Integrated Unitarity
for Scattering Amplitudes

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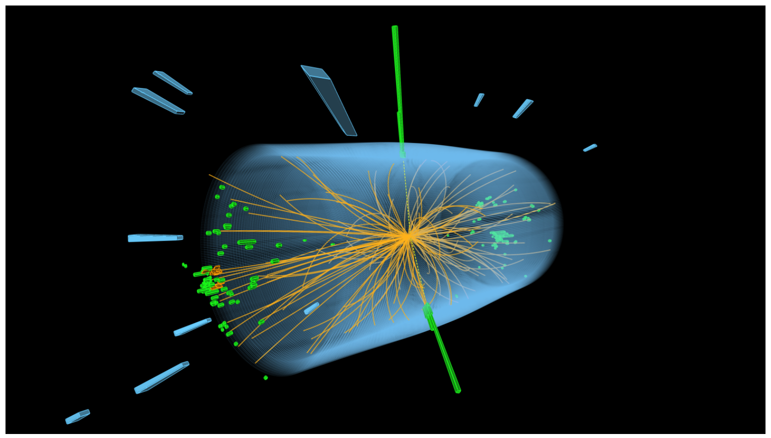
Presentation plan

- Overview (motivation, state-of-the-art)
- Background (cuts, discontinuities, dispersion relation)
- Integrated Unitarity (derivation, example, application)
- Outlook (Multivariate Integrated Unitarity)
- Outro (what's next?)

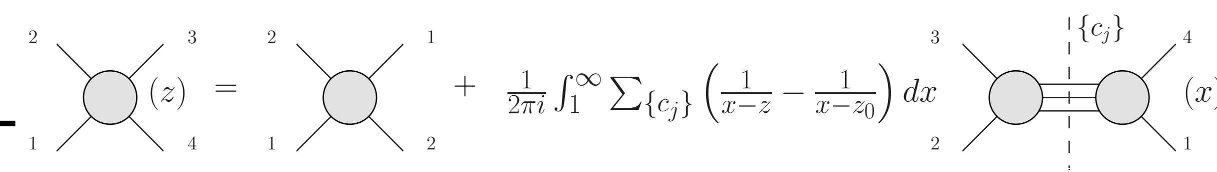
Overview

Overview : motivation

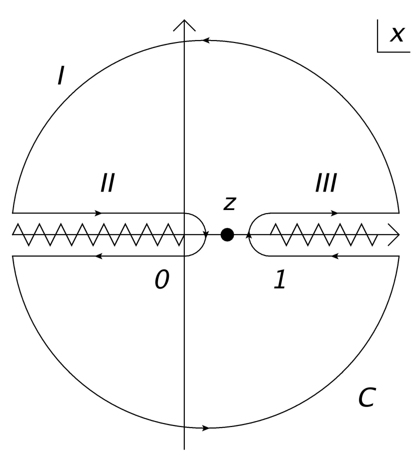
precision HEP



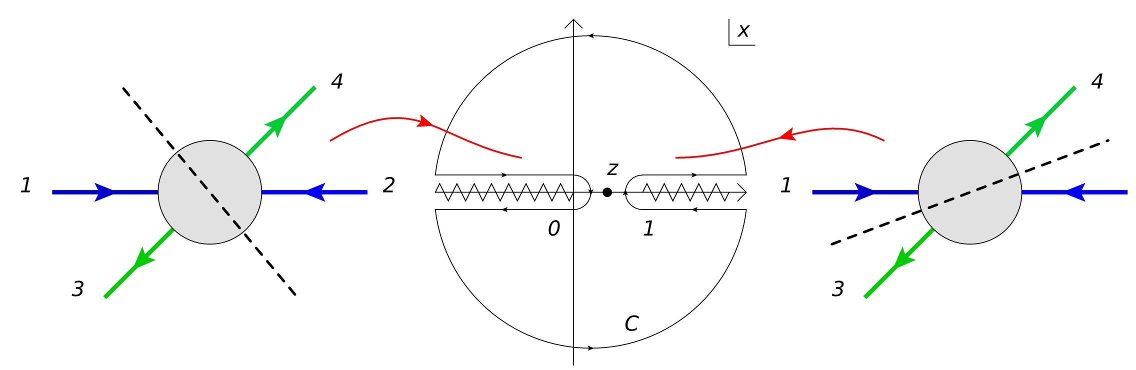
Scattering Amplitudes



mathematical structure

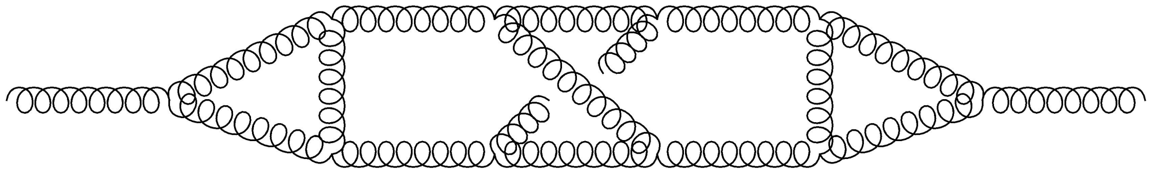


useful properties

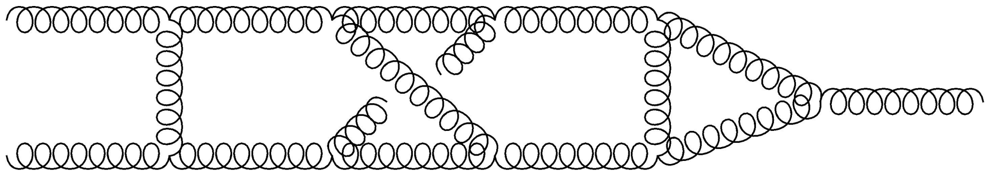


Overview : QCD amplitudes frontier

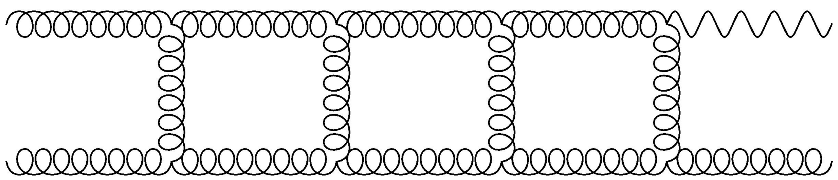
- 2-point : 5-loop (β function)
  [Herzog, Ruijl, Ueda, Vermaseren, Vogt 1701.01404]



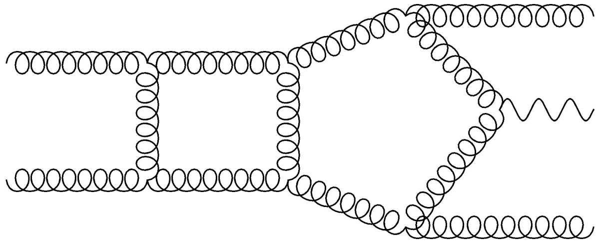
- 3-point : 4-loop (form factors)
  [Lee, Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser 2202.04660]



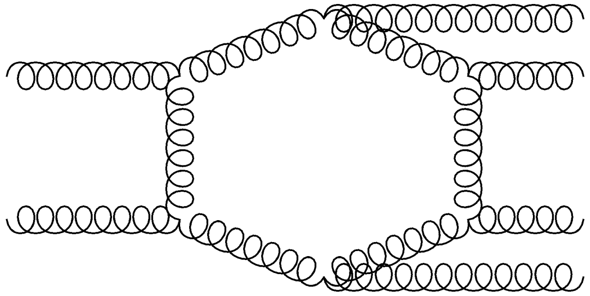
- 4-point : 3-loop ($\leq$1-off shell)
  [**PB**, Caola, Chakraborty, Gambuti, Manteuffel, Tancredi]
  [Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi]



- 5-point : 2-loop (multiscale)
  [Abreu, Badger, Buccioni, Chawdhry, Chicherin, Cordero, Czakon, Devoto, Gambuti, Ita, Manteuffel, Mitov, Page, Peraro, Poncelet, Sotnikov, Tancredi, Zoia, ...]



- $\geq$6-point : 1-loop (numerical, automated)
  [Blackhat, GoSam, MadLoops, OpenLoops (Buccioni), Rocket (Zanderighi), ...]



Overview : modern methods for QCD amplitudes

- form factor decomposition ~ project Feynman diagrams onto independent tensor structures
  [Garland, Gehrmann, Glover, Koukoutsakis, Remiddi 0206067]

- Generalized Unitarity ~ constrain coefficients of Feynman Integrals with cuts
  [Badger, Frellesvig, Zhang 1310.1051]

- Numerical Unitarity ~ constrain over finite fields the coefficients of Master Integrals with cuts
  [Abreu, Cordero, Ita, Jaquier, Page, Zeng 1703.05273]

- …

Integrated Unitarity ~ Generalized Unitarity @ integrated level

compatible with all above methods

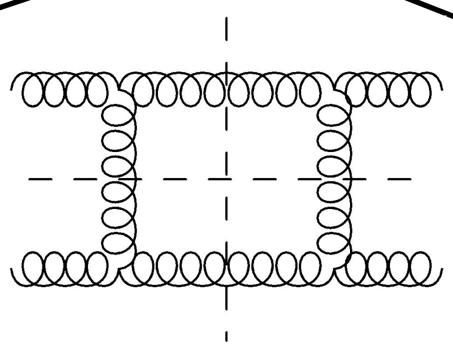
Overview : Integrated VS Generalized Unitarity

consider a toy 1-loop amplitude

$$\mathscr{A}^{(1)} \sim \int \frac{d^d k}{(2\pi)^d} \frac{\mathcal{N}(k)}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)} = r_{1234} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)} + \text{subsectors}$$

cut all 4 propagators

$$\text{Cut}_{1234}\,\mathscr{A}^{(1)} \sim \int \frac{d^d k}{(2\pi)^d} \mathscr{A}_1^{(0)}(k)\; \mathscr{A}_2^{(0)}(k)\; \mathscr{A}_3^{(0)}(k)\; \mathscr{A}_4^{(0)}(k)\, \delta^+(\mathscr{D}_1(k))\, \delta^+(\mathscr{D}_2(k))\, \delta^+(\mathscr{D}_3(k))\, \delta^+(\mathscr{D}_4(k)) = r_{1234} \int \frac{d^d k}{(2\pi)^d} \delta^+(\mathscr{D}_1(k))\, \delta^+(\mathscr{D}_2(k))\, \delta^+(\mathscr{D}_3(k))\, \delta^+(\mathscr{D}_4(k))$$



Generalized Unitarity

can compute integral coefficient

$$r_{1234} = \frac{1}{2} \sum_{\text{2 cut solns } k^*} \mathscr{A}_1^{(0)}(k^*)\; \mathscr{A}_2^{(0)}(k^*)\; \mathscr{A}_3^{(0)}(k^*)\; \mathscr{A}_4^{(0)}(k^*)$$

scalar integral remains to be computed

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathscr{D}_1(k)\,\mathscr{D}_2(k)\,\mathscr{D}_3(k)\,\mathscr{D}_4(k)}$$

Integrated Unitarity

compute cut amplitude

$$\text{Cut}_{1234}\,\mathscr{A}^{(1)} = r_{1234} \int \frac{d^d k}{(2\pi)^d} \delta^+(\mathscr{D}_1(k))\, \delta^+(\mathscr{D}_2(k))\, \delta^+(\mathscr{D}_3(k))\, \delta^+(\mathscr{D}_4(k))$$

can reconstruct whole amplitude

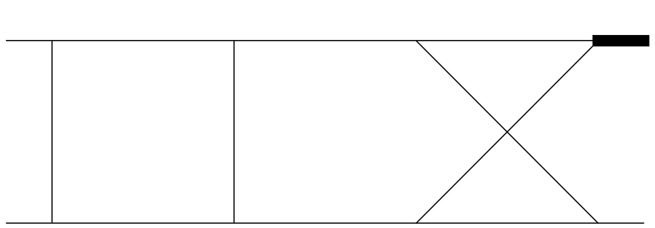
$$\mathscr{A}^{(1)}(s,u) \sim \int_0^\infty \frac{ds'}{s'-s} \int_0^\infty \frac{du'}{u'-u} \text{Cut}_{1234}\,\mathscr{A}^{(1)}(s',u')$$

Overview : Feynman integrals frontier

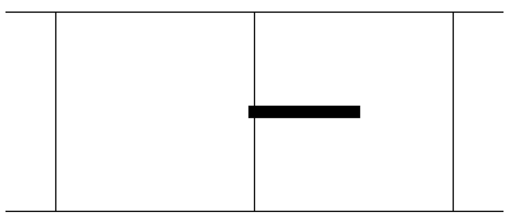
amplitude = f(color, helicity, **kinematics**)

here, we focus on analytic properties

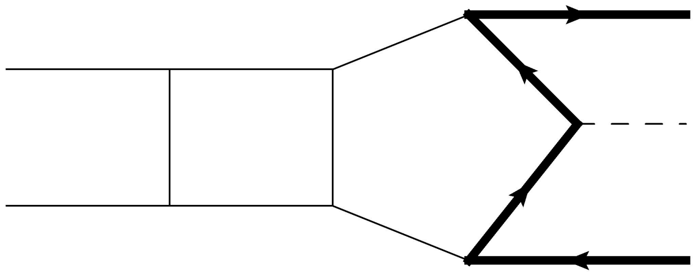
[Henn, Lim, Bobadilla 2302.12776]



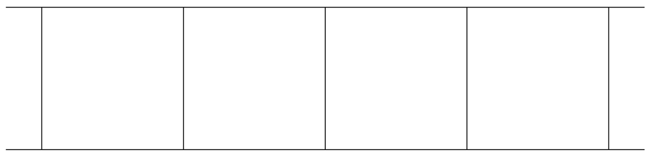
[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2306.15431]



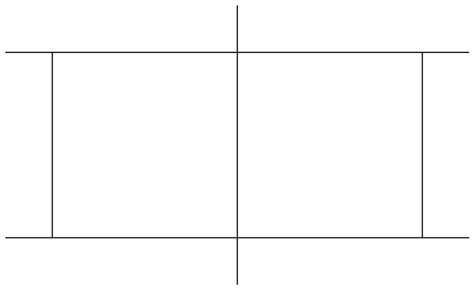
[Cordero, Figueiredo, Kraus, Page, Reina 2312.08131]



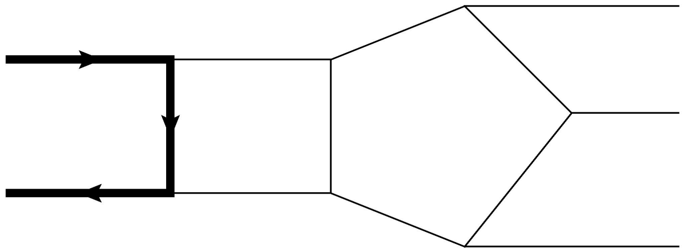
now available !



[Henn, Matijašić, Miczajka, Peraro, Xu, Zhang 2403.19742]



[Badger, Becchetti, Giraudo, Zoia 2404.12325]



Overview : analytic properties frontier

modern developments in analytic properties of amplitudes

- differential equations for cut integrals

  [Abreu, Britto, Duhr, Gardi, Grönqvist, Matthew]

- monodromy for PolyLog discontinuities

  [Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu]

- dispersion relation applications

  [Tancredi, Remiddi, Primo]

- modern analytic S-matrix program

  [Mizera, Telen, Hannesdottir, Caron-Huot, Giroux, Fevola]

Background

Background : cuts

- integral definition :

$$I_{\{n_i\}} = \int \left(\prod_{l=1}^{L} D^d k_l \right) \prod_{i=1}^{N} \mathscr{D}_i^{-n_i} \qquad D^d k_l = e^{\epsilon \gamma_E} \frac{d^d k_l}{i \pi^{d/2}}$$

- cut propagator :

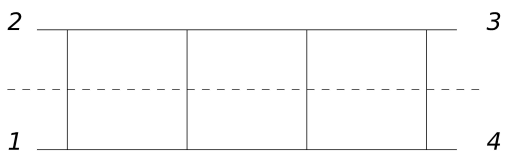
$$\frac{1}{\mathscr{D} + i\varepsilon} \to 2\pi i\, \delta^+(\mathscr{D}) \qquad \delta^+(q^2) = \delta(q^2)\, \theta(q_0)$$

- cut integral :

$$\mathrm{Cut}_u I_{\{n_i\}} = \sum_{\{c_j\} \in \mathscr{C}_u} I_{\{n_i\};\{c_j\}} = \sum_{\{c_j\} \in \mathscr{C}_u} \int \left(\prod_{l=1}^{L} D^d k_l \right) \left(\prod_{i \notin \{c_j\}} \mathscr{D}_i^{-n_i} \right) \prod_{m \in \{c_j\}} \delta_{1,n_m} 2\pi i\, \delta^+(\mathscr{D}_m)$$

- less subsectors :

$$/2^C$$



- Integration-By-Parts identities (IBPs) :
  [Laporta 0102033]

$$\int \left(\prod_{l=1}^{L} D^d k_l \right) \frac{\partial}{\partial k_l^\mu} \left(q^\mu \prod_{i=1}^{N} \mathscr{D}_i^{-n_i} \right) = 0$$

[Chetyrkin, Tkachov 1981]

$$p_{j\,\mu}\, p_{l\,\nu} \left(p_n^\nu \frac{\partial}{\partial p_{n,\mu}} - p_n^\mu \frac{\partial}{\partial p_{n,\nu}} \right) I_{\{n_i\}} = 0$$

[Gehrmann, Remiddi 9912329]

- Differential Equations (DEQ) : $\partial_{x_n} M_i(\vec{x}, \epsilon) = A_{ij}(\vec{x}, \epsilon)\, M_j(\vec{x}, \epsilon)$ $\xrightarrow{\text{canonical}}$ $\partial_{x_n} M_i^c(\vec{x}, \epsilon) = \epsilon\, A_{ij}^c(\vec{x})\, M_j^c(\vec{x}, \epsilon)$

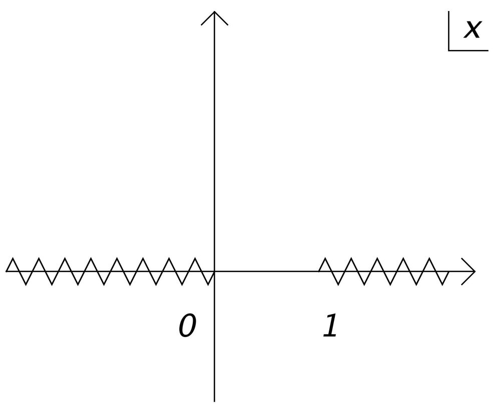
Master Integrals (MIs) → [Kotikov 1991] [Henn 1304.1806]

→ can solve cut integrals with DEQ

Background : discontinuities

- discontinuity : $\mathrm{Disc}_0 f(x) = f(x + i\epsilon) - f(x - i\epsilon)$

- from now on, focus on specific kinematics : $x = -\frac{t}{s}$ 4-point massless



- branch cuts :

$t > 0$	$u > 0$	$s > 0$
$x < 0$	$x > 1$	pushed to ∞

- Harmonic Polylogarithms (HPLs) : $G(\alpha_n, \ldots, \alpha_1; x) = \int_0^x \frac{dz}{z - \alpha_n} G(\alpha_{n-1}, \ldots, \alpha_1; z)$ $\alpha_k \in \{0,1\}$

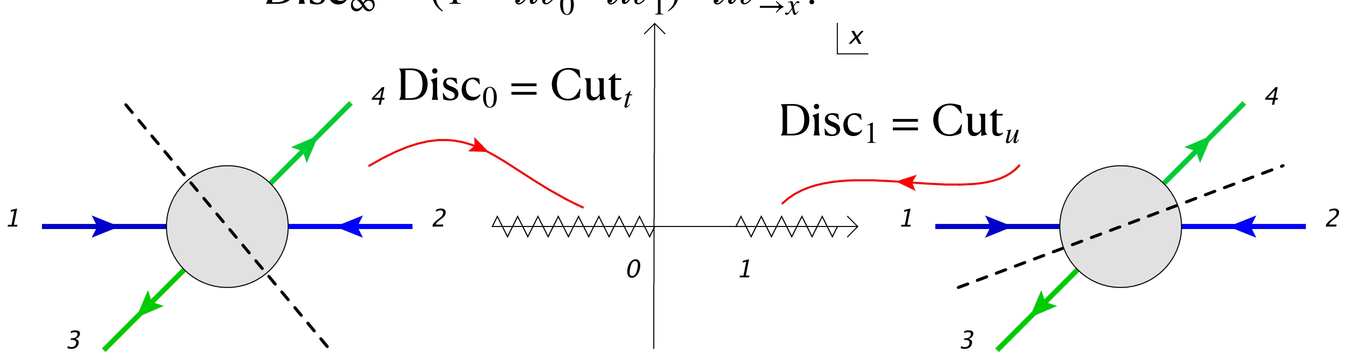
  [Remiddi, Vermaseren 9905237]

- discontinuities of HPLs algorithmic from monodromy matrices :

  [Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

$$\mathrm{Disc}_0 = (1 - \mathcal{M}_0) \cdot \mathcal{M}_{\to x}\,,$$
$$\mathrm{Disc}_1 = -\,(1 - \mathcal{M}_1) \cdot \mathcal{M}_{\to x}\,,$$
$$\mathrm{Disc}_\infty = (1 - \mathcal{M}_0 \cdot \mathcal{M}_1) \cdot \mathcal{M}_{\to x}\,.$$

- unitarity :



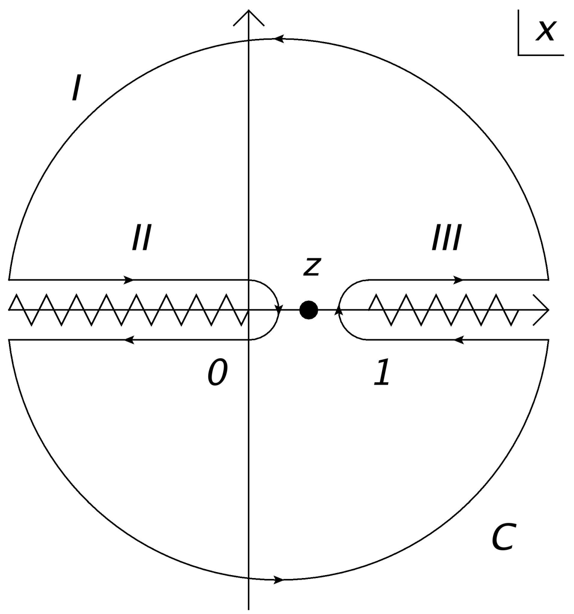
$(\mathrm{Disc}_\infty = \mathrm{Cut}_s)$

Background : dispersion relation

Cauchy's integral formula

$$\mathscr{A}(z) = \frac{1}{2\pi i}\oint_C \frac{\mathscr{A}(x)dx}{x-z}$$

[Cutkosky, Mandelstam, Eden, Landshoff, Olive, Polkinghorne, Remiddi, van Neerven, Kniehl, Sirlin]



piecewise contour

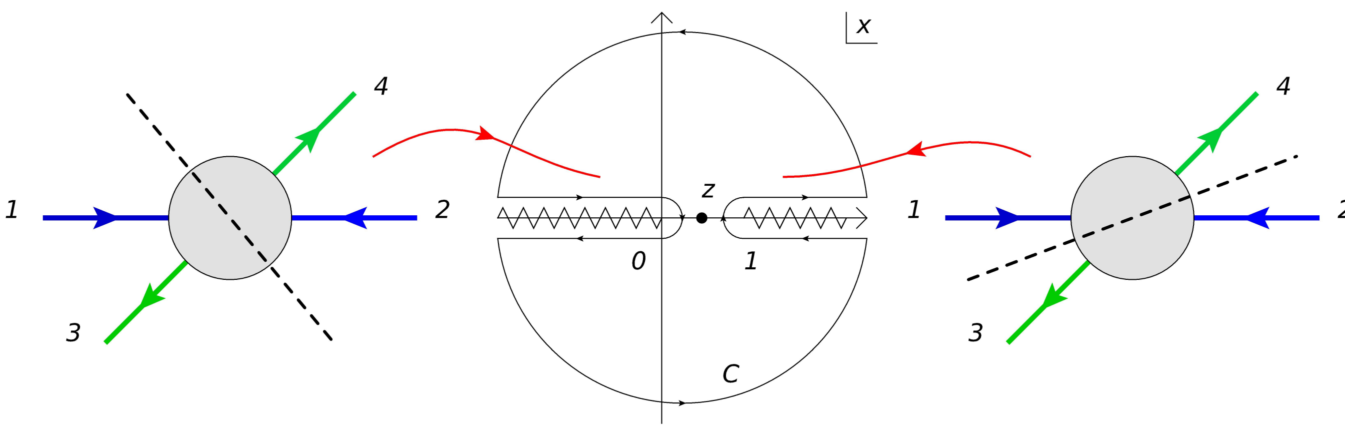
$$\mathscr{A}(z) = c_\infty + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Disc}_0 + \int_1^\infty \mathrm{Disc}_1\right)\frac{\mathscr{A}(x)dx}{x-z}$$

cancel constant from infinite arc

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Disc}_0 + \int_1^\infty \mathrm{Disc}_1\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

unitarity

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$



Integrated Unitarity

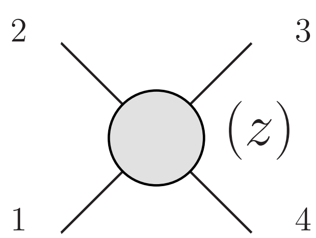
Integrated Unitarity

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

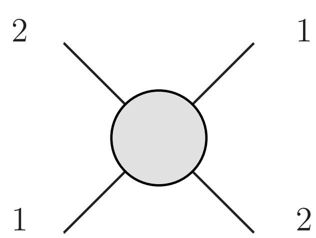
expressing cuts as phase space integrals

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\Bigg(\int_0^\infty \sum_{\{c_i\}\in\mathscr{C}_t}\int d\mathrm{PS}_{t,\{c_i\}}\,\mathscr{A}_{t,\{c_i\},L}\,\mathscr{A}^*_{t,\{c_i\},R}$$
$$+\int_1^\infty \sum_{\{c_j\}\in\mathscr{C}_u}\int d\mathrm{PS}_{u,\{c_j\}}\,\mathscr{A}_{u,\{c_j\},L}\,\mathscr{A}^*_{u,\{c_j\},R}\Bigg)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)dx$$

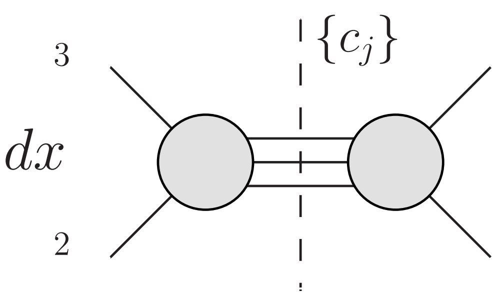
diagrammatically, in the planar case



$(z) \; = $

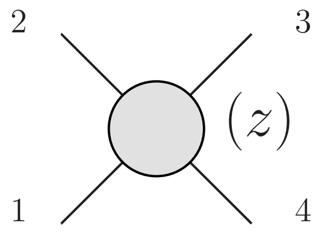


$$+ \; \frac{1}{2\pi i}\int_1^\infty \sum_{\{c_j\}}\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)dx$$

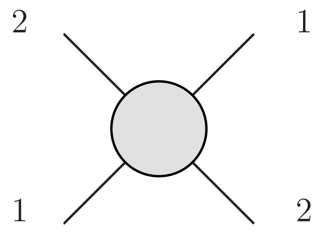


(x)

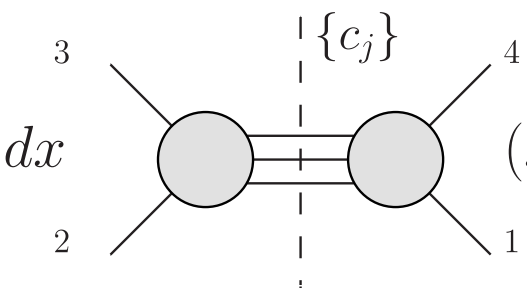
Integrated Unitarity : properties



$$(z) =$$



$$+ \frac{1}{2\pi i}\int_1^\infty \sum_{\{c_j\}} \left(\frac{1}{x-z} - \frac{1}{x-z_0}\right) dx$$



$$(x)$$

- Generalized Unitarity @ integrated level : constrains both MIs and their coefficients with cuts
- algorithmic in dimReg : cut canonical differential equations
- compatible with other amplitude methods
- less subsectors : by a factor of 2^{# cut propagators}
- less MIs : simpler DEQ
- simpler IBPs : propagator powers {0,-1,-2,…} unsupported on a cut

Integrated Unitarity : 3 methods

$$\mathscr{A}(z) = \mathscr{A}_0 + \frac{1}{2\pi i}\left(\int_0^\infty \mathrm{Cut}_t + \int_1^\infty \mathrm{Cut}_u\right)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)\mathscr{A}(x)dx$$

A. explicit integration : convergent e.g. for canonical MIs

subtraction terms needed for full amplitude

$$\mathscr{A}(x) \to S(x)\ \mathscr{A}(x) \qquad S(x) = \frac{(1-x)^p x^q}{(x-z_1)^r} \qquad -\mathop{\mathrm{Res}}_{x\to z_1} \mathscr{A}(x)\, S(x)\left(\frac{1}{x-z} - \frac{1}{x-z_0}\right)$$

B. ansatz reconstruction with 2 discontinuities + # evaluations :

$$\begin{cases} \mathrm{Disc}_0\, \mathscr{A} & = \mathrm{Cut}_t\, \mathscr{A} \\ \mathrm{Disc}_1\, \mathscr{A} & = \mathrm{Cut}_u\, \mathscr{A} \\ \mathscr{A}(z_i) & = \mathscr{A}_i \end{cases}$$

$$\mathscr{A}(z) = \epsilon^{\#} \sum_{n\geq 0} \epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} r_{n,\vec{\alpha}}(z)\, G(\overrightarrow{\alpha}, z)$$

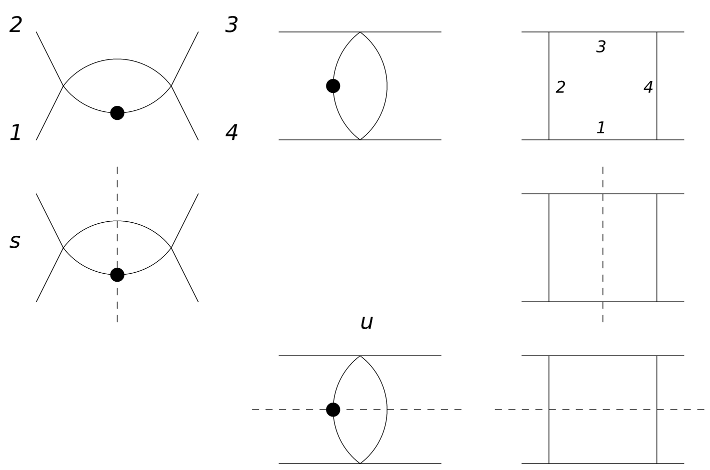
↑ unknowns

C. ansatz reconstruction with 3 discontinuities :

$$\begin{cases} \mathrm{Disc}_0\, \mathscr{A} & = \mathrm{Cut}_t\, \mathscr{A} \\ \mathrm{Disc}_1\, \mathscr{A} & = \mathrm{Cut}_u\, \mathscr{A} \\ \mathrm{Disc}_\infty\, \mathscr{A} & = \mathrm{Cut}_s\, \mathscr{A} \end{cases}$$

$$\mathscr{A}(z) = \epsilon^{\#} \sum_{n\geq 0} \epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} r_{n,\vec{\alpha}}(z)\, G(\overrightarrow{\alpha}, z)$$

Integrated Unitarity : example



cut MIs

$M_i^c \in \{\epsilon(2\epsilon-1)\, I_{1,0,1,0}\,, \epsilon(2\epsilon-1)\, I_{0,1,0,1}\,, \epsilon^2(x-1)\, I_{1,1,1,1}\}$

$\mathrm{Cut}_s M_i^c \in \{\epsilon(2\epsilon-1)\, I_{1,0,1,0;1,3},\ 0,\ \epsilon^2(x-1)\, I_{1,1,1,1;1,3}\}$

$\mathrm{Cut}_u M_i^c \in \{0,\ \epsilon(2\epsilon-1)\, I_{0,1,0,1;2,4}, \epsilon^2(x-1)\, I_{1,1,1,1;2,4}\}$

$\mathrm{Cut}_t M_i^c \in \{0,\ 0,\ 0\}$

cut DEQ

$\partial_x M_i^c(x,\epsilon) = \epsilon\, A_{ij}^c(x)\, M_j^c(x,\epsilon)$

$\partial_x \mathrm{Cut}_s M_i^c(x,\epsilon) = \epsilon\, A_{ij}^{c,s}(x)\, \mathrm{Cut}_s M_j^c(x,\epsilon)$

$\partial_x \mathrm{Cut}_u M_i^c(x,\epsilon) = \epsilon\, A_{ij}^{c,u}(x)\, \mathrm{Cut}_u M_j^c(x,\epsilon)$

$\partial_x \mathrm{Cut}_t M_i^c(x,\epsilon) = 0$

$$A^c(x) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\frac{1}{1-x}} & \mathbf{0} \\ \mathbf{\frac{2}{x}} & \mathbf{\frac{2}{x}} + \mathbf{\frac{2}{1-x}} & \frac{1}{x} + \frac{1}{1-x} \end{pmatrix}$$

3 methods

unknowns

$$M_i^c(z) = \epsilon^{-2} \sum_{n\geq 0} \epsilon^n \sum_{\vec{\alpha}\,:\,|\vec{\alpha}|\leq n} c_{n,\vec{\alpha}}\, G(\vec{\alpha}, z) \qquad \alpha_k \in \{0,1\}$$

A. explicit integration

$$M_i^c(z) = M_{i,0}^c + \frac{1}{2\pi i}\int_1^\infty \left(\frac{dx}{x-z} - \frac{dx}{x}\right) \mathrm{Cut}_u M_i^c(x)$$

✔

B. ansatz + 2cuts + 1pt

ansatz | computed

$$\begin{cases} \mathrm{Disc}_0 M_i^c & = \mathrm{Cut}_t M_i^c \\ \mathrm{Disc}_1 M_i^c & = \mathrm{Cut}_u M_i^c \\ M_i^c(0) & = M_{i,0}^c \end{cases}$$

✔

C. ansatz + 3cuts

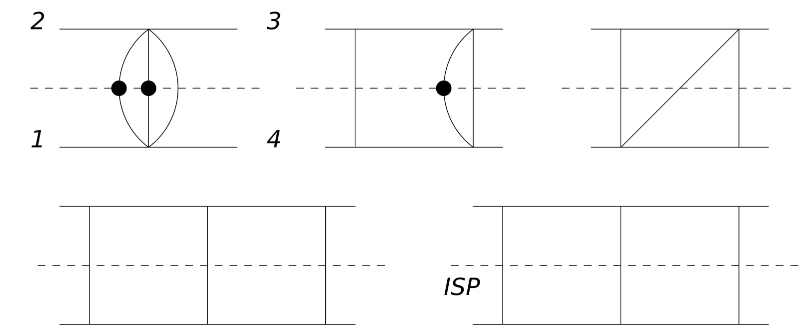
$$\begin{cases} \mathrm{Disc}_0 M_i^c & = \mathrm{Cut}_t M_i^c \\ \mathrm{Disc}_1 M_i^c & = \mathrm{Cut}_u M_i^c \\ \mathrm{Disc}_\infty M_i^c & = \mathrm{Cut}_s M_i^c \end{cases}$$

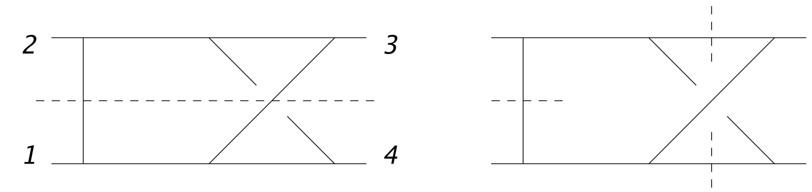
✔

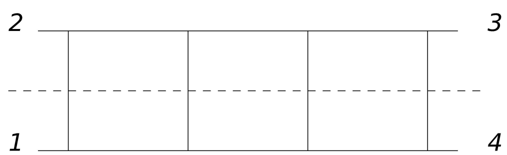
Integrated Unitarity : further checks

Master Integrals (from DEQ)

- 2-loop planar (#MIs : 8 → 5) ✓
- 2-loop nonplanar (#MIs : 12 → 2+6) ✓
- 3-loop planar ladder (#MIs : 26 → 17) ✓



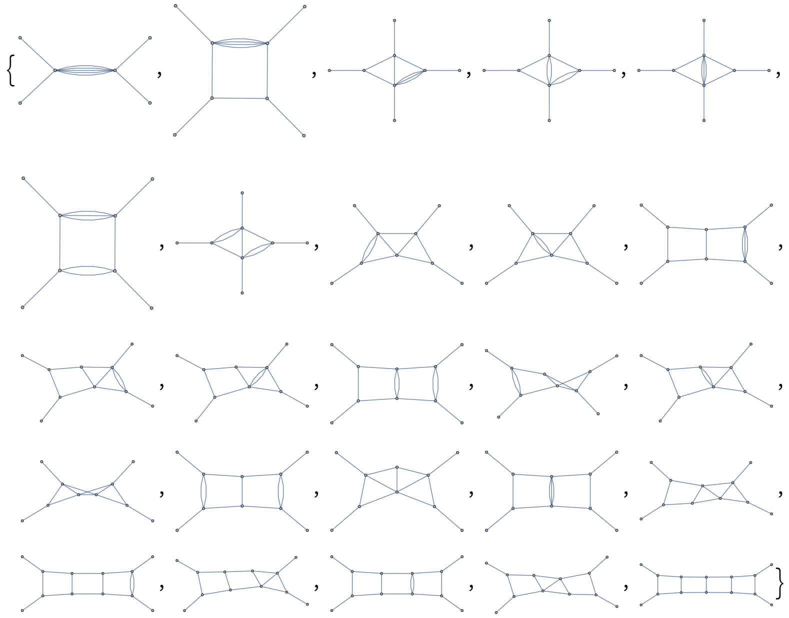




Amplitudes (from form factor method)

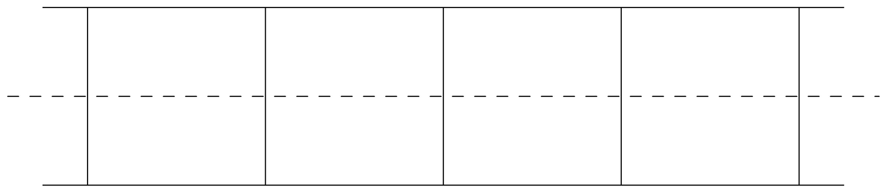
- 1-loop gg → gg (#INTs : /~2 per cut) ✓
- 2-loop gg → gg planar (#INTs : /~8 per cut) ✓
- 2-loop gg → gg nonplanar (#INTs : /~4 per cut) ✓

Integrated Unitarity : four-loop ladder



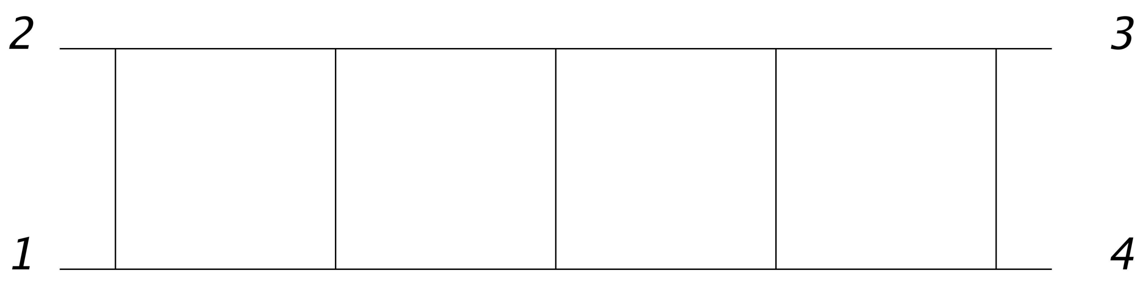
procedure

- 22 generalized propagators = 13 denominators + 9 ISPs
- cut u : 5 propagators
- IBP : 59 MIs $\mathrm{Cut}_u M_i^c$ ($\texttt{LiteRed} + \texttt{Kira}$)
- canonical DEQ : $A_{ij}^c(x) = \dfrac{a_{ij}}{x} + \dfrac{b_{ij}}{1-x}$ ($\texttt{CANONICA} + \texttt{MultivariateApart} + \texttt{FiniteFlow}$)
- canonical general solution : $M_u^c(x,\epsilon) = \mathbb{P}e^{\epsilon \int A^c(x) dx} M_{u,0}^c(\epsilon)$ ($\texttt{PolyLogTools}$ + in-house)
- regularity constraints on BCs $M_{u,0}^c(\epsilon)$: 59 → 5 (in-house)
- 5 remaining BCs : weight 7 ($\texttt{AMFlow}$)
- method B : $\mathrm{Disc}_0 M_i^c = \mathrm{Cut}_t M_i^c = 0 \quad \& \quad \mathrm{Disc}_1 M_i^c = \mathrm{Cut}_u M_i^c$ (in-house)



fixed all HPL coefficients to weight 8

Integrated Unitarity : four-loop ladder



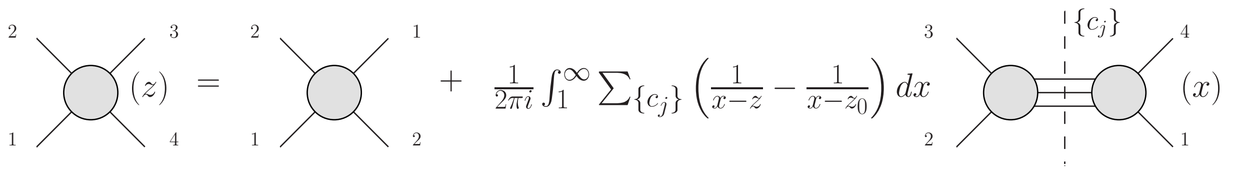
[illegible]

Conclusions

Conclusions

Integrated Unitarity :

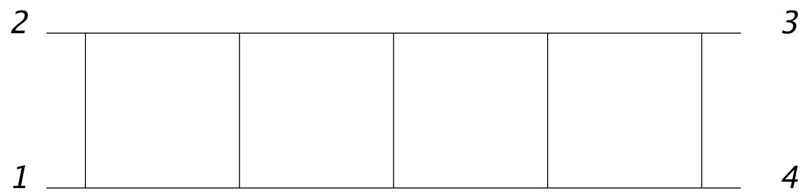
- ~ Generalized Unitarity @ integrated level
- allows algorithmic usage of dispersion relation



- beneficial for IBP computational complexity
- requires understanding of analytic structure

trade-off

- formulated for massless 4-point kinematics
- checked for 2-loop nonplanar amplitude
- provided new 4-loop ladder integral result



Outlook

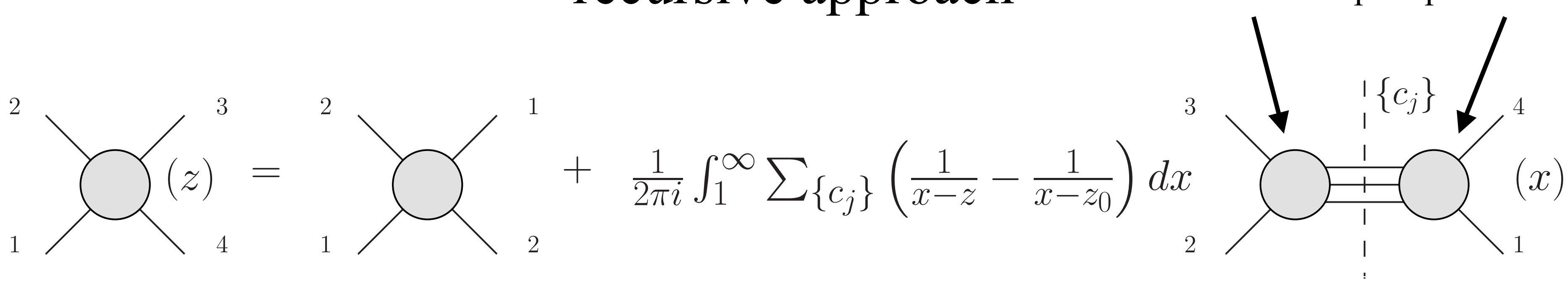
kinematic limits

e.g. $\mathrm{Disc}_1 \mathscr{A} = \mathrm{Cut}_u \mathscr{A}$ alone gives u→0 limit at fixed Log accuracy to any subleading power

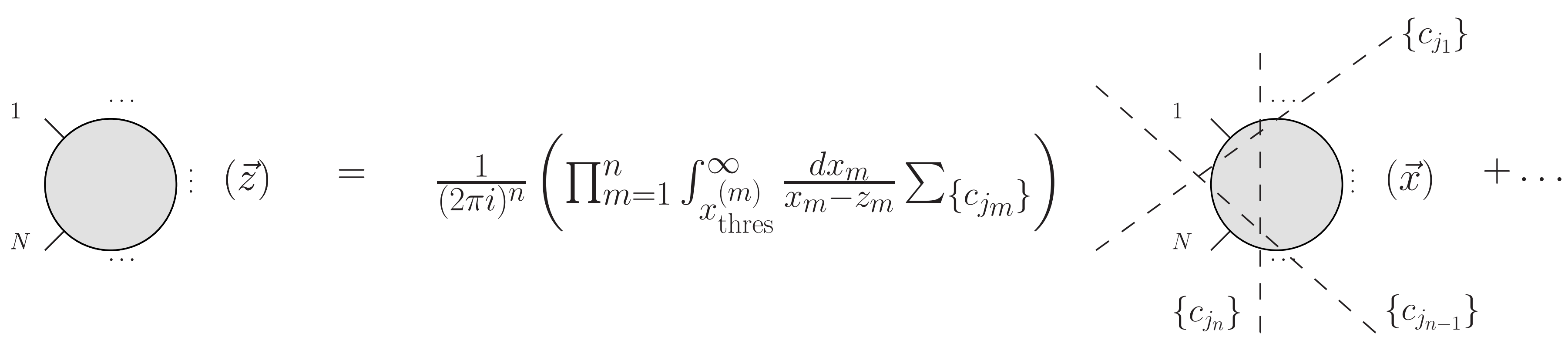
$$\lim_{x\to 1} \mathscr{A}(x) \sim c_{1,1}(x)\, G(1,1,x) + c_{0,1}(x)\, G(0,1,x) + c_{\dots,0}(x)\, G(\dots,0,x)$$

Leading Log → Next-to-Leading Log → suppressed

recursive approach



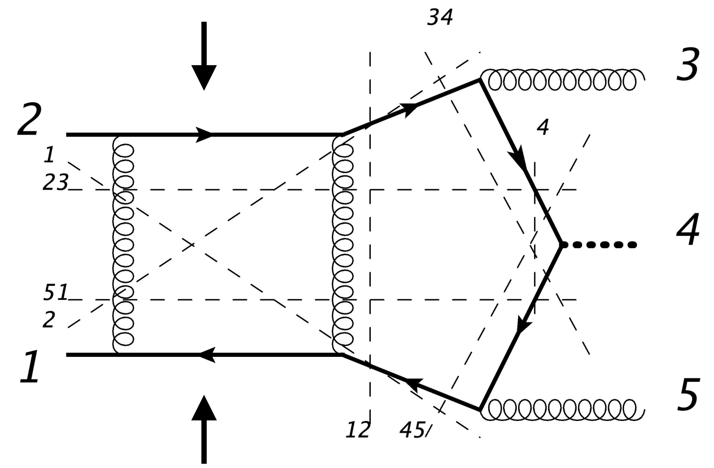
Multivariate Integrated Unitarity



Outlook into Multivariate Integrated Unitarity

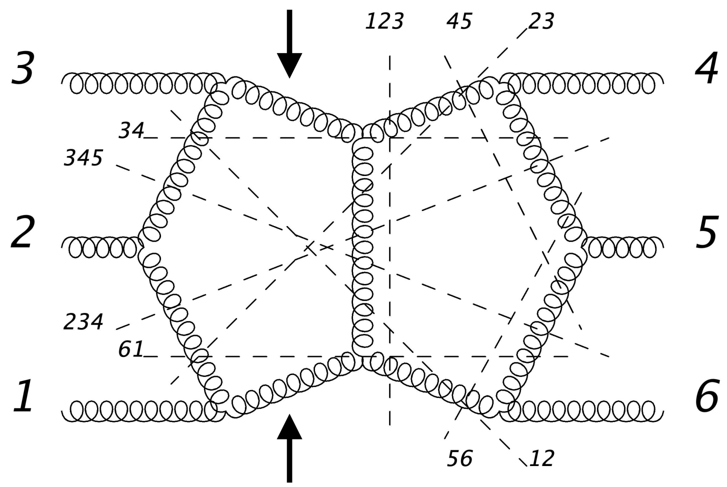
examples

$pp \to t\bar{t}H$ @ 2 loops



uncut

$gg \to gggg$ @ 2 loops



properties

- many cut propagators → simpler IBPs
- iterative Cauchy formula → multivariate complex analysis
- Landau singularities → nontrivial analytic structure

  [Helmer, Papathanasiou, Tellander 2402.14787] [Correia 2212.06157]

- Steinmann relations → possible simplifications

  [Caron-Huot, Dixon, McLeod, Hippel 1609.00669]

THANK YOU

Appendix

Appendix : monodromies

following :

[Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

- Harmonic Polylog : $G(\alpha_n, \ldots, \alpha_1; x)$ $\qquad \alpha_k \in \{0,1\}$
- vector of derivatives : $$\mathscr{V}_i = \begin{cases} 1 & \text{if } i = 0\,, \\ (-1)^{\#\text{ of }1}\, G(\alpha_{n+1-i}, \ldots, \alpha_n, x) & \text{if } n \geq i > 0 \end{cases}$$
- connection matrix : $\omega_{ij} = \dfrac{dx}{x - \alpha_{n-i}}\, \delta_{i+1,j}$ $\qquad$ s.t. $\qquad d\mathscr{V} = \mathscr{V} \cdot \omega$
- variation matrix : $\mathscr{M}_\gamma = \mathscr{P}\, e^{\int_\gamma \omega}$ $\qquad$ collects all $n+1$ solutions for $\mathscr{V}$
- general solution : $$\left(\mathscr{M}_{\to x}\right)_{ij} = \sum_{k=0}^{n} (-1)^{\#\text{ of }1}\, G(\alpha_{n-i}, \ldots, \alpha_{n-i-k+1}, x)\, \delta_{i+k,j}$$ $$G(x) = 1$$
- monodromy matrices : $$\mathscr{M}_0 = \mathscr{M}_{\circlearrowleft_0}\,,$$ $$\mathscr{M}_1 = \mathscr{M}_{\to 1}\, \mathscr{M}_{\circlearrowleft_1}\, \mathscr{M}_{\to 1}^{-1}\,.$$
- discontinuities : $$\text{Disc}_0 = (1 - \mathscr{M}_0) \cdot \mathscr{M}_{\to x}\,,$$ $$\text{Disc}_1 = -\,(1 - \mathscr{M}_1) \cdot \mathscr{M}_{\to x}\,,$$ $$\text{Disc}_\infty = (1 - \mathscr{M}_0 \cdot \mathscr{M}_1) \cdot \mathscr{M}_{\to x}\,.$$

Appendix : ansatz matching

- example ansatz : $c_{1,1}\,G(1,1;x) + c_{1,0}\,G(1,0;x) + c_{0,1}\,G(0,1;x) + c_{0,0}\,G(0,0;x) + c_1\,G(1;x) + c_0\,G(0;x) + c$
- impose e.g. $\text{Disc}_0 = 0$: $c_{1,1}\,G(1,1;x) + 0 + c_{0,1}\,G(0,1;x) + 0 + c_1\,G(1;x) + 0 + c$
- $\text{Disc}_1 = 2\pi i\,(2\,G(1;x) + 3\,G(0;x) + 5\,\pi i)$ $= c_{1,1}\,(2\pi i\,(-G(1;x) + i\pi)) + c_{0,1}\,(-2\pi i\,G(0,x)) + c_1\,(-2\pi i) + 0$
- now only constant unconstrained : $-2\,G(1,1;x) - 3\,G(0,1;x) - 7\,i\pi\,G(1;x) + c$
- impose fixed value e.g. ζ_2 at x=0 : $-2\,G(1,1;x) - 3\,G(0,1;x) - 7\,i\pi\,G(1;x) + \zeta_2$