{g,i,I}(\{p_i\}) \right\rangle \Bigg) \Bigg] + \frac{\sqrt{1-x}}{x}\frac{q \cdot \epsilon^*(p_i, \sigma_{n+1})}{q \cdot p_i}\mathbf{T}_i^{c_{n+1}} \left| M^{(0)}(\{p_i\}) \right\rangle$$

$a_i = q,\ a_{n+1} = \bar{q}$

$$\left| M^{(0)}(\{k_i\}_{i=1}^{n+1}) \right\rangle = \mathbf{Split}^{(0)}_{i,n+1 \leftarrow i}(k_i, k_{n+1}, p_i) \left| M^{(0)}(\{p_i\}) \right\rangle + \sqrt{x(1-x)} \left(\frac{1}{x} \left| S^{(0)}_{\bar{q},i}(\{p_i\}) \right\rangle + \left| C^{(0)}_{\bar{q},i}(\{p_i\}, q) \right\rangle + \frac{x}{1-x} \left| \bar{S}^{(0)}_{\bar{q},i}(\{p_i\}) \right\rangle + \sum_I \left(\frac{1}{x_I - x} - \frac{1}{x_I} \right) \left| R^{(0)}_{\bar{q},i,I}(\{p_i\}) \right\rangle \right)$$

Numerical checks

$$\Delta_{\mathrm{LP/NLP}} \equiv \frac{1}{N} \sum_{\substack{\text{singular} \\ \text{colour flows } \{c\} \\ \text{helicities } \{\sigma\}}} \left| \frac{\left[\left\langle \{c,\sigma\} \middle| M_g^{(1)} \right\rangle - \left\langle \{c,\sigma\} \middle| M_g^{(1)} \right\rangle_{\mathrm{LP/NLP}} \right]_{\mathcal{O}(\epsilon^0)}}{\left[\left\langle \{c,\sigma\} \middle| M_g^{(1)} \right\rangle \right]_{\mathcal{O}(\epsilon^0)}} \right|$$



Recola + Cuttools

Conclusions

- What about the proof ?

Expansion-by-regions + comparison of CDR poles in generic analytic form

- Astonishing simplifications of collinear-enhanced contributions
  - compare with the jet-function from 1503.05156 where convolutions are missing !

$$J^{\nu(1)}(p,n,k\,;\epsilon) \;=\; (2p\cdot k)^{-\epsilon}\left[\left(\frac{2}{\epsilon}+4+8\epsilon\right)\left(\frac{n\cdot k}{p\cdot k}\,\frac{p^{\nu}}{p\cdot n}-\frac{n^{\nu}}{p\cdot n}\right)-(1+2\epsilon)\,\frac{\mathrm{i}\,k_{\alpha}\Sigma^{\alpha\nu}}{p\cdot k}+\left(\frac{1}{\epsilon}-\frac{1}{2}-3\epsilon\right)\frac{k^{\nu}}{p\cdot k}+(1+3\epsilon)\left(\frac{\gamma^{\nu}\,\not{n}}{p\cdot n}-\frac{p^{\nu}\,\not{k}\,\not{n}}{p\cdot k\,p\cdot n}\right)\right]$$

  - or with the QED results for jet-functions in convolutions from 2008.01736

derivative of collinear amplitude

$$J^{(1)\mu\nu}_{(f\gamma)}(x,p,k) = -\,\frac{e^2}{16\pi^2}\left(\frac{-2\,p^+k^-}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(\epsilon)\,[x\,(1-x)]^{-\epsilon}\,\bar{u}(p)\Big\{2\,(1-x)\,\eta^{\mu\nu}-\frac{\epsilon}{1-\epsilon}\,x\,\gamma^{\nu}\gamma^{\mu}+2\,(1-2\,x)\,\frac{k^+}{k^-}n^{\mu}n^{\nu}$$
$$-\,2\,(1-2\,x)\,\bar{n}^{\mu}n^{\nu}+\frac{1}{k^-}\left[x\,\gamma^{\mu}\not{k}\,n^{\nu}+2\,\frac{\epsilon}{1-\epsilon}\,x\,k^{\mu}n^{\nu}+\frac{\epsilon}{1-\epsilon}\,x\,\gamma^{\nu}\not{k}\,n^{\mu}-2\,(1-x)\,n^{\mu}k^{\nu}\right]\Big\}\,,$$

$$J^{(1)\mu\nu\rho}_{(f\partial\gamma)}(x,p,k) = -\,\frac{e^2\,p^+}{8\pi^2}\left(\frac{-2\,p^+k^-}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{1-\epsilon}\,[x\,(1-x)]^{1-\epsilon}\,\bar{u}(p)\,n^{\nu}\left(\eta_{\perp}^{\mu\rho}-\frac{n^{\mu}k_{\perp}^{\rho}}{k^-}\right)\,.$$

- Bonus result for tree-level subleading collinear limits