# Soft photons

# and

# Next-to-Leading-Power corrections

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# Outline

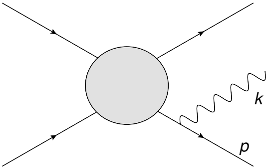
# Introduction

# Power corrections

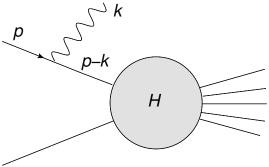
**Perturbative** calculations are the cornerstone of theoretical predictions in hep
E.g. expansion in $\alpha$: **LO, NLO, NNLO, ...**

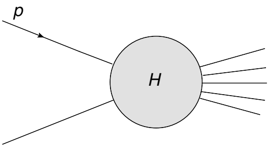
Large separation of scales
$\rightarrow$ expansion in other small parameters : **LP, NLP, NNLP, ...**

E.g. $k \ll p$ (**soft** expansion )



# Soft factorization

$\mathcal{A} =$ 


$\mathcal{H} =$ 


$$\boxed{\underbrace{\mathcal{A}(p_1,\ldots,p_n,k)}_{\text{radiative}} = \underbrace{\mathcal{S}(k)}_{\text{soft factor}} \times \underbrace{\mathcal{H}(p_1,\ldots,p_n)}_{\text{non-radiative}}}$$

$$\begin{aligned}
\mathcal{S} \quad = \quad & \overrightarrow{\mathcal{S}^{(0)}_{\mathrm{LP}} + \mathcal{S}^{(1)}_{\mathrm{LP}}} + \mathcal{S}^{(2)}_{\mathrm{LP}} + \ldots \\
& + \mathcal{S}^{(0)}_{\mathrm{NLP}} + \mathcal{S}^{(1)}_{\mathrm{NLP}} + \mathcal{S}^{(2)}_{\mathrm{NLP}} + \ldots \\
& + \mathcal{S}^{(0)}_{\mathrm{NNLP}} + \mathcal{S}^{(1)}_{\mathrm{NNLP}} + \mathcal{S}^{(2)}_{\mathrm{NNLP}} + \ldots \\
& + \ldots
\end{aligned}$$

More interesting structure for $n$ emissions: $\mathcal{S} = \exp(\mathcal{W})$

# History extends to present days (both at LP and NLP)

Soft Factorization has a long history

- LP (QED) [Bethe-Heitler 1934, Bloch-Nordsieck 1937, Yenni-Fratuushi-Suura 1961]
- LP (gravity) [Weinberg 1965]
- NLP (QED-tree) [Low 1958, Burnett-Kroll 1967].

up to more recent times

- LP (QCD) tree [Berends-Giele 1988]
  1-loop[Catani-Grazzini 2000]
  2-loop[Duhr-Gehrmann 2013]
  3-loop[Herzog-Ma-Mistlberger-Suresh 2023, Chen-Luo-ManYan-Zhou 2023]
- LP (massless QED) 3-loop [Ma-Sterman-Venkata 2023]
- NLP (QCD, massless QED) [DelDuca 1990, Casali 2014, Bern-Davies-Nohle2014, Larkoski-Neill-Stewart 2014, DB-Laenen-Magnea-Melville-Vernazza2015, Beneke-Broggio-Jaskiewicz-Vernazza 2019, Liu-Mecaj-Neubert-Wang 2021, Ravindran-Sankar-Tiwari 2022, Sterman-Vogelsang 2023, Czakon-Eschment-Schellenberger 2023 + many others!]
- NLP (QED) 1-loop [Engel,Signer,Ulrich 2021] all-orders [Engel 2023]
- NLP, NNLP gravity [White 2011, Cachazo-Strominger 2014, Beneke-Hager-Szafron 2022 + many others! ]

# METHODS

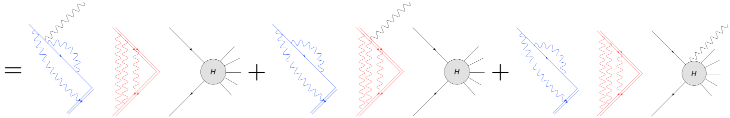
- Effective field theories

$$\mathcal{L}_{\text{SCET}} = \mathcal{L}_{\text{LP}} + \mathcal{L}_{\text{NLP}}$$

$$\sigma_{\text{DY-LP}} = H \times S\,, \qquad \sigma_{\text{DY-NLP}} = H \otimes J \otimes \bar{J} \otimes \tilde{S}$$

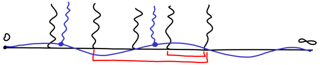
- Diagrammatic factorization

$$\mathcal{A}^{\mu} = \mathcal{J}^{\mu} \times \mathcal{S} \times \mathcal{H} \quad + \quad \mathcal{J} \times \mathcal{S}^{\mu} \times \mathcal{H} \quad + \quad \mathcal{J} \times \mathcal{S} \times \mathcal{H}^{\mu}$$



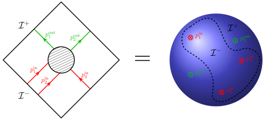

- Worldline formalism

$$\langle \frac{\hat{\not p} + \not A(\hat{x})}{(\hat{\not p} - \not A(\hat{x}))^2 - m^2} \rangle \sim \int dT \int d\theta \int \mathcal{D}\psi \int \mathcal{D}x\mathcal{D}p\, e^{-i\int dt\,\left(p\cdot\dot{x} + \frac{i}{2}\psi\cdot\dot{\psi} - \text{Den} - \frac{\theta}{T}\text{Num}\right)}\,.$$




- On-shell methods, celestial methods, ...

# WHY DO WE CARE TO GO BEYOND LP?

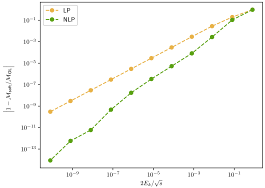


Understanding **infrared structure** of QFTs

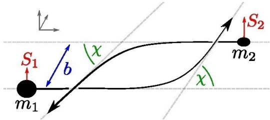
Soft gluon **resummation** beyond LP

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty} \sum_{m=0}^{2n-1} \alpha_s^n \left[ \underbrace{a_{nm} \left( \frac{\log^m(\xi)}{\xi} \right)_+}_{\textbf{LP}(LL,NLL,NNLL,\dots)} + \underbrace{b_{nm} \log^m(\xi)}_{\textbf{NLP}(LL,NLL,NNLL,\dots)} + \mathcal{O}(\xi) \right]$$

Efficient computation of
**differential** distributions




Gravitational waves
**PM expansion**



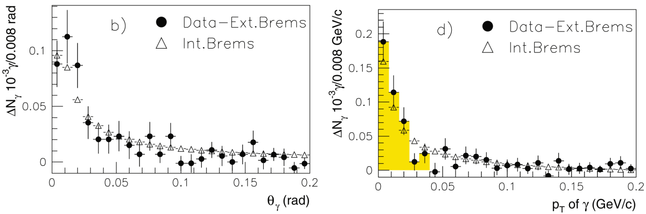

In this talk I focus on the **soft-photon bremsstrahlung**. Why?

[Image credits: Antonelli, Kavanagh, Khalil, Steinhoff, Vines PRL 125, 011103, Strominger arXiv:1703.05448, Engel, Signer, Ulrich JHEP 04(2022)097]

# WHY SOFT PHOTONS? 1. DEFINITION

Numerically, when is a photon soft?

- **What is the resolution in energy-momentum one has to achieve for NLP effects to be measurable?** When is the LP approx valid? Is that compatible with the experimental uncertainties?
- In QCD resummation one is blind to energy-momentum of the **undetected** gluon. In the bremsstrahlung case the photon is **detected**. Soft photon spectra give direct access to NLP effects.
- Soft theorems formulated with $k \to 0$, but photon spectra necessarily have $k \neq 0$.
- Question relevant also for more formal investigations and for condensed media [Landau-Pomeranchuk 1953, Feal-Vazquez 2018]




[Image credit: DELPHI collaboration, Eur. Phys. J. C (2008) 57: 499–514]

# WHY SOFT PHOTONS? 2. EXPERIMENTAL DISCREPANCIES

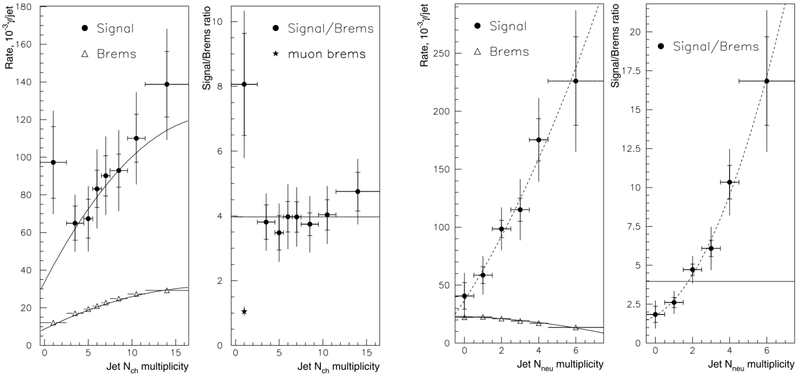
| Experiment | Collision Energy | Photon $k_T$ | Photon/Brem Ratio |
|---|---|---|---|
| $K^+p$, CERN,WA27, BEBC (1984) | 70 GeV/c | $k_T <$ 60 MeV/c | 4.0 $\pm$0.8 |
| $K^+p$, CERN,NA22, EHS (1993) | 250 GeV/c | $k_T <$ 40 MeV/c | 6.4 $\pm$1.6 |
| $\pi^+p$, CERN,NA22, EHS (1997) | 250 GeV/c | $k_T <$ 40 MeV/c | 6.9 $\pm$1.3 |
| $\pi^-p$, CERN,WA83,OMEGA (1997) | 280 GeV/c | $k_T <$ 10 MeV/c | 7.9 $\pm$1.4 |
| $\pi^+p$, CERN,WA91,OMEGA (2002) | 280 GeV/c | $k_T <$20 MeV/c | 5.3 $\pm$0.9 |
| $pp$, CERN,WA102,OMEGA (2002) | 450 GeV/c | $k_T <$20 MeV/c | 4.1 $\pm$0.8 |
| $e^+e^-\rightarrow$hadrons, CERN,DELPHI with hadron production (2010) | $\sim$91 GeV(CM) | $k_T <$60 MeV/c | 4.0 |
| $e^+e^-\rightarrow\mu^+\mu^-$, CERN,DELPHI with no hadron production (2008) | $\sim$91 GeV(CM) | $k_T <$60 MeV/c | 1.0 |

[Table taken from Cheuk-Yin Wong, arXiv:1404.0040. See also Martha Spyropoulus-Stassinaki, CF 2002, V. Perepelitsa, for the DELPHI Collaboration, Nonlin. Phenom. Complex Syst. 12, 343 (2009) ]

# Why soft photons? 2. Experimental discrepancies

DELPHI data for hadronic $Z$ decays

Photon range: 200 MeV $< \omega_k <$ 1 GeV, $p_t$ (w.r.t. jet) $<$ 80 MeV




[Abdallah et al. Eur. Phys. J. C (2010) 67: 343–366]

Only LP term used for comparison. Natural to wonder what is the impact of NLP corrections.

# Why soft photons? 3. Consistency of LBK theorem

$$\pi(p_a)\pi(p_b) \to \pi(p_1')\pi(p_2')\gamma(k)$$

Two seemingly different versions of the theorem[Lebiedowicz, Nachmann, Szczurek 2021]

$$\mathcal{A}^{\mu}_{\text{Low}} = e\mathcal{H}(s_L', t_2)\left[\frac{p_a^{\mu}}{p_a\cdot k} - \frac{p_1'^{\mu}}{p_1'\cdot k}\right] + e\frac{\partial\mathcal{H}(s_L', t_2)}{\partial s_L'}\left[p_b^{\mu} - \frac{p_b\cdot k}{p_a\cdot k}p_a^{\mu} + p_2'^{\mu} - \frac{p_2'\cdot k}{p_1'\cdot k}p_1'^{\mu}\right]$$

$$\mathcal{A}^{\mu}_{\text{V2}} = e\mathcal{H}(s_L, t)\left[\frac{p_a^{\mu}}{p_a\cdot k} - \frac{p_1'^{\mu}}{p_1'\cdot k}\right] + 2e\frac{\partial\mathcal{H}(s_L, t)}{\partial s_L}\left[p_b^{\mu} - \frac{p_b\cdot k}{p_a\cdot k}p_a^{\mu}\right]$$
$$- 2e\frac{\partial\mathcal{H}(s_L, t)}{\partial t}\left[(p_a - p_1)\cdot k - p_a\cdot l_1\right]\left[\frac{p_a^{\mu}}{p_a\cdot k} - \frac{p_1^{\mu}}{p_1\cdot k}\right]$$

where

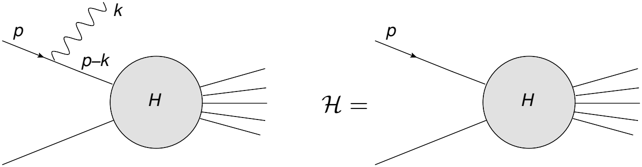
$$\begin{aligned}
&p_a + p_b = p_1 + p_2 = p_1' + p_2' + k\,,\\
&s = p_a\cdot p_b + p_1\cdot p_2, \qquad t = (p_b - p_2)^2\\
&s_L = p_a\cdot p_b + p_1'\cdot p_2', \qquad t_2 = (p_b - p_2')^2\,,\\
&l_1 = p_1 - p_1'\,, \qquad l_2 = p_2 - p_2'\,.
\end{aligned}$$

- Disagreement at NLP
- However, LBK theorem passed several non-trivial tests, e.g. NNLO DY [DB-Laenen-Magnea-Melville-Vernazza-White 2015] $\mathcal{O}(\Lambda)$ in $t\bar{t}$ [Makarov-Melnikov-Nason-Ozcelik 2023], ...

# Review of LBK theorem

# DIAGRAMMATICS (LP)

Goal: $\underbrace{\mathcal{A}(p_1,\ldots,p_n,k)}_{\text{radiative}} = \underbrace{\mathcal{S}(k)}_{\text{soft factor}} \times \underbrace{\mathcal{H}(p_1,\ldots,p_n)}_{\text{non-radiative}}$

$\mathcal{A} =$  $\mathcal{H} =$



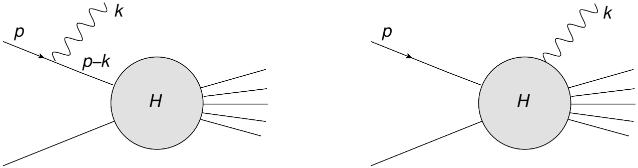
$\mathcal{A} = H(p-k)\,\frac{(\not p - \not k)}{(p-k)^2}\,(Q\,\epsilon^*(k)\cdot\gamma)\,u(p)$ $\qquad$ $\mathcal{H} = H\;u(p)$

At **LP**, take the leading term for $k \to 0$ (**eikonal** approximation):

$$\mathcal{A} = \mathcal{S}_{LP}\,\mathcal{H}\,, \qquad \mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \eta_i \frac{\epsilon^*(k)\cdot p_i^\mu}{p_i \cdot k}$$

- insensitive to spin of hard emitter
- hard particles do not recoil ($k \to 0$)
- insensitive to the short distance physics i.e. non radiative amplitude $\mathcal{H}$

# Diagrammatics (NLP)




- External emission: expand up to $\mathcal{O}(k)$

$$
\begin{aligned}
\mathcal{A}^{\mu}_{\rm ext}(p) &= H(p-k)\,\frac{(\not{p}-\not{k})}{(p-k)^2}\,(Q\,\gamma^{\mu})u(p) \\
&= Q\,H(p)\left(\frac{p^{\mu}}{p\cdot k} + \frac{k^{\mu}}{2p\cdot k} - \frac{k^2 p^{\mu}}{2(p\cdot k)^2} - \frac{i k_{\nu}\sigma^{\mu\nu}}{p\cdot k}\right) u(p) \\
&\quad + \; Q\,\frac{p^{\mu}}{p\cdot k} k^{\nu}\, \underbrace{\frac{\partial H(p-k)}{\partial k_{\nu}}\bigg|_{k=0}}_{-\frac{\partial H(p)}{\partial p_{\nu}}} u(p) \; + \; \mathcal{O}(k)
\end{aligned}
$$

- Expansion performed assuming r.h.s. a $f(k)$, not restricted to the physical constraint $\sum_i p_i = k$. But of course after expanding it we are only interested in the value of $f(k)$ on momentum conservation surface!
- Alternative: do not leave mom-cons. surface and parametrize $p_i(k)$ (note derivatives on spinors). Parametrization $p_i(k)$ not unique.

# DIAGRAMMATICS (NLP)

- Internal emission: use Ward identity $k_\mu(\mathcal{A}^\mu_{\text{ext}} + \mathcal{A}^\mu_{\text{int}}) = 0$

$$\mathcal{A}^\mu_{\text{int}} = \sum_i Q_i \frac{\partial H(p^i)}{\partial p^i_\mu} u(p^i) + \underbrace{\Delta^\mu}_{\mathcal{O}(k)}$$

- Adding $\mathcal{A}^\mu_{\text{ext}}$ and $\mathcal{A}^\mu_{\text{int}}$:

$$\begin{aligned}
\mathcal{A}^\mu &= \sum_i Q_i \frac{p_i^\mu}{p_i \cdot k} \mathcal{H}(p_1 \ldots p_n) \\
&+ \sum_i Q_i \left( \frac{k^\mu}{2p \cdot k} - \frac{k^2 p^\mu}{2(p \cdot k)^2} - \frac{i k_\nu \sigma^{\mu\nu}}{p \cdot k} \right) \mathcal{H}(p_1 \ldots p_n) \\
&+ \sum_i Q_i \underbrace{\left( -\frac{p_i^\mu k^\nu}{p_i \cdot k} \frac{\partial}{\partial p_i^\nu} + \frac{\partial}{\partial p_i^\mu} \right)}_{-\frac{k_\nu}{p_i \cdot k} \underbrace{\left( p_i^\mu \frac{\partial}{\partial p_i^\nu} - p_i^\nu \frac{\partial}{\partial p_i^\mu} \right)}_{\equiv L^{\mu\nu}}} \mathcal{H}(p_1 \ldots p_n)
\end{aligned}$$

$L^{\mu\nu}$ is the angular momentum generator of the Lorentz group

# LBK THEOREM (NLP)

This is the **sub-leading** soft theorem, known as **Low-Burnett-Kroll theorem**:
[Low 1958 (scalar emitters), Burnett-Kroll 1968 (spin $\frac{1}{2}$ emitters, conjecture for generic spin), Bell-VanRoyen 1969 (generic spin), Cachazo-Strominger 2014]

$$\mathcal{A}(p_1, \ldots, p_n, k) = (\mathcal{S}_{LP} + \mathcal{S}_{NLP})\, \mathcal{H}(p_1, \ldots, p_n)\,,$$

$$\mathcal{S}_{LP} = \sum_{i=1}^{n} \eta_i Q_i \frac{\epsilon^*(k) \cdot p_i}{p_i \cdot k}\,, \quad \mathcal{S}_{NLP} = \sum_{i=1}^{n} \eta_i Q_i \frac{\epsilon^*_\mu(k) k_\nu (\sigma^{\mu\nu} + L^{\mu\nu})}{p_i \cdot k}$$

- corrections to the strict limit $k \to 0$: **small recoil** of the emitter taken into account
- sensitive to the **spin** of the emitter (e.g. $\sigma^{\mu\nu} = 0$ for scalars, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$for spin 1/2, etc.)
- orbital angular momentum $L^{\mu\nu}$ is sensitive to the short distance interactions in $\mathcal{H}$ (hard lines do not start from a pointlike vertex)
- NLP corrections here are valid only at the tree-level

# From amplitudes to Cross-sections

At amplitude level two NLP contributions:

- Spin $\sigma^{\mu\nu}$
- Orbital $L^{\mu\nu}$ i.e. derivatives

**Squaring** and **summing over polarizations**, spin contribution becomes also a derivative. Crucial identity e.g. for leg $p_1$ (neglecting $\sim k^\mu$):

$$\frac{\not{k}\gamma^\mu}{p_1\cdot k}(\not{p}_1+m)+(\not{p}_1+m)\frac{\gamma^\mu\not{k}}{p_1\cdot k} \;=\; -\gamma^\mu+\frac{p_1^\mu}{p_1\cdot k}\not{k} \;=\; G_1^{\mu\nu}\frac{\partial}{\partial p_1^\nu}(\not{p}_1+m)$$

Then, traditional LBK with derivatives reads

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2=\sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \rightarrow \textbf{LP}$$

$$+\sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p_\mu^i}{p_i\cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \rightarrow \textbf{NLP}$$

# Ambiguities at NLP

Problem: momentum conservation
l.h.s. $\sum_i p_i = k$ VS $\sum_i p_i = 0$ on the r.h.s. $\to$ difference for finite $k \neq 0$

Let us replace in $\mathcal{H}(p_1, \ldots, p_n)$

$$\boxed{p_i \to \tilde{\mathbf{p}}_{\mathbf{i}}(\mathbf{k}) = p_i + \mathbf{c_i} k + \mathcal{O}(k^2)}$$

$\mathbf{c_i}$ are **arbitrary** coefficients $\Longrightarrow$ Is LBK invariant at NLP?

$$\overline{|\mathcal{A}(p_1, \ldots, p_n, k)|}^2 = \sum_{ij=1}^{n} (-\eta_i \eta_j Q_i Q_j) \frac{p_i \cdot p_j}{p_i \cdot k \, p_j \cdot k} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_{\mathbf{1}}, \ldots, \tilde{\mathbf{p}}_{\mathbf{n}})|}^2$$
$$+ \sum_{ij=1}^{n} (-\eta_i \eta_j Q_i Q_j) \frac{p^i_\mu}{p_i \cdot k} \xi_j \left( \eta^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j \cdot k} \right) \frac{d}{dp_j^\nu} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_{\mathbf{1}}, \ldots, \tilde{\mathbf{p}}_{\mathbf{n}})|}^2$$

LBK theorem is invariant if $\mathbf{c_i}$ dependence **cancels up to NNLP** corrections.

# Ambiguities at NLP

First Taylor expand in $k$

$$\overline{|\mathcal{H}(\tilde{\mathbf{p}}_1,\dots,\tilde{\mathbf{p}}_n)|}^2 = \overline{|\mathcal{H}(p_1,\dots,p_n)|}^2 + k^\mu \sum_i \mathbf{c_i} \frac{\partial}{\partial p^i_\mu} \overline{|\mathcal{H}(p_1,\dots,p_n)|}^2 + \mathcal{O}(k^2)$$

Then impose momentum conservation $k = \sum_i p_i$

$$\frac{d}{dp^\nu_j} \overline{|\mathcal{H}(\tilde{\mathbf{p}}_1,\dots,\tilde{\mathbf{p}}_n)|}^2 = \frac{\partial}{\partial p^\nu_j} \overline{|\mathcal{H}(p_1,\dots,p_n)|}^2 + g^{\mu\nu}\xi_j \sum_i \mathbf{c_i} \frac{\partial}{\partial p^i_\mu} \overline{|\mathcal{H}(p_1,\dots,p_n)|}^2 + \mathcal{O}(k)$$

Plug this into LBK with $\tilde{p}_i$
$\rightarrow$ we get original LBK (with $p_i$) + remainder term that depends on $c_i$

$$\begin{aligned} R(\mathbf{c_i}) &= \sum_{ij=1}^{n} (-\eta_i\eta_j Q_i Q_j) \frac{p_i \cdot p_j}{p_i \cdot k \, p_j \cdot k} k^\mu \sum_m \mathbf{c_m} \frac{\partial}{\partial p^m_\mu} \overline{|\mathcal{H}(p_1,\dots,p_n)|}^2 \\ &\quad + \sum_{ij=1}^{n} (-\eta_i\eta_j Q_i Q_j) \frac{p^i_\mu}{p_i \cdot k} \xi_j \left( \eta^{\mu\nu} - \frac{p^\mu_j k^\nu}{p_j \cdot k} \right) \xi_j \sum_m \mathbf{c_m} \frac{\partial}{\partial p^\nu_m} \overline{|\mathcal{H}(p_1,\dots,p_n)|}^2 + \mathcal{O}(1) \\ &= \mathbf{0} + \mathcal{O}(1) = \mathbf{NNLP} \end{aligned}$$

$\Longrightarrow$ LBK is invariant at NLP under momentum transformation

# Ambiguities at NLP

Two cases in particular are relevant:

- $\mathbf{c_i = 0}$ (i.e. unphysical momenta, as in original LBK)
- $\sum_i \mathbf{c_i = -1} \implies \sum_i p_i = k, \sum_i \tilde{p}_i = 0$ (i.e. momentum conservation restored)

By virtue of the invariance, the two cases are equivalent at NLP. Hence, traditional LBK with unphysical momenta is consistent.

More generally, there are an infinite number of (formally equivalent at NLP) versions of the theorem, that differ by NNLP terms.

Here invariance shown under momenta transformation. The more general invariance of LBK under

$$\mathcal{H} \to \mathcal{H} + \Delta \quad \text{with} \quad \Delta(p_i)\delta(p_i) = 0 \tag{1}$$

can be proven [Balsach, DB, Kulesza 2023] which holds also e.g. for constant amplitudes.

Key aspect: choice of functional dependence for $\mathcal{H}$ yields the version of the theorem. Cancellation of NLP ambiguities between 2 terms of the theorem.

# Ambiguities at NLP

A simple example: $\pi(p_a)\pi(p_b) \to \pi(p_1')\pi(p_2')\gamma(k)$

Functional dependence:

- $\mathcal{H}(s_L, t)$ as in $\mathcal{A}^\mu_{\mathrm{V2}}$[Lebiedowicz, Nachmann, Szczurek 2021]
- $\mathcal{H}(s_L', t_2) = \mathcal{H}(s_L, t) + \delta s_L' \frac{\partial \mathcal{H}}{\partial s_L} + \delta t_2 \frac{\partial \mathcal{H}}{\partial t}$ as in $\mathcal{A}^\mu_{\mathrm{Low}}$ [Low 1958]

with some algebra

$$\delta s_L' = -(p_a + p_b)\cdot k + \mathcal{O}\left(k^2\right) ,$$

$$\delta t_2 = -2\left[(p_a - p_1)\cdot k - p_a \cdot l_1\right] + \mathcal{O}\left(k^2\right) .$$

one can see that $\mathcal{A}^\mu_{\mathrm{Low}} = \mathcal{A}^\mu_{\mathrm{V2}} + \mathcal{O}(k^2)$

$$\mathcal{A}^\mu_{\mathrm{Low}} = e\mathcal{H}(s_L', t_2)\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1'^\mu}{p_1'\cdot k}\right] + e\frac{\partial \mathcal{H}(s_L', t_2)}{\partial s_L'}\left[p_b^\mu - \frac{p_b\cdot k}{p_a\cdot k}p_a^\mu + p_2'^\mu - \frac{p_2'\cdot k}{p_1'\cdot k}p_1'^\mu\right]$$

$$\mathcal{A}^\mu_{\mathrm{V2}} = e\mathcal{H}(s_L, t)\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1'^\mu}{p_1'\cdot k}\right] + 2e\frac{\partial \mathcal{H}(s_L, t)}{\partial s_L}\left[p_b^\mu - \frac{p_b\cdot k}{p_a\cdot k}p_a^\mu\right]$$
$$- 2e\frac{\partial \mathcal{H}(s_L, t)}{\partial t}\left[(p_a - p_1)\cdot k - p_a\cdot l_1\right]\left[\frac{p_a^\mu}{p_a\cdot k} - \frac{p_1^\mu}{p_1\cdot k}\right]$$

The 2 versions are equivalent up to NNLP corrections. Similar analysis done by [Fadin-Khoze 2024]

# AMBIGUITIES AT NLP

- Traditional LBK is consistent at NLP
- Many forms of traditional LBK (all equivalent up to NNLP)
- consistent $\neq$ efficient. Some form of the theorem might be more efficient for a numerical implementation (NNLP effects can be visible in photon spectra since $k \neq 0$ )
- in particular, is there a form where the non-radiative process can be computed with unambiguous physical momenta?

# LBK theorem with shifted kinematics

# From derivatives to shifts

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \rightarrow \textbf{LP}$$

$$+\sum_{ij}(-\eta_i\eta_j Q_i Q_j)\frac{p^i_\mu}{p_i\cdot k}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2 \quad \rightarrow \textbf{NLP}$$

Exploit the fact that derivatives are generators of translations:

$$f(x+\epsilon) = f(x) + \epsilon\,\frac{d}{dx}f(x)$$

$\rightarrow$ convert derivatives into **shifted momenta**

[DelDuca, Laenen, Magnea, Vernazza, White 2017, van Beekveld-Beenakker-Laenen-White 2020, Bonocore, Kulesza 2021, van Beekveld-Danish-Laenen-Pal-Tripathi-White 2023]

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \sum_{i,j=1}^{n} -\eta_i\eta_j\frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}\left(1-\sum_j \delta p_j^\nu\frac{\partial}{\partial p_j^\nu}\right)\overline{|\mathcal{H}(p_1,\ldots,p_n)|}^2$$

# From derivatives to shifts

LBK with shifted kinematics:

$$\overline{|\mathcal{A}(p_1,\ldots,p_n,k)|}^2 = \underbrace{\sum_{i,j=1}^{n} -\eta_i\eta_j Q_i Q_j \frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}}_{\textbf{LP}\text{ factor!}} \overline{|\mathcal{H}(p_1+\delta p_1,\ldots,p_n+\delta p_n)|}^2$$

$$\delta p_j^\nu = \eta_j \xi_j Q_j \left(\sum_{k,l} \eta_k\eta_l Q_k Q_l \frac{p_k\cdot p_l}{(p_k\cdot k)(p_l\cdot k)}\right)^{-1} \sum_i \left(\frac{\eta_i Q_i p_{i\mu}}{k\cdot p_i}\right)\left(\eta^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j\cdot k}\right)$$

Note that

$$\delta p_i = \mathcal{O}(k) \qquad \sum_i \delta p_i = -k \qquad p_i\cdot\delta p_i = 0$$

# From derivatives to shifts

Simple case: **2 charged particles**

$$|\mathcal{A}(p_1, p_2, k)|^2 = \left( \sum_{i,j=1}^{2} -\eta_i \eta_j Q_i Q_j \frac{p_i \cdot p_j}{p_i \cdot k \, p_j \cdot k} \right) |\mathcal{H}(p_1 + \delta p_1, p_2 + \delta p_2)|^2 \qquad (2)$$

where

$$\delta p_1^\mu = \frac{1}{2} \left( -\frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\mu + \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\mu - k^\mu \right)$$
$$\delta p_2^\mu = \frac{1}{2} \left( \frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\mu - \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\mu - k^\mu \right)$$

Immediate to see

$$\begin{aligned} \delta p_i = \mathcal{O}(k) \quad &\rightarrow \quad \text{LBK is NLP} \\ \delta p_1 + \delta p_2 = -k \quad &\rightarrow \quad \text{momentum is conserved} \\ p_i \cdot \delta p_i = 0 \quad &\rightarrow \quad \text{on shell?} \end{aligned}$$

# From derivatives to shifts

$$p_i \cdot \delta p_i = 0 \implies (p_i + \delta p_i)^2 = m^2 + \mathcal{O}(k^2) = m^2 + \text{NNLP}$$

Momenta are on-shell at NLP, hence theorem consistent at NLP

However, **masses do get shifted** by a NNLP amount!

$$(\delta p_j)^2 = Q_j^2 \left( \sum_{k,l} \eta_k \eta_l Q_k Q_l \frac{p_k \cdot p_l}{(p_k \cdot k)(p_l \cdot k)} \right)^{-1} \neq 0 \,,$$

i.e. with shifts we recovered momentum conservation, but momenta are off-shell at NNLP $\rightarrow$ problem for numerical implementations, where $k \neq 0$

Is there a LBK formulation fulfilling both **momentum conservation** AND **on-shell** condition exactly (i.e. not just at NLP)?

# LBK theorem with modified shifted kinematics

# Modified shifts

LBK theorem works at NLP
$\Longrightarrow$ freedom to introduce **spurious NNLP** terms in the shifts.

We would like shifts $\delta p_i$ to

(i) conserve momentum exactly, i.e.

$$\sum_i \xi_i \delta p_i + k = 0 \,,$$

(ii) not shift the masses exactly, i.e.

$$(p_i + \delta p_i)^2 = m_i^2 \,,$$

(iii) reduce to old shifts up to NNLP corrections, i.e.

$$\delta p_j^\nu = \eta_j \xi_j Q_j \left( |\mathcal{S}_{\mathrm{LP}}|^2 \right)^{-1} \sum_i \left( \frac{\eta_i Q_i}{k \cdot p_i} \right) \left( p_i^\nu - \frac{p_i \cdot p_j}{p_j \cdot k} k^\nu \right) + \mathcal{O}\left(k^2\right) .$$

Is this possible?

# Modified shifts

Consider the ansatz

$$\delta p_i^\mu = \sum_j A_{ij}^{\mu\nu} p_{j\nu} + B_i^{\mu\nu} k_\nu \ ,$$

and determine coefficients $A_{ij}^{\mu\nu}$ and $B_i$ by imposing conditions (i)-(iii).

$\rightarrow$ conditions not too constraining, many solutions for $\delta p_i$. But we seek a single solution!

- restrict our ansatz

$$\delta p_i^\mu = \sum_j A \eta_i \xi_i Q_i \frac{\eta_j Q_j}{k \cdot p_j} p_{j\nu} + B_i^{\mu\nu} k_\nu$$

- impose $p_{j\nu}$ and $k_\nu$ to be linear independent
- verify that solution has correct behaviour for $k \rightarrow 0$

# Modified shifts

Result: [Balsach, DB, Kulesza]

$$\delta p_i^\mu = A\eta_i\xi_i Q_i \sum_j \frac{\eta_j Q_j}{k\cdot p_j} p_{j\nu} G_i^{\nu\mu} - \frac{1}{2}\frac{A^2 Q_i^2 |\mathcal{S}_{\rm LP}|^2}{p_i\cdot k} k^\mu \,,$$

with

$$A = \frac{1}{\chi}\left(\sqrt{1+\frac{2\chi}{|\mathcal{S}_{\rm LP}|^2}}-1\right) \qquad \chi = \sum_i \frac{\xi_i Q_i^2}{p_i\cdot k}\,.$$

$$|\mathcal{S}_{\rm LP}|^2 = \sum_{i,j}\eta_i\eta_j Q_i Q_j \frac{p_i\cdot p_j}{(p_i\cdot k)(p_j\cdot k)}$$

- Momentum is **conserved** (exactly)
- Momenta are **on-shell** (exaclty)
- Shifts are $\mathcal{O}(k) \implies$ equivalent to traditional LBK at NLP

$\implies$ This form of LBK allow computation of non-radiative process $\mathcal{H}$ with most general-purpose event generators

Price to pay: **spurious NNLP** terms in the shifts

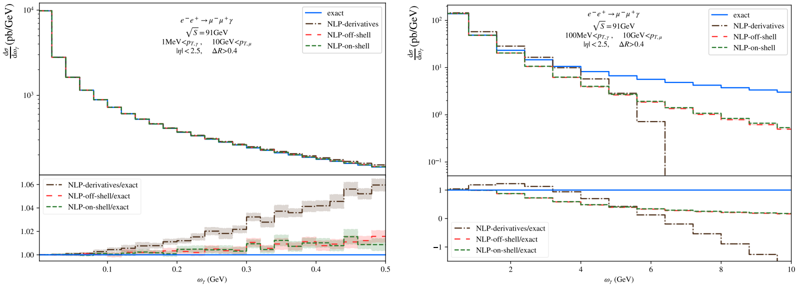
# Three versions of (tree-level) LBK

- All theoretically consistent at NLP
- NNLP ambiguities contained in all three versions (“scheme” dependence)
- When spectra are computed numerically, NNLP effects are visible
- Which version is more efficient and versatile? Which has more predictive power?
- Once we select the best NLP method, what is resolution in momentum we need for NLP to be measurable?

# Results for $e^+e^- \to \mu^+\mu^-\gamma$ and $pp \to \mu^+\mu^-\gamma$

# THREE VERSIONS OF (TREE-LEVEL) LBK

Results for $e^+e^- \to \mu^+\mu^-\gamma$ [Balsach, DB, Kulesza]



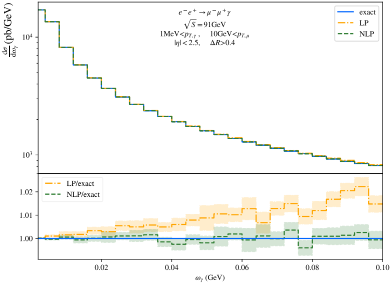

Note

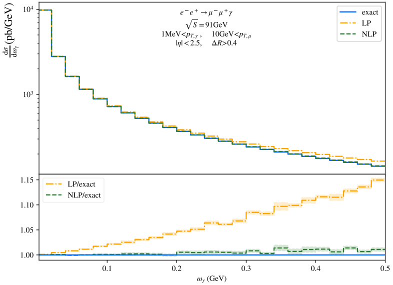
- non-radiative amplitude can be computed analytically (used here for derivatives and off-shell shifts)
- exact means tree-level with no soft expansion
- estimation of NNLP effects

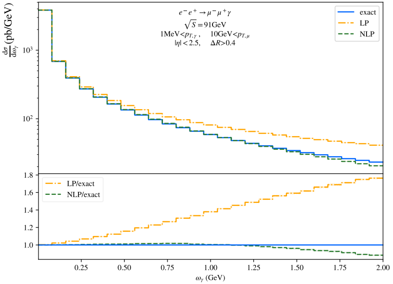
On-shell shifts work better. Used later as NLP

# LP VS NLP

## $e^+e^- \to \mu^+\mu^-\gamma$: (c.m.) $\omega$ distributions [Balsach, DB, Kulesza]

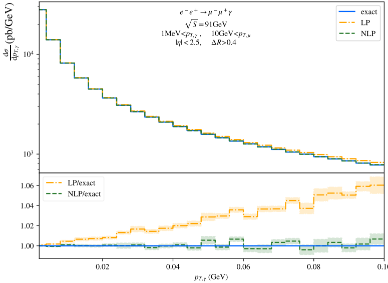


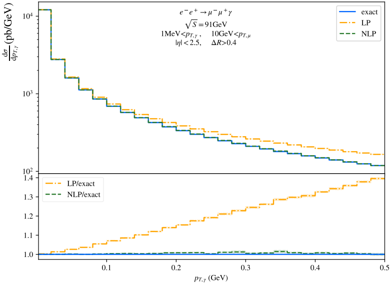



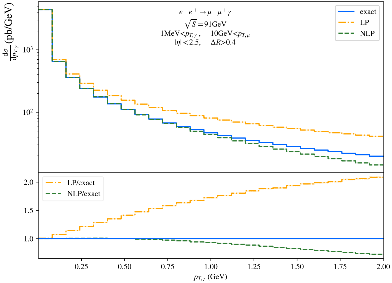



# LP VS NLP

## $e^+e^- \to \mu^+\mu^-\gamma$: $p_t$ distributions [Balsach, DB, Kulesza]

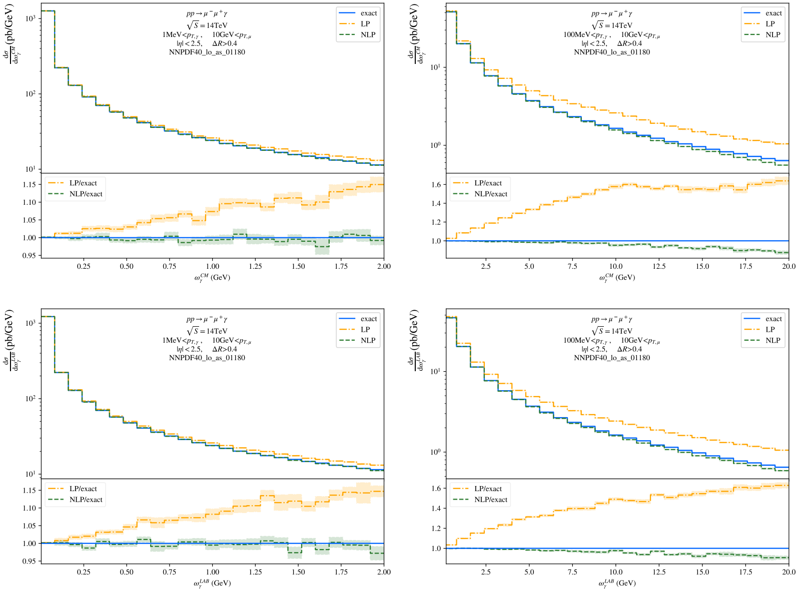


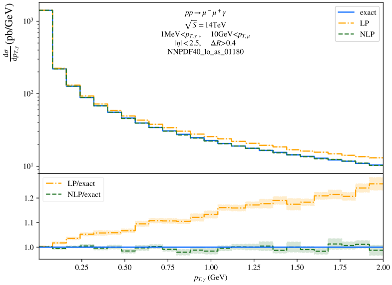
# LP VS NLP

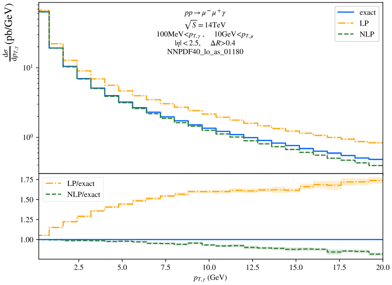
$pp \to \mu^+\mu^-\gamma$: (c.m. and lab) $\omega$ distributions [Balsach, DB, Kulesza]



# LP VS NLP

$pp \to \mu^+\mu^-\gamma$: $p_t$ distributions [Balsach, DB, Kulesza]



# Loop corrections to LBK theorem

# NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

- LP soft photon theorem **does not** receive corrections at **one-loop**.

$$\epsilon^*_\mu(k)\mathcal{A}^\mu = \mathcal{S}_{LP}\ \mathcal{A}_n\ , \qquad \mathcal{A}_n = \mathcal{A}_n^{(0)}, \mathcal{A}_n^{(1)}, \mathcal{A}_n^{(2)}, \ldots$$

$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k}\ ,$$

- at NLP, soft theorems **do** receive **one-loop corrections**.[Bern,Davies,Nohle 2014, He,Huang,Wen 2014, Larkoski,Neill, Stewart 2014, DB,Laenen,Magnea,Vernazza,White 2014]

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(0)} = (\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree})\ \mathcal{A}_n^{(0)}\ ,$$

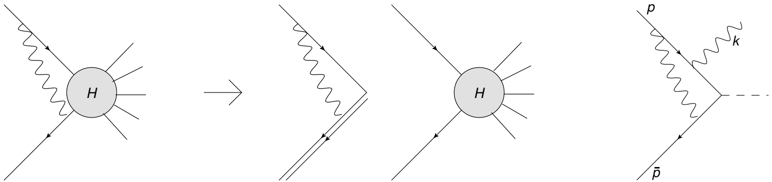
$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(1)} = (\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree})\ \mathcal{A}_n^{(1)} + \mathbf{?}\ ,$$

$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k}\ , \qquad \mathcal{S}_{NLP-tree} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*_\mu(k)k_\nu(\sigma^{\mu\nu} + L^{\mu\nu})}{p_i\cdot k}$$

Various sources of correction. E.g. soft region in the massive case [Engel,Signer,Ulrich 2021]. In the high energy limit, it is interesting to look at the **massless limit** (crucial for the massless parton model) and the **collinear region**

# NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

Virtual collinear effects are captured by **radiative jet functions** $J^\mu$ [DelDuca 1990, DB, Laenen, Magnea, Vernazza, White 2014, Gervais 2017, Beneke, Garny, Szafron, Wang 2018, Laenen, Damste, Vernazza, Waalewijn, Zoppi 2020, Liu, Neubert, Schnubel, Wang 2021].




In particular, the one-loop quark radiative jet function in dimensional regularization (with $d = 4 - 2\epsilon$ and $\bar{\mu}$ the $\overline{\text{MS}}$ scale) reads [DB,Laenen,Magnea,Melville,Vernazza,White,2015]

$$J^{\mu(1)} = \left(\frac{\bar{\mu}^2}{2p\cdot k}\right)^{\epsilon} \left[ \left(\frac{2}{\epsilon} + 4 + 8\epsilon\right)\left(\frac{n\cdot k}{p\cdot k}\frac{p^\mu}{p\cdot n} - \frac{n^\mu}{p\cdot n}\right) - (1+2\epsilon)\frac{ik_\alpha S^{\alpha\mu}}{p\cdot k} + \left(\frac{1}{\epsilon} - \frac{1}{2} - 3\epsilon\right)\frac{k^\mu}{p\cdot k} + (1+3\epsilon)\left(\frac{\gamma^\mu \not{n}}{p\cdot n} - \frac{p^\mu}{p\cdot k}\frac{\not{k}\not{n}}{p\cdot n}\right)\right] + \mathcal{O}(\epsilon^2, k)$$

# NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

Thus, the next-to-soft theorem (i.e. LBK theorem) receives a **logarithmic correction**:

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(0)} = \left(\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree}\right)\mathcal{A}_n^{(0)} \ ,$$

$$\epsilon^*_\mu(k)\mathcal{A}^{\mu(1)} = \left(\mathcal{S}_{LP} + \mathcal{S}_{NLP-tree}\right)\mathcal{A}_n^{(1)} + \left(\sum_i \epsilon^*_\mu(k)\, q_i\, J_i^{\mu(1)}\right)\mathcal{A}_n^{(0)} \ ,$$

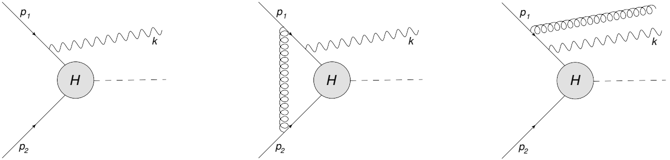
$$\mathcal{S}_{LP} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*(k)\cdot p_i}{p_i\cdot k} \ , \qquad \mathcal{S}_{NLP-tree} = \sum_{i=1}^{n} Q_i \frac{\epsilon^*_\mu(k)k_\nu(\sigma^{\mu\nu} + L^{\mu\nu})}{p_i\cdot k}$$

$$\left(\sum_i \epsilon^*_\mu(k)\, q_i\, J_i^{\mu(1)}\right)\mathcal{A}_n^{(0)} = \frac{2}{p_1\cdot p_2}\left[\sum_{ij}\left(\frac{1}{\epsilon} + \log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)\right) q_j\, p_i\cdot k\, \frac{p_j\cdot\epsilon}{p_j\cdot k}\right]\mathcal{A}_n^{(0)}$$

- Note that amplitude is IR divergent $\epsilon \to 0$
- $\log(\omega_k)$ corrections to soft theorems in QED also discussed (mainly classically) by Laddha-Sahoo-Sen. Here however more standard approach (i.e. dim.reg.) to regularization of soft and collinear divergences, which allows implementation in the massless limit required in QCD partonic calculations.

# NLP BREMSSTRAHLUNG WITH QCD CORRECTIONS

IR divergences ($1/\epsilon$) cancel by adding real emission diagram:




The **soft photon** emission from the loop with a **collinear gluon** is captured by the radiative jet function $J^\mu$ (note here the **mixed QED-QCD** effect)
The corresponding contribution is what is needed for a process with a single quark-antiquark pair in the **massless limit** such as

- $e^+e^- \to q\,\bar{q}\,\gamma$
- $p\,p \to \mu^+\mu^-\gamma$
- ...

For processes with more than two colored particles situation more subtle (but structure is similar)

# NLP bremsstrahlung with QCD corrections

The soft photon bremsstrahlung at $\mathcal{O}(\alpha_s)$ becomes

$$\boxed{\frac{d\sigma_{\text{NLP}}}{d^3k} = \frac{d\sigma_{\text{LP+(NLP-tree)}}}{d^3k} + \frac{\alpha_s}{4\pi}\frac{d\sigma_{\text{NLP-}J}}{d^3k}\,,}$$

where

$$\boxed{\frac{d\sigma_{\text{NLP-}J}}{d^3k} = \frac{\alpha}{(2\pi)^2}\frac{1}{\omega_k}\int d^3p_3\ldots d^3p_n\left(\sum_{i=1}^{2}\eta_i\frac{8\log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)}{p_i\cdot k}\right)d\sigma_H(p_1,\ldots,p_n)}$$

- **Correction of order** $\boxed{\alpha_s\log\left(\frac{\bar{\mu}^2}{2p_i\cdot k}\right)}$ **to LP spectrum** $\frac{d\sigma}{d\omega_k}$ **hence particularly enhanced for small** $\omega_k$ **and small** $k_t$
- **expecially relevant for hadrons (since for leptons -** $\alpha \ll \alpha_s$, $m \to 0$**)**

# Conclusions

- General interest in soft photons at NLP (numerical definition, experimental anomalies, consistency of LBK theorem)
- Different formulations of (tree-level) LBK theorem (derivatives, off-shell shifts, on-shell shifts) are all theoretically consistent and formally equivalent at NLP
- Different formulations correspond to reshuffling of NNLP effects, which might be numerically relevant (scheme choice) $\Longrightarrow$ not all formulations equally efficient
- New LBK formulation with on-shell shifted kinematics allows standard event generation for non-radiative process
- Numerical results show resolution in energy/momentum for NLP effects to be visible