MAX-PLANCK-INSTITUT
FÜR PHYSIK

Zero-jettiness soft function at N3LO in QCD

in collaboration with Daniel Baranowski, Maximilian Delto, Andrey Pikelner and Kirill Melnikov

Based on *2111.13594* & *2204.09459* & *2401.05245* and work in preparation.

Chen-Yu Wang | 2024-05-11 | Ringberg 2024

Outline

Motivation

- The ever-increasing experimental precision at the LHC and the HL-LHC in the future demands percent level precision from the theoritical side. *ATLAS 2019; CMS 2021*
- On the theoretical side many N3LO calculations and phenomenology results are available.
- Computing differential cross-section requires subtracting infrared divergences in the phase space:
  - Slicing:
    - q_T subtraction scheme *Catani and Grazzini 2007*
    - N-jettiness subtraction scheme *Boughezal, Focke, et al. 2015; Gaunt et al. 2015*
  - Subtraction:
    - CoLoRFull *Somogyi et al. 2005*
    - Antenna *Gehrmann-De Ridder et al. 2005*
    - STRIPPER *Czakon 2010*
    - Nested soft-collinear subtraction *Caola et al. 2017*
    - Local analytic sector subtraction *Magnea et al. 2018*
  - Project-to-Born *Cacciari et al. 2015*
  - ...

Motivation

- To obtain differential cross sections, one can use slicing to extract and cancel infrared divergences properly:

$$\sigma(O) = \int_0 \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau} = \int_0^{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau} + \int_{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau}.$$

  - q_T subtraction scheme *Catani and Grazzini 2007*
  - N-jettiness subtraction scheme *Boughezal, Focke, et al. 2015; Gaunt et al. 2015*
- q_T subtraction scheme is available up to N3LO *Li and Zhu 2017; Ebert et al. 2020b; Luo et al. 2020*
- N-jettiness factorization theorem derived in SCET *Stewart et al. 2010a,b*

$$\lim_{\tau \to 0} \mathrm{d}\sigma(O) = B \otimes B \otimes \sum_i J_i \otimes S_N \otimes H \otimes \mathrm{d}\sigma_{\mathrm{LO}} + \mathcal{O}(\tau).$$

  - Beam function B @ N3LO *Ebert et al. 2020a; Baranowski et al. 2023*
  - Jet function J @ N3LO *Banerjee et al. 2018; Brüser et al. 2018*
  - Soft function S_N @ N2LO *Hornig et al. 2011; Kelley et al. 2011; Monni et al. 2011; Boughezal, Liu, et al. 2015; Bell et al. 2018; Campbell et al. 2018; Jin and Liu 2019*

Definition

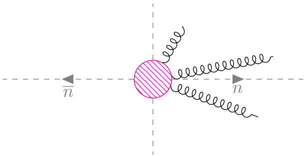
- Zero-jettiness is defined as

$$\tau = \sum_{i=1}^{m} \min_{j\in 1,2}\left[\frac{2q_j\cdot k_i}{Q_j}\right] = \sum_{i=1}^{m}\min\{\alpha_i,\beta_i\}.$$

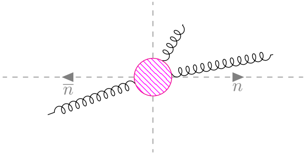
where min(...) can be written out using the **Heaviside θ function**:

$$\delta\left(\tau - \sum_{i=1}^{m}\min\{\alpha_i,\beta_i\}\right) = \delta\left(\tau - \beta_1 - \beta_2 - \ldots\right)\theta(\alpha_1-\beta_1)\theta(\alpha_2-\beta_2)\ldots$$
$$+\,\delta\left(\tau - \alpha_1 - \beta_2 - \ldots\right)\theta(\beta_1-\alpha_1)\theta(\alpha_2-\beta_2)\ldots$$
$$+\ldots.$$

Sudakov decomposition: $k_i = \frac{\alpha_i}{2}n + \frac{\beta_i}{2}\overline{n} + k_{\perp,i}$, where $\alpha_i = k_i\cdot\overline{n}$, $\beta_i = k_i\cdot n$, and $n\cdot\overline{n} = 2$.



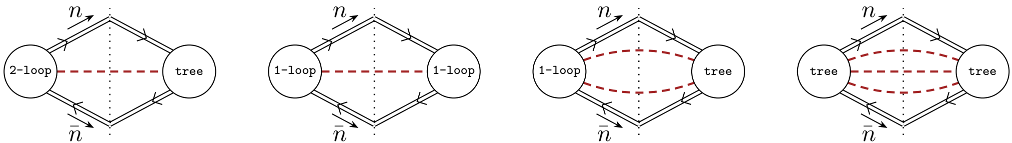
nnn configuration



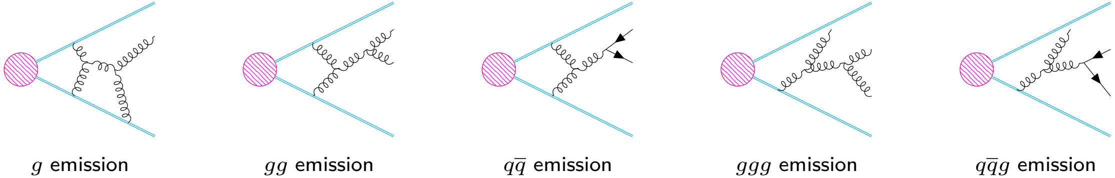
$nn\overline{n}$ configuration

Definition: Amplitude @ N3LO

- The limit $\tau \to 0$ corresponds to the soft limit of the squared amplitude - eikonal rules
- Need to include all possible real and virtual corrections to the amplitude squared



- Possible to combine different measurement function terms into unique configurations
- Perform integration over highly non-trivial region



Color structures decomposition

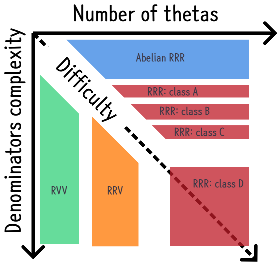
Full result splitting according to the number of soft emissions

$$S_{N^3LO} = S_{\text{RRR}} + S_{\text{RRV}} + S_{\text{RVV}}$$

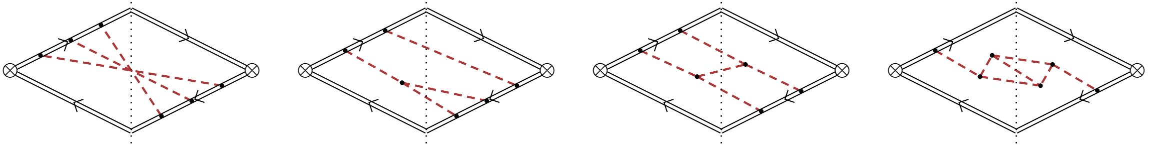
	C_R^3	$C_R^2 n_f T_F$	$C_R^2 C_A$	$C_R (n_f T_F)^2$	$C_R C_F n_f T_F$	$C_R C_A n_f T_F$	$C_R C_A^2$
S_{RRR}	$+$	$+$	$+$		$+$	**?**	**?**
S_{RRV}			$+$	$+$	$+$	$+$	$+$
S_{RVV}						$+$	$+$
S_{N^3LO}	$+$	$+$	$+$	$+$	$+$	$+$	$+$
Poles	yes	yes	yes	yes	yes	**?**	**?**

All poles in $S^{(3)}$ can be fixed from RGE and NNLO result with known anomalous dimensions.

Relative complexity of ingredients



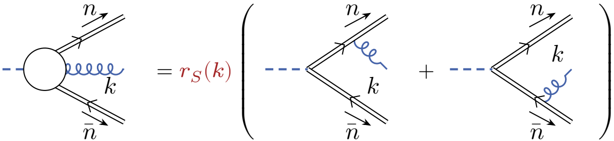
- For each soft emission we have one Heaviside θ-function in the measurement function that complicates the integration.
- Most complicated one-loop sub-integrals in the RRV case make direct integration impossible.
- Most complicated denominators in RRR case make direct integration impossible.
- Unregulated divergencies in the RRR case.



$$A \qquad B \sim \frac{1}{k_1 \cdot k_2} \qquad C \sim \frac{1}{(k_1 \cdot k_2)(k_1 \cdot k_3)} \qquad D \sim \frac{1}{(k_1 + k_2 + k_3)^2}$$

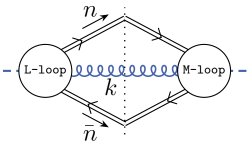
Single-real corrections

Two-loop corrections $r_S^{(2)}$ to single gluon emission soft current are known exactly in ε. [Duhr,Gehrmann'13]



$$r_S(k) = 1 + \sum_{l=1}^{\infty} A_s^l \left[\frac{-(n\cdot\bar{n})}{2(k\cdot n)(k\cdot\bar{n})} \right]^{l\varepsilon} r_S^{(l)}$$

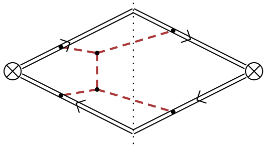
$$w_{L,M}(k) = \text{Re}\left[J_L^\dagger(k) J_M(k) \right] =$$



Two contributions from different hemisphere emissions need to be integrated, $S_g^{(3)} = s_{2,0} + s_{1,1} + s_{0,2}$

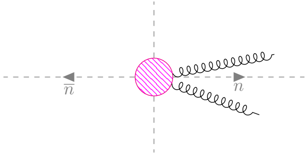
$$s_{l,m} = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \delta^+\left(k^2\right) \left[\delta(\tau - k\cdot n)\theta(k\cdot\bar{n} - k\cdot n) + \delta(\tau - k\cdot\bar{n})\theta(k\cdot n - k\cdot\bar{n})\right] w_{l,m}(k)$$

Double-real corrections

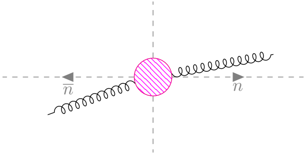


- Results for one-loop soft current are known. [Zhu'20][Czakon et al.'22]
- RRV result for gg final state is known. [Chen,Feng,Jia,Liu'22]
- Recalculation including $q\bar{q}$ final state. [Baranowski et al.'24]

- We need to integrate squared amplitude in the soft limit with constraints from δ and θ functions.
- Only two different configurations nn and $n\bar{n}$ contribute.

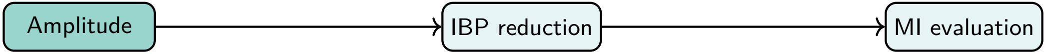


nn configuration
$\delta\left(\tau-\beta_1-\beta_2\right)\theta(\alpha_1-\beta_1)\theta(\alpha_2-\beta_2)$



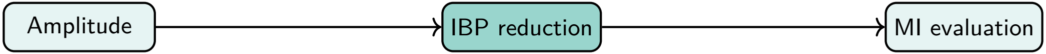
$n\bar{n}$ configuration
$\delta\left(\tau-\beta_1-\alpha_2\right)\theta(\alpha_1-\beta_1)\theta(\beta_2-\alpha_2)$

RRV: Procedures



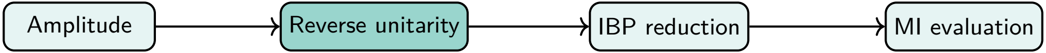
- Amplitude: generate from scratch $S = \sum_i C_i I_i$.

RRV: Procedures



- Amplitude: generate from scratch $S = \sum_i C_i I_i$.
- IBP reduction: $\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} \left[p_\mu \frac{1}{\prod_i D_i} \right] = 0.$ [Chetyrkin,Tkachov'81]

RRV: Procedures



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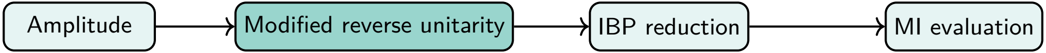
  - Reverse unitarity: transform δ functions to denominators [Anastasiou,Melnikov'02]

$$\delta(p^2 - m^2) = \frac{1}{2\pi} \left[\frac{i}{p^2 - m^2 + i\varepsilon} - \frac{i}{p^2 - m^2 - i\varepsilon} \right].$$

RRV: Procedures



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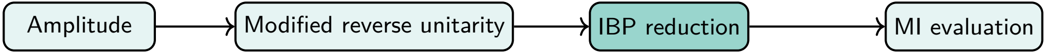
  - How to deal with θ functions? $\Longrightarrow$ **Modified reverse unitarity**

RRV: Procedures



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RRV: Procedures

Amplitude → Modified reverse unitarity → IBP reduction → MI evaluation

- Amplitude: generate from scratch $S = \sum_i C_i I_i$.
- IBP reduction: $\int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} \left[p_\mu \frac{1}{\prod_i D_i} \right] = 0.$ [Chetyrkin,Tkachov'81]
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$$\delta(p^2 - m^2) = \frac{1}{2\pi} \left[\frac{i}{p^2 - m^2 + i\varepsilon} - \frac{i}{p^2 - m^2 - i\varepsilon} \right].$$

  - How to deal with θ functions? $\Longrightarrow$ **Modified reverse unitarity**
- Master integral evaluation: $S = \sum_i C_i' I_i'$.

What if direct integration is impossible? $\Longrightarrow$ **Solve differential equations w.r.t. auxiliary parameters**

Modified reverse unitarity

- IBP identities can be constructed for properly regularized integrals:

$$\int \mathsf{d}^d k_i\, \mathsf{d}^d l\, \frac{\partial}{\partial k_{1\mu}} \left[v_\mu f(k_i, l) \theta_1(\ldots)\theta_2(\ldots) \right] = 0, \qquad \frac{\partial}{\partial \alpha_1} \theta(\alpha_1 - \beta_1) = \delta(\alpha_1 - \beta_1),$$

  which generates two kinds of contributions:

$$\int \mathsf{d}^d k_i\, \mathsf{d}^d l \left\{ \frac{\partial}{\partial k_{1\mu}} \left[v_\mu f(k_i, l) \right] \theta_1 \theta_2 + f'(k_i, l) \delta_1 \theta_2 \right\} = 0$$

- The **homogenous** term corresponds to the normal IBP identites without θ functions.
- The **inhomogenous** term introduces new families and requires **partial fraction decomposition**.
  - Auxiliary families with δ functions in place of θ functions are required.

Modified reverse unitarity

$$\frac{\partial}{\partial k_{1\mu}}\left[v_\mu f(k_i,l)\right]\theta_1\theta_2 + f'(k_i,l)\delta_1\theta_2 = 0$$

nn configuration	$n\bar{n}$ configuration
$\theta\theta$	$\theta\theta$

- $f_i = \theta$ or δ:

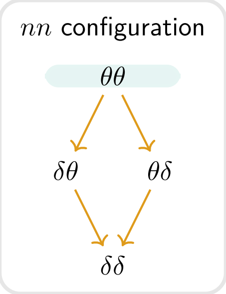
  nn configuration: $\delta\left(\tau - \beta_1 - \beta_2\right) f_1(\alpha_1 - \beta_1) f_2(\alpha_2 - \beta_2)$

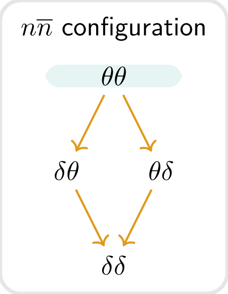
  $n\bar{n}$ configuration: $\delta\left(\tau - \beta_1 - \alpha_2\right) f_1(\alpha_1 - \beta_1) f_2(\beta_2 - \alpha_2)$

- Starting from the amplitude with measure $\theta\theta$,

Modified reverse unitarity

$$\frac{\partial}{\partial k_{1\mu}} \left[v_\mu f(k_i, l)\right] \theta_1\theta_2 + f'(k_i, l)\delta_1\theta_2 = 0$$





- $f_i = \theta$ or δ:

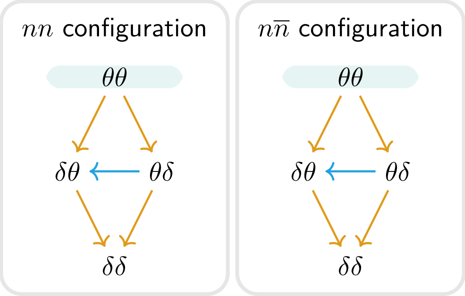
  nn configuration: $\delta\left(\tau - \beta_1 - \beta_2\right) f_1(\alpha_1 - \beta_1) f_2(\alpha_2 - \beta_2)$

  $n\bar{n}$ configuration: $\delta\left(\tau - \beta_1 - \alpha_2\right) f_1(\alpha_1 - \beta_1) f_2(\beta_2 - \alpha_2)$

- Starting from the amplitude with measure $\theta\theta$,
  - **IBP identities** connect measures with fewer θ functions.

Modified reverse unitarity

$$\frac{\partial}{\partial k_{1\mu}}\left[v_\mu f(k_i,l)\right]\theta_1\theta_2 + f'(k_i,l)\delta_1\theta_2 = 0$$



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- Starting from the amplitude with measure $\theta\theta$,
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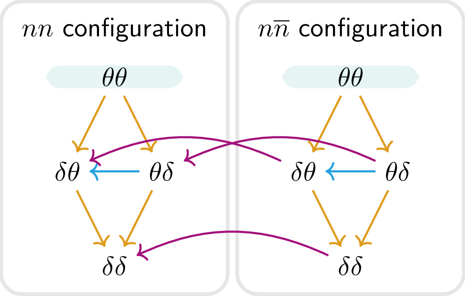

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- Starting from the amplitude with measure $\theta\theta$,
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- More **symmetry relations** between configurations.

Modified reverse unitarity

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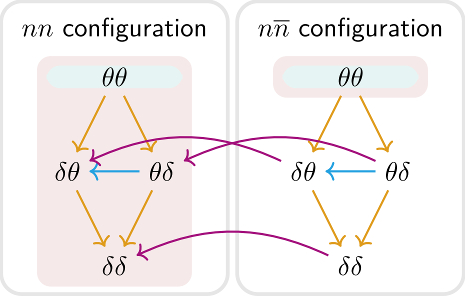

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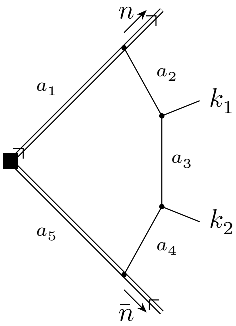
- Starting from the amplitude with measure $\theta\theta$,
  - **IBP identities** connect measures with fewer θ functions.
  - **symmetry relations** connect measures with different permutations.
- More **symmetry relations** between configurations.

- Generate the IBP system manually.
- Solve the system with `Kira` and `FireFly` (`reduce_user_defined_system`). [Klappert et al.'21]

RRV: Master integrals calculation



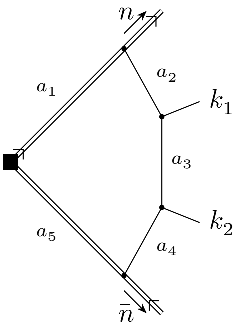
- Number of MIs after IBP reduction of both configurations:

$\delta\delta$	$\delta\theta + \theta\delta$	$\theta\theta$
8	36	15

- Direct integration is possible, except pentagons and boxes with $a_3 = 0$.
- **Differential equation in auxiliary parameters** for most complicated integrals:

$$\partial_z J(\varepsilon, z) = M(\varepsilon, z) J(\varepsilon, z), \qquad I_i(\varepsilon) = \int \mathrm{d}z \, J_i(\varepsilon, z).$$

RRV: Master integrals from differential equations



- For $\delta\delta$ integrals, we introduce auxiliary paremeter $z = 2k_1 \cdot k_2$:

$$I_{\delta\delta} = \int \mathsf{d}(k_1 \cdot k_2) f(k_1 \cdot k_2) = \int_0^1 \mathsf{d}z\, J(z).$$

- For integrals involving θ functions, we introduce variable z_i for each θ function through its integral representation:

$$\theta(b-a) = \int_0^1 \mathsf{d}z\, b\delta(zb-a)$$

and the master integrals can be obtained through

$$I_{\delta\theta} = \int_0^1 \mathsf{d}z\, J(z), \quad I_{\theta\theta} = \int_0^1 \mathsf{d}z_1 \int_0^1 \mathsf{d}z_2\, J(z_1, z_2).$$

RRV: Master integrals from differential equations

- For all auxiliary integrals it is possible to find alternative basis of integrals, such ε dependence of the DE system matrix factorizes completely: $M(\varepsilon, z) \to \varepsilon A(z)$. [Henn'13]
- Straightforward solution for integrals in canonical basis in terms of GPLs
- Simpler boundary conditions fixing due to known general form of expansion near singular points

$$g(z) = z^{a_1+b_1\varepsilon}\left(c_1 + \mathcal{O}(z)\right) + z^{a_2+b_2\varepsilon}\left(c_2 + \mathcal{O}(z)\right) + \ldots$$

- Construction of subtraction terms to remove endpoint singularities in final integration

$$I = \int_0^1 \mathrm{d}z\, J(z) = \int_0^1 \underbrace{\left[J(z) - z^{a_i+b_i\varepsilon} j_0(z) - (1-z)^{a_k+b_k\varepsilon} j_1(z)\right]}_{\varepsilon-\text{expanded}} \mathrm{d}z$$

$$+ \int_0^1 \underbrace{\left(z^{a_i+b_i\varepsilon} j_0(z) - (1-z)^{a_k+b_k\varepsilon} j_1(z)\right)}_{\varepsilon-\text{exact}} \mathrm{d}z$$

RRV: Results

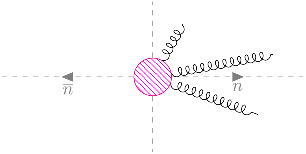
- Results for separate configurations contain $\text{Li}_4(1/2)$, but sum of two configurations has ζ_n only with maximal transcendental weight 6
- From N^{3}LO RRV + N^{2}LO result its possible extract large n_f contribution to the renormalized soft function

Non-logarithmic part of the Laplace space result

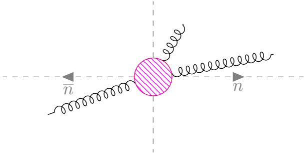
$$\tilde{S}^{(3)}_{\text{nl}} = C_R \left(n_f T_F\right)^2 \left[\frac{265408}{6561} - \frac{400}{243}\pi^2 - \frac{51904}{243}\zeta_3 + \frac{328}{1215}\pi^4 + \mathcal{O}\left(\frac{1}{n_f}\right)\right]$$

Triple real corrections

- Recalculated input for eikonal factor with partial fractioning and topology mapping:
  - $ggg = ggg + gc\bar{c}$, coincides with known expression in physical gauge [Catani,Colferai,Torrini'19]
  - $gq\bar{q}$ in agreement with [Del Duca,Duhr,Haindl,Liu'23]



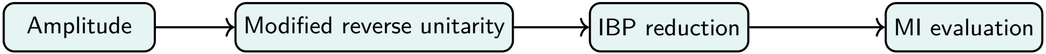
nnn configuration
$\delta\left(\tau-\beta_1-\beta_2-\beta_3\right)\theta(\alpha_1-\beta_1)\theta(\alpha_2-\beta_2)\theta(\alpha_3-\beta_3)$



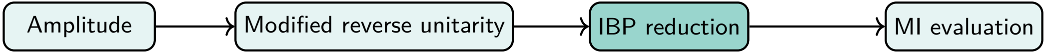
$nn\overline{n}$ configuration
$\delta\left(\tau-\beta_1-\beta_2-\alpha_3\right)\theta(\alpha_1-\beta_1)\theta(\alpha_2-\beta_2)\theta(\beta_3-\alpha_3)$

- Same hemisphere result for ggg final state is known. [Baranowski et al.'22]

RRR: Procedures



RRR: Procedures



- IBP reduction: unregulated integrals in the IBP system $\Longrightarrow$ wrong reduction

RRR: Procedures

Amplitude → Modified reverse unitarity → IBP reduction with ν → MI evaluation

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  - **Additional analytic regulator $\left(\prod_{i=1}^{3} \min\{\alpha_i, \beta_i\}\right)^{\nu}$ is required.**
  - Can we get rid of it?

RRR: Procedures

Amplitude → Modified reverse unitarity → IBP reduction with ν → **MI evaluation**

- IBP reduction: unregulated integrals in the IBP system $\Longrightarrow$ wrong reduction
  - **Additional analytic regulator $\left(\prod_{i=1}^{3} \min\{\alpha_i, \beta_i\}\right)^{\nu}$ is required.**
  - Can we get rid of it?
- Master integral evaluation: complicated denominator

$$\frac{1}{(k_1+k_2+k_3)^2} \sim \frac{1}{2k_1\cdot k_2 + 2k_2\cdot k_3 + 2k_3\cdot k_1}$$

$\Longrightarrow$ **add a mass-like auxiliary parameter m^2 and evaluate with DE**

$$\frac{1}{(k_1+k_2+k_3)^2 + m^2}$$

Unregulated integrals

- Not all integrals appearing during IBP reduction are regulated dimensionally.

$$J = \int \frac{\mathrm{d}\Phi_3 \delta(\tau - \beta_{123})\theta(\alpha_1 - \beta_1)\delta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)}{(k_1 \cdot k_3)(\alpha_1 + \alpha_2)\alpha_3\beta_1}$$

$$\propto \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \beta_2^{-2\varepsilon} \frac{\delta(1-\beta_{123})}{(\xi_1 + \xi_3 + 2\sqrt{\xi_1\xi_3}\cos\theta_{13})(\beta_1 + \beta_2\xi_1)\beta_3\beta_1}$$

where $\xi_i = \beta_i/\alpha_i$. In the following region

$$\xi_1 \sim \xi_3 \sim \beta_1 \sim \lambda$$

we have

$$J \sim \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \delta(1 - \xi_1 - \xi_3 - \beta_1) \int \frac{\mathrm{d}\lambda}{\lambda} \left(\frac{\lambda^2}{\lambda\lambda}\right)^{-\varepsilon} \times \cdots$$

Integration over λ diverges.

Unregulated integrals

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$$\propto \int \left(\prod_{i=1}^{3} \mathrm{d}\Omega_i^{d-2}\right) \mathrm{d}\xi_1 \mathrm{d}\xi_3 \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \left(\frac{\beta_1^2\beta_3^2}{\xi_1\xi_3}\right)^{-\varepsilon} \beta_2^{-2\varepsilon} \frac{\delta(1-\beta_{123})(\beta_1\beta_2\beta_3)^\nu}{(\xi_1 + \xi_3 + 2\sqrt{\xi_1\xi_3}\cos\theta_{13})(\beta_1 + \beta_2\xi_1)\beta_3\beta_1}$$

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Integration over λ regulated by ν.

Unregulated integrals

- Requires 3 real emissions to produce non-vanishing contribution.
- Additional analytic regulator to the integration measure $\mathrm{d}\Phi_3 \left(\prod_{i=1}^{3} \min\{\alpha_i, \beta_i\}\right)^{\nu}$ is required.
- IBP identities need to be modified:

$$\frac{\partial}{\partial \beta_1} \beta_1^{\nu} = \beta_1^{\nu} \frac{\nu}{\beta_1}.$$

  Similar to the modified reverse unitarity $\Longrightarrow$ auxiliary families & partial fraction decomposition.
- IBP reduction with additional regulator possible but more complicated, especially for auxiliary integrals $J(m^2)$.
  - reduction time, file size, basis choice, ...
  - Works for nnn configuration but too expensive for $nn\bar{n}$ configuration.
  - Can we get rid of the regulator while obtaining a correct IBP reduction?

Unregulated integrals

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  Similar to the modified reverse unitarity $\Longrightarrow$ auxiliary families & partial fraction decomposition.
- Observation: **unregularized integrals only appear in the IBP system not in the amplitude**.
- Filter the regulated IBP system:
  - Generate IBP relations with the analytic regulator.
  - Filter away all unregulated integrals in the IBP system. $\Longleftarrow$ **requires proper power counting**
  - Set ν to 0.
- Correct and fast reduction.

Evaluation of $1/k_{123}^2$ integrals

- The problematic denominator $1/k_{123}^2$

$$\frac{1}{(k_1+k_2+k_3)^2} \sim \frac{1}{2k_1\cdot k_2 + 2k_2\cdot k_3 + 2k_3\cdot k_1}$$

  involves 3 dot products $\Longrightarrow$ **add a mass-like auxiliary parameter m^2 and take the limit $m^2 \to \infty$**

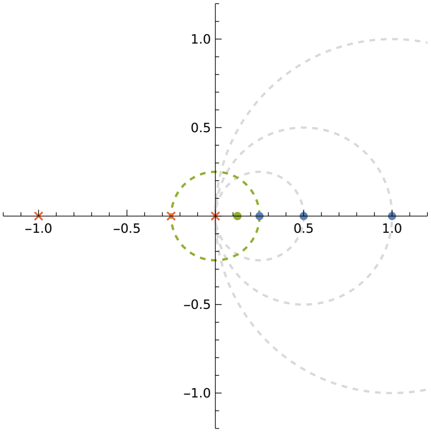
$$\partial_{m^2} J(\varepsilon, m^2) = M(\varepsilon, m^2) J(\varepsilon, m^2), \qquad I_i(\varepsilon) = \lim_{m^2\to 0} J_i(\varepsilon, m^2) = \lim_{m^2\to 0} \int \mathrm{d}\Phi \, \frac{1}{k_{123}^2 + m^2} \frac{\cdots}{\cdots}.$$

- Solve the differential equations **numerically** from $m^2 \to \infty$ to $m^2 \to 0$ [Liu et al.'18][Chen et al.'22]
- Reconstruct **analytical** expression from numerical data.

Evaluation of $1/k_{123}^2$ integrals

$$J = \int \mathrm{d}\Phi_{\delta\theta\theta}^{nnn} \frac{1}{(k_{123}^2 + m^2)(k_2 \cdot \overline{n})}$$

m^2 plane



- System size:
  - ~ 150 integrals for nnn configuration.
  - ~ 650 integrals for $nn\overline{n}$ configuration.
- **Boundary conditions** at $m^2 \to \infty$ involves several regions as the Heaviside functions allow α_i to be large:

$$J\big|_{m^2\to\infty} = \begin{cases} (m^2)^0 & \text{with } \alpha_1, \alpha_2, \alpha_3 \ll m^2 \\ (m^2)^{-\varepsilon} & \text{with } \alpha_1, \alpha_i \ll m^2, \text{ while } \alpha_j \sim m^2 \\ (m^2)^{-2\varepsilon} & \text{with } \alpha_1 \ll m^2, \text{ while } \alpha_2, \alpha_3 \sim m^2 \end{cases}$$

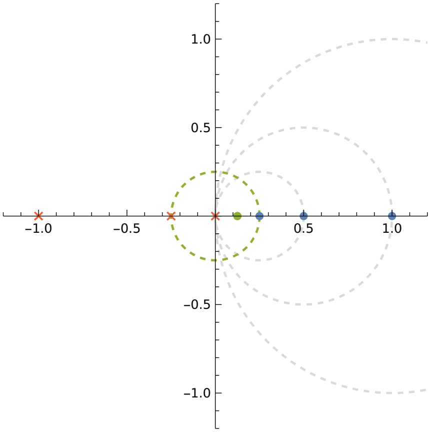
- The problematic denominator simplifies at the boundary:

$$k_{123}^2 + m^2 \sim \begin{cases} m^2 & (m^2)^0 \\ \alpha_j(\beta_1 + \beta_i) + m^2 & (m^2)^{-\varepsilon} \\ (\alpha_2 + \alpha_3)\beta_1 + 2k_2 \cdot k_3 + m^2 & (m^2)^{-2\varepsilon} \end{cases}$$

Evaluation of $1/k_{123}^2$ integrals

$$J = \int \mathrm{d}\Phi_{\delta\theta\theta}^{nnn} \frac{1}{(k_{123}^2 + m^2)(k_2 \cdot \overline{n})}$$

m^2 plane



- Analytic continuate to the neighborhood of **physical point** $m^2 = 0$.
- For $m^2 \geq 0$, $J_i(\varepsilon, m^2)$ is real $\Rightarrow$ consistency check to the solution.
- Matching at the **physical point** $m^2 = 0$:

$$J = \sum_{i,j,k} c_{ijk}(\varepsilon)(m^2)^{i+j\varepsilon} \ln^k m^2$$

- I corresponds to $\lim_{m^2 \to 0} J(\varepsilon, m^2) = c_{000}(\varepsilon)$.
- Finally we reconstruct the analytical expression.

Direct integration of master integrals and boundary conditions

- We have calculated ~ 130 integrals without $1/k_{123}^2$ denominator and ~ 100 boundary conditions by direct integration with `HyperInt` [Panzer'15]
- Summary of used techniques
  - Change variables to satisfy all constraints from δ and θ functions
  - Perform as many integrations as possible in terms of ${}_2F_1$ and F_1 functions with known argument transforms
  - Do remaining integrations in terms of ${}_pF_q$ functions if possible
  - For final integral representation with minimal number of integrations and minimal set of divergencies we construct subtraction terms
  - Integrand with all divergencies subtracted is expanded in ε and integrated term by term with `HyperInt`
  - Subtraction terms are integrated in the same way

Current status

Full result splitting according to the number of soft emissions

$$S_{N^3LO} = S_{\text{RRR}} + S_{\text{RRV}} + S_{\text{RVV}}$$

	C_R^3	$C_R^2 n_f T_F$	$C_R^2 C_A$	$C_R(n_f T_F)^2$	$C_R C_F n_f T_F$	$C_R C_A n_f T_F$	$C_R C_A^2$
S_{RRR}	+	+	+		+	?	?
S_{RRV}			+	+	+	+	+
S_{RVV}						+	+
S_{N^3LO}	+	+	+	+	+	+	+
Poles	yes	yes	yes	yes	yes	?	?

All poles in $S^{(3)}$ can be fixed from RGE and NNLO result with known anomalous dimensions.

Remaining steps to complete calculation

- **Numerical checks**
  - Extensive numerical checks of separate amplitude terms and/or master integrals are needed for RRR integrals with $1/k_{123}^2$.
  - Very complicated due to high degree of divergencies for both MB and sector decomposition approaches.
- **More orders in ε**
  - Once agreement for poles is found we need to produce higher ε-order expansions from DE.
  - Requires deeper expansions for boundary conditions.
- **More digits for reconstructing analytical expression**
  - When all needed ε-expansions are known we need to solve DE numerically to high precision to reconstruct analytical result.
  - Same hemisphere ggg amplitude requires ~ 1800 digits to discover the expression involving GPLs with sixth root of unity up to weight 6.
  - Full expression is expected to be simpler, i.e. less digits required for reconstruction.

Conclusion

- N3LO QCD corrections are crucial to the percent level phenomenology at LHC and HL-LHC.
- We are ready to produce final numbers for renormalized N^{3}LO zero-jettiness soft function.
- Extensive numerical checks of the most complicated triple-real emission contributions is underway.
- Efficient **reduction techniques for integrals with Heaviside θ functions** applicable for phase-space integrals with loops and additional regulators.
- Powerful method for evaluating complicated RRR integrals by **solving differential equations in auxiliary parameter numerically** with high precision and calculated all needed boundary conditions

Analytic regulator

- Although the soft function itself is regularized dimensionally, we found that an additional regulator is required to obtain a correct result

$$\mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} \to \mathrm{d}\Phi^{nnn}_{f_1 f_2 f_3} (k_1 \cdot n)^\nu (k_2 \cdot n)^\nu (k_3 \cdot n)^\nu .$$

- The amplitude reduces to

$$S^{nnn}_{ggg} = \sum_\alpha c_\alpha(\nu) I^\nu_\alpha + \nu \sum_\alpha \tilde{c}_\alpha(\nu) \overline{I}^\nu_\alpha ,$$

where two of the $\overline{I}^\nu_\alpha$ are $1/\nu$-divergent.
- For integrals without $1/k^2_{123}$ denominator, we can proceed as before and obtain analytical results.
- For integrals with $1/k^2_{123}$ denominator, we now have two limits to take:

$$I(\epsilon) = \lim_{\nu \to 0} \lim_{m \to 0} J(\epsilon, \nu, m).$$

We find that these two limits do commute, thus we can set $\nu = 0$ beforhand and solve the equation.

Backup