Local Unitarity

Building a numerical collider

Valentin Hirschi

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www.alphaloop.ch

In coll. with:

Z. Capatti, B. Ruijl, M. Fraaije, L. Huber

TUM Seminar- Munich

June 5, 2024

{ ANARICAL }

$$\log(2)$$

Expression

$$\int_1^2 dx \frac{1}{x}$$

{ ANARICAL }

$$\log(2)$$

EXPRESSION

$$\int_1^2 dx \frac{1}{x}$$

ANALYTICAL ?

LAY MAN CLASSIFICATION

NUMERICAL ?

{ ANARICAL }

$$\log(2)$$

EXPRESSION

$$\int_1^2 dx \frac{1}{x}$$

ANALYTICAL ?

LAY MAN CLASSIFICATION

NUMERICAL ?

$$\sim \sum_{i=1}^{N} \frac{(1-x)^i}{-i}\Big|_{x=2}$$

IMPLEMENTATION

$$\sim \frac{1}{N}\sum^{N} \frac{1}{(1+\mathrm{rdm}())}$$

{ ANARICAL }

Analytical ?	Lay man classification	Numerical ?
$\log(2)$	Expression	$\int_1^2 dx \frac{1}{x}$
$\sim \sum_{i=1}^{N} \frac{(1-x)^i}{-i}\Big\vert_{x=2}$	Implementation	$\sim \frac{1}{N}\sum^{N} \frac{1}{(1+\mathrm{rdm}())}$
$t \propto D \log(D)^2$	Complexity for "D" accurate digits	$t \propto D^2$

Overview of Local Unitarity

One-dimensional toy example

- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \qquad\qquad \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$

One-dimensional toy example

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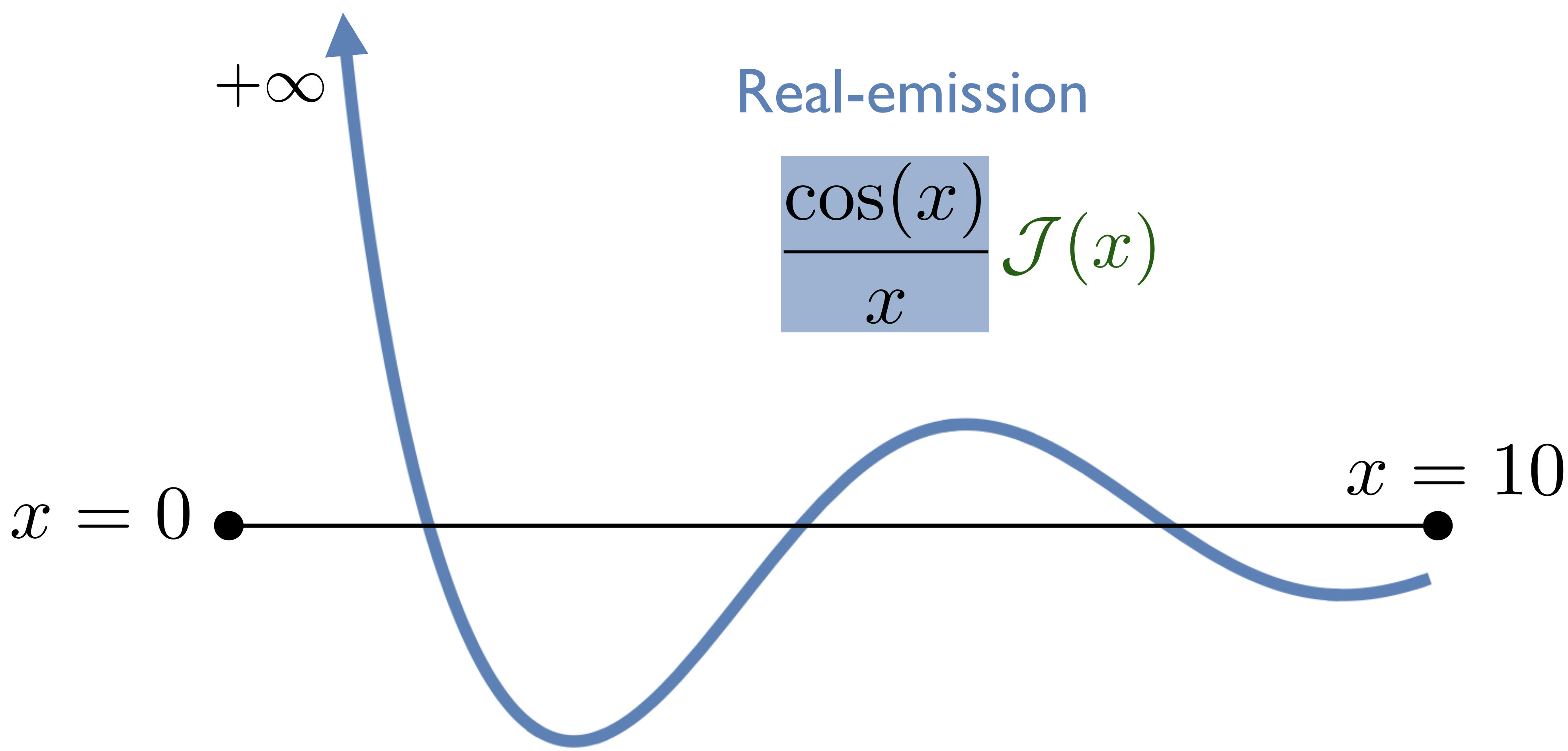

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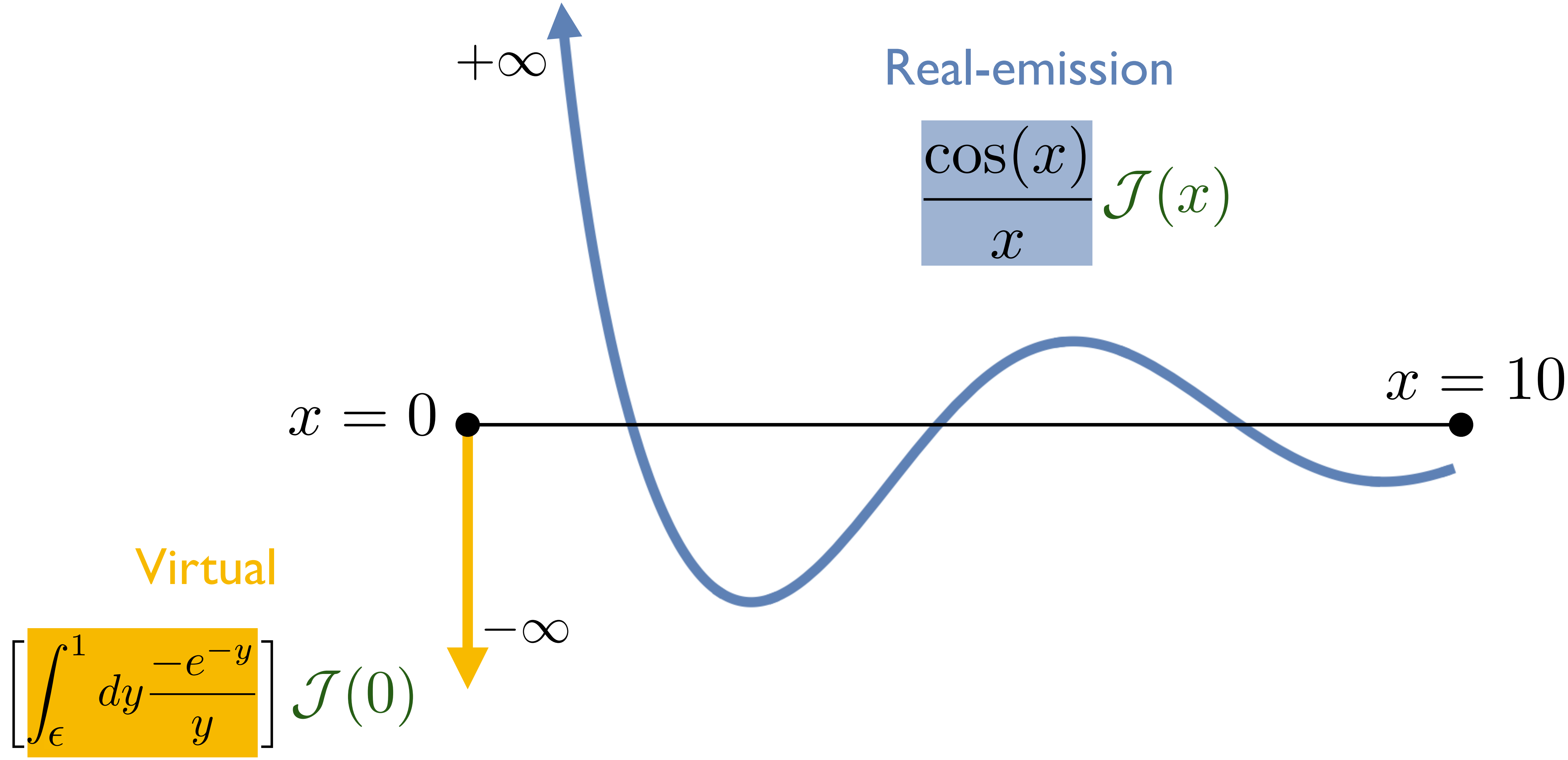

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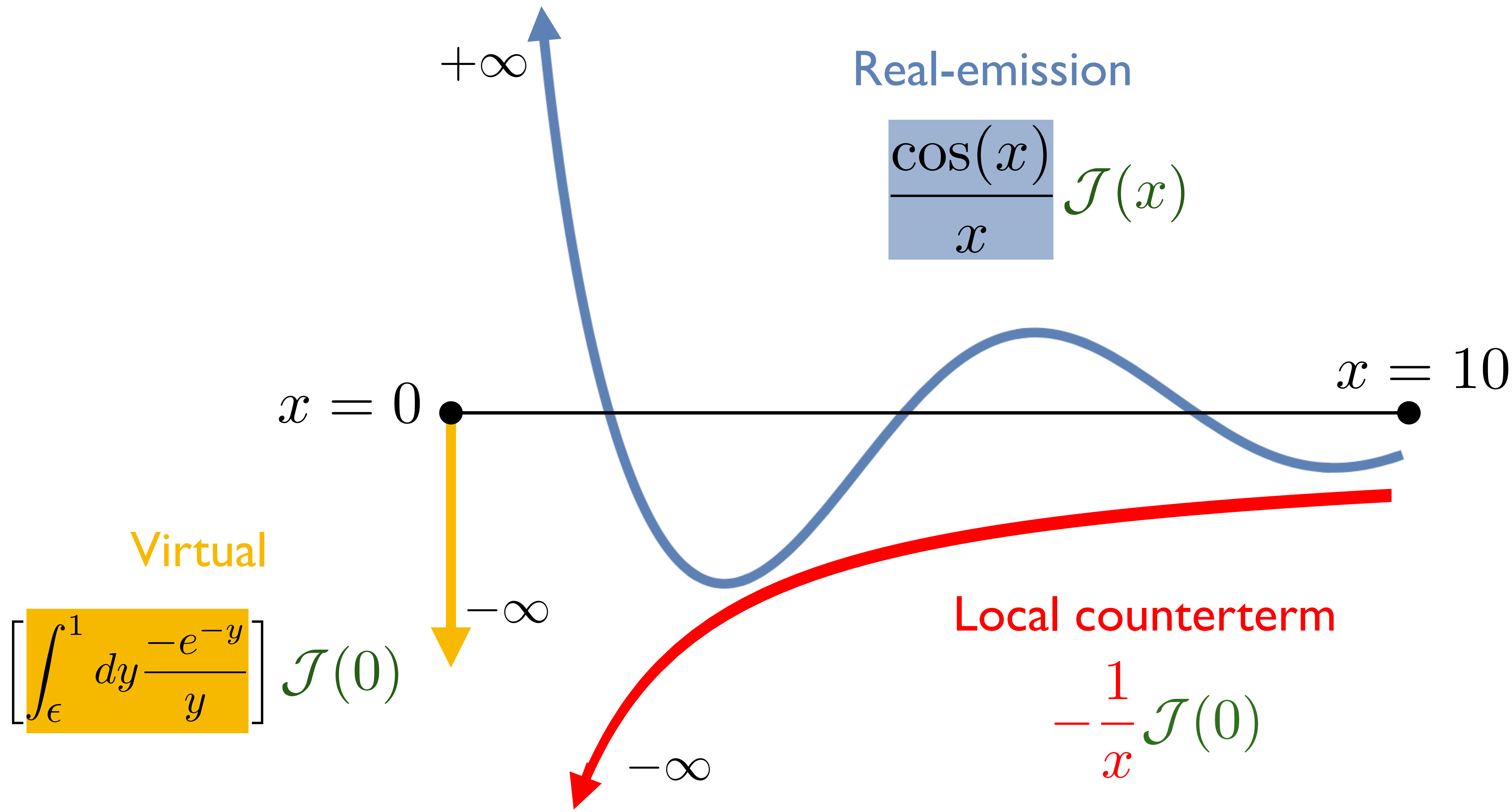
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\boxed{\frac{\cos(x)}{x}} \mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0) \right] + \left[\boxed{\int_0^1 dy \frac{-e^{-y}}{y}} \right] \mathcal{J}(0)$$



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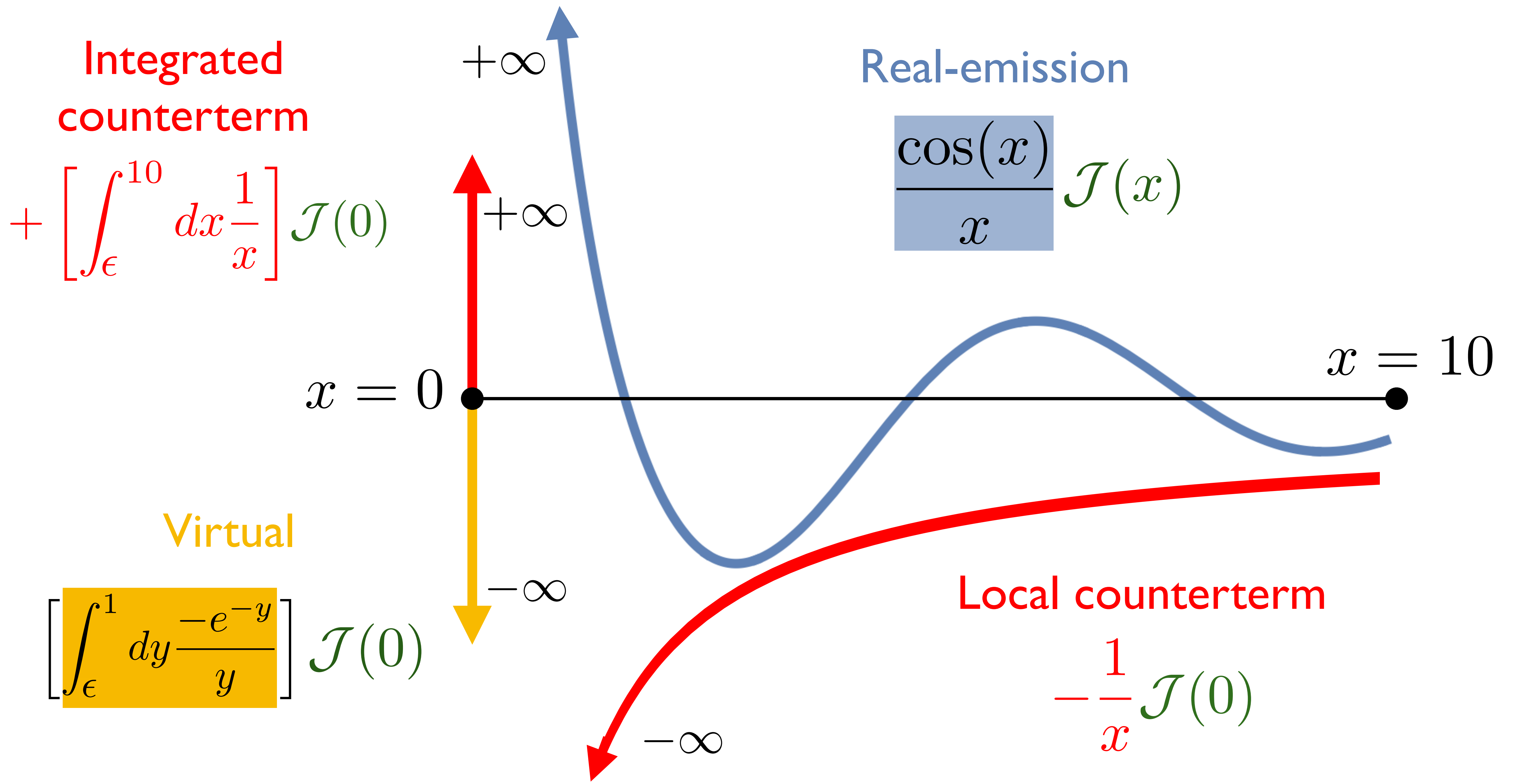

One-dimensional toy example

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y}\right]$$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) \qquad \right]$$

One-dimensional toy example

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y}\right]$$

$$\Big\downarrow \text{" LU "} \; y = x/10 \Big\downarrow$$

$$\left[\int_0^{10} dx \frac{-e^{-x/10}}{x}\right]$$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{e^{-x/10}}{x}\mathcal{J}(0)\right]$$

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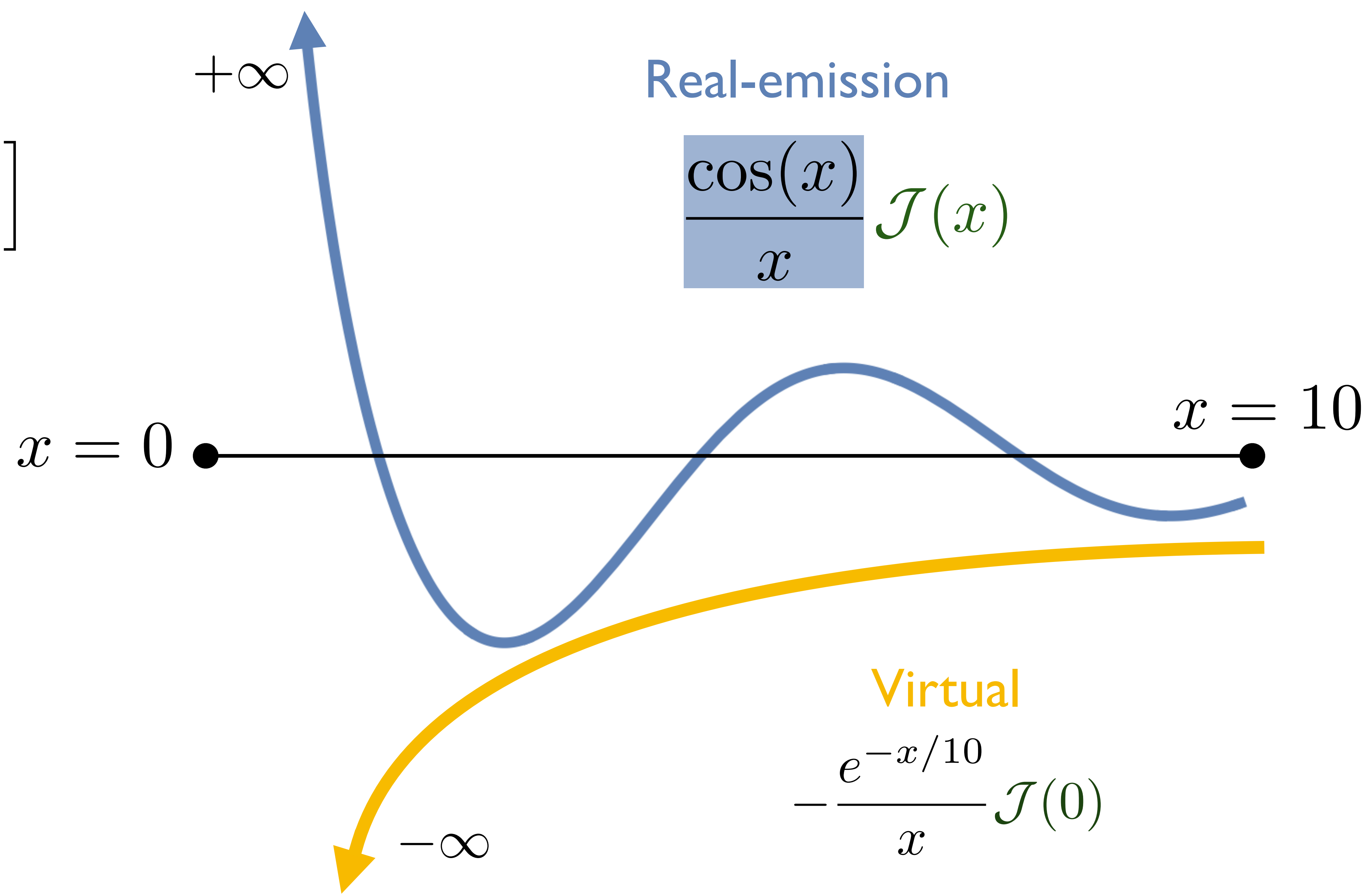
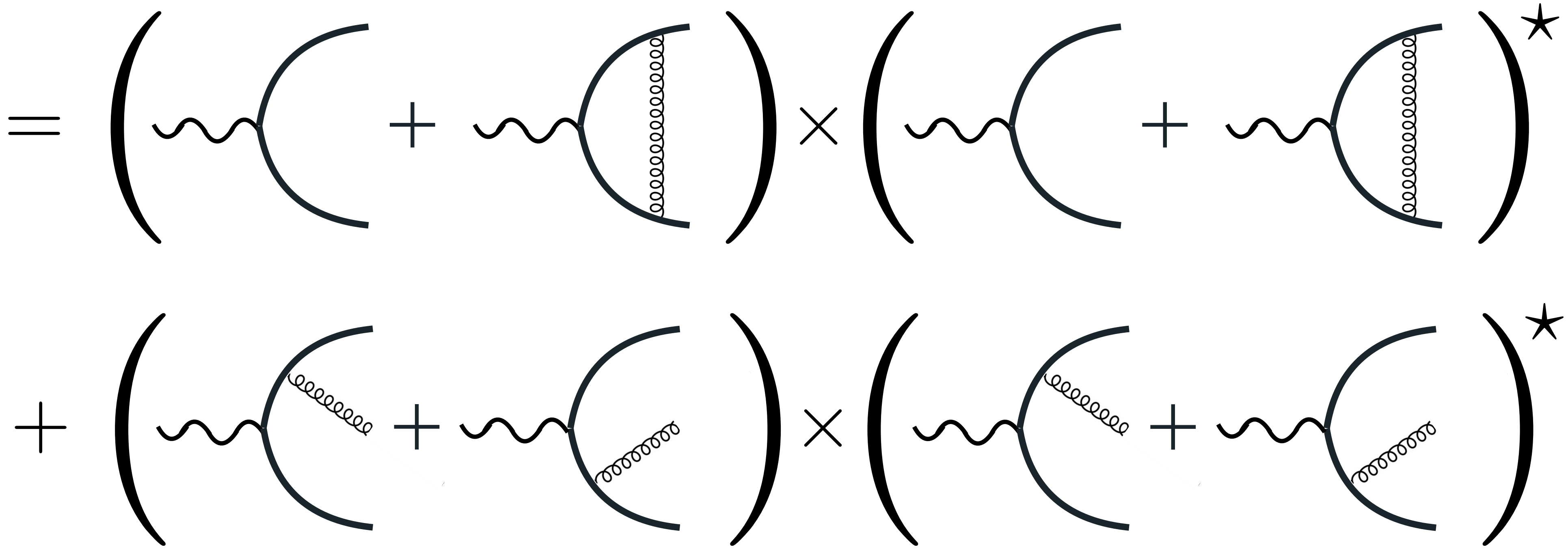


Remedy: forward-scattering graphs

$$\sigma_{\gamma^\star \to d\bar{d}} =$$



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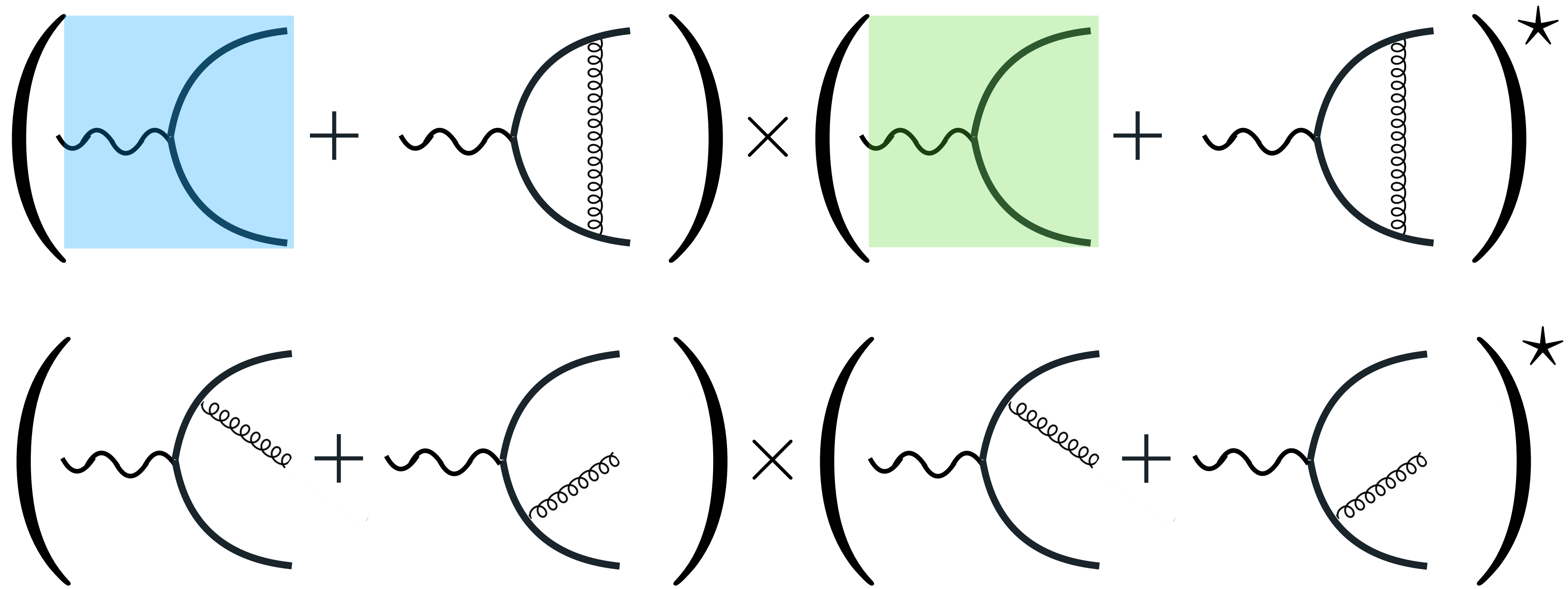

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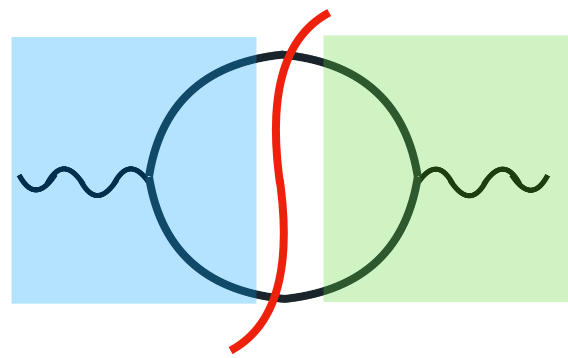

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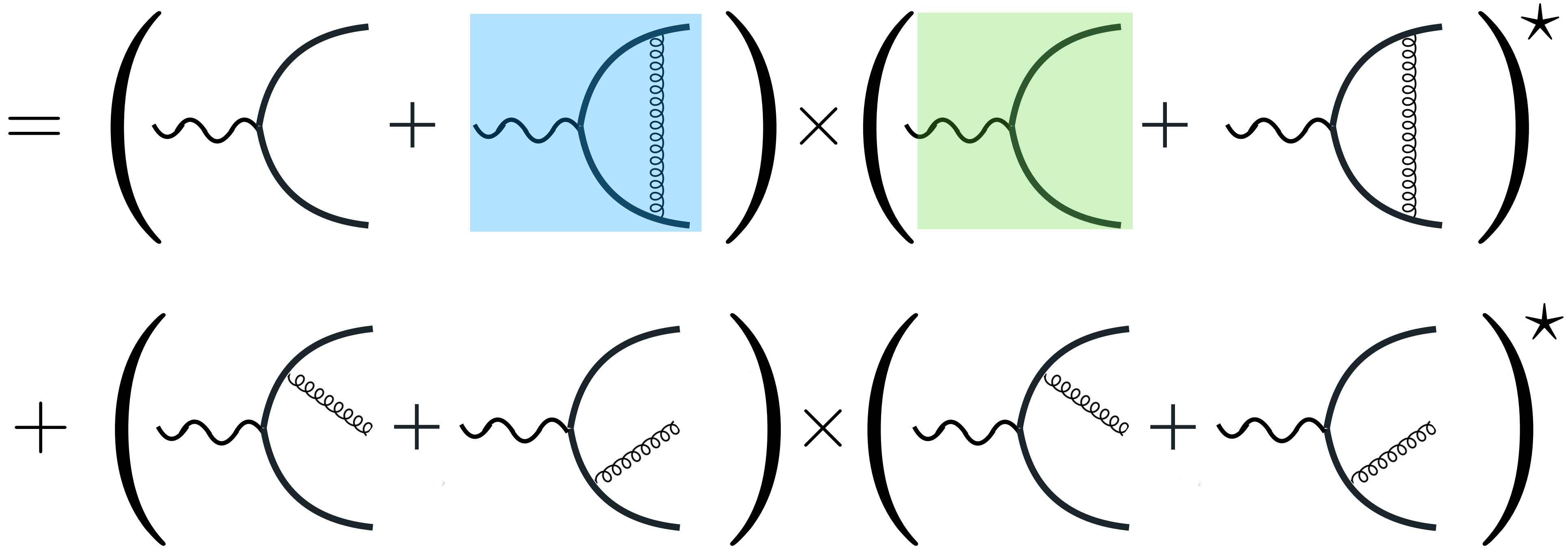

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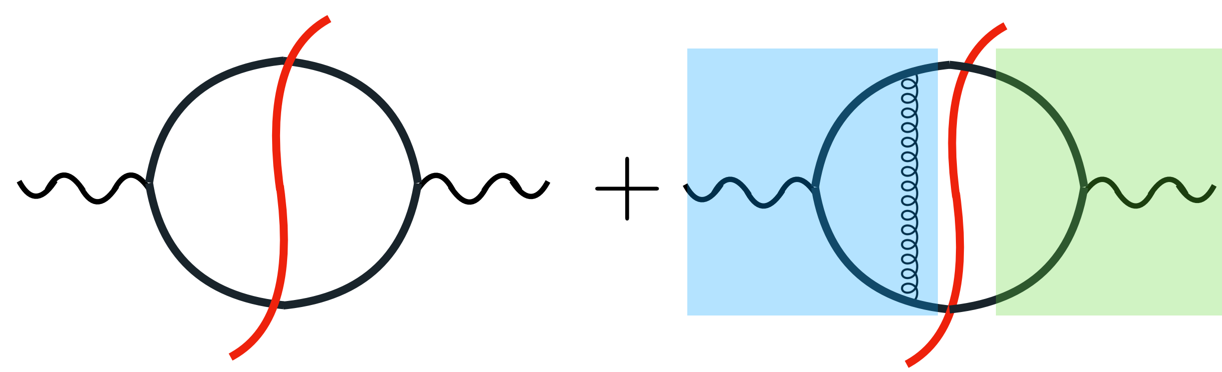

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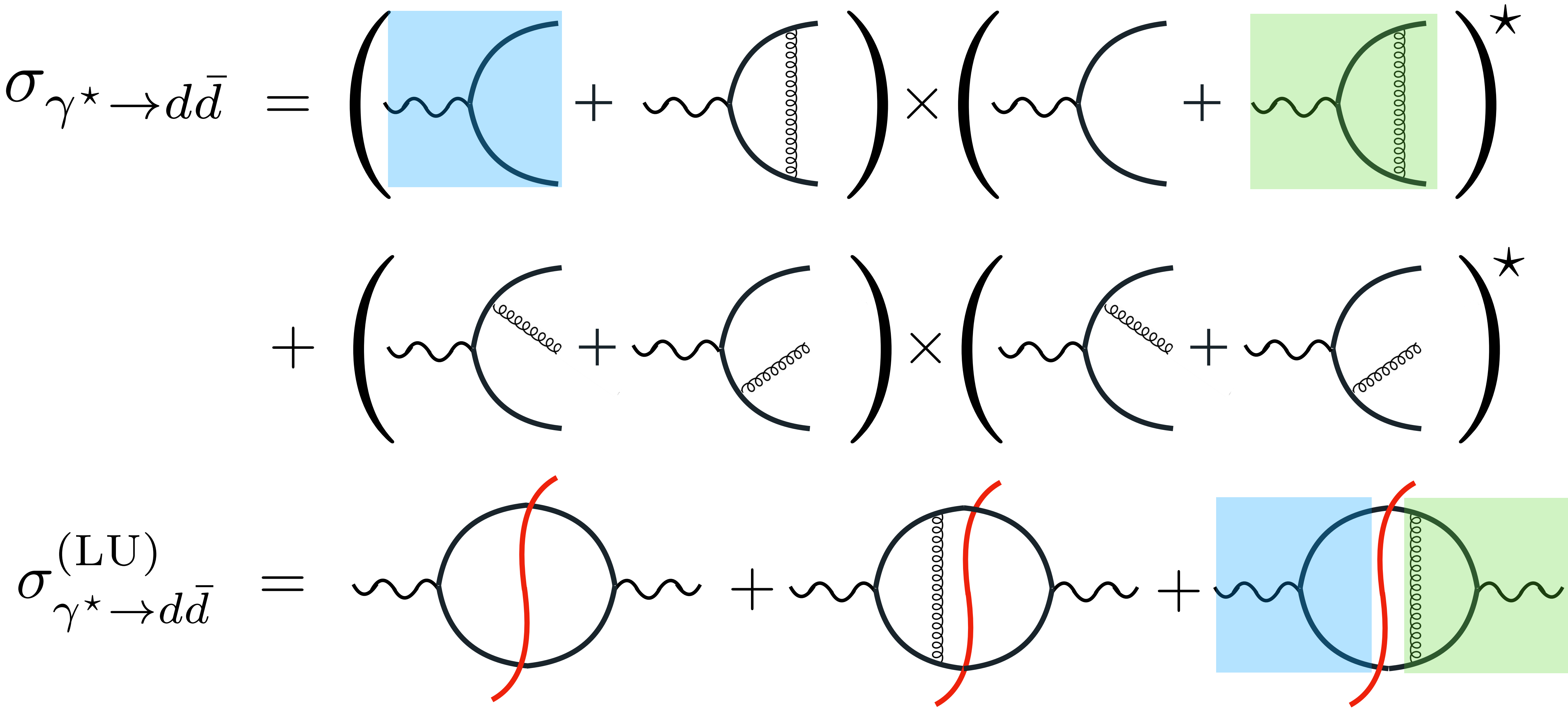

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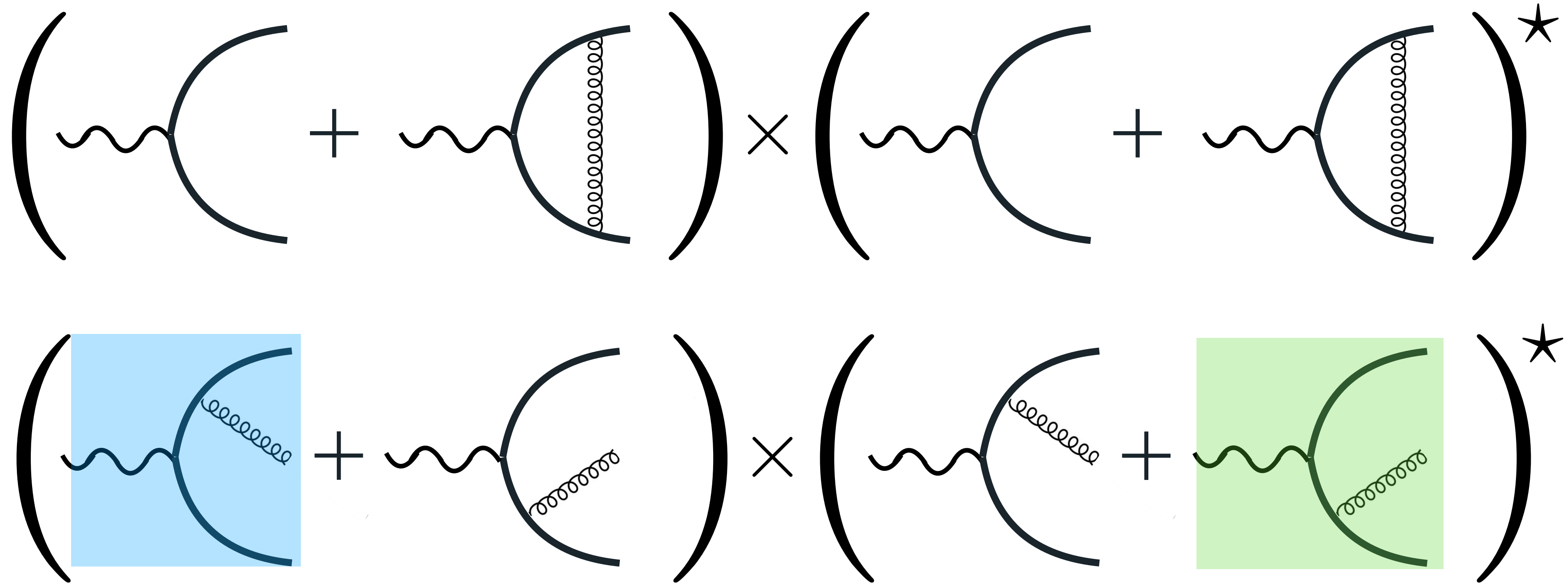

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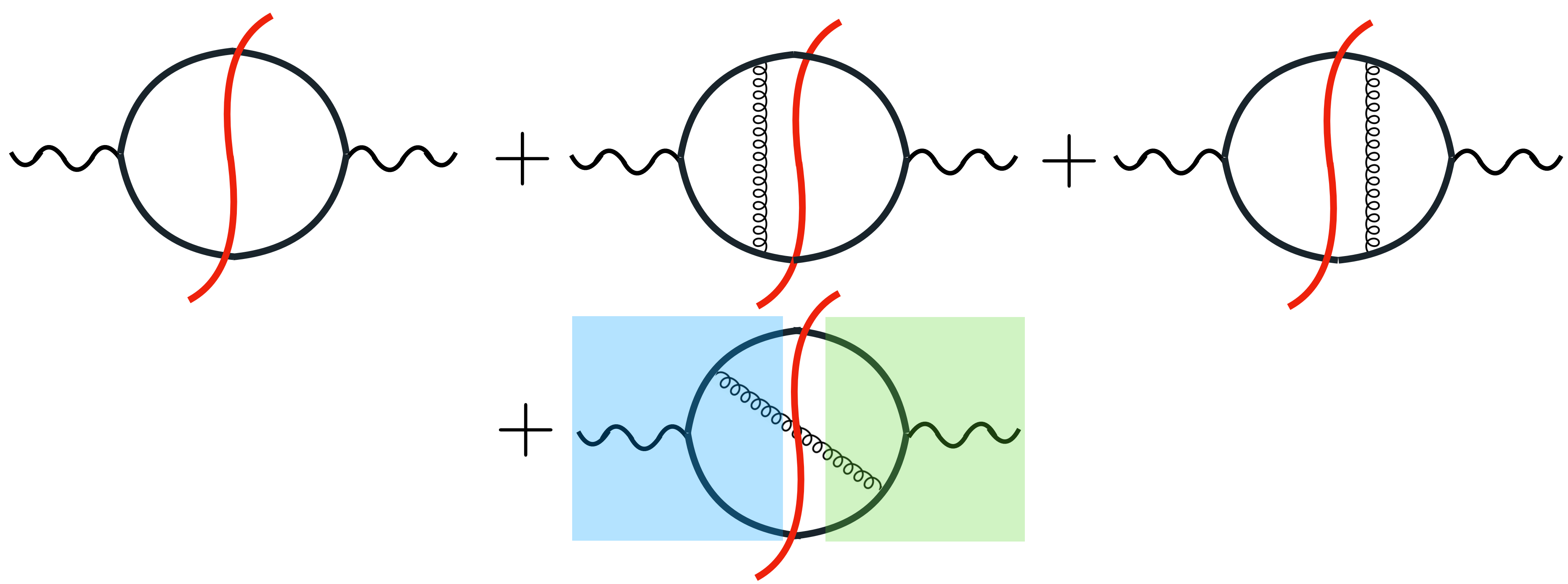

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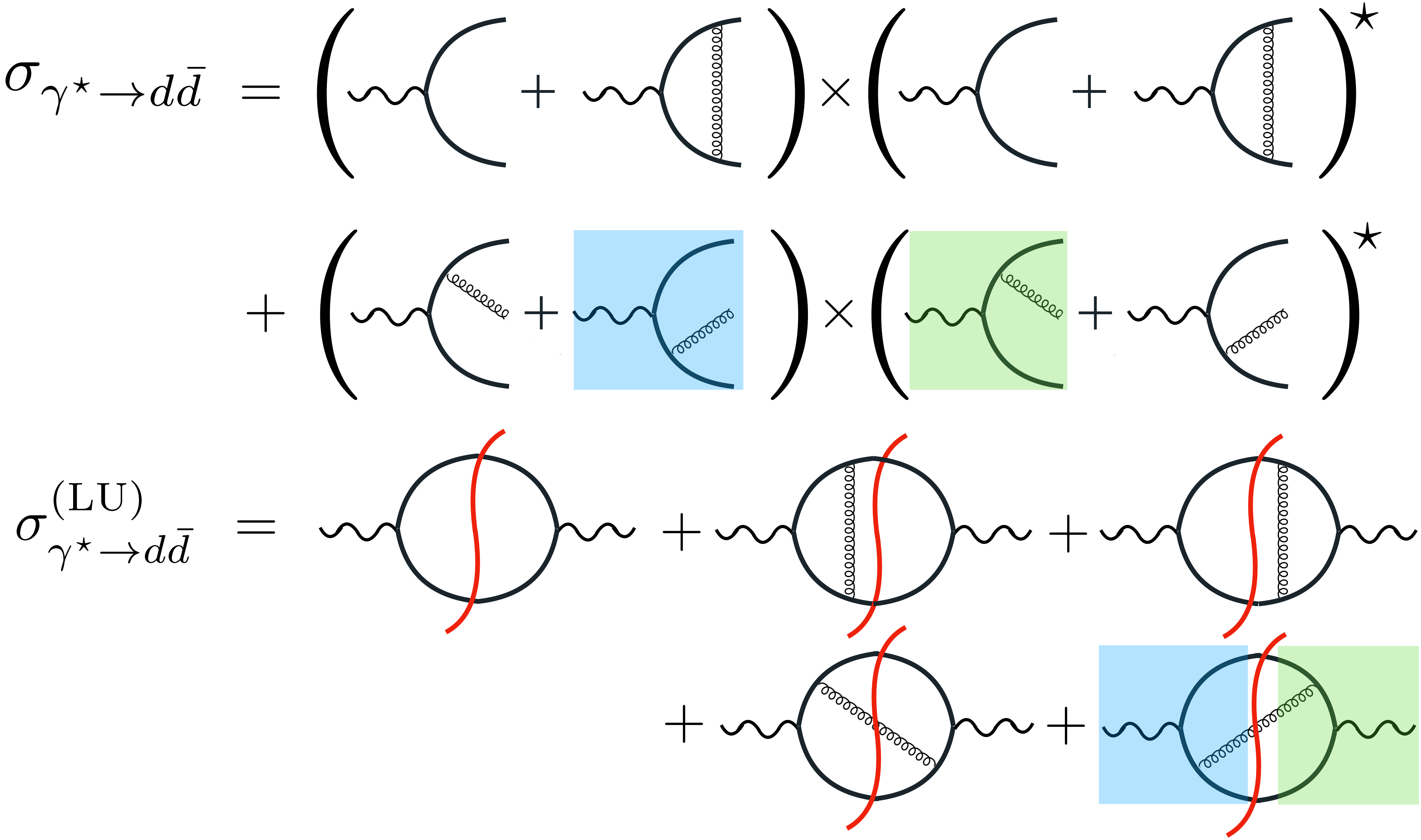

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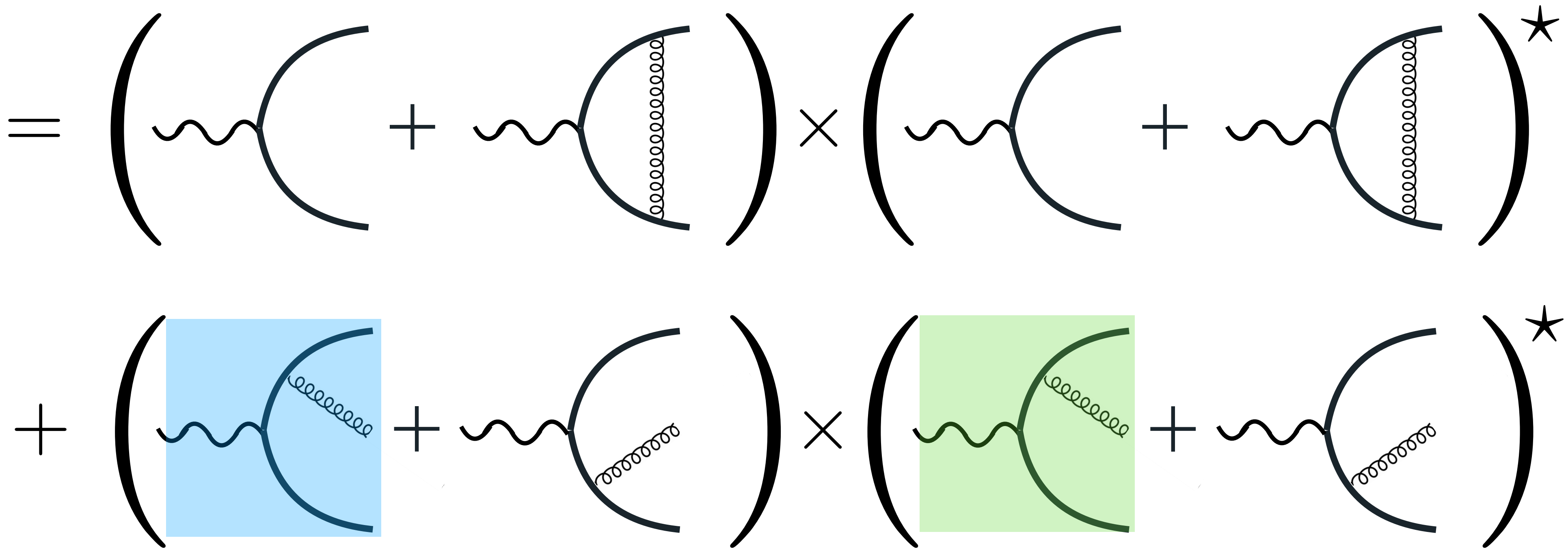

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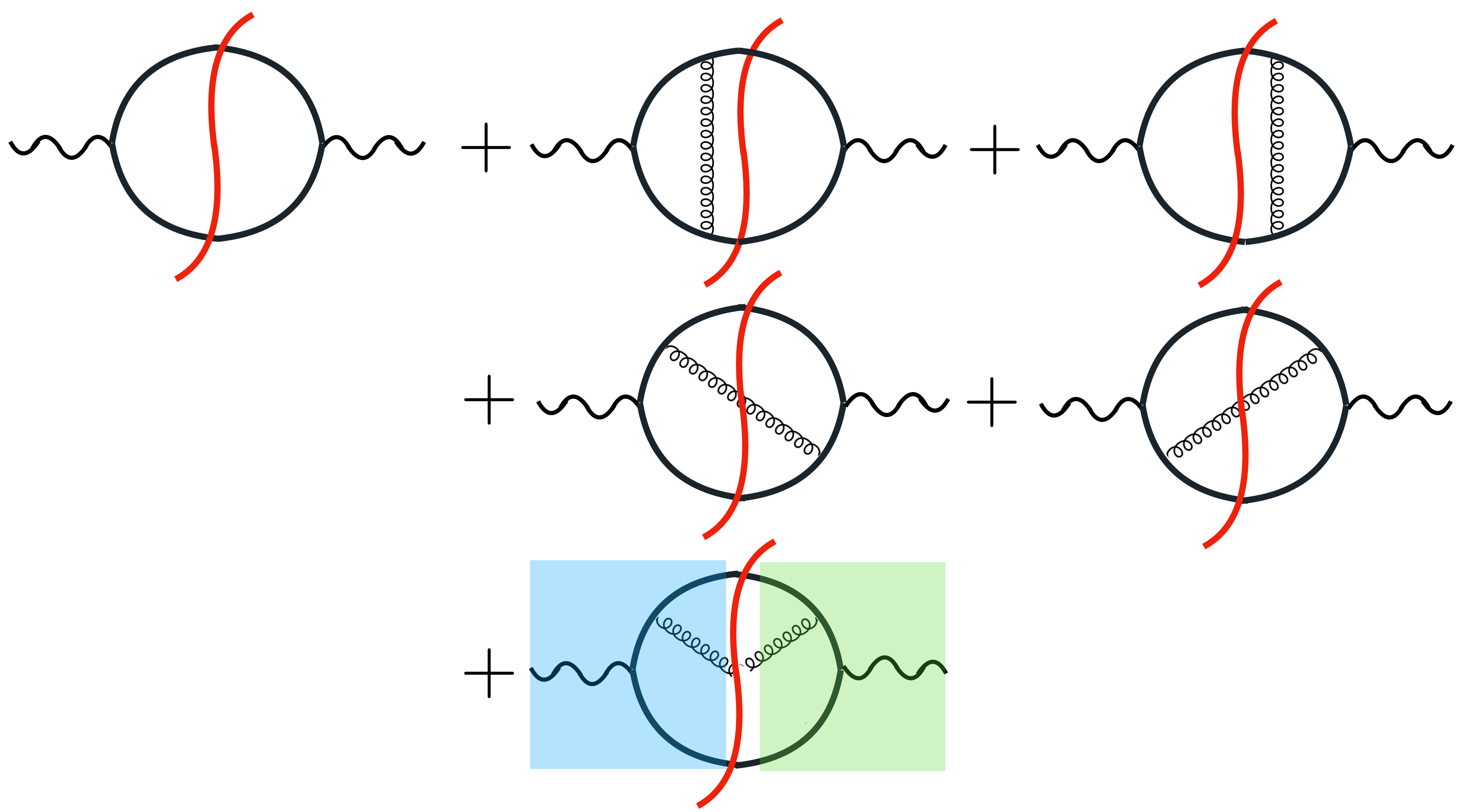

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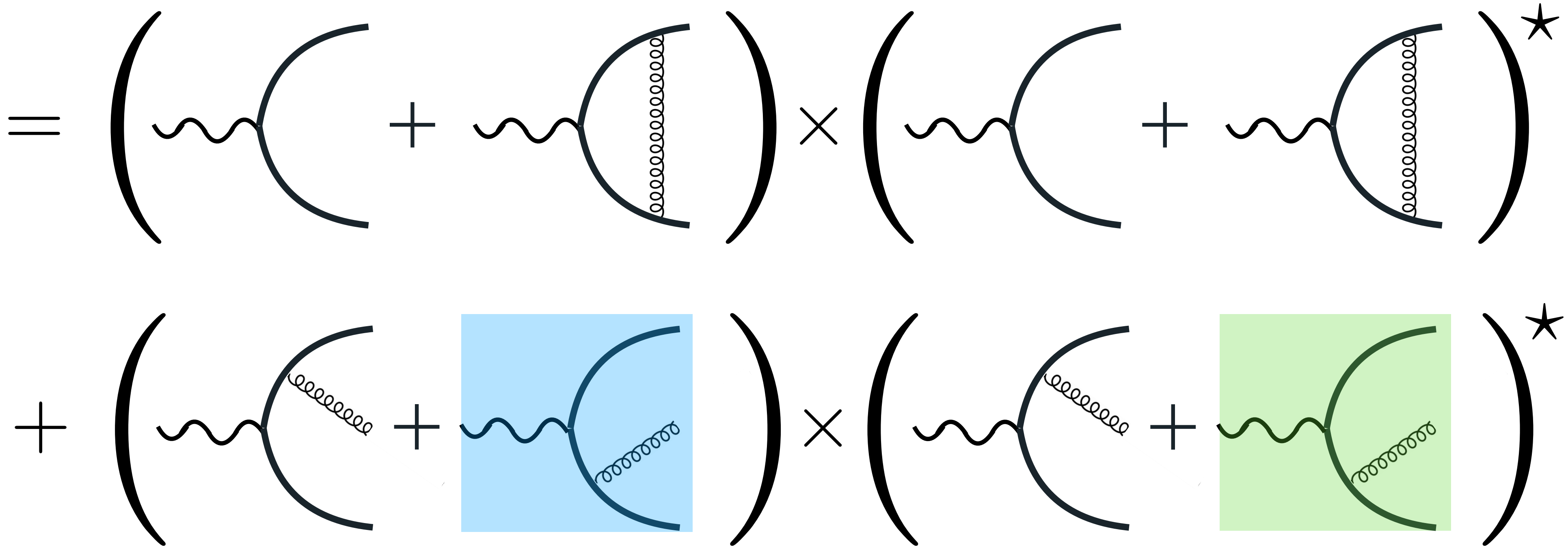

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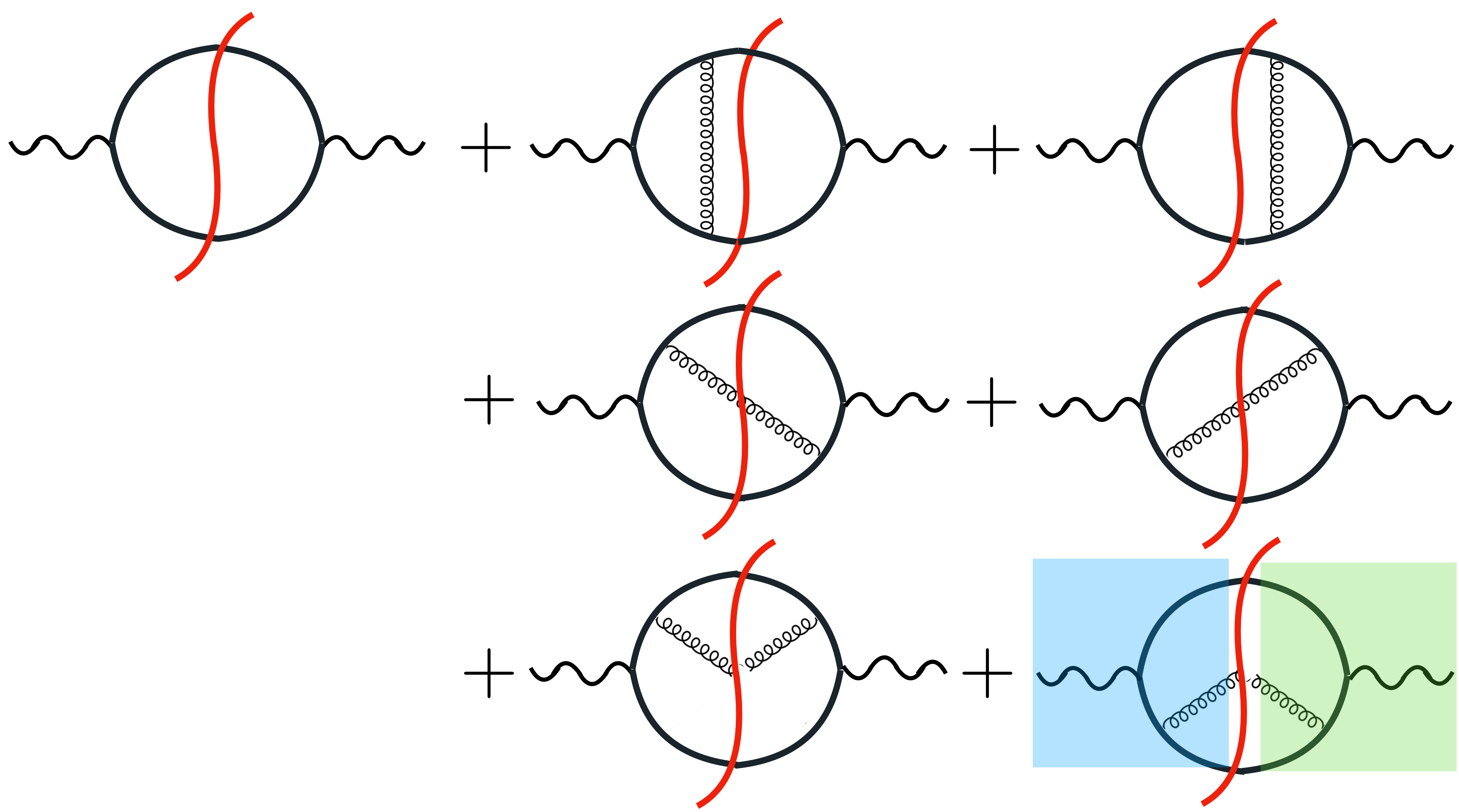

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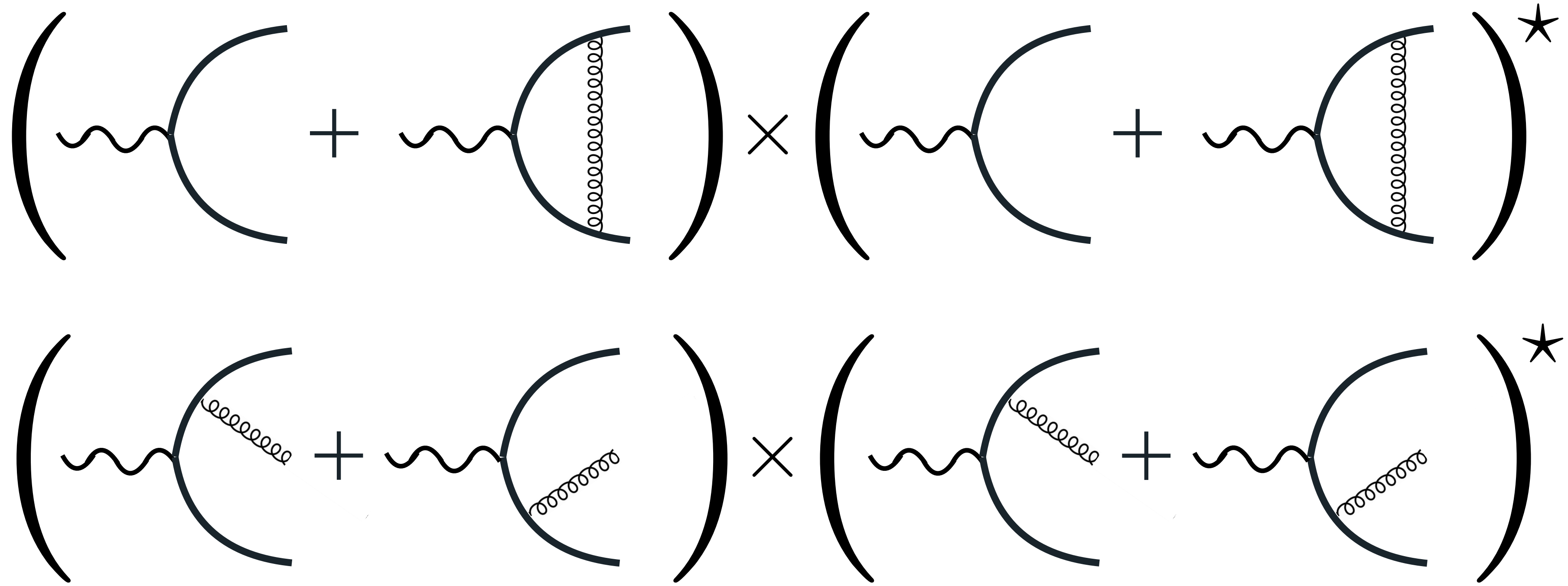

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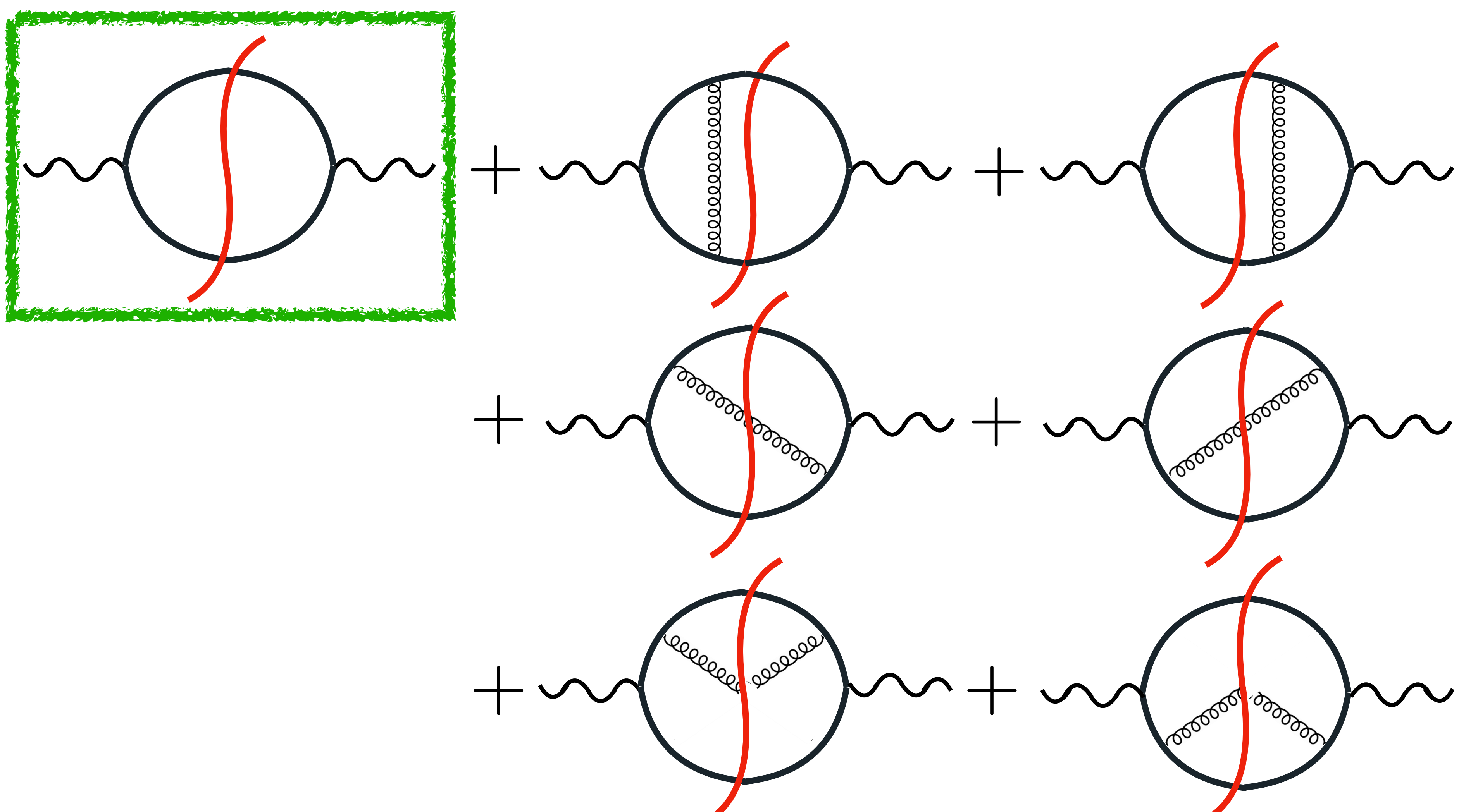
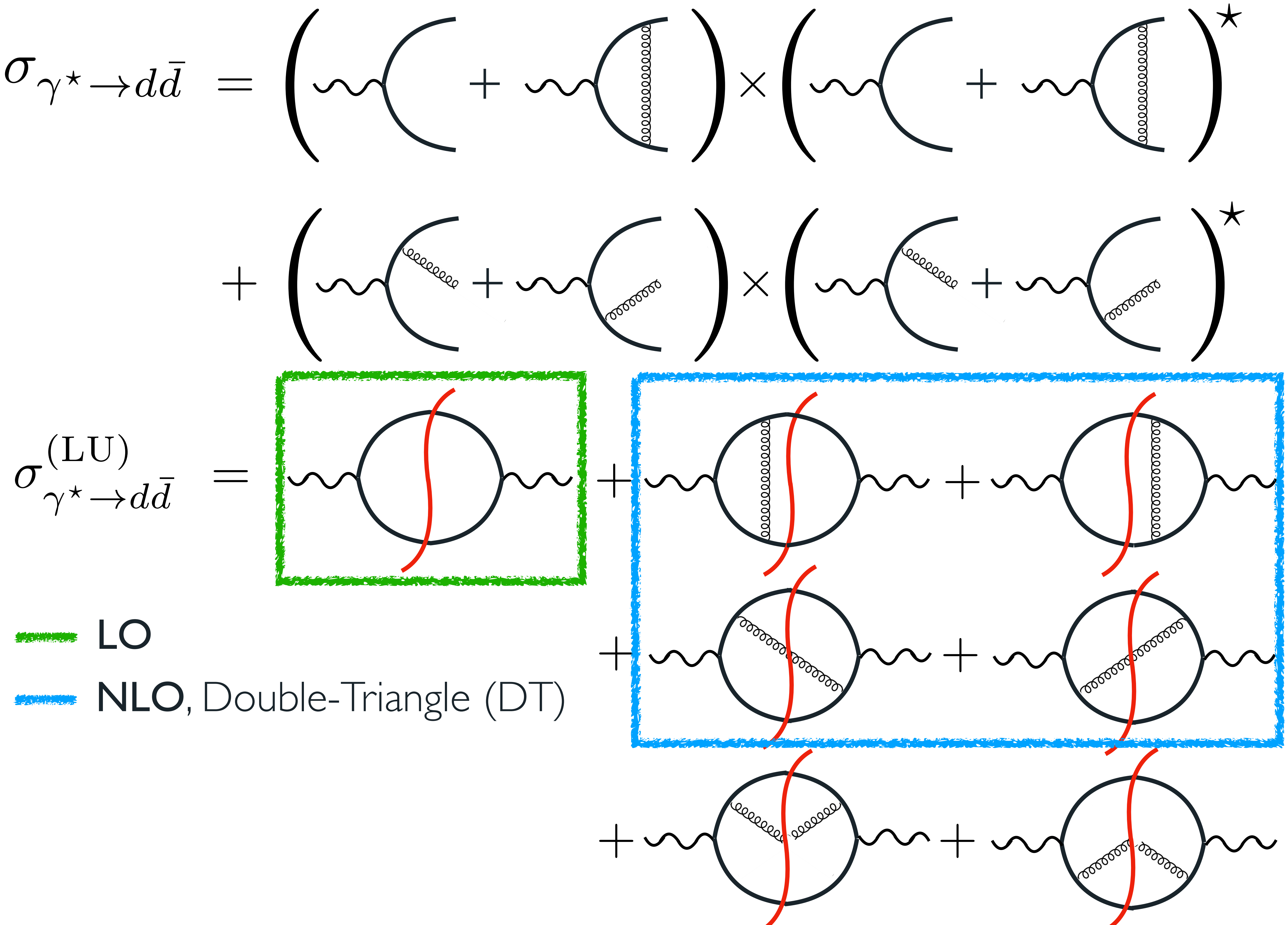


LO

REMEDY: FORWARD-SCATTERING GRAPHS



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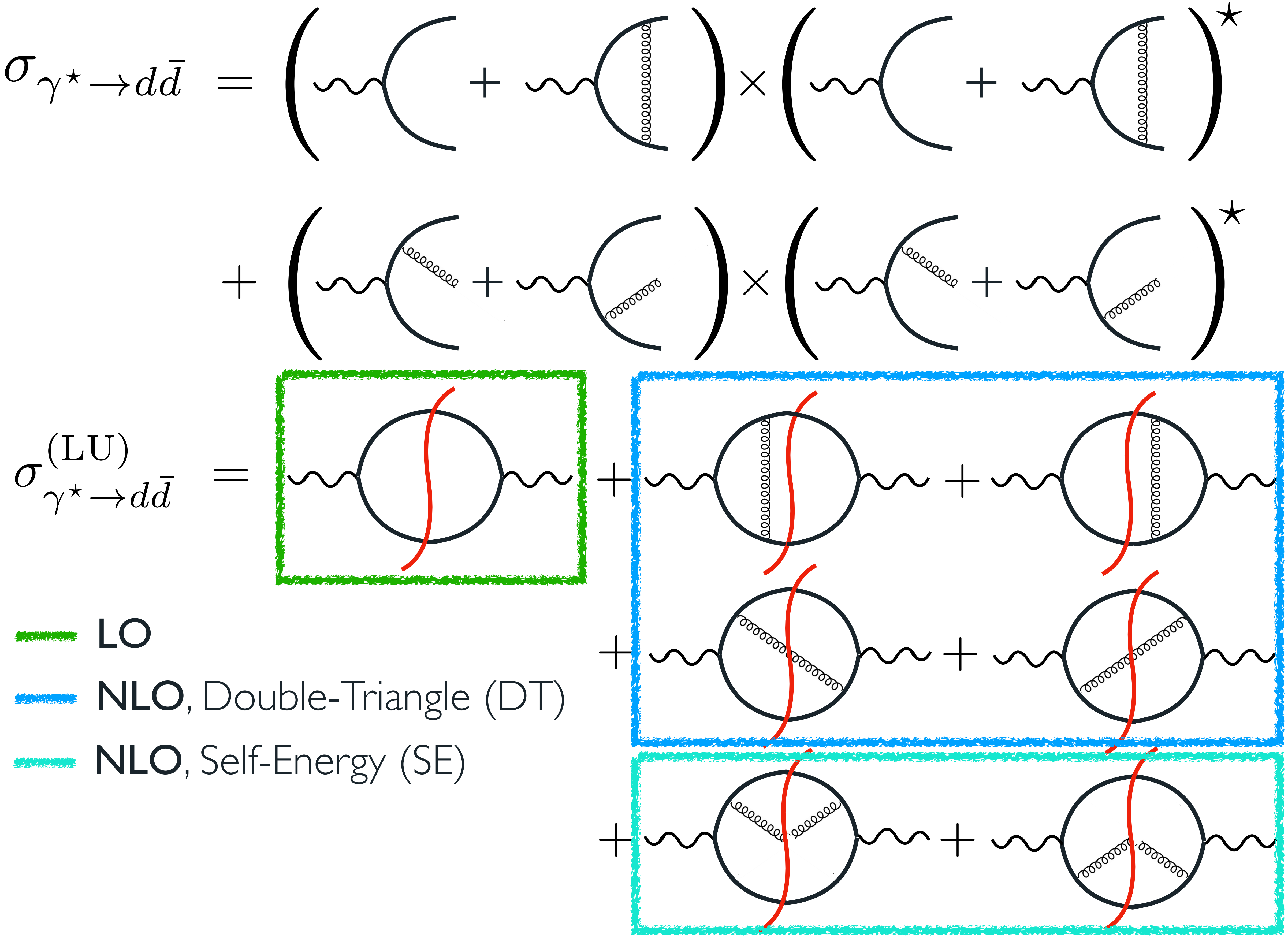

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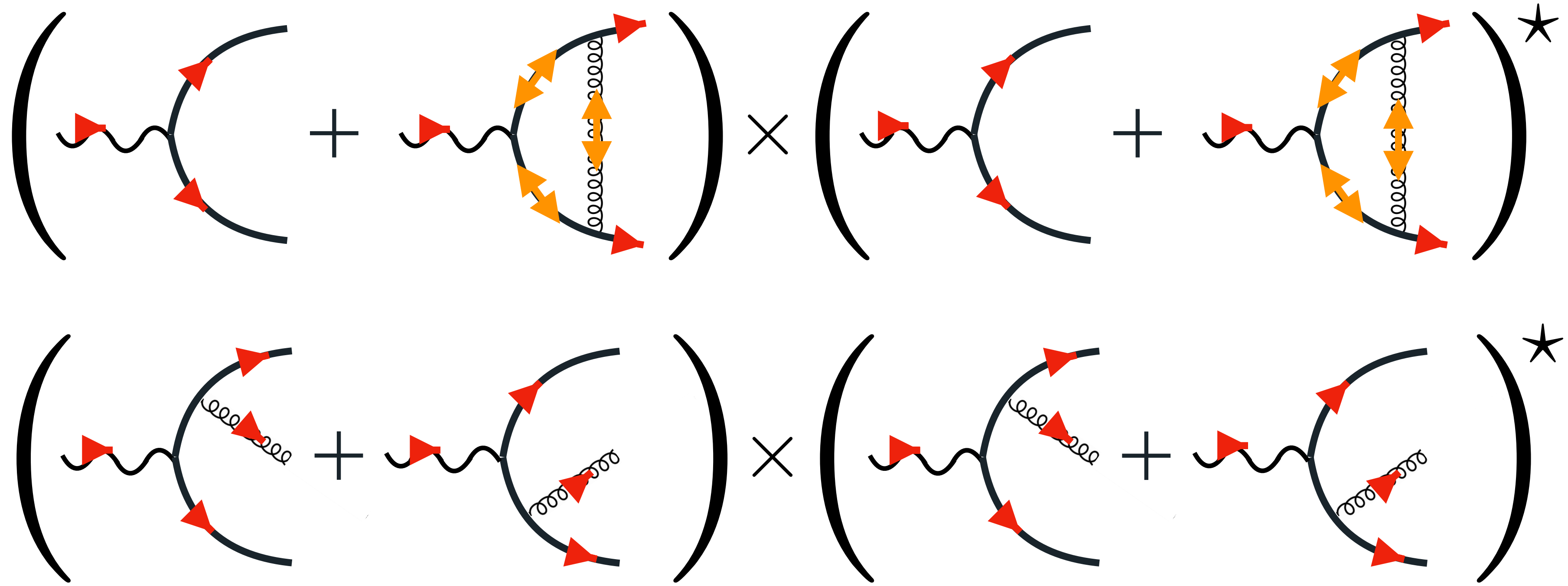

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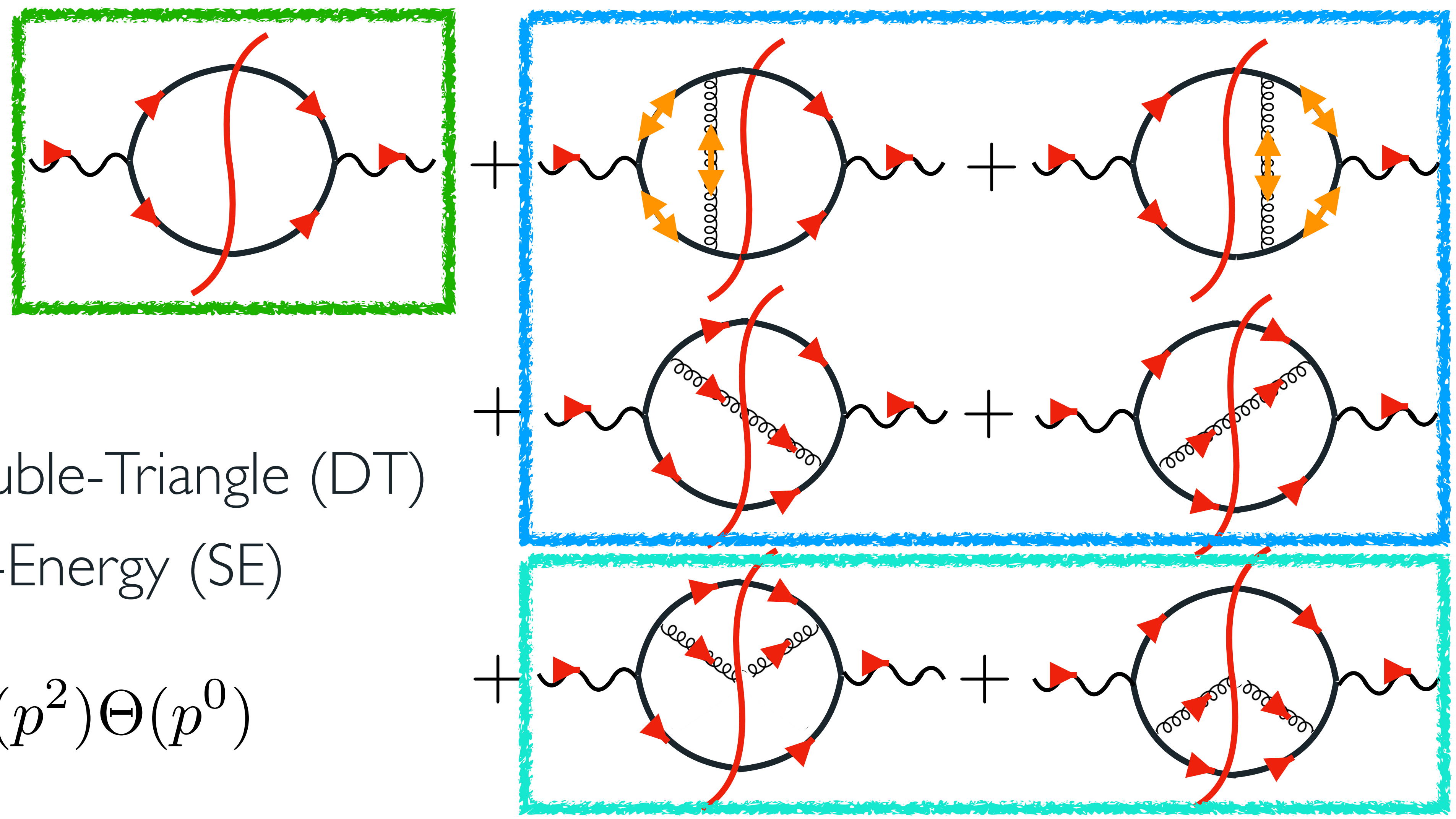
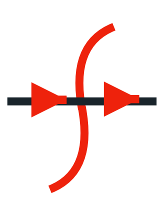
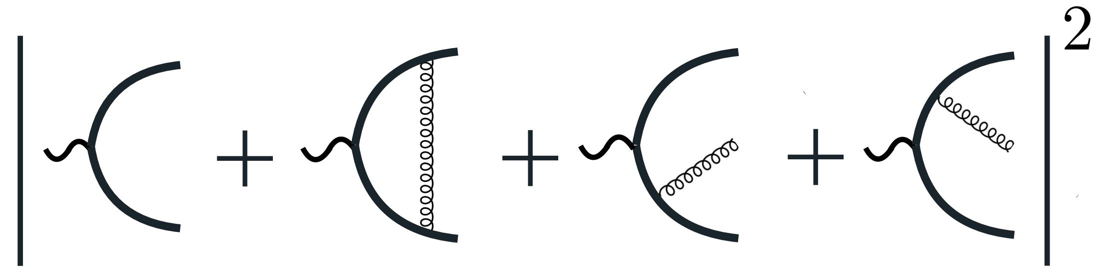


- LO
- NLO, Double-Triangle (DT)
- NLO, Self-Energy (SE)

 $\equiv \dfrac{p^2}{2p^0}\delta(p^2)\Theta(p^0)$

Local Unitarity: A conceptual shift

$$\sigma^{\text{(normal)}}_{\gamma^\star \to d\bar{d}} = \int \Pi^{\text{(phase-space)}}$$



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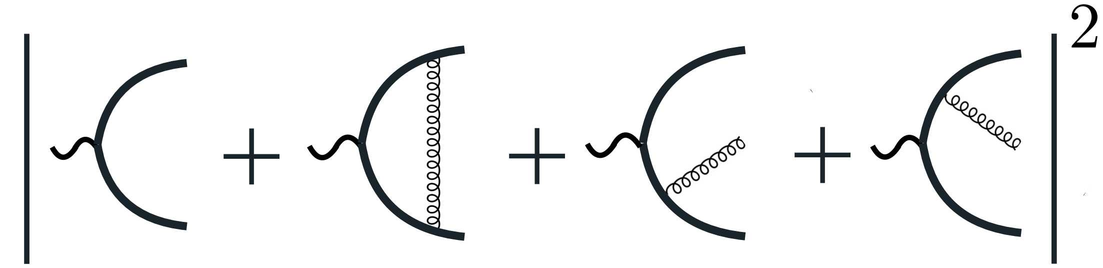

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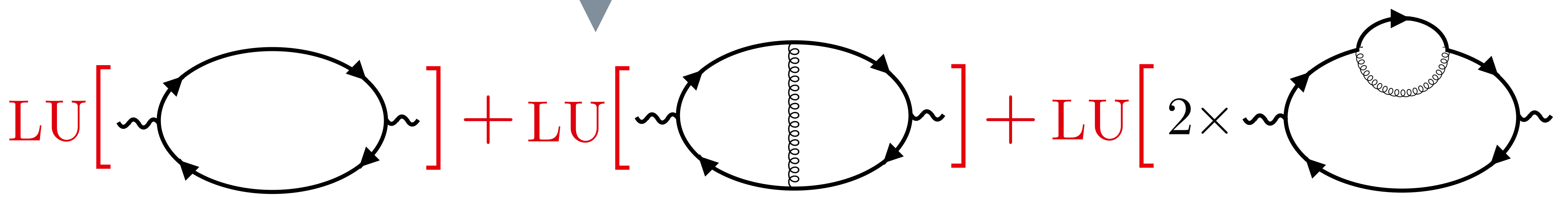
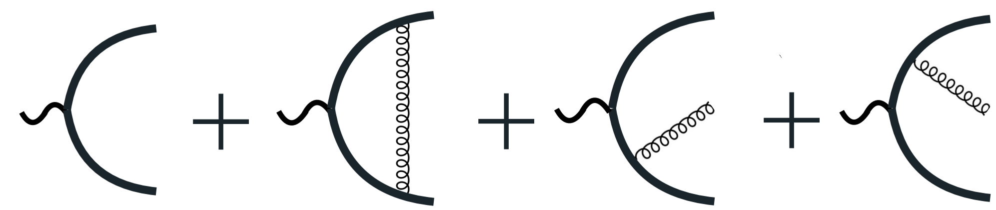
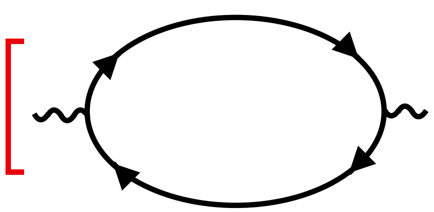


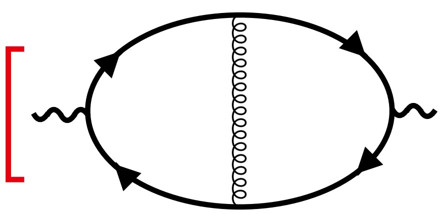
Local Unitarity: A conceptual shift

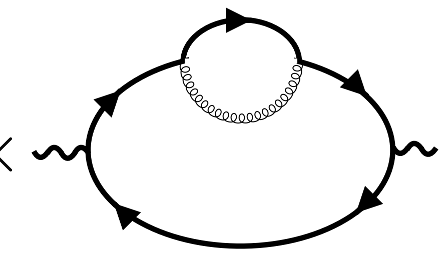
$$\sigma^{(\text{normal})}_{\gamma^\star \to d\bar{d}} = \int \Pi^{(\text{phase-space})} \Big| \quad \Big|^2$$



$$\sigma^{(\text{LU})}_{\gamma^\star \to d\bar{d}} = \text{LU}\Big[\quad\Big] + \text{LU}\Big[\quad\Big] + \text{LU}\Big[2\times\quad\Big]$$





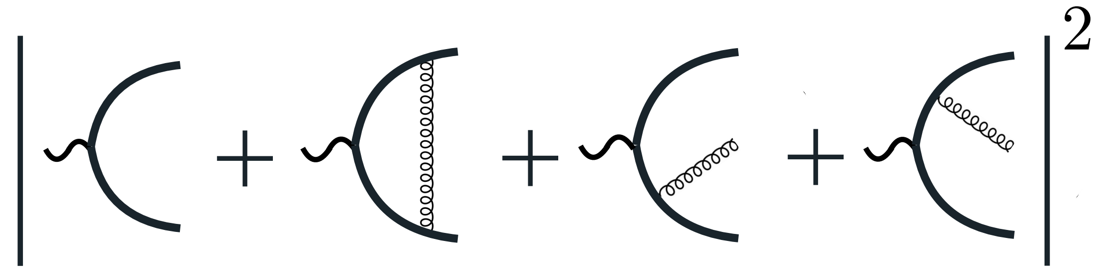


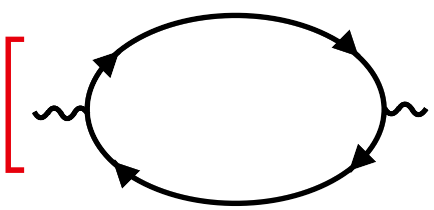
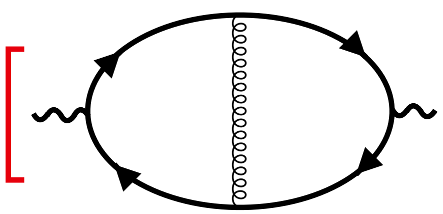
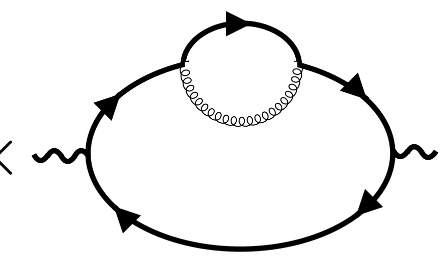
$$\sum_{c\in\{RRR,RRV,RVV,\ldots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|^2_{\text{truncated}}$$

IR-subtraction numerical $d = 4$ analytic $d = 4 - 2\epsilon$

Local Unitarity: A conceptual shift

$$\sigma^{(\text{normal})}_{\gamma^\star \to d\bar{d}} = \int \Pi^{(\text{phase-space})}$$



$$\sigma^{(\text{LU})}_{\gamma^\star \to d\bar{d}} = \text{LU}[$$  $$] + \text{LU}[$$  $$] + \text{LU}[2\times$$  $$]$$

$$\sum_{c\in\{RRR,RRV,RVV,\dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|^2_{\text{truncated}}$$

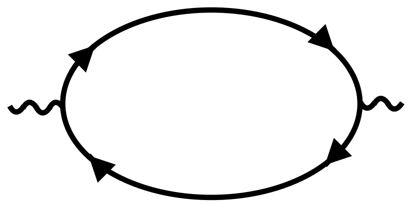
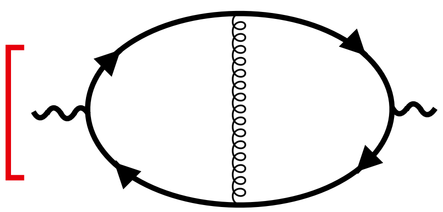
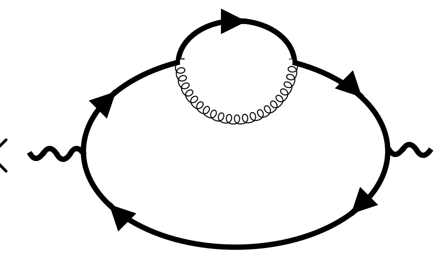
IR-subtraction numerical $d = 4$ analytic $d = 4 - 2\epsilon$

$$\sum_{j=1}^{n_{\text{supergraphs}}} \int \Pi\, g_j^{(\text{LU})}$$

numerical $d = 4$

NO IR-subtraction

So what? It's just the optical theorem no?

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[\, 2\times$  $\Big]$

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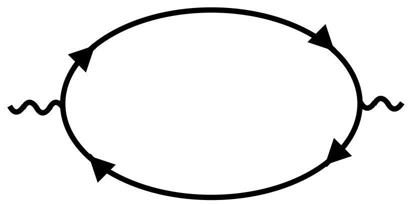
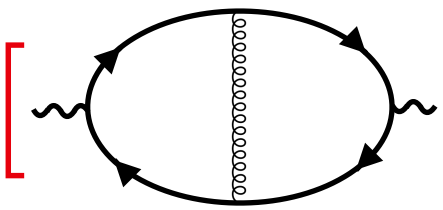
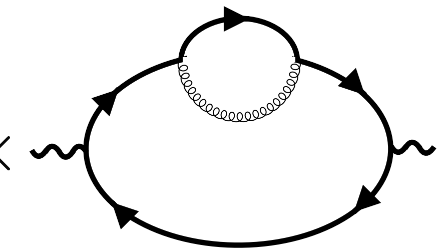
$\sigma^{(\mathrm{LU})}_{\gamma^\star\to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[2\times$  $\Big]$

- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$

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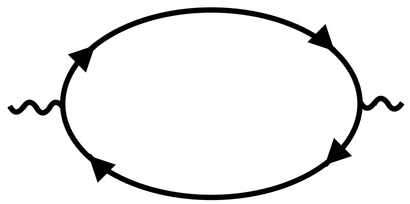
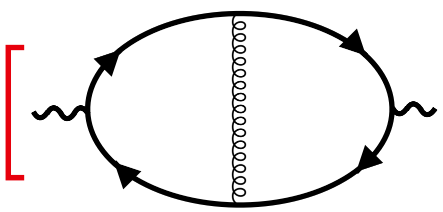
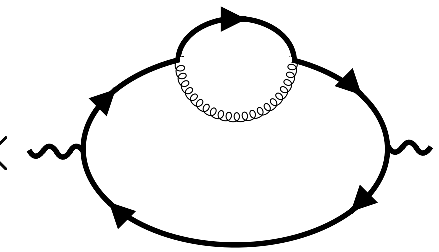
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[2\times$  $\Big]$

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- The **differential** version of LU is our new paradigm: **Local Unitarity**

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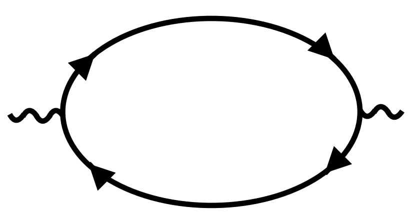
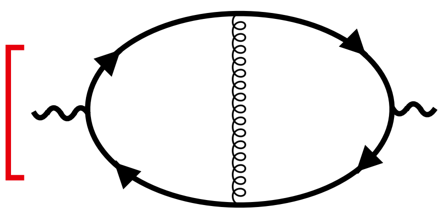
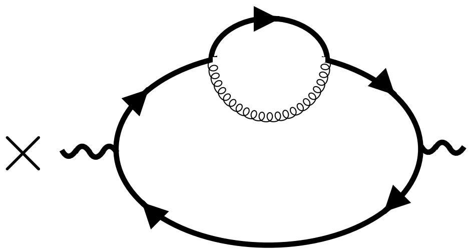
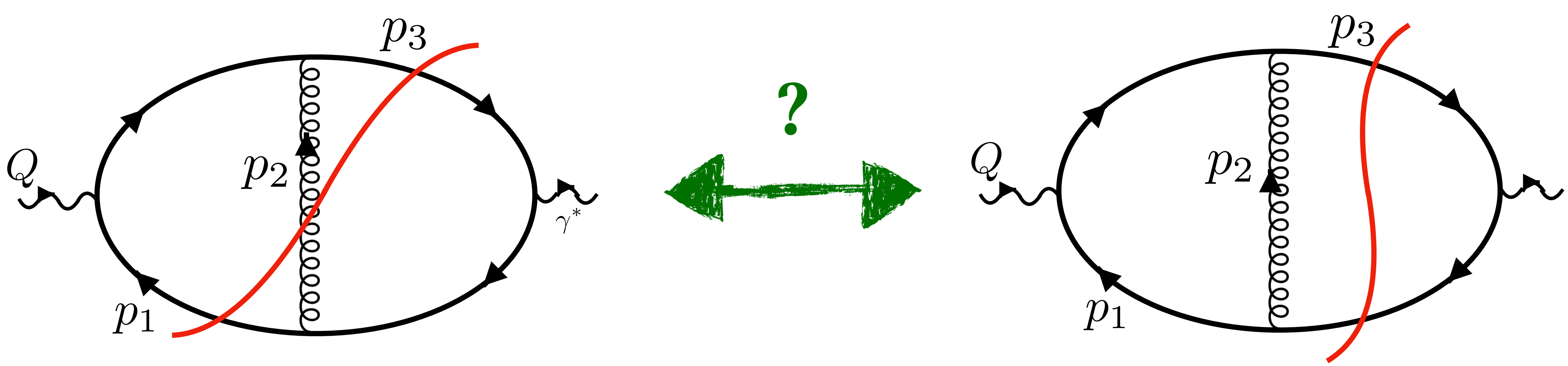
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[2\times$  $\Big]$

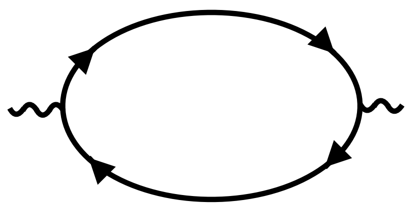
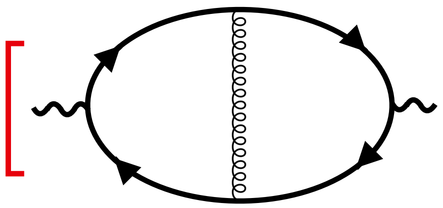
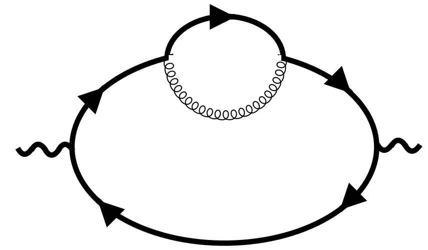
- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$
- The **differential** version of LU is our new paradigm: **Local Unitarity**
- Each supergraph is **individually locally finite**, how you ask ?



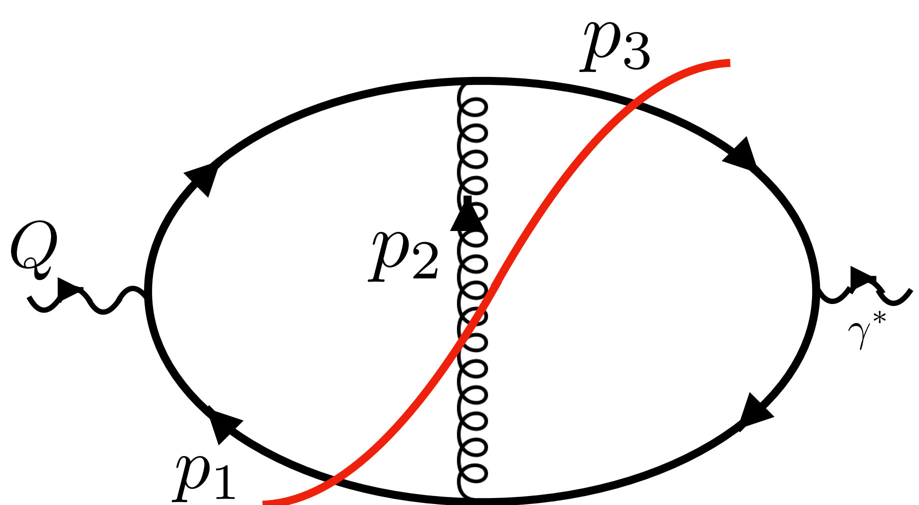
$\propto \delta^{(+)}(p_3^2)\delta^{(-)}(p_1^2)\delta^{(-)}(p_2^2) \longleftrightarrow \propto \delta^{(+)}(p_3^2)\delta^{(-)}((p_1+p_2)^2)$

?

So what? It's just the optical theorem no?

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\big[$  $\big] + \mathrm{LU}\big[$  $\big] + \mathrm{LU}\big[\, 2\times$  $\big]$

- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$
- The **differential** version of LU is our new paradigm: **Local Unitarity**
- Each supergraph is **individually locally finite**, how you ask ?



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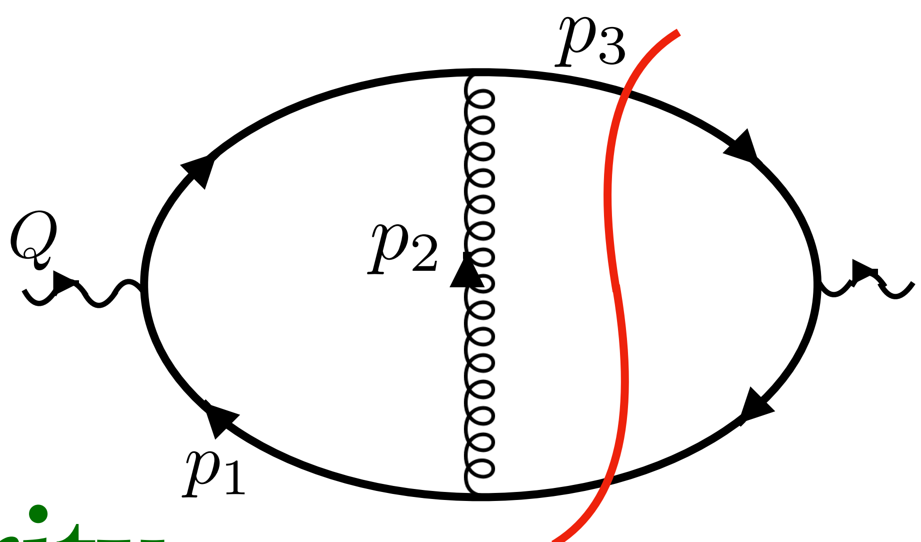
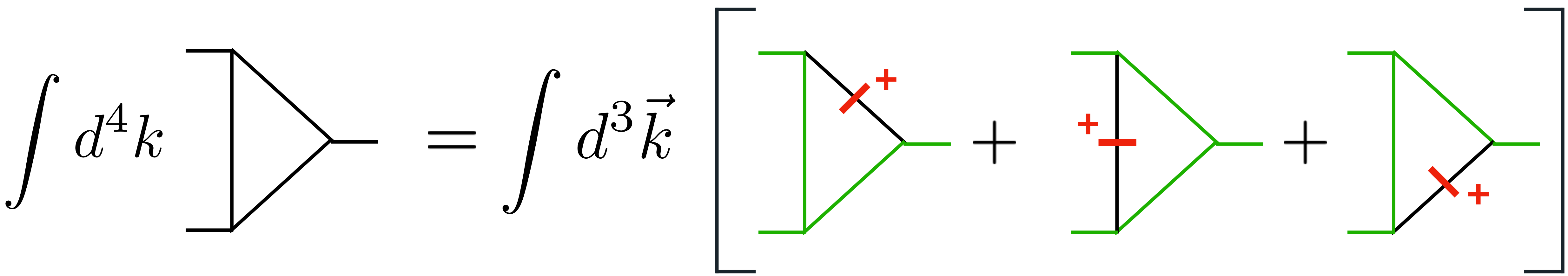


Local Unitarity

$\propto \delta^{(+)}(p_3^2)\delta^{(-)}(p_1^2)\delta^{(-)}(p_2^2)$ $\propto \delta^{(+)}(p_3^2)\delta^{(-)}((p_1+p_2)^2)$

?

Main ingredient: Loop Tree Duality



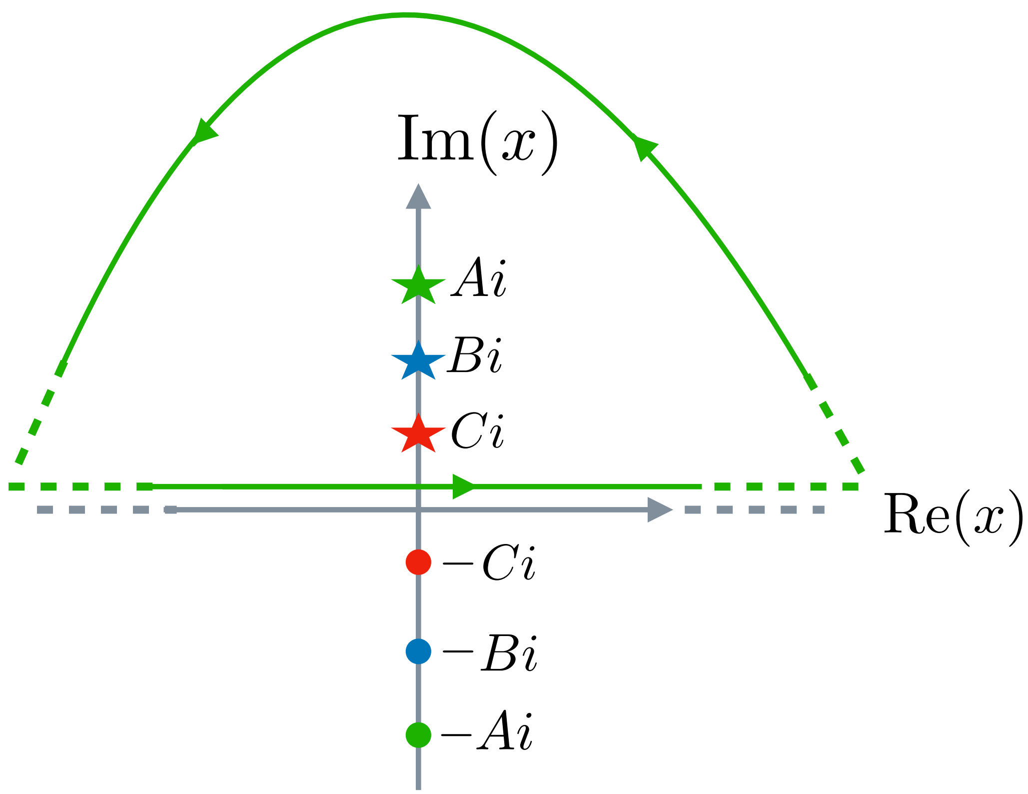
(closely related to the Feynman-tree theorem
And Time-Ordered Perturbation Theory)

(ONE)-LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx F(x) \qquad F(x) = \frac{1}{x^2+A^2}\frac{1}{x^2+B^2}\frac{1}{x^2+C^2}$$

$$F(x) = \frac{1}{(x-Ai)(x+Ai)}\frac{1}{(x-Bi)(x+Bi)}\frac{1}{(x-Ci)(x+Ci)}$$

$$(\mathrm{Assumptions} \to \{A>0, B>0, C>0\})$$



Cauchy: $\left(R(x^\star) \equiv \mathrm{Res}(F, x = x^\star) \right)$

$$I = (-2\pi i)[R(Ai) + R(Bi) + R(Ci)]$$

What does it correspond to for a one-loop integral?

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

(ONE-)LOOP TREE DUALITY

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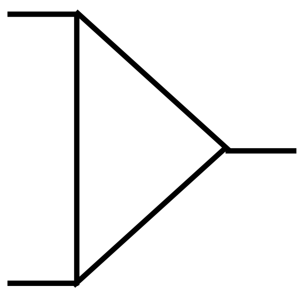
Pole selected for each propagator

Then integrate the energy component using residue theorem



$$= \int \mathrm{d}^3\vec{k} \left[\mathrm{Res}_1\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_2\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_3\left[\frac{N}{D_1 D_2 D_3}\right]\right]$$

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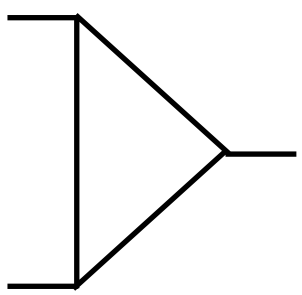
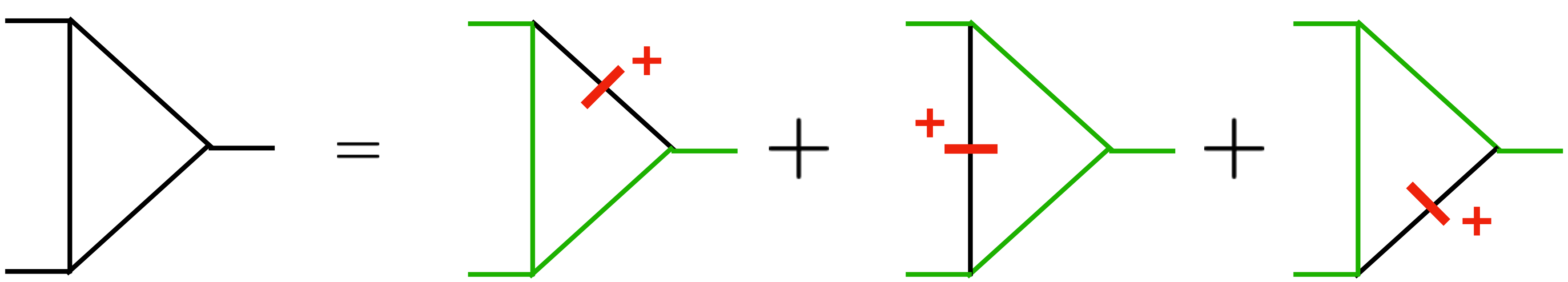


$$= \int \mathrm{d}^3\vec{k} \left[\mathrm{Res}_1\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_2\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_3\left[\frac{N}{D_1 D_2 D_3}\right]\right]$$

Residues can be represented as cuts:



(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

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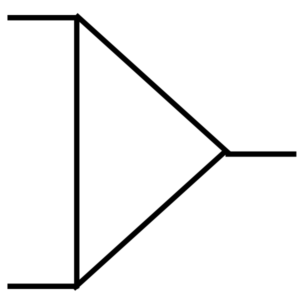

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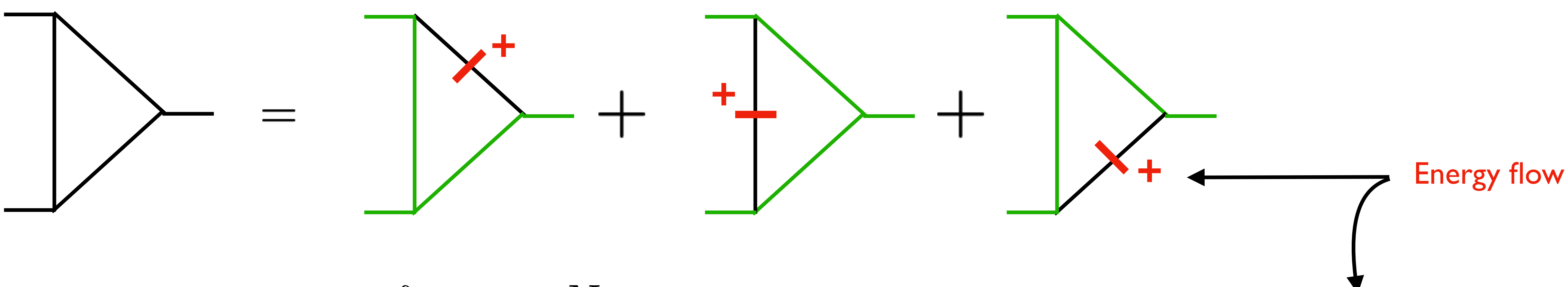
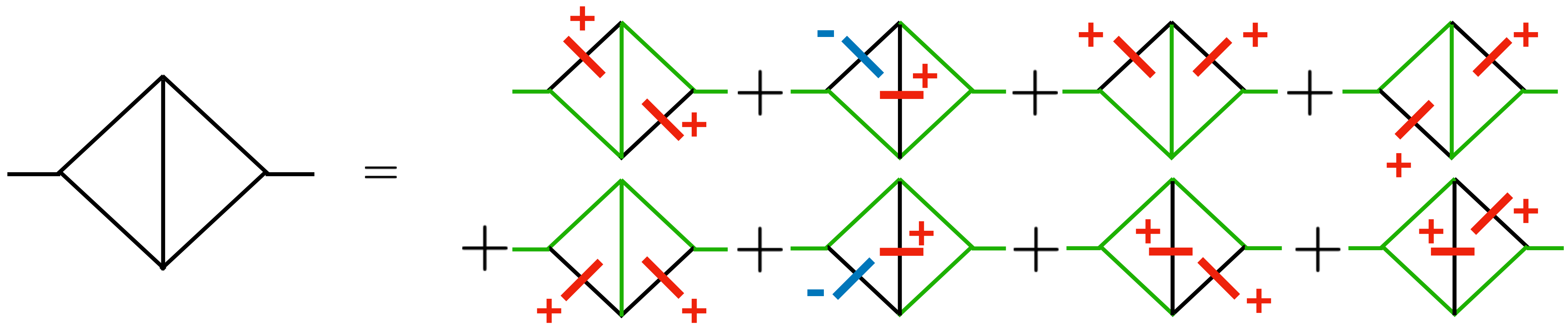


$$= \int \mathrm{d}^4 k \frac{N}{D_1 D_2 D_3} \left(D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3) \right)$$

(Multi-)Loop Tree Duality

Applying LTD to a two-loop double-triangle: one residue per spanning tree



Interplay of momentum conservation and causal prescription is key to obtain the energy flow

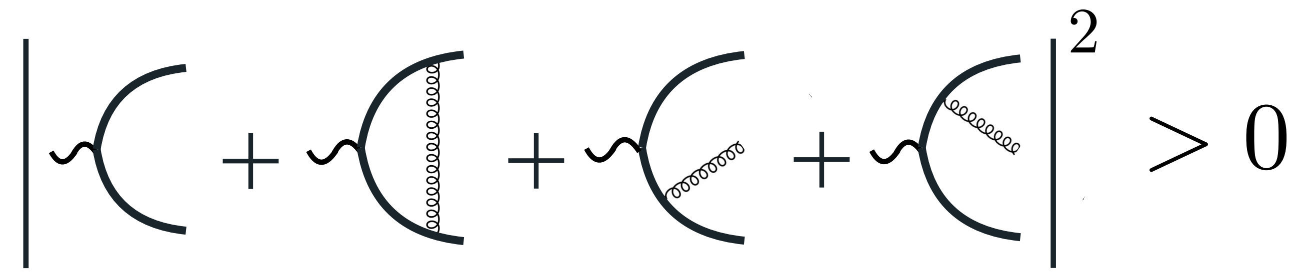
- **Distributional identities:** [Bierenbaum, Catani, Draggiotis, Rodrigo, arxiv: 1007.0194]
- **Averaging procedure:** [Runkel, Scór, Vesga, Weinzierl, arxiv: 1902.02135]
- **Iterative procedure:** [Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138]
- **Manifestly causal:** [Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 2009.05509]
- **Cross-Free Family** (the best 3D repr.!) [Capatti, arxiv: 2211.09653]

Codes : [https://github.com/apelloni/cLTD]

[https://bitbucket.org/wjtorresb/lotty]

{ Pick your candy: cannot all be manifest }

Positivity



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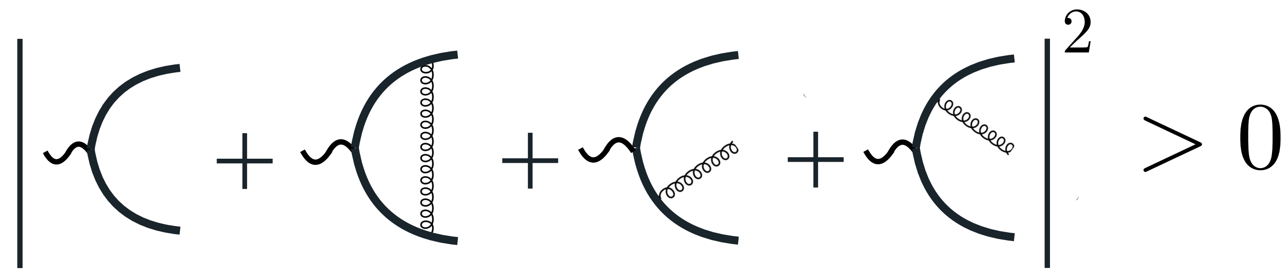

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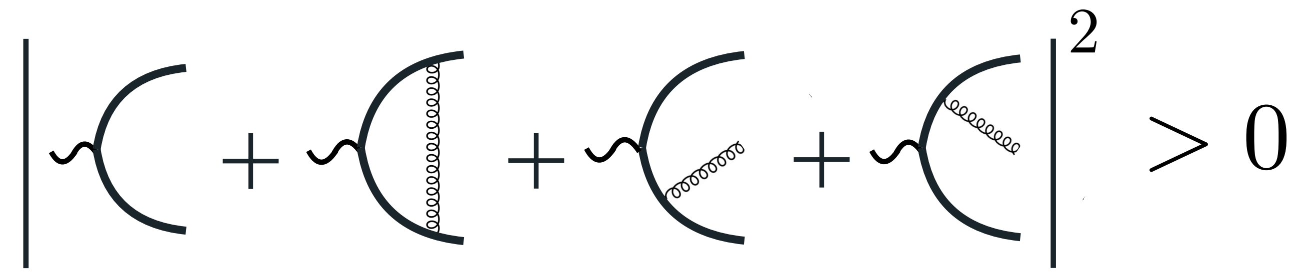

Lorentz Invariance

$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

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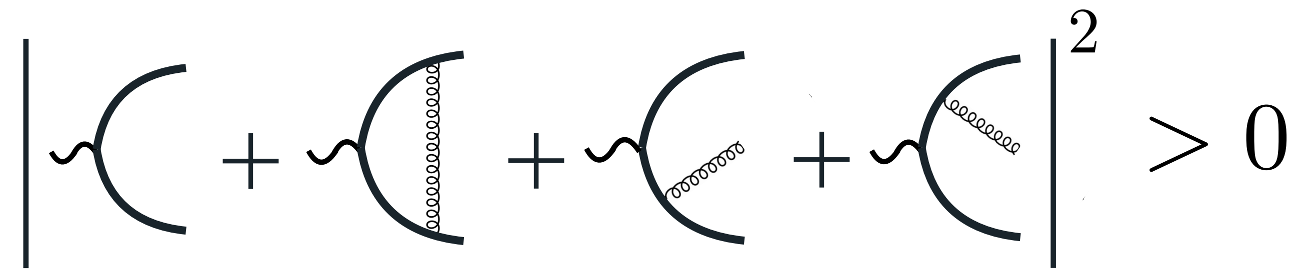

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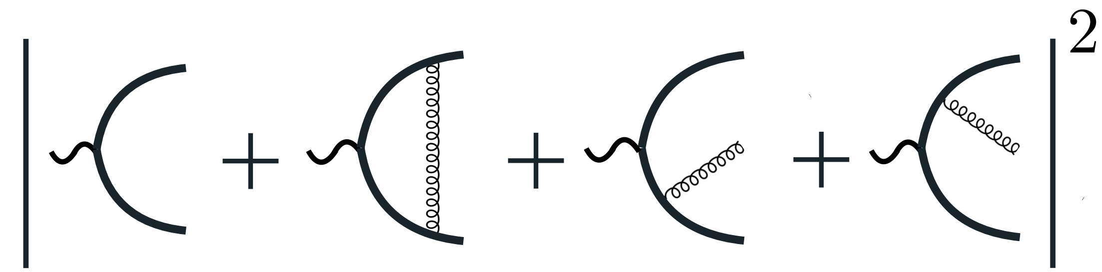

$$> 0$$

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$$k^{\mu}\mathcal{M}_{\mu} = 0$$

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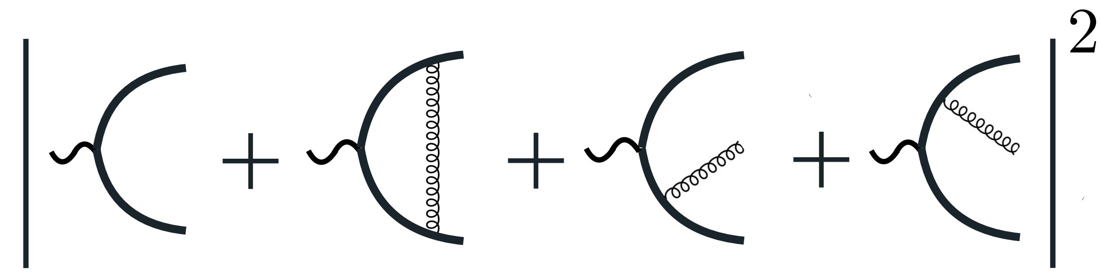

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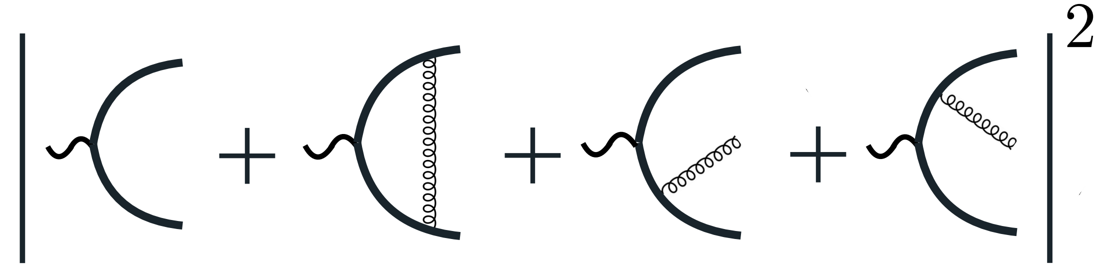

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Unitarity

$$\mathrm{i}(T^{\dagger} - T) = T^{\dagger}T$$

{ Pick your candy: cannot all be manifest }

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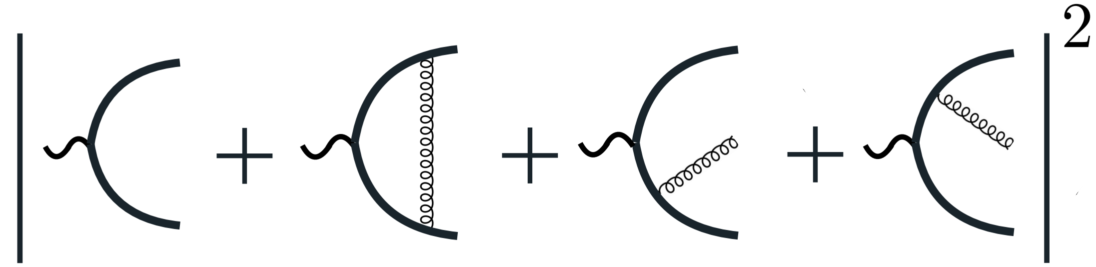

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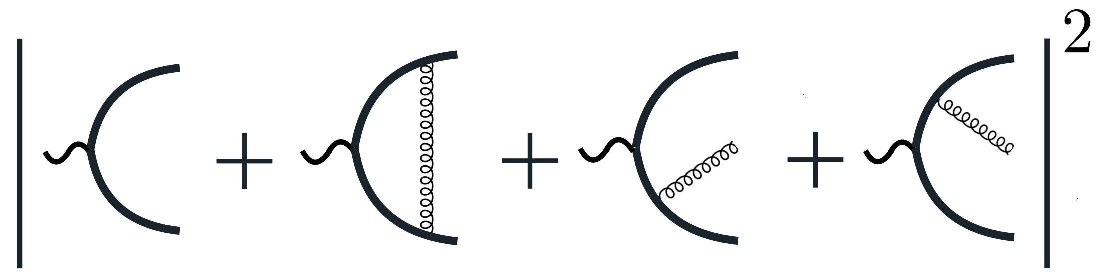

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$\boxed{\text{Unitarity}}$

$$\mathrm{i}(T^{\dagger} - T) = T^{\dagger}T$$

$\boxed{\text{Locally finite}}$

$$\lim_{k\,\rightarrow\,\text{soft,colli,UV}} I(k) = \mathcal{O}(1)$$

Four types of singularities :

Thresholds | Contour-deformation or thres. subtraction

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Thresholds	Contour-deformation or thres. subtraction
Infrared	Local Unitarity

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Integrable	(tropical) sampling

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Thresholds	Contour-deformation or thres. subtraction
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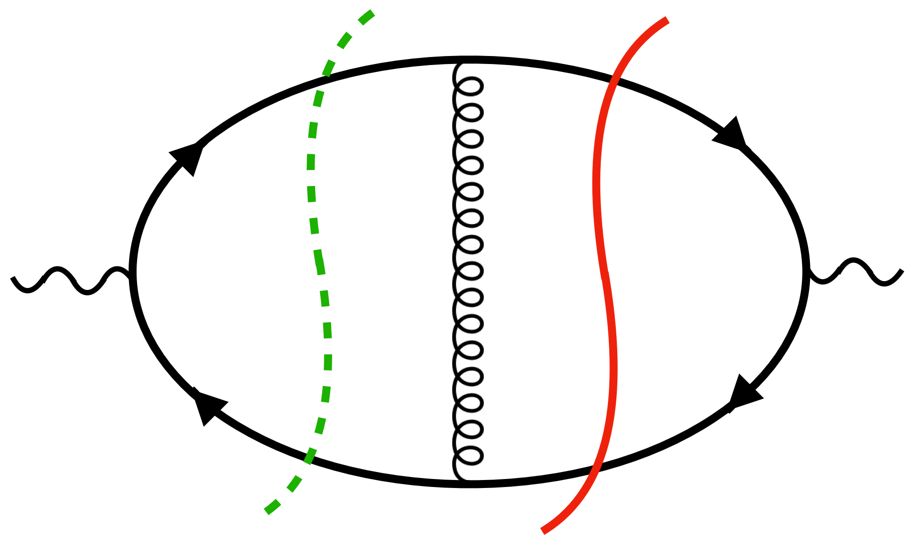
Integrable	(tropical) sampling
Ultraviolet	Local BPHZ

Thresholds

$$E_1 = \sqrt{|\vec{k}|^2 + m^2 - i\epsilon}$$

$$q = (Q^0, \vec{q})$$



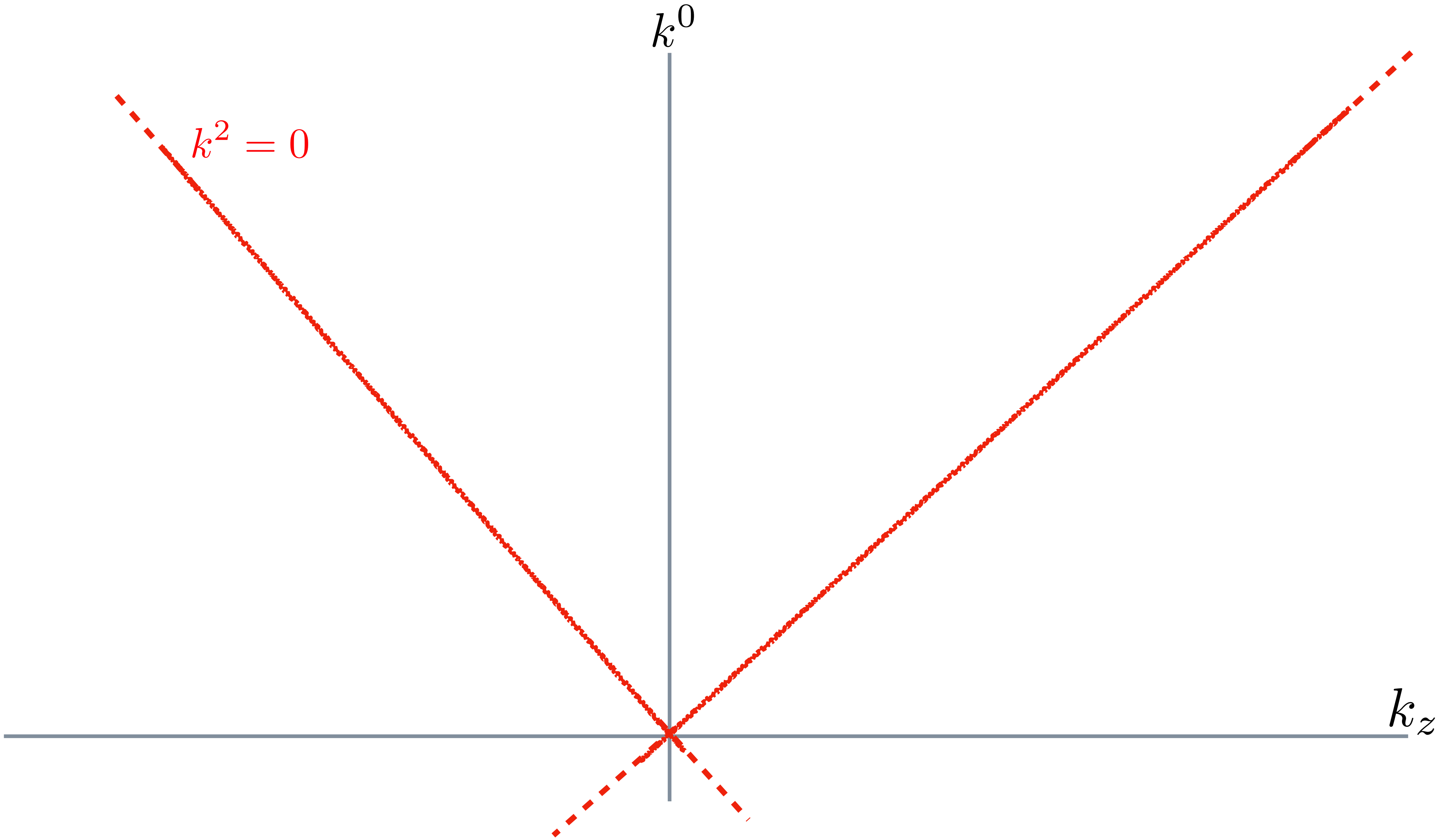
$$E_2 = \sqrt{|\vec{k} - \vec{q}|^2 + m^2 - i\epsilon}$$

$$\int d^3\vec{k}\; I^{(\text{Local Unitarity})} \supset \int d^3\vec{k}\; \frac{1}{E_1 E_2 E_3} \left(\frac{1}{(E_1 + E_2 - Q^0)(E_1 + E_2 + Q^0)} \right)$$

$$\eta(\vec{k}) = E_1 + E_2 - Q^0 \overbrace{=}^{\vec{Q}=0 \;\; m=0} 2|\vec{k}| - Q^0$$

Singular surfaces in Minkowski space

$$\int d^4k \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$



The integrand is singular along each of the coloured surface

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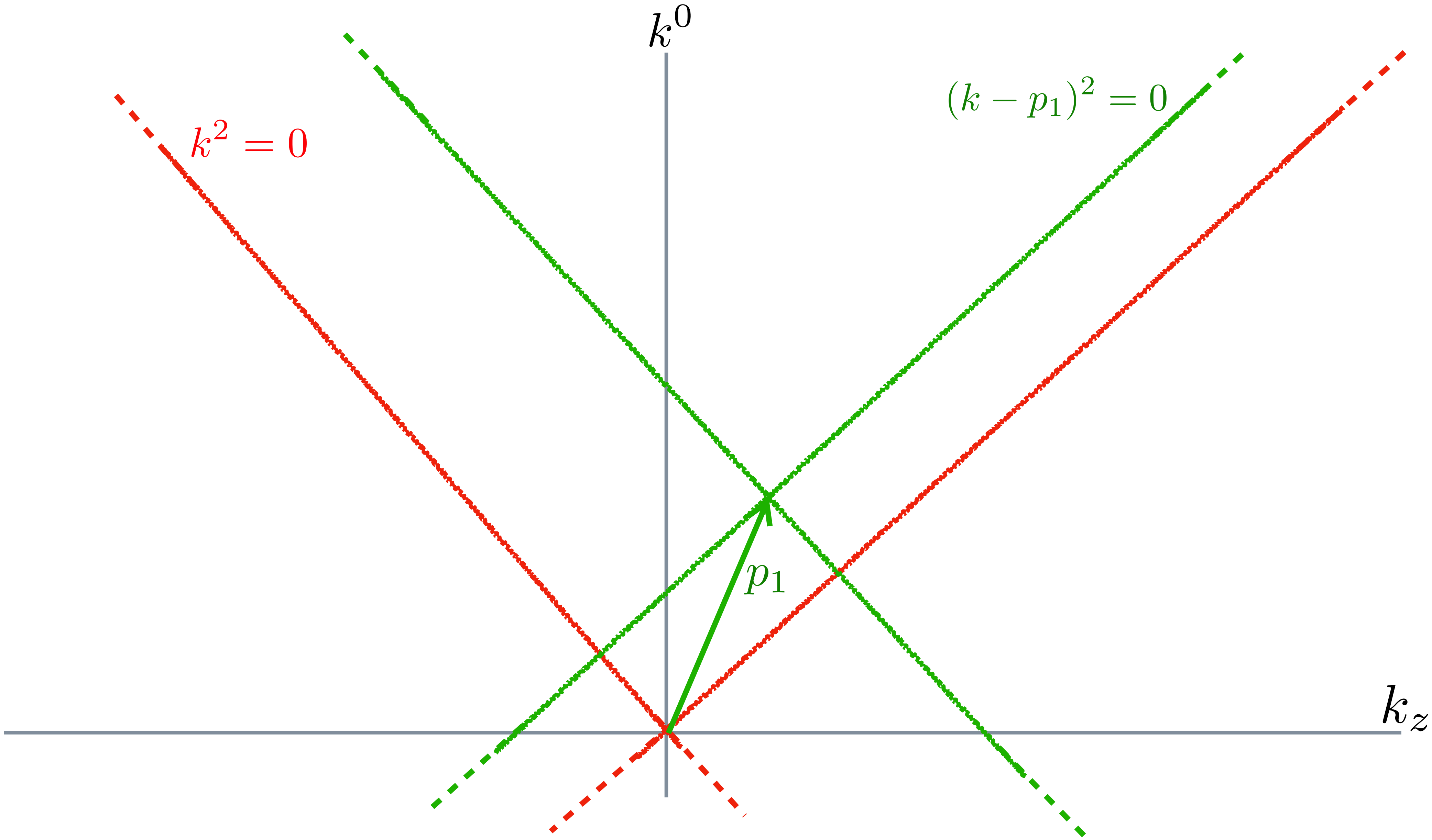

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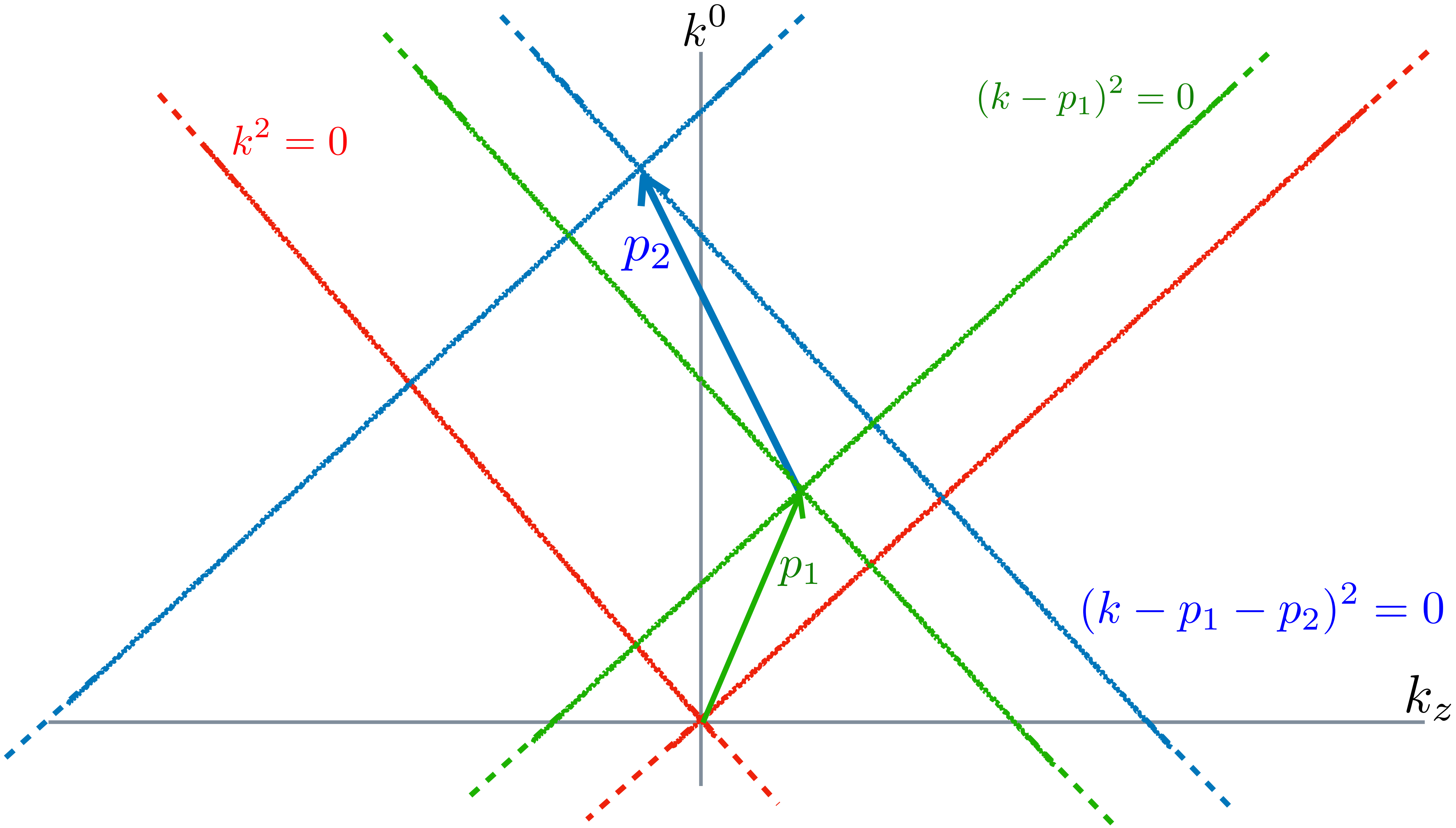

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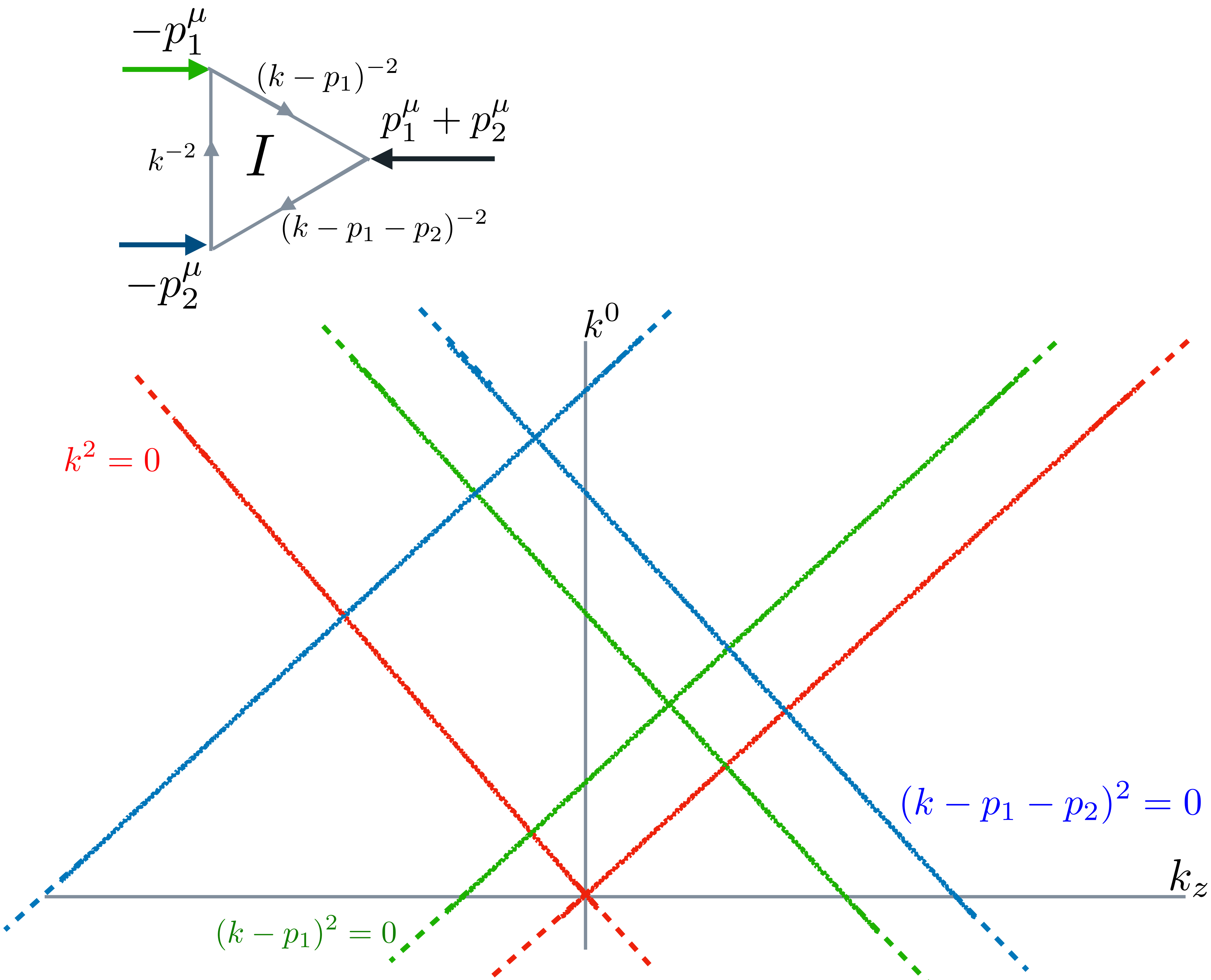
Singular surfaces of the LTD representation

Analytically integrate over the loop energies using Cauchy's theorem (LTD):



$p_i{}^2 > 0 \quad \forall i$

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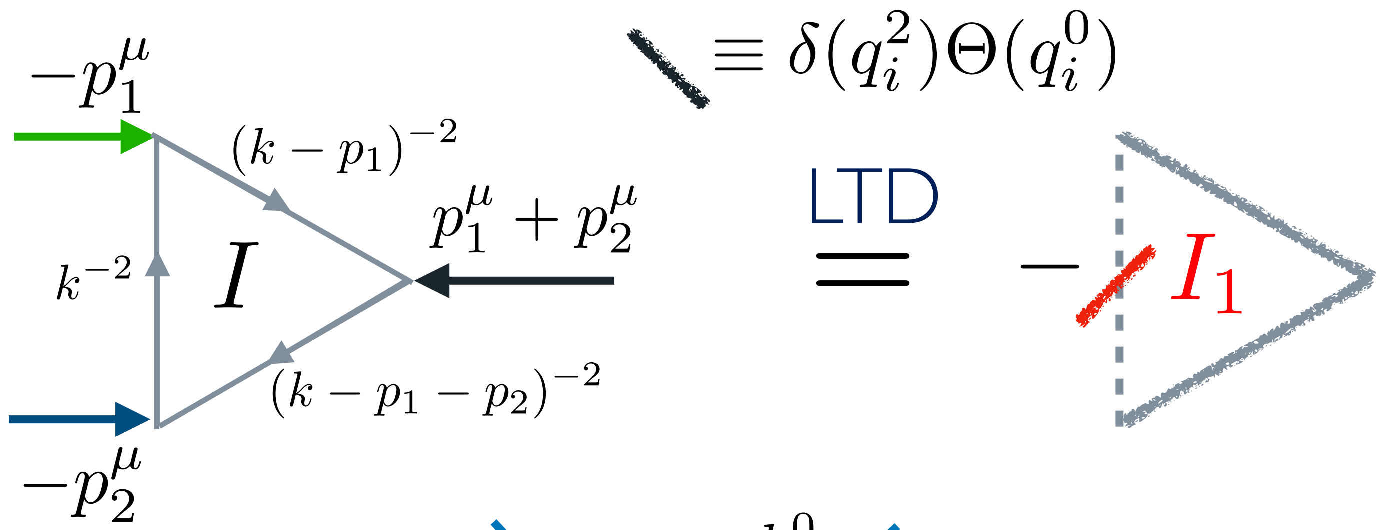

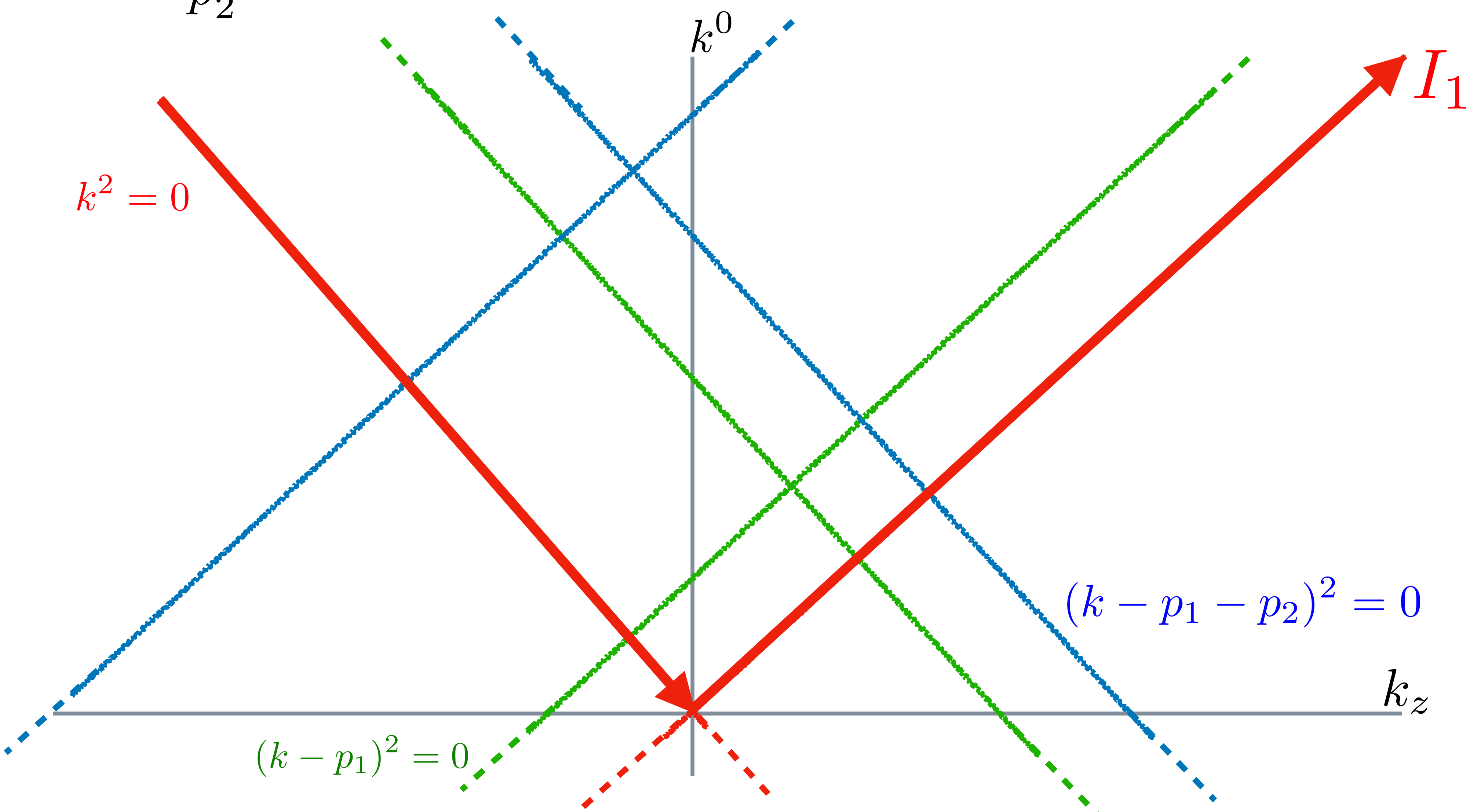

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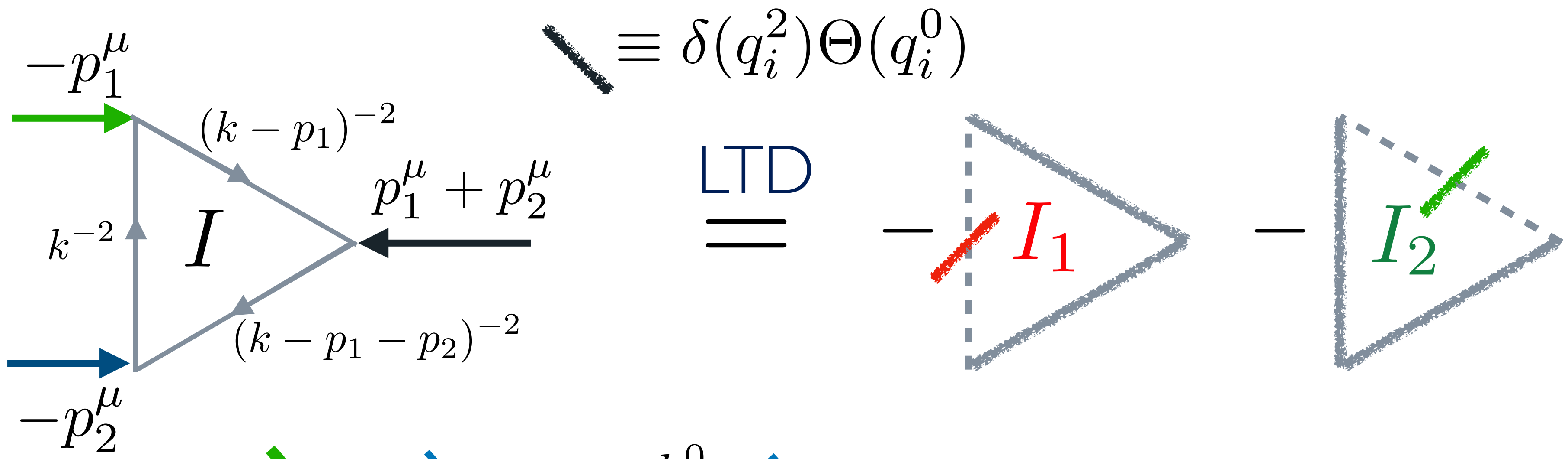

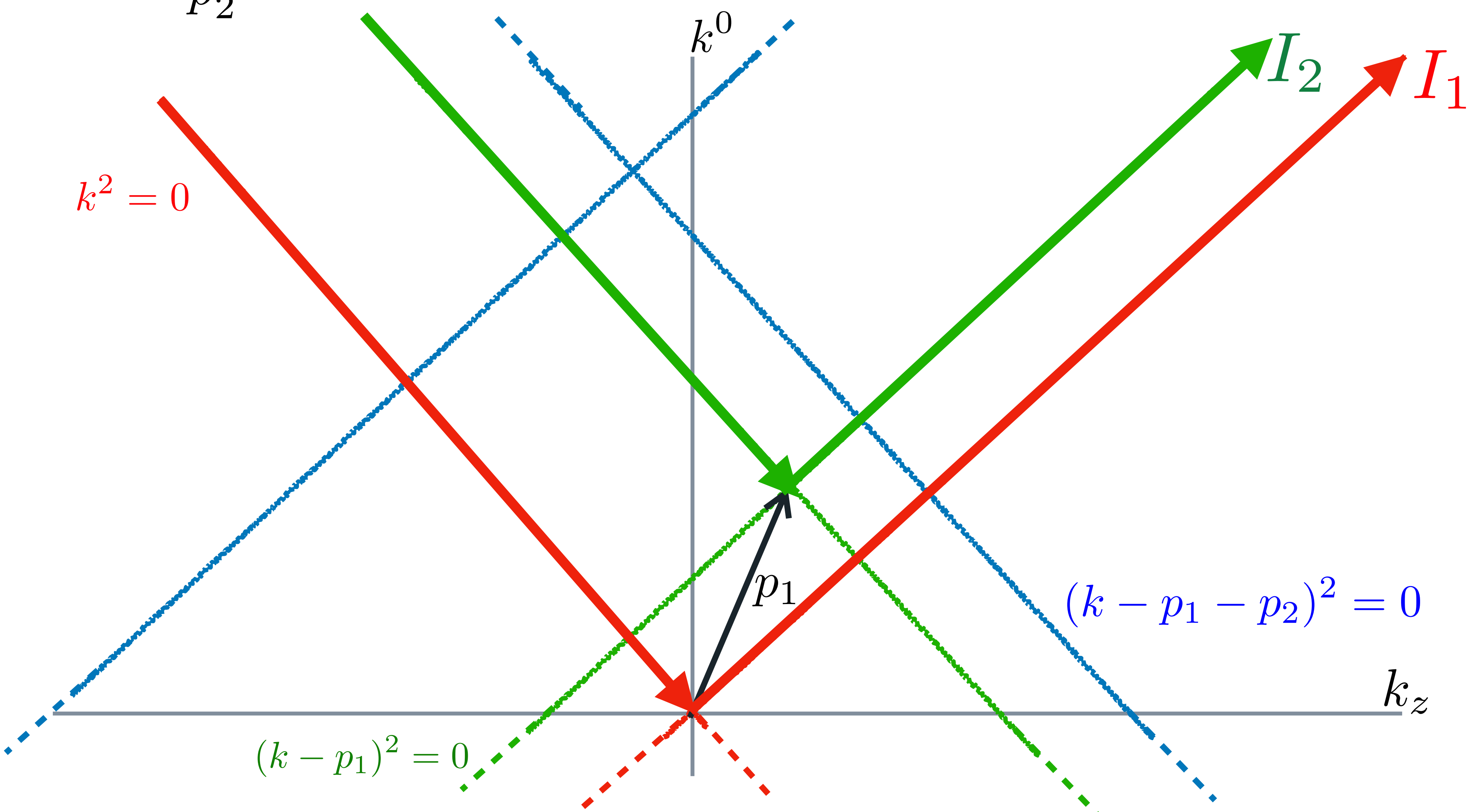

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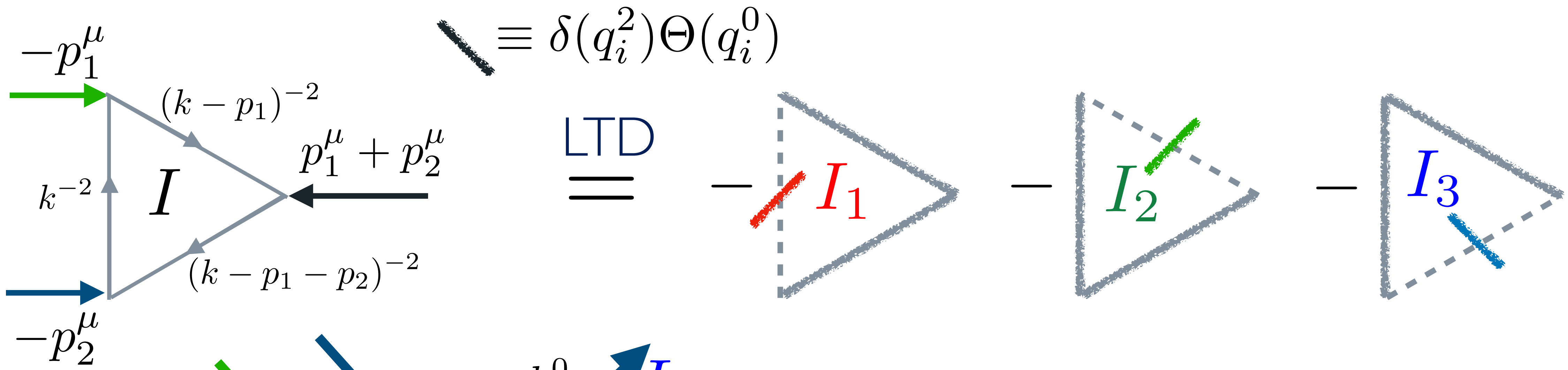

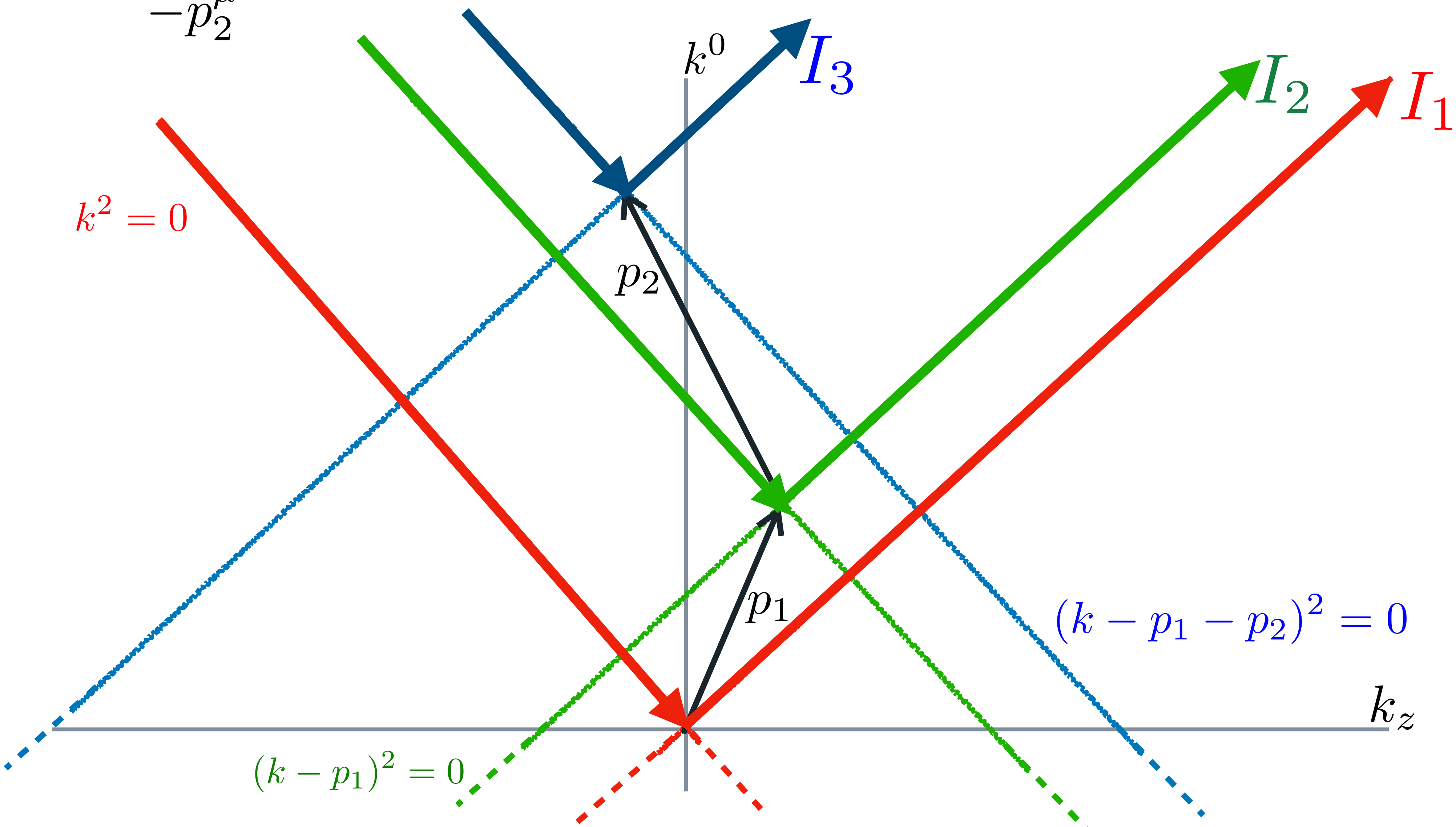

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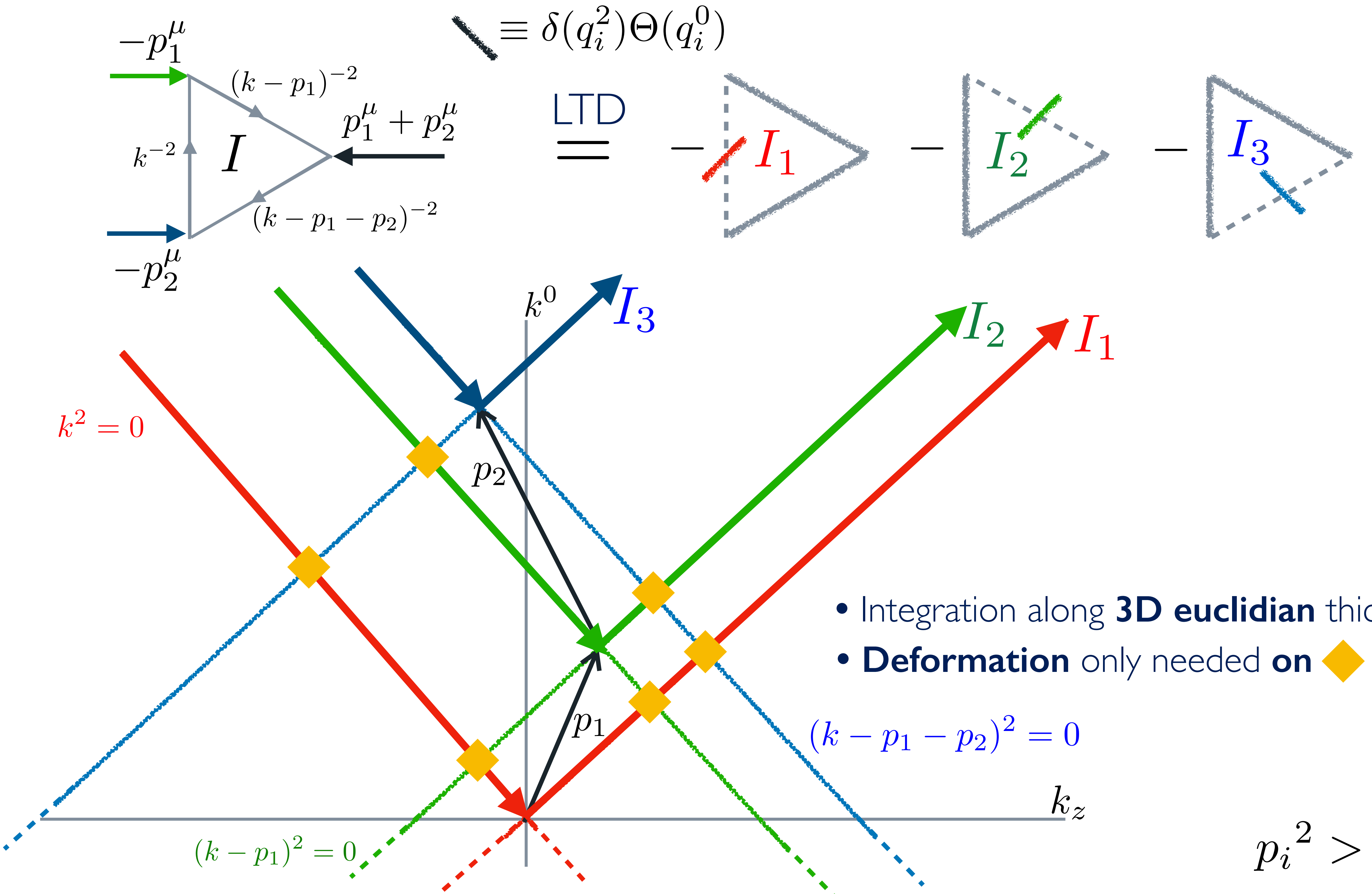

- Integration along **3D euclidian** thick lines
- **Deformation** only needed **on** ◆ !

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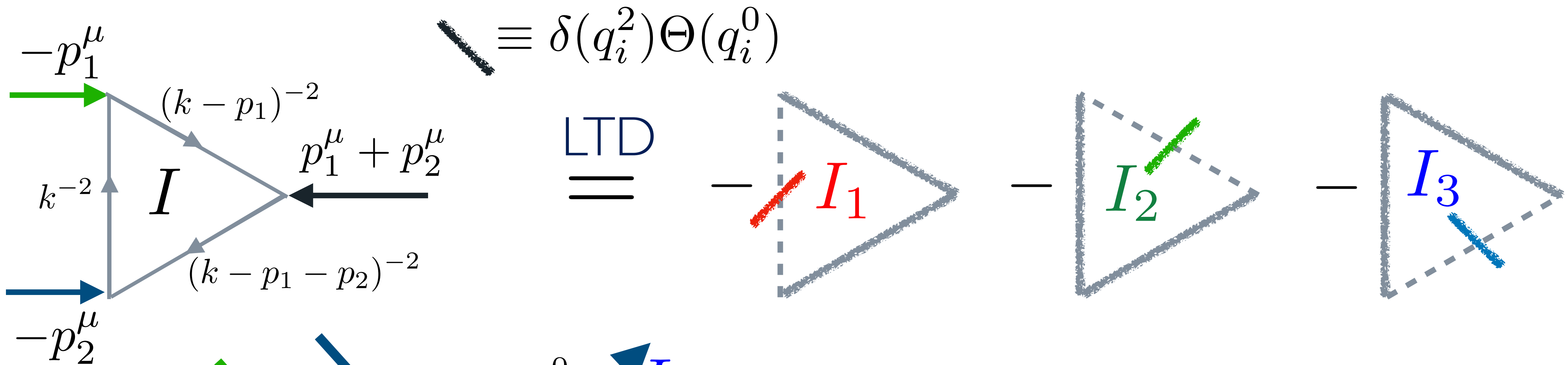

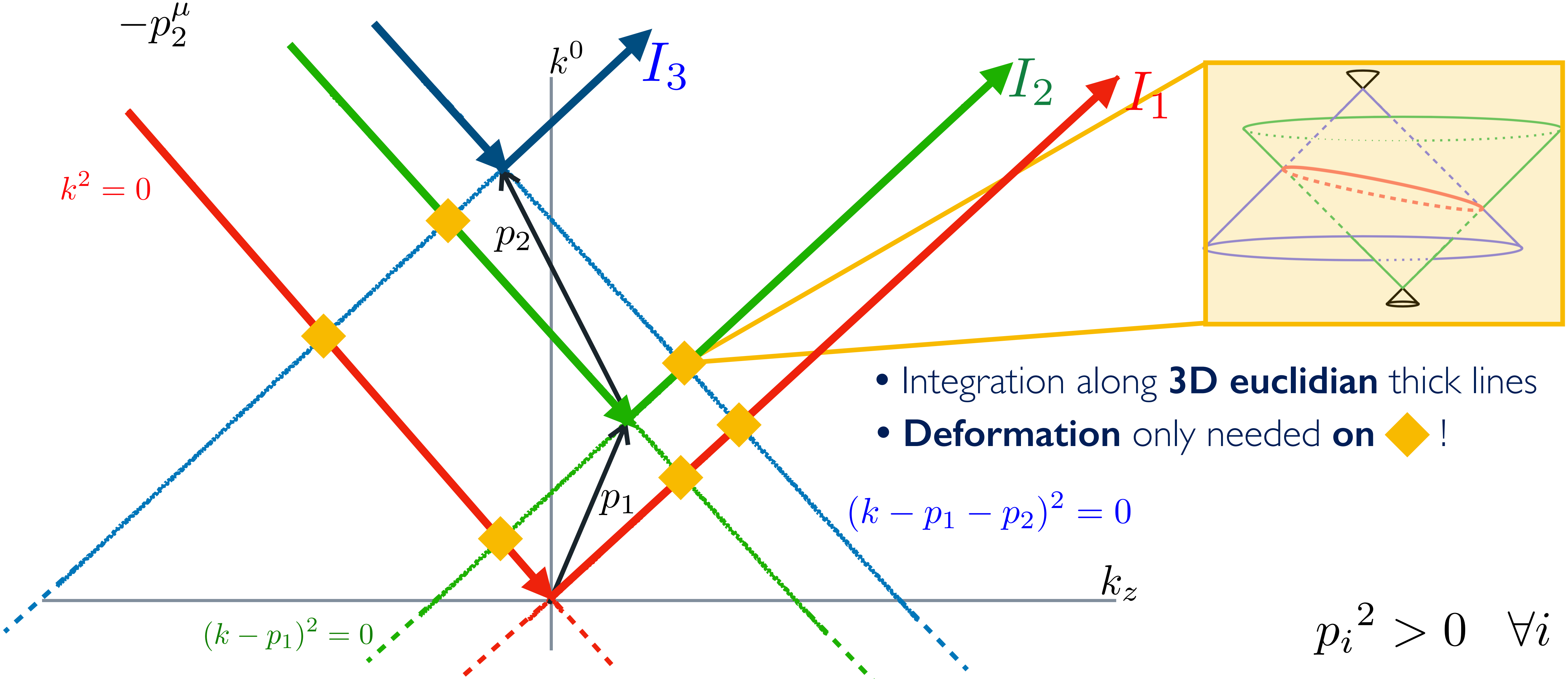

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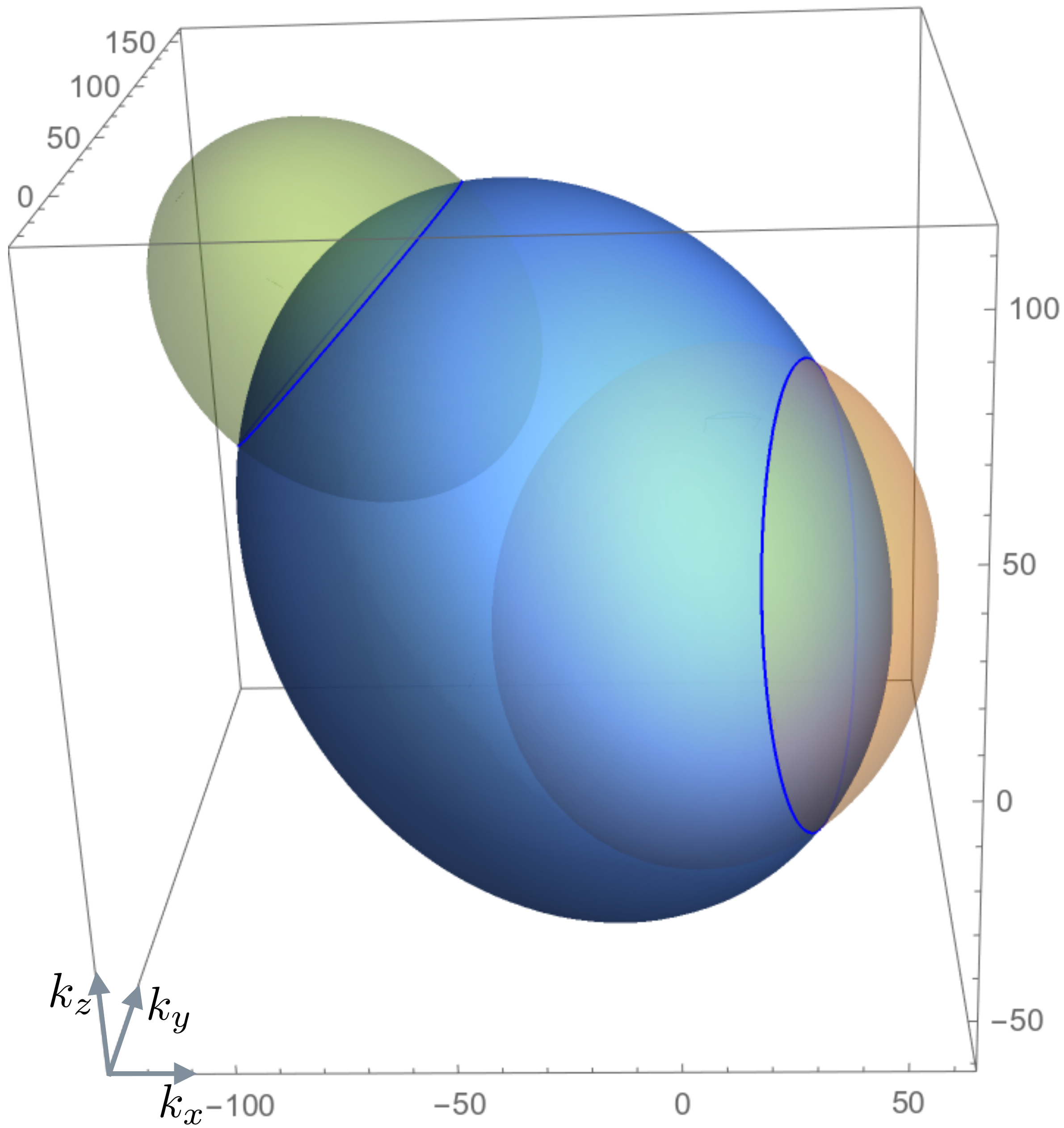
Singular surfaces - 2D ellipsoids



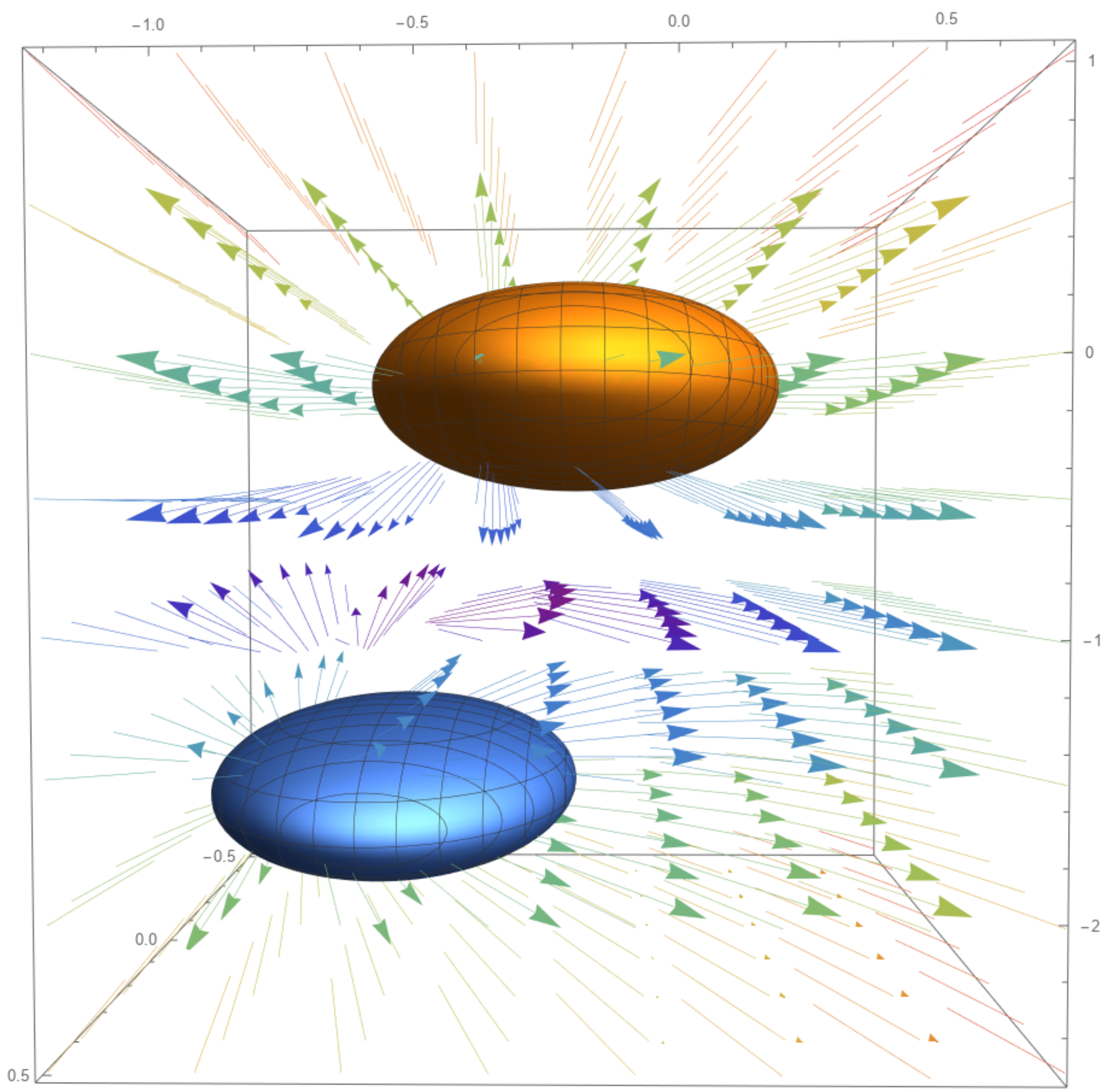
- General **one-loop** ellipsoid **expression**:

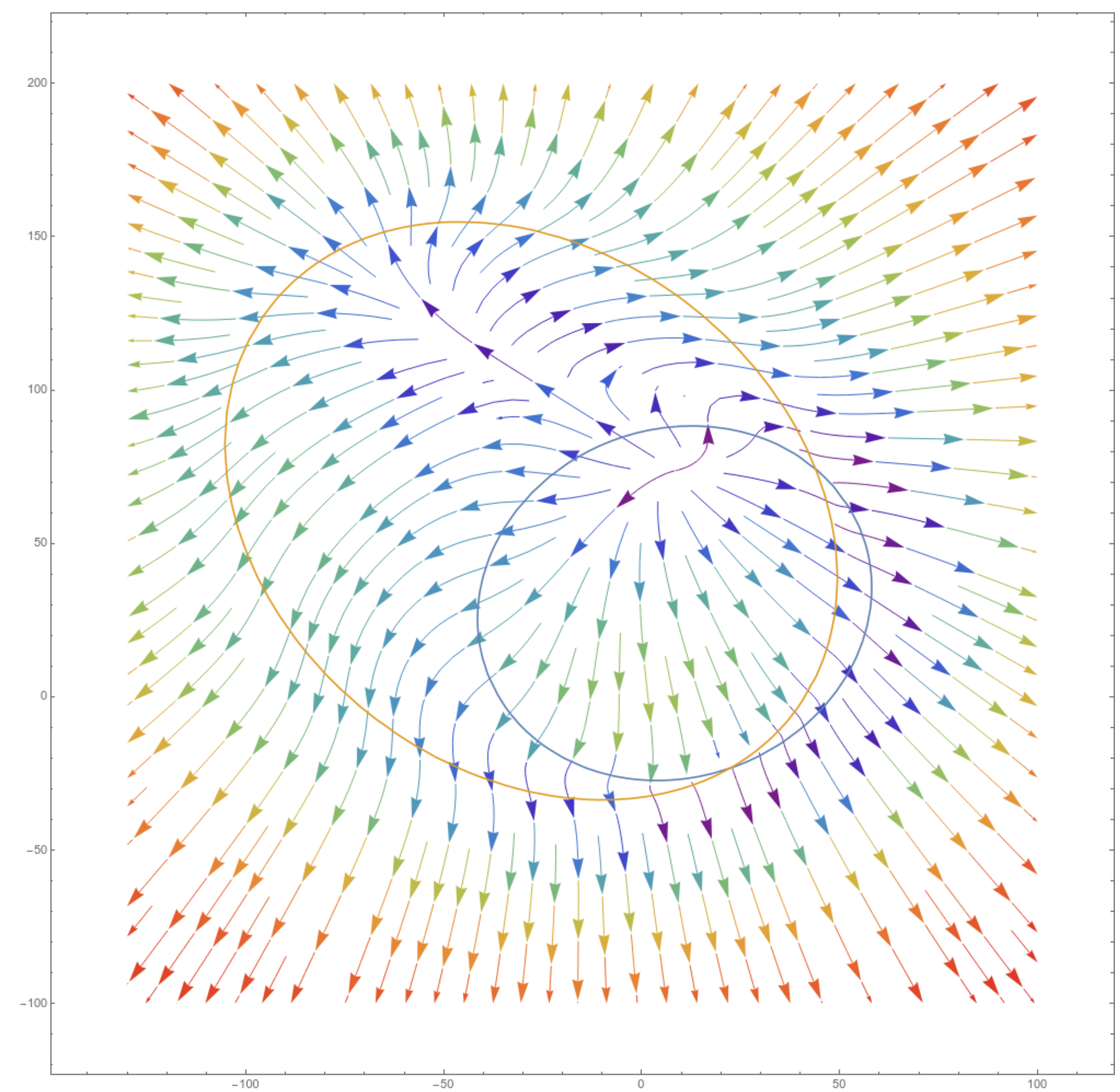
$$E_{ij}(\vec{k}) = \sqrt{\left(\vec{k}+\vec{p_i}\right)^2 + m_i^2 - i\delta} + \sqrt{\left(\vec{k}+\vec{p_j}\right)^2 + m_j^2 - i\delta} - p_i^0 + p_j^0$$

Deforming around singular 2D-ellipsoids

Deformation: $\vec{k} \to \vec{k} - i\vec{\kappa}$

Causal prescription imposes: $\vec{\kappa} \cdot \vec{n}_{E_{ij}} > 0$





Deforming around singular 2D-ellipsoids

How to construct such a field? For example for this case:

[Capatti, VH, Kermanschah, Pelloni, Ruijl]
[arxiv:1906.06138]

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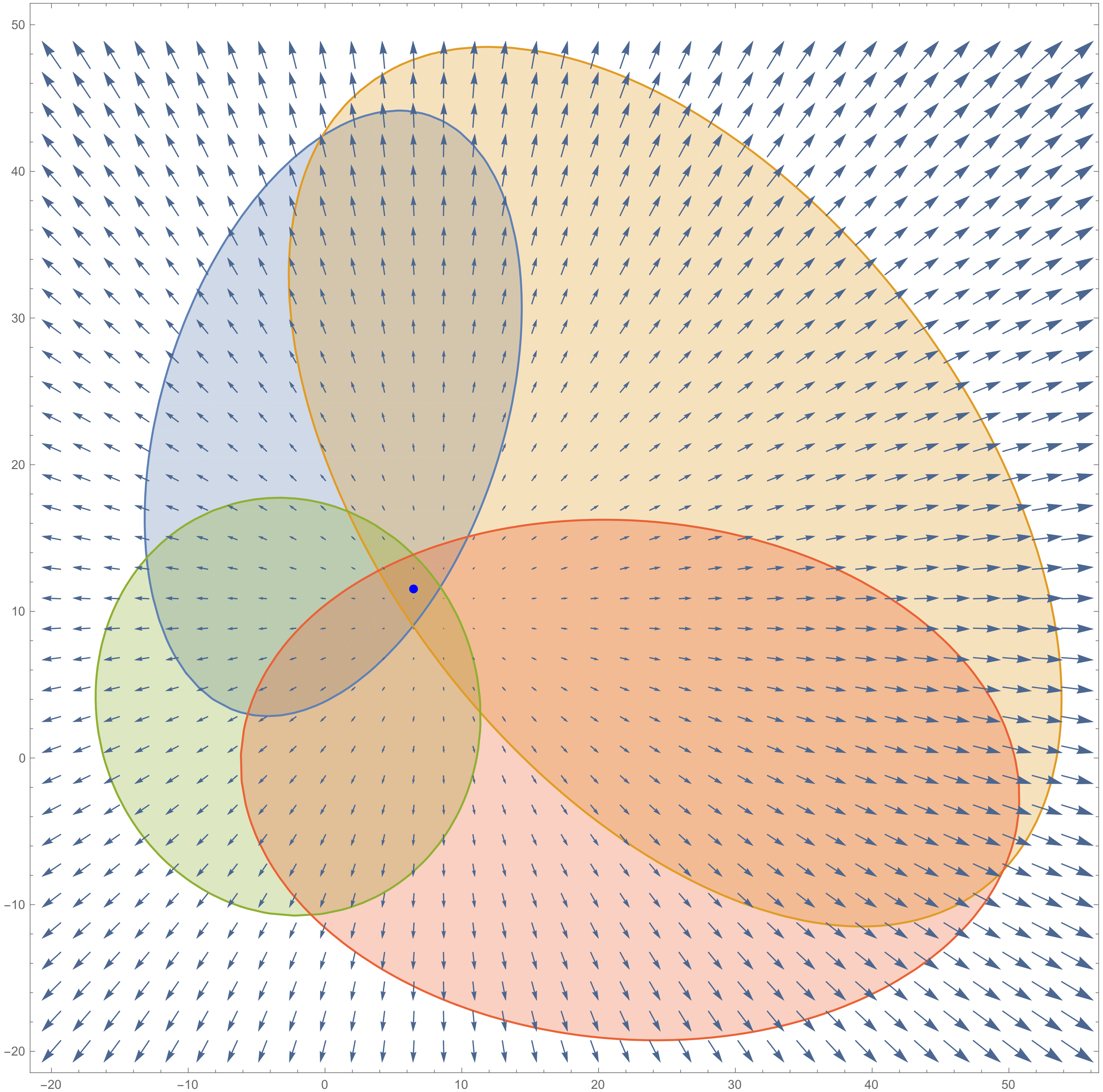
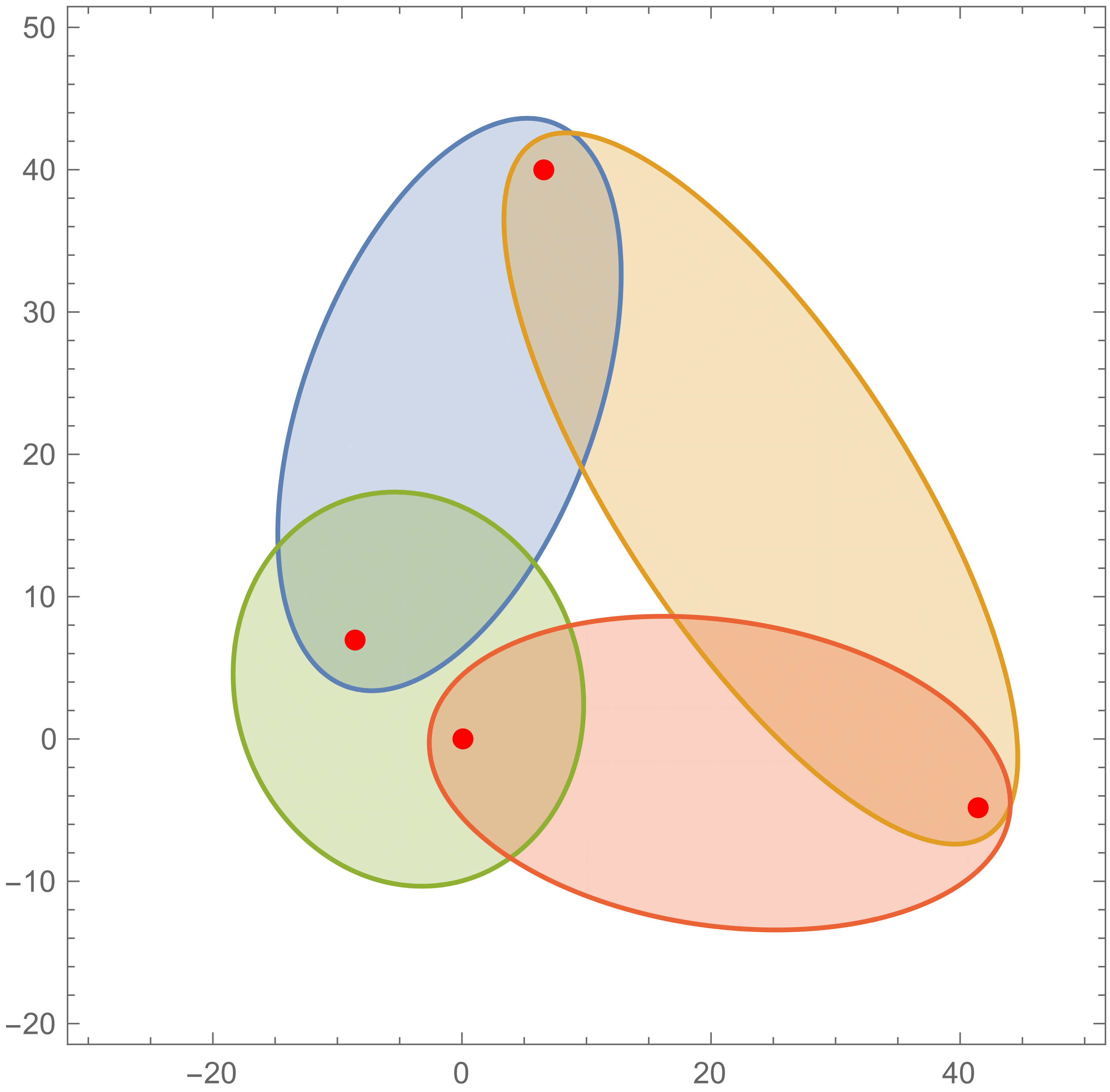


A **radial field** centered in the inside of all ellipsoids!

[Capatti, VH, Kermanschah, Pelloni, Ruijl]
[arxiv:1906.06138]

Deforming around singular 2D-ellipsoids

But then what if there is no point in the inside of **all ellipsoids** (`Box4E` example) ?



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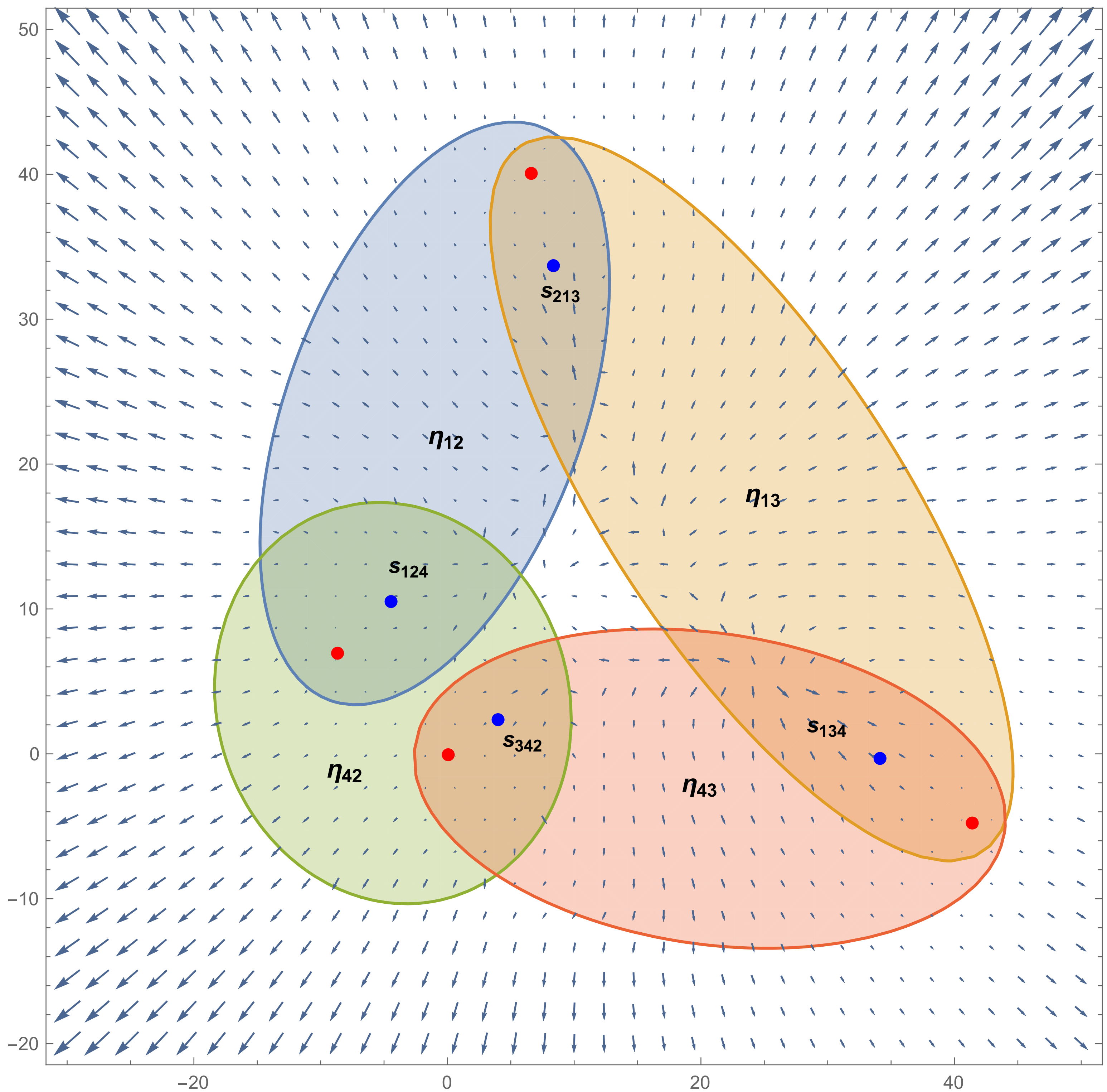
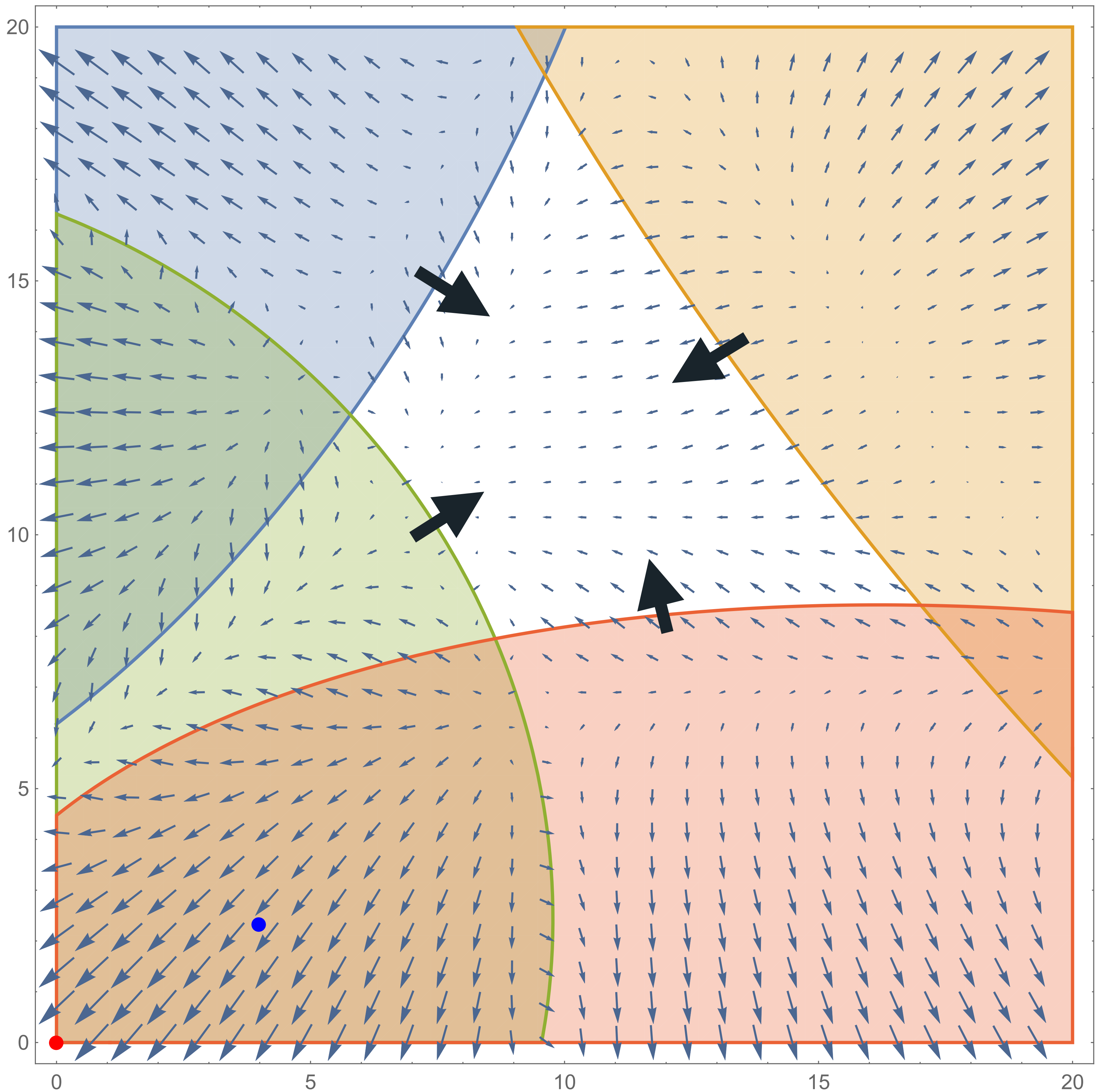


Deforming around singular 2D-ellipsoids

But then what if there is no point in the inside of **all ellipsoids** (`Box4E` example) ?



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0}$$

$$\stackrel{p_1^\mu = (2,\vec{0})}{=} \frac{1}{2|\vec{k}| - 2} \propto \frac{1}{\sqrt{k_x^2 + k_y^2} - 1}$$

[D. Kermanschah, arXiv : 2110.06869]

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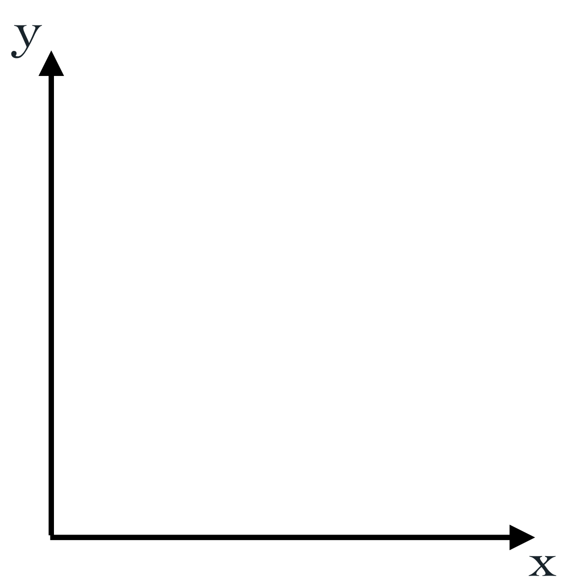

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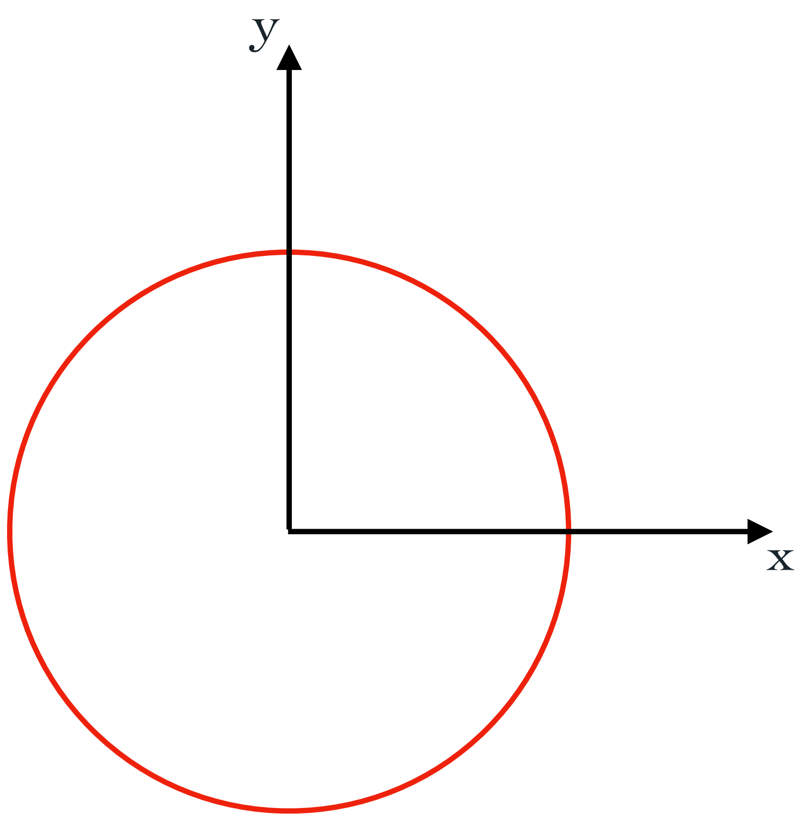

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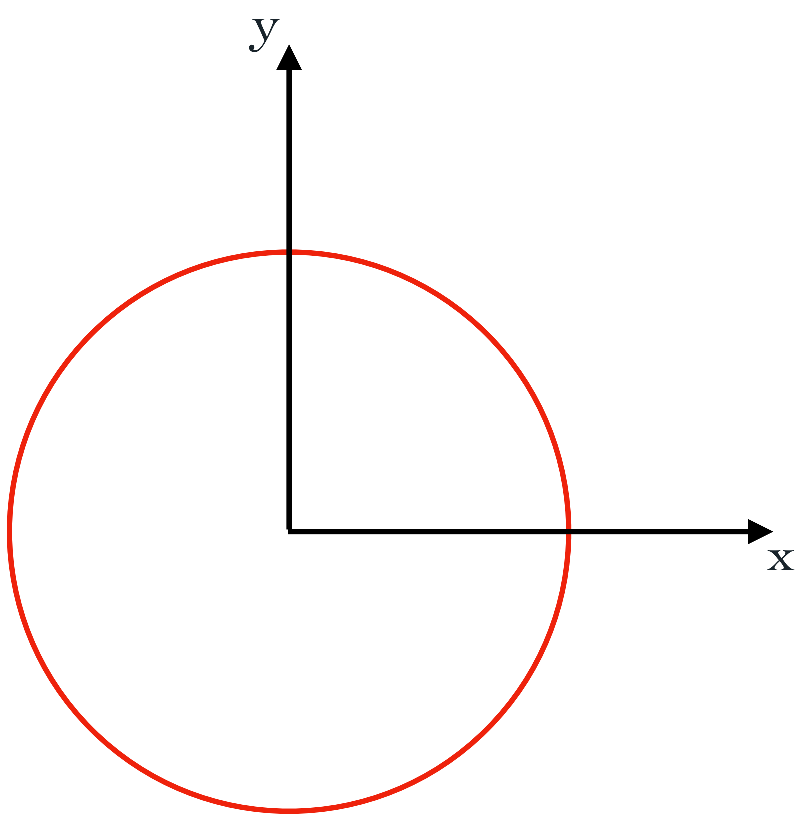

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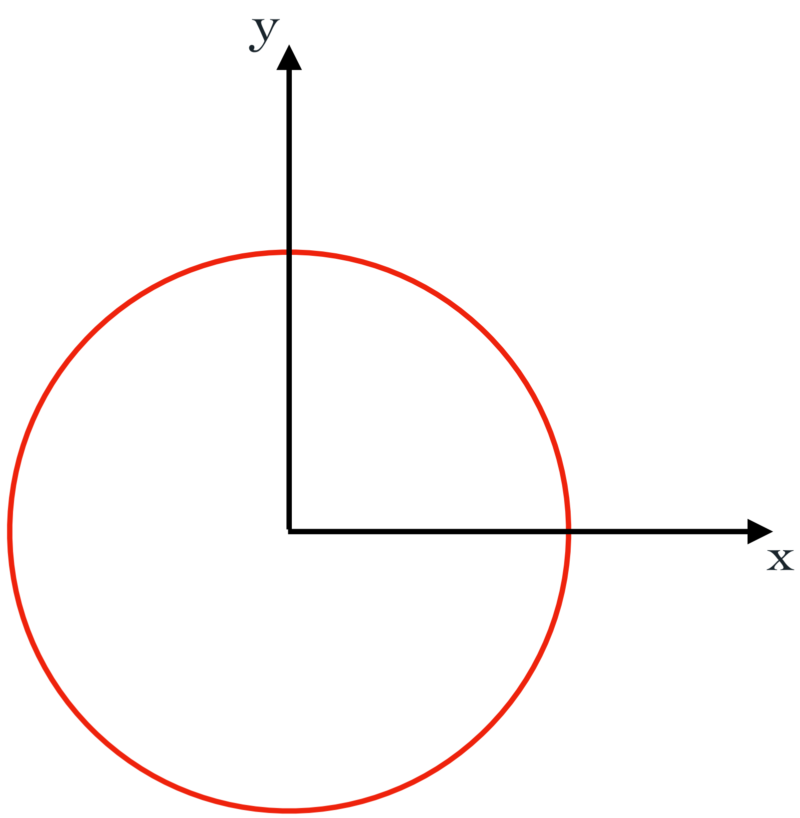
Local threshold counterterm

$$= \lim_{\delta \to 0^+} \frac{4}{\pi} \left[\int_0^\infty dr \left(\frac{1}{r^2 + 1} \right) \frac{1}{1 - r \pm \mathrm{i}\delta} - \int_{1-\Delta}^{1+\Delta} dr \left(\frac{1}{(1)^2 + 1} \right) \frac{1}{1 - r \pm \mathrm{i}\delta} \right]$$



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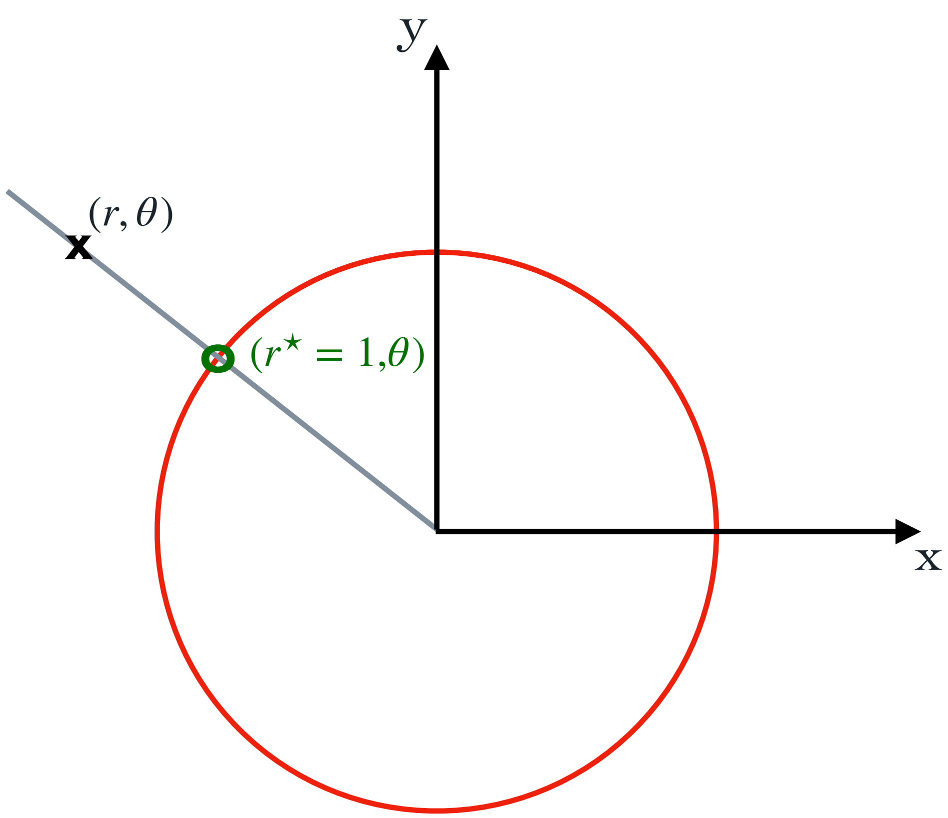

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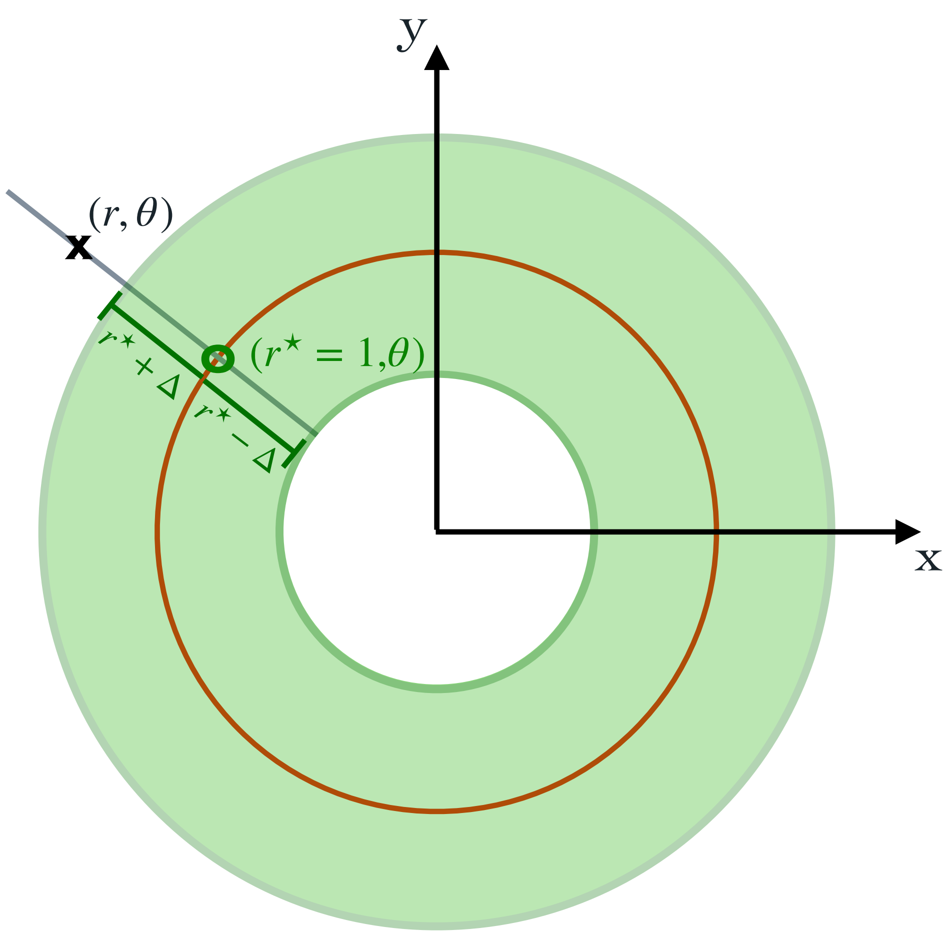

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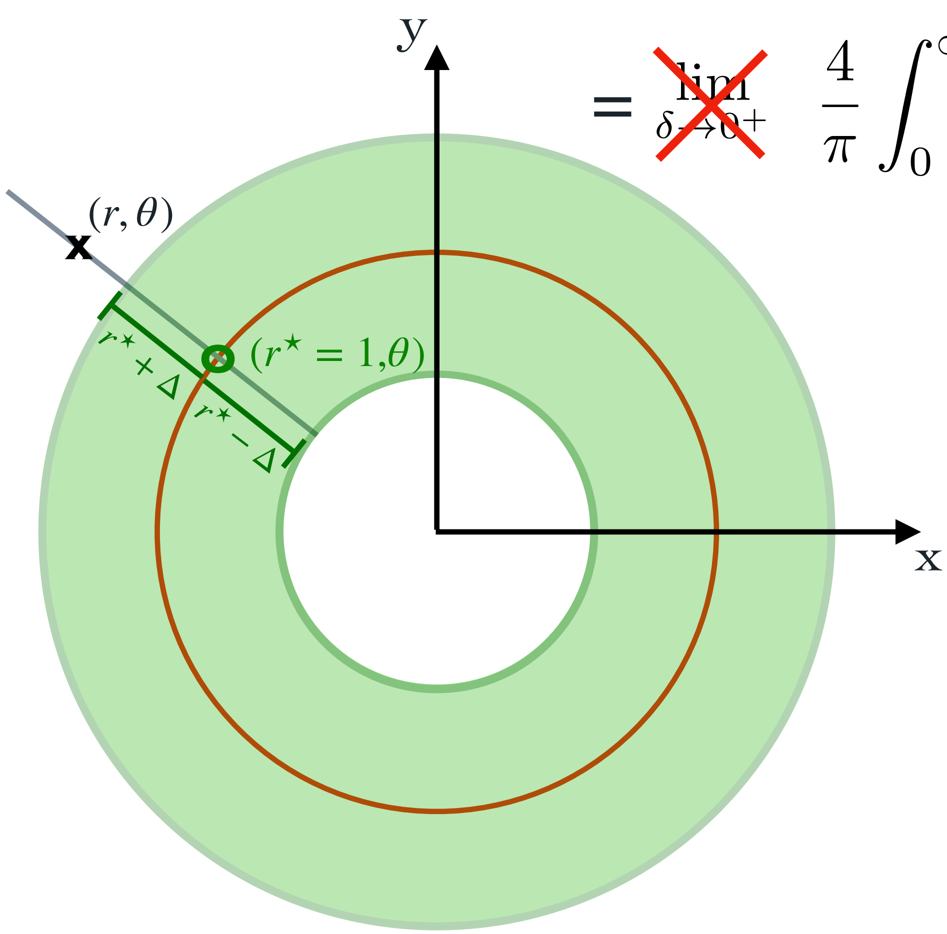
$\delta = 0$ can be taken safely !



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Local threshold counterterm

Integrated threshold counterterm

$$= \lim_{\delta\to0^+}\frac{4}{\pi}\left[\int_0^\infty dr\left(\frac{1}{r^2+1}\right)\frac{1}{1-r\pm i\delta} - \int_{\boxed{1-\Delta}}^{\boxed{1+\Delta}} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm i\delta}\right] + \frac{4}{\pi}\int_{1-\Delta}^{1+\Delta} dr\left(\frac{1}{(1)^2+1}\right)\frac{1}{1-r\pm i\delta}$$

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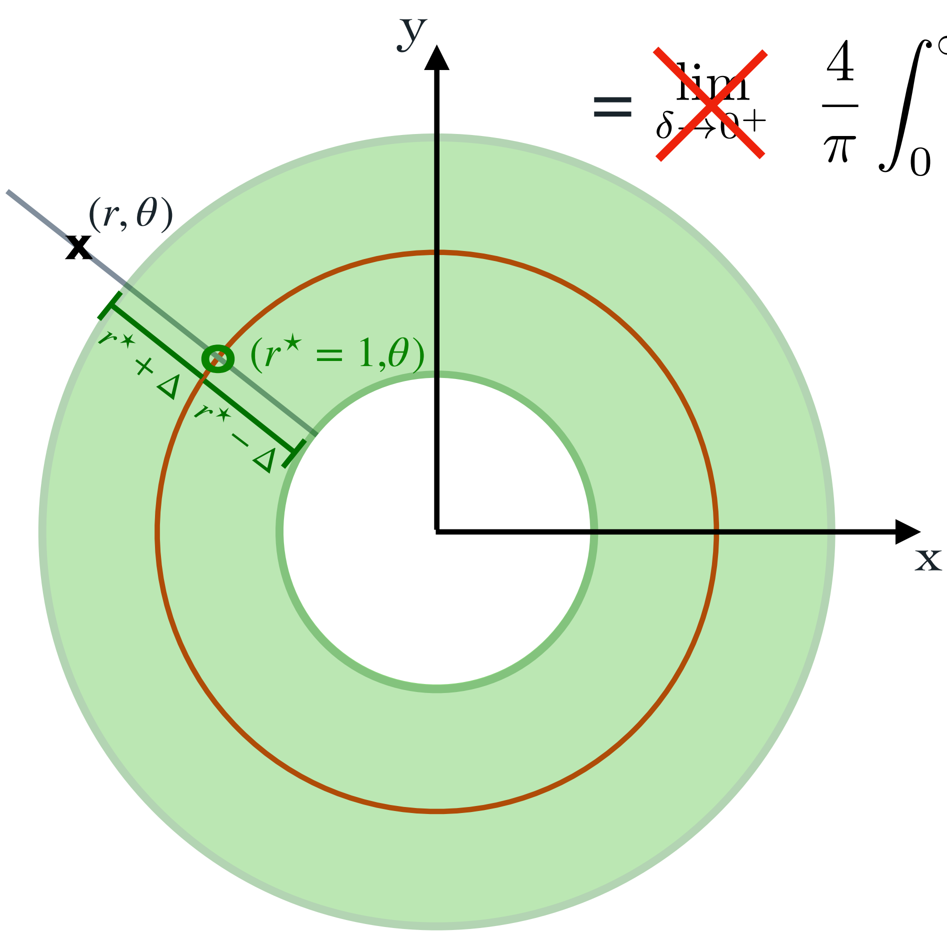

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Local threshold counterterm — Integrated threshold counterterm

$$= \lim_{\delta \to 0^+} \frac{4}{\pi} \left[\int_0^\infty dr \left(\frac{1}{r^2+1} \right) \frac{1}{1 - r \pm i\delta} - \int_{\boxed{1-\Delta}}^{\boxed{1+\Delta}} dr \left(\frac{1}{(1)^2 + 1} \right) \frac{1}{1 - r \pm i\delta} \right] + \frac{4}{\pi} \int_{1-\Delta}^{1+\Delta} dr \left(\frac{1}{(1)^2+1} \right) \frac{1}{1 - r \pm i\delta}$$

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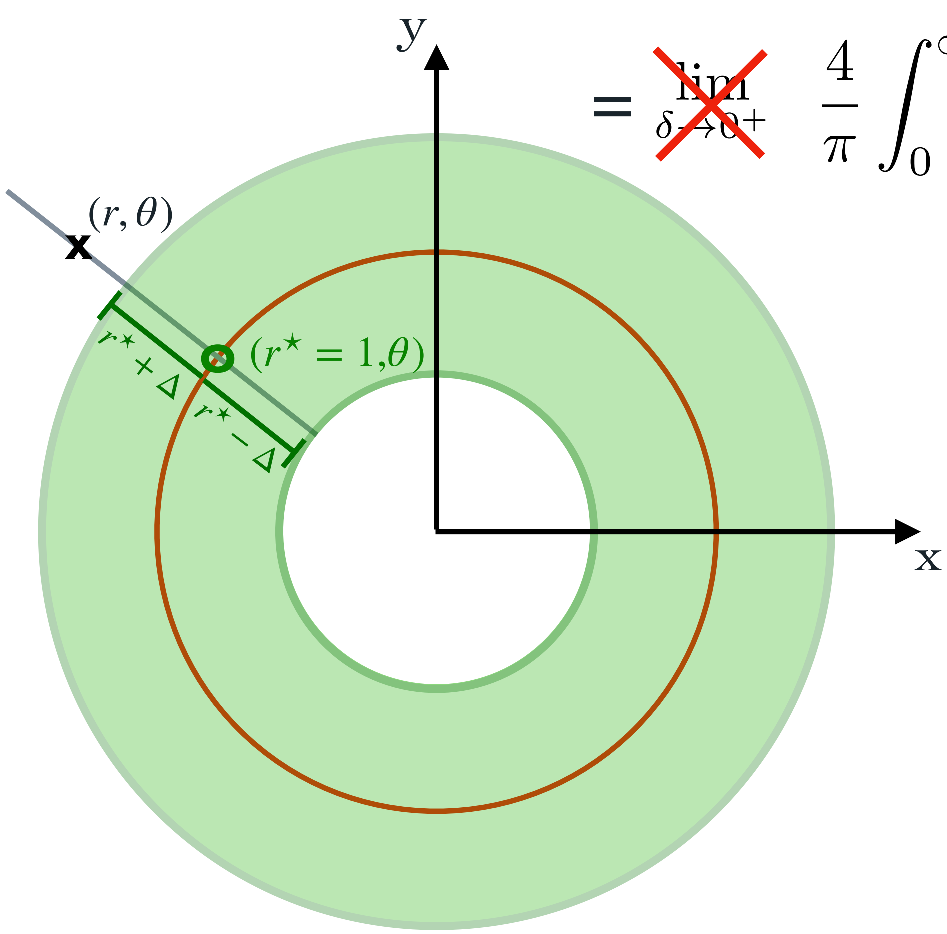
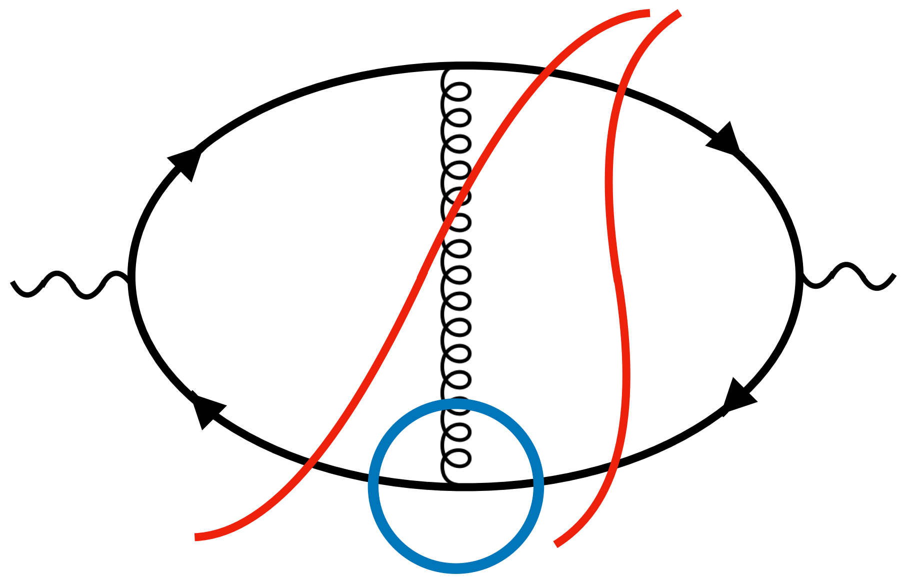
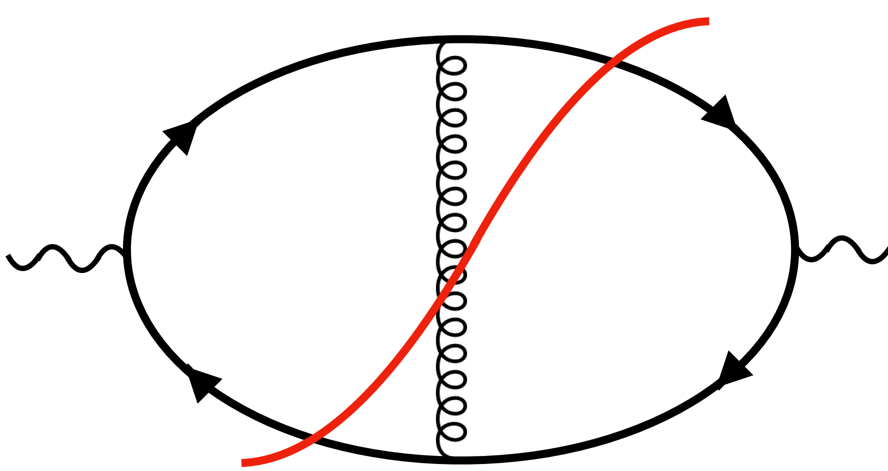


$$+ \lim_{\delta \to 0^+} \int_{1-\Delta}^{1+\Delta} dr \frac{4}{\pi} \frac{1}{2} \frac{1}{r - 1 \pm i\delta} \overset{\Delta \leq 1}{=} \underbrace{\frac{2}{\pi} \mathrm{PV}\left[\frac{1}{r - 1 \pm i\delta}\right]}_{0} \boxed{\mp 2\,\mathrm{i}}$$

Easy to compute since Principal Value is zero by construction !

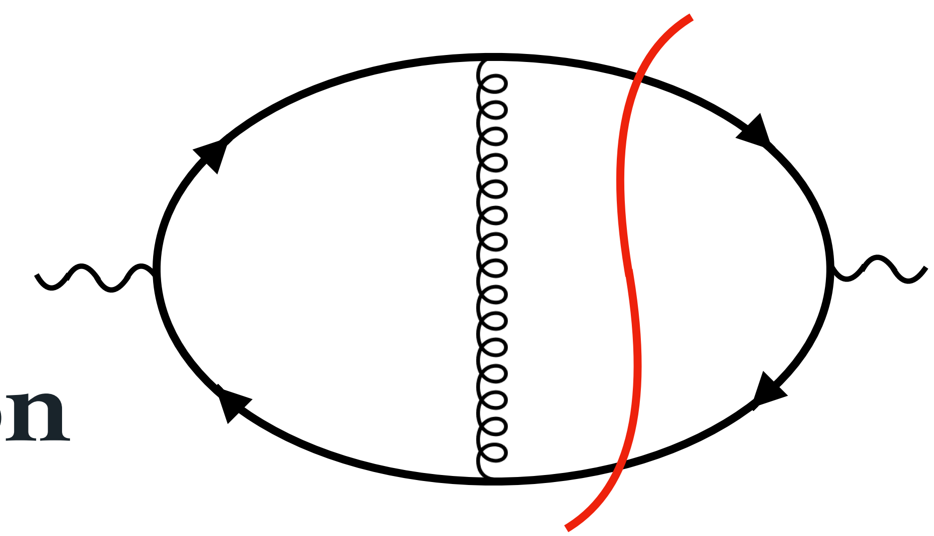
Local Unitarity





local

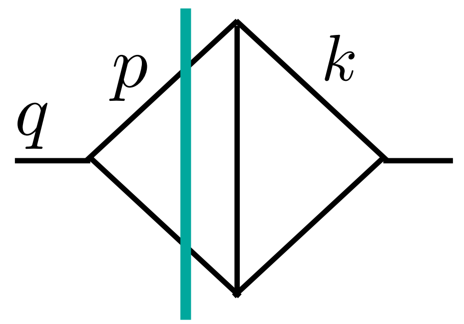
cancellation



Locality Unitarity

We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

$$\frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\mathrm{d}^4k\,\delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0)$$



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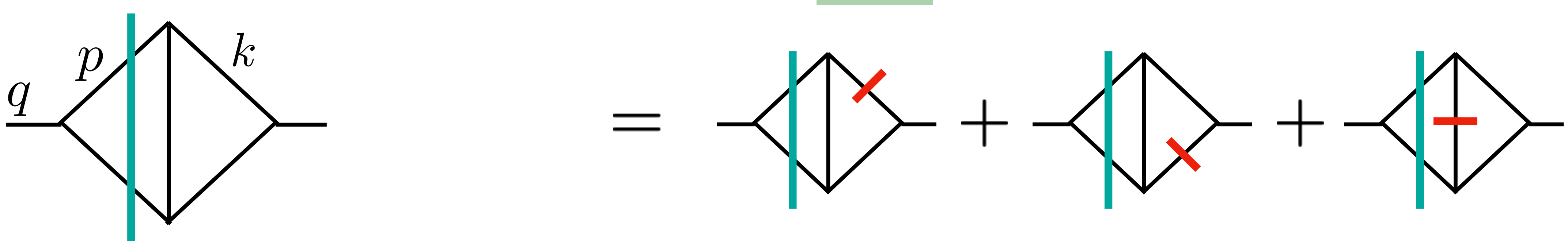

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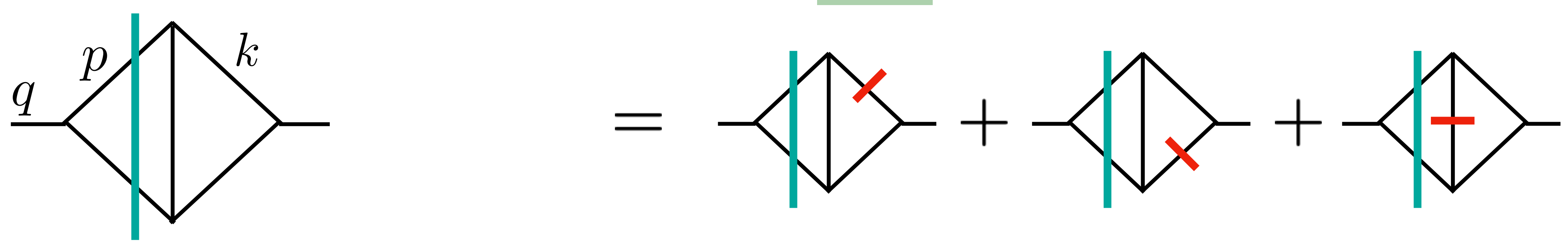
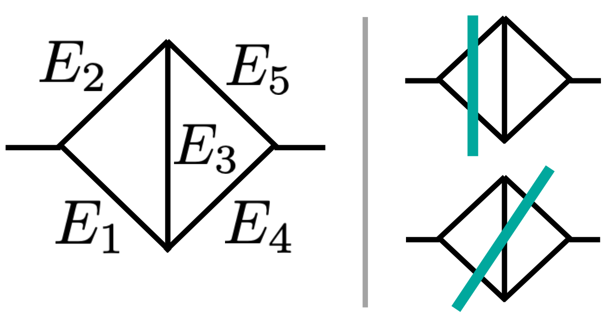


But the measure is **not yet fully aligned**:



$$= \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\,\left(\delta(E_1+E_2-Q_0)f_{\mathrm{virt}}+\delta(E_1+E_3+E_5-Q_0)f_{\mathrm{real}}\right)$$

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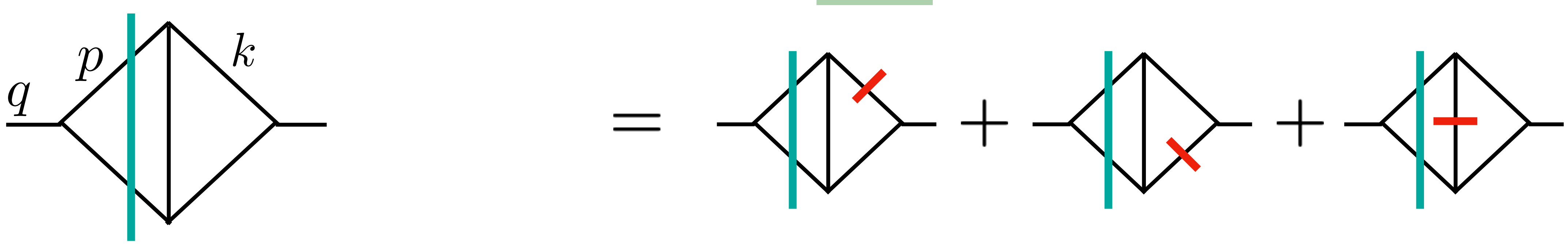

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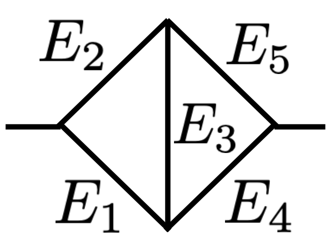
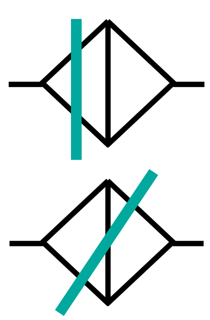




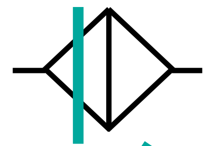
$$= \int \boxed{\mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}}\,\big(\delta(\underbrace{E_1+E_2-Q_0}_{\eta_{\mathrm{v}}(\vec{k},\vec{p})})f_{\mathrm{virt}}+\delta(\underbrace{E_1+E_3+E_5-Q_0}_{\eta_{\mathrm{r}}(\vec{k},\vec{p})})f_{\mathrm{real}}\big)$$

(on-shell energies: $E_i(\vec{k}_i)=\sqrt{\vec{k}_i^2+m_i^2-i\delta}$)

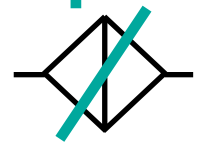
Causal flow

Causal flow

The measure now differs only in the **delta enforcing on shell energy conservation**



$$\sim \delta(E_1 + E_2 - Q_0)$$



$$\sim \delta(E_1 + E_3 + E_5 - Q_0)$$

Objective: find a common variable to solve both deltas.

CAUSAL FLOW

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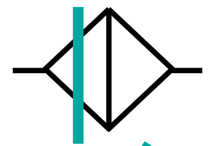 $$\sim \delta(E_1 + E_2 - Q_0)$$

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A different perspective on the usual phase space mapping problem

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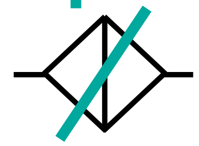 $\sim \delta(E_1 + E_3 + E_5 - Q_0)$

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Solution: introduce an auxiliary variable in which to solve the delta

$$\delta(|\vec{k}| - Q_0) \xrightarrow[\vec{k} \to t\vec{k}]{} \delta(t|\vec{k}| - Q_0) \longrightarrow t = \frac{Q_0}{|\vec{k}|}$$

Soper,
arXiv: 9804454 (1998)

Soper,
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ZC, Hirschi, Pelloni, Ruijl
arXiv: 2010.01068 (2020)

General FSR cancellations
For N to M N^{k}LO processes

Causal flow

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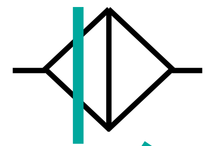

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General FSR cancellations
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A toy example:

$$\int \mathrm{d}^3\vec{k}\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$

Causal flow : toy integral

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$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k}) \qquad \text{using} \qquad 1 = \int \mathrm{d}t\, h(t)$$

Causal flow : toy integral

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$ using $$1 = \int \mathrm{d}t\, h(t)$$

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, t^3\, h(t)\, \delta(t|\vec{k}| - Q_0) f(t\vec{k})$$ using $$\vec{k} \to t\vec{k}$$

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$$= \int \mathrm{d}^3\vec{k}\, \frac{Q_0^3}{|\vec{k}|^4}\, h(Q_0/|\vec{k}|)\, f(Q_0\vec{k}/|\vec{k}|)$$ with $$t^\star = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

Causal flow : toy integral

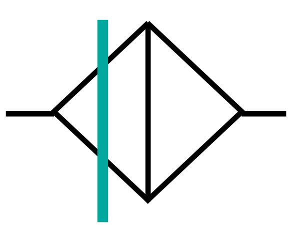
$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0)\, f(\vec{k})$$ using $$1 = \int \mathrm{d}t\, h(t)$$

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, t^3\, h(t)\, \delta(t|\vec{k}| - Q_0)\, f(t\vec{k})$$ using $$\vec{k} \to t\vec{k}$$

$$= \int \mathrm{d}^3\vec{k}\, \frac{Q_0^3}{|\vec{k}|^4}\, h(Q_0/|\vec{k}|)\, f(Q_0\vec{k}/|\vec{k}|)$$ with $$t^\star = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

When applying this construction to LU we get:



$$= \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\delta(E_1 + E_2 - Q_0) f_{\mathrm{virt}} = \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\, g_{\mathrm{v}}(t^\star_{\mathrm{v}})$$

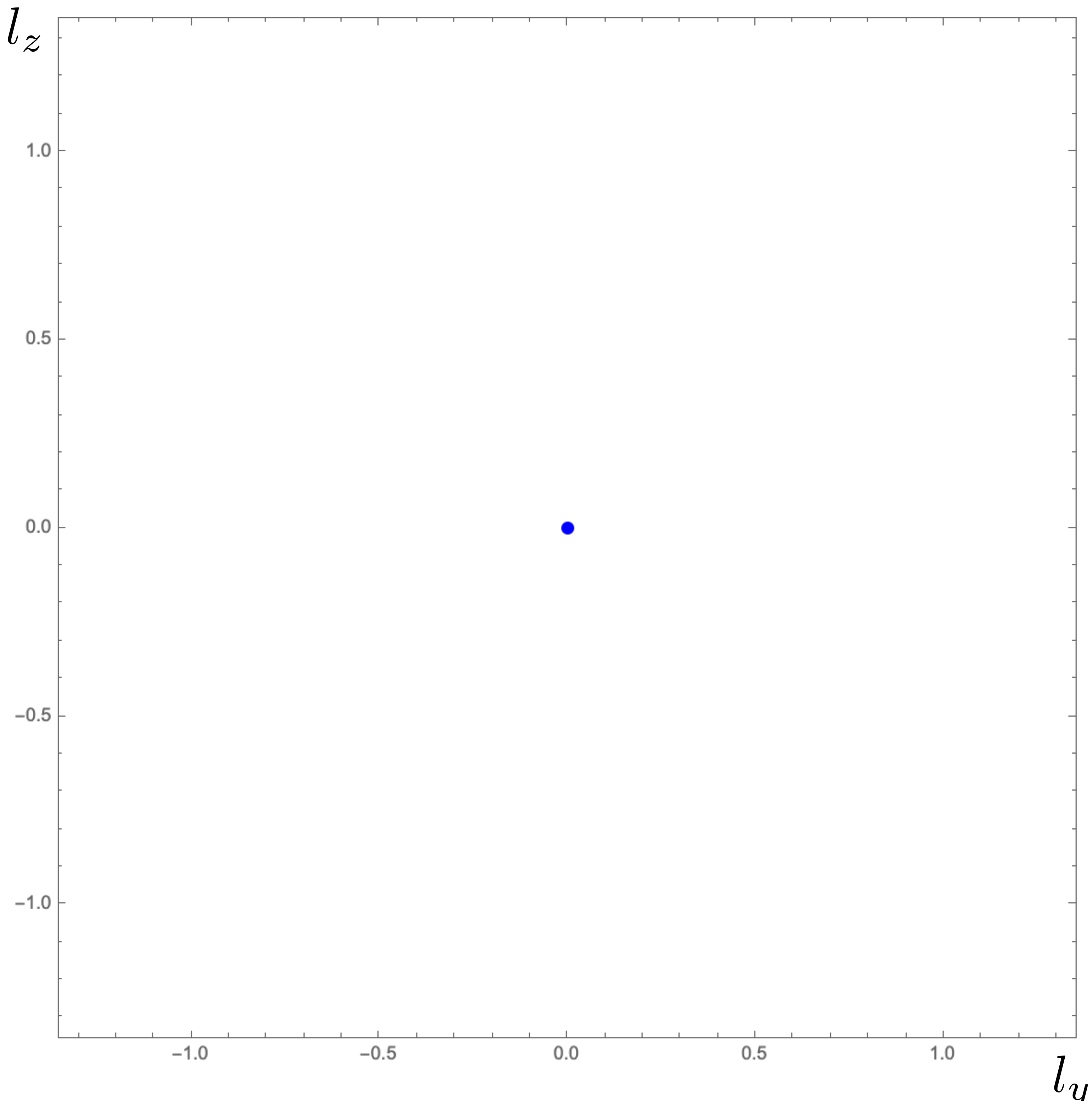
where $t^\star_{\mathrm{v}} = t^\star_{\mathrm{v}}(\vec{k},\vec{p}) = \dfrac{Q_0}{E_1 + E_2}$

$$(\vec{p},\vec{k}) \to \vec{\phi}(t,(\vec{p},\vec{k}))$$

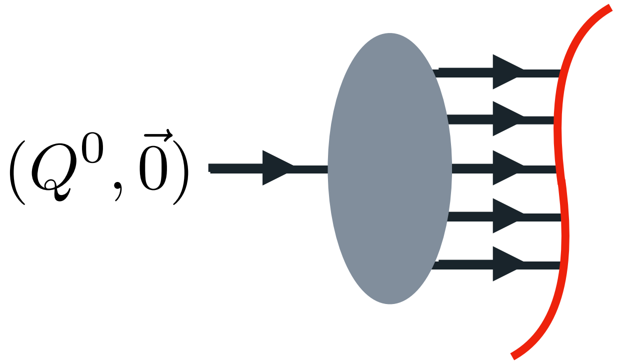
"Causal flow" is called like this because it is the generalisation of the Soper, derive from a contour deformation field satisfying the causal constraints.

$$\begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0,(\vec{k},\vec{l})) = (\vec{k},\vec{l}) \end{cases}$$

Locality Unitarity: visualisation

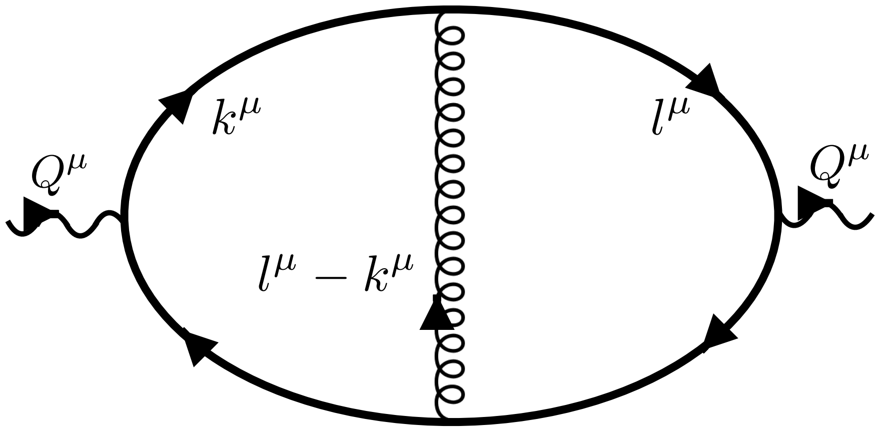


$\left(\vec{k}, \vec{l} \right) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$



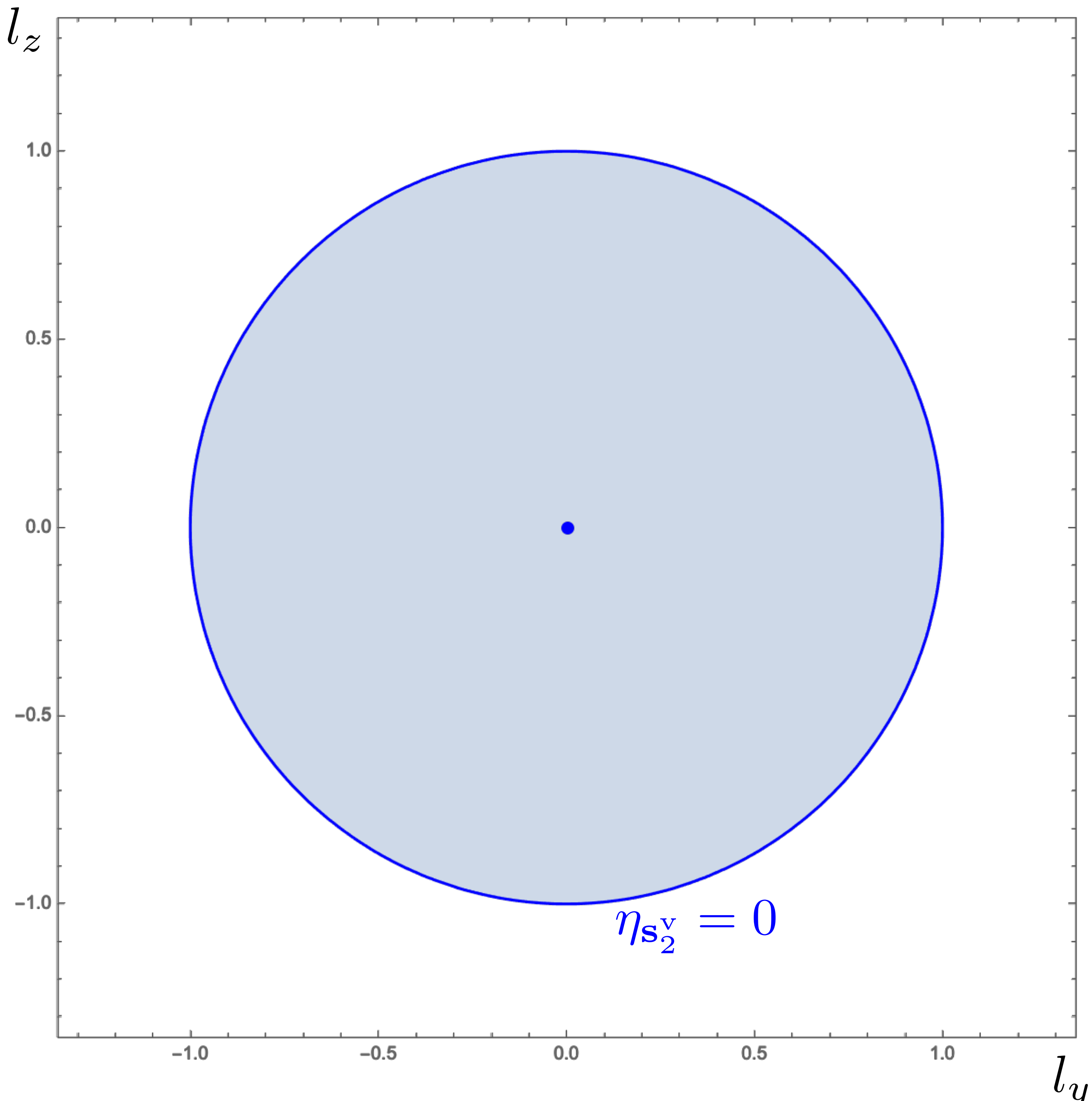
$$Q^\mu = (2, 0, 0, 0)$$

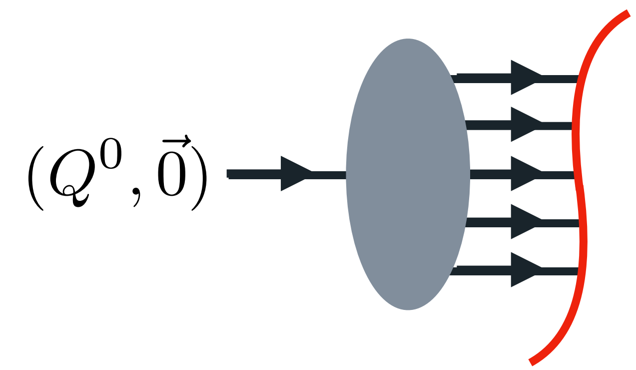
$$(\vec{k}, \vec{l}) = \left(\ (0, 0.5, 0.5), \ (0, l_y, l_z) \ \right)$$



— = Cutkosky cut ≡ threshold

Locality Unitarity: visualisation





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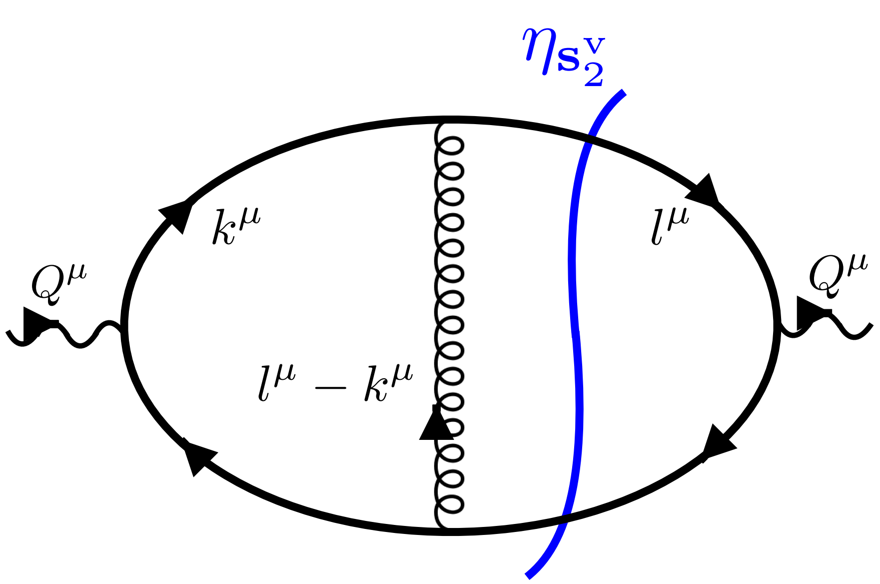

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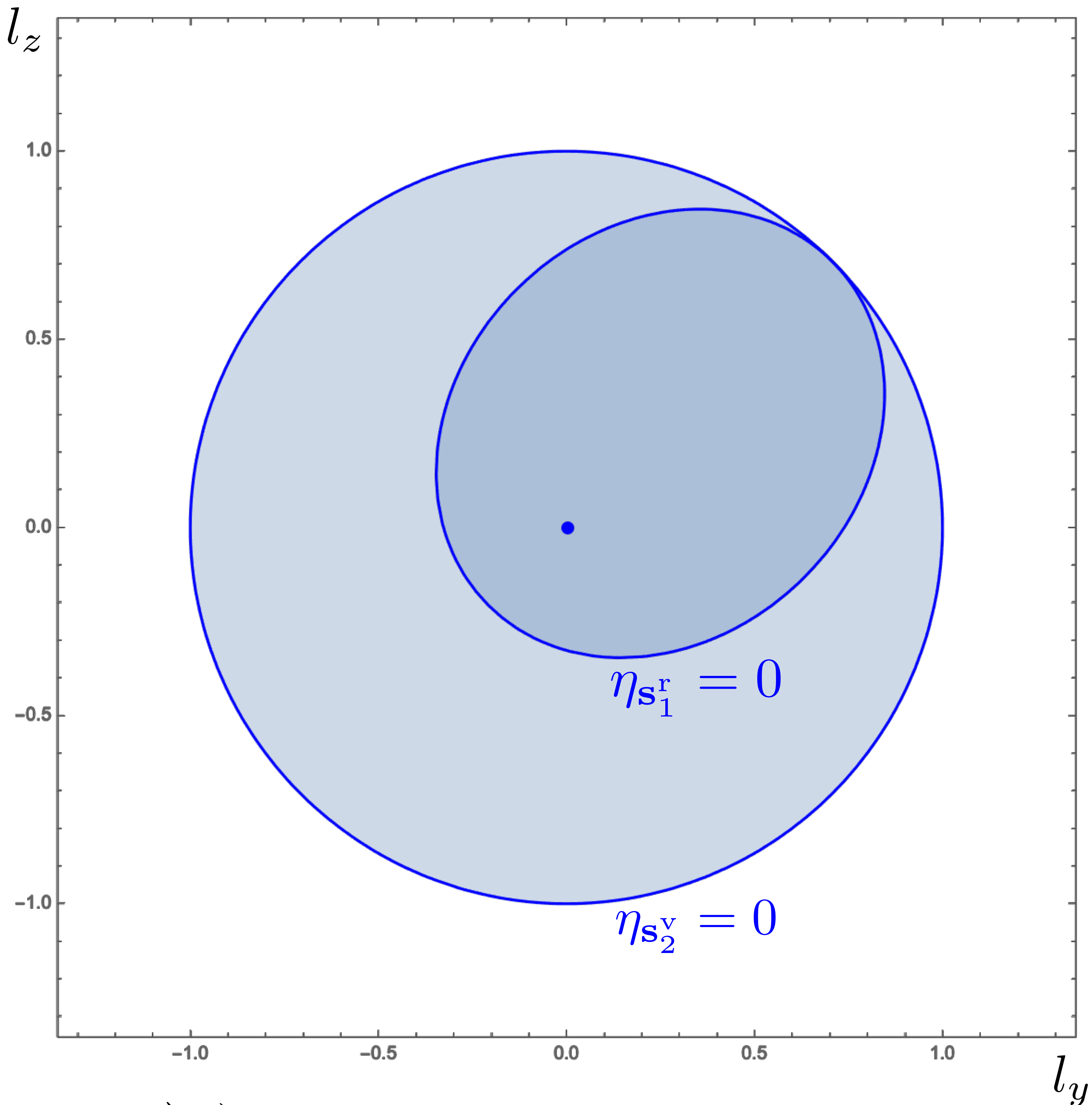

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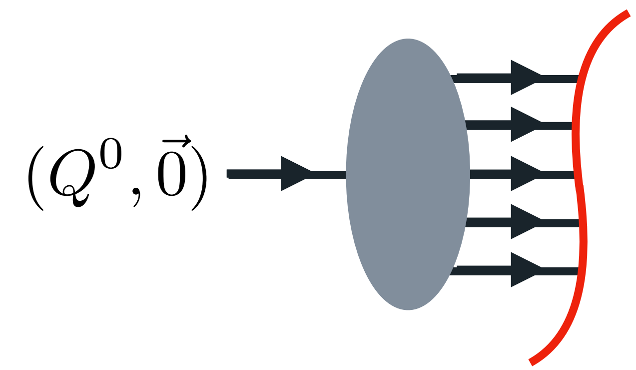

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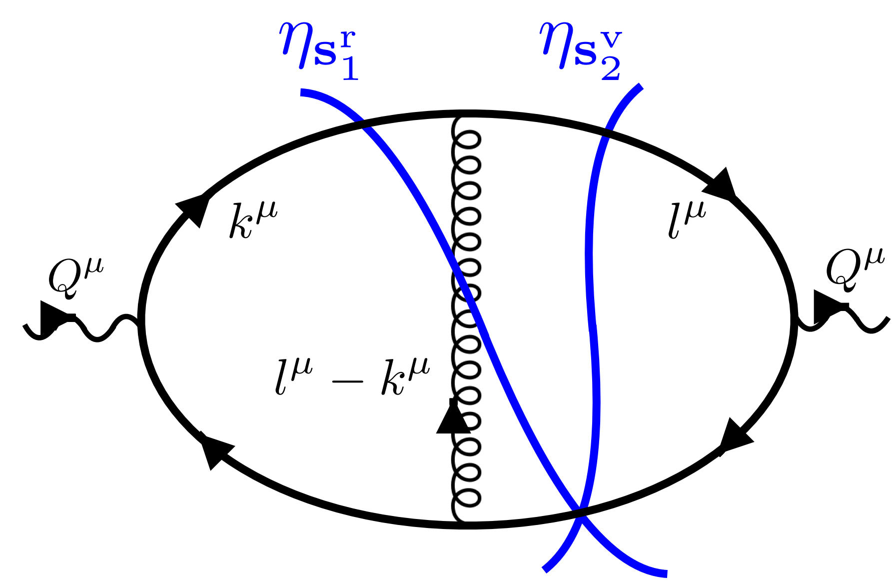

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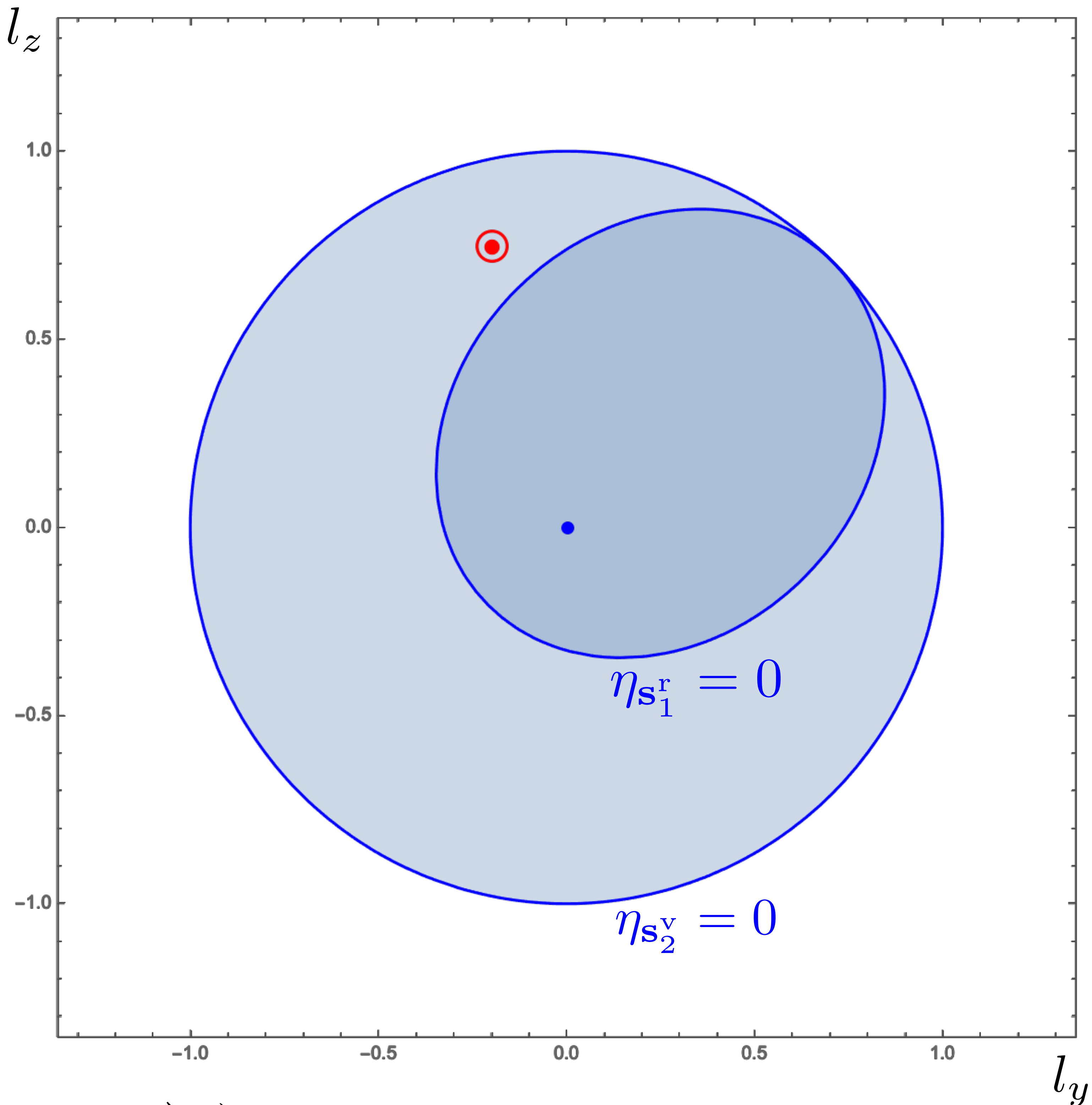
Locality Unitarity: visualisation



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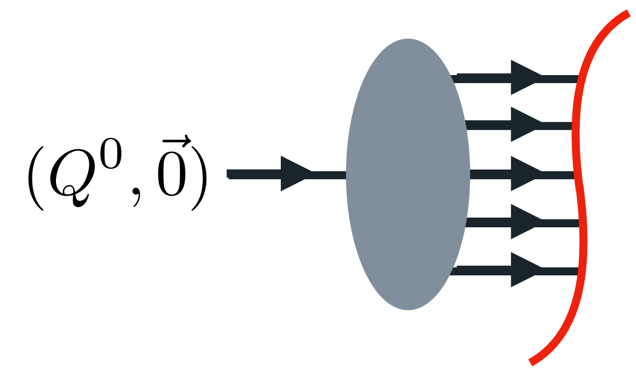

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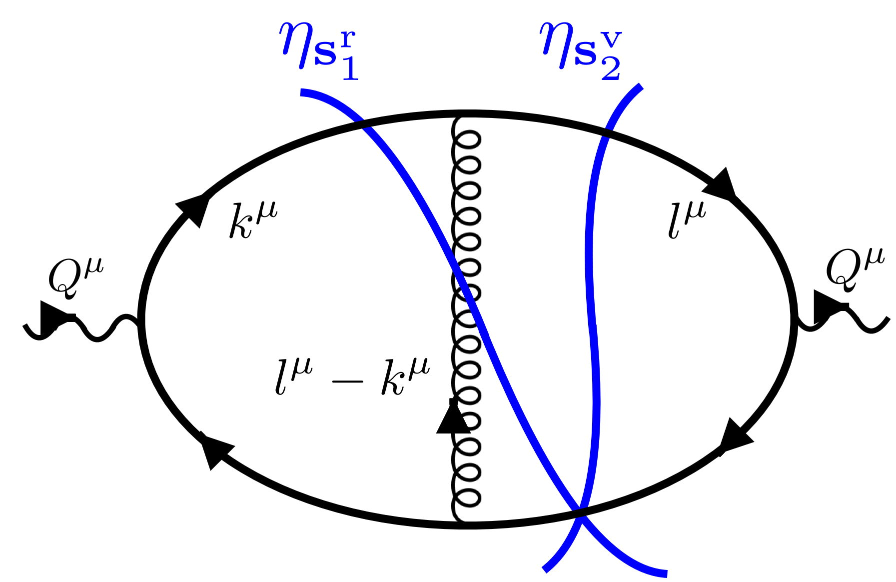

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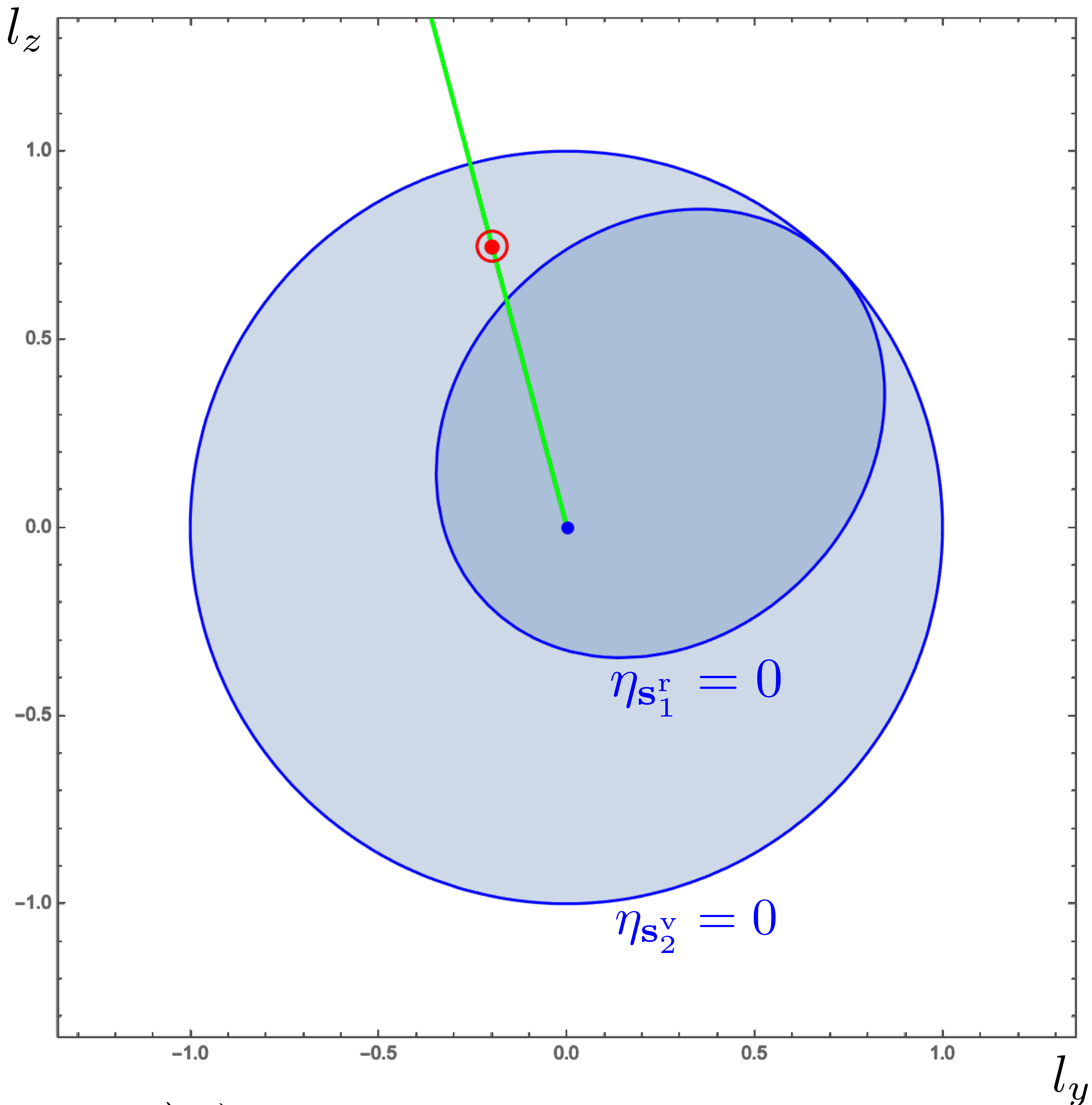

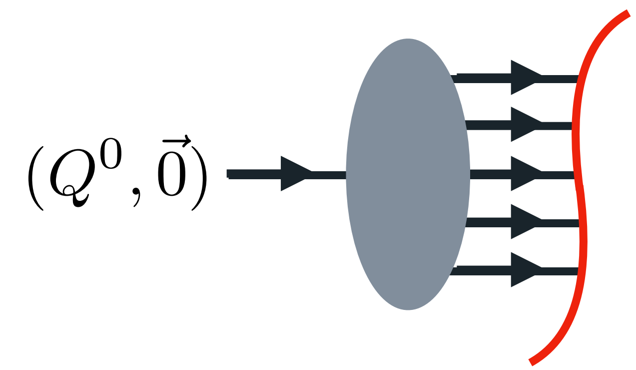

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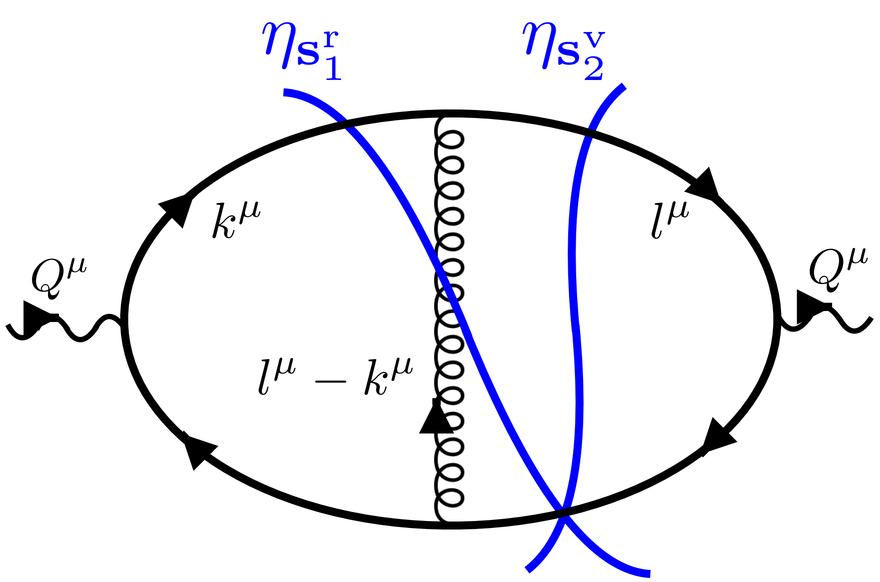

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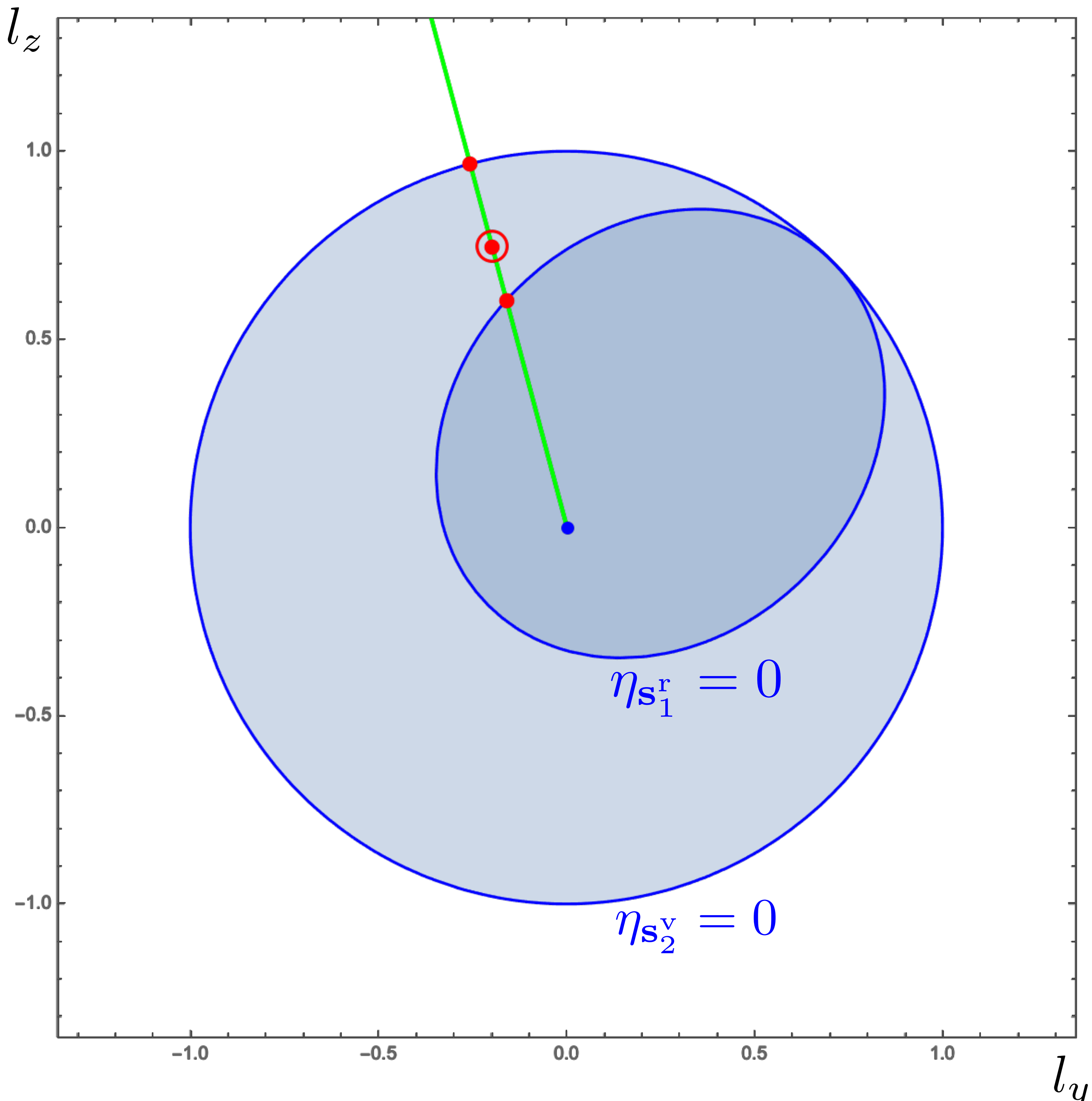

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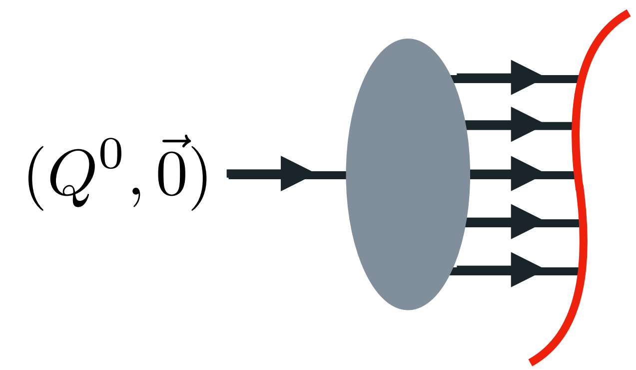

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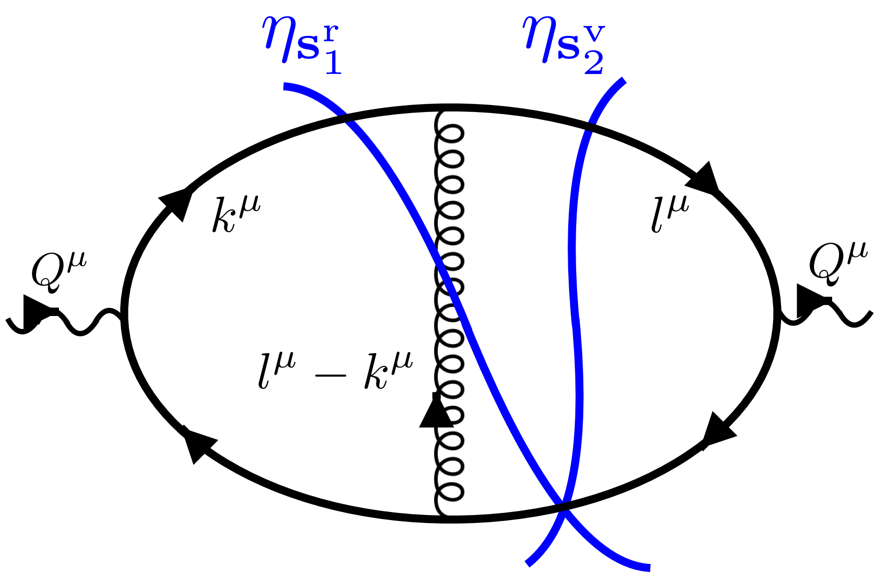

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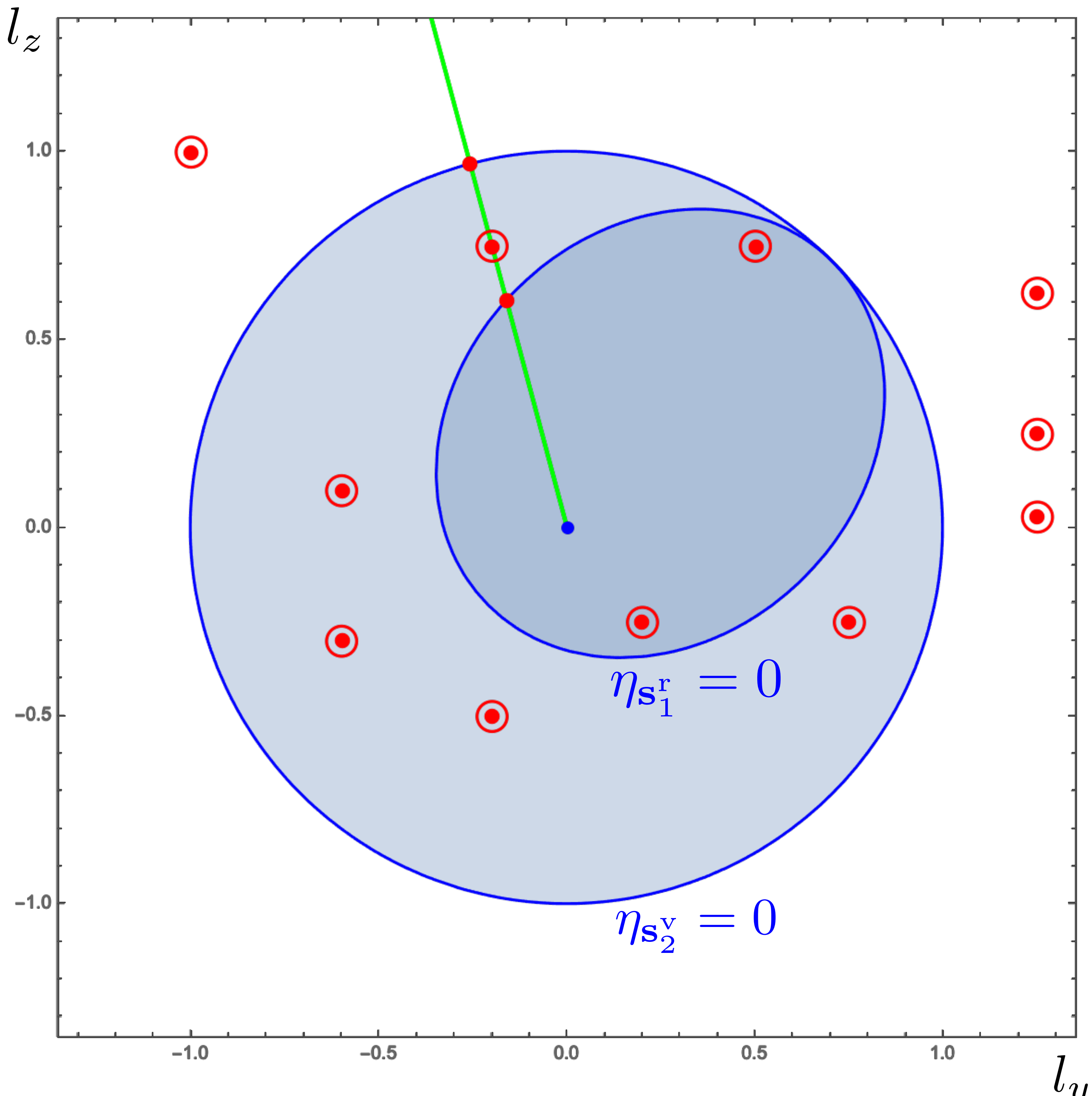
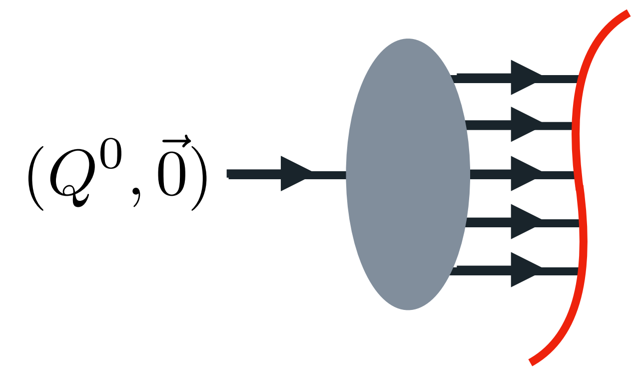
Locality Unitarity: visualisation





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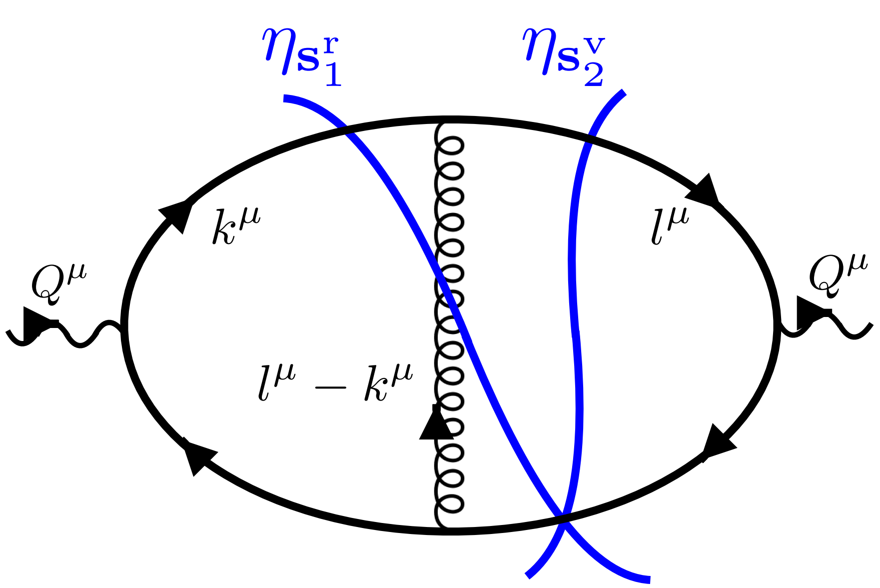

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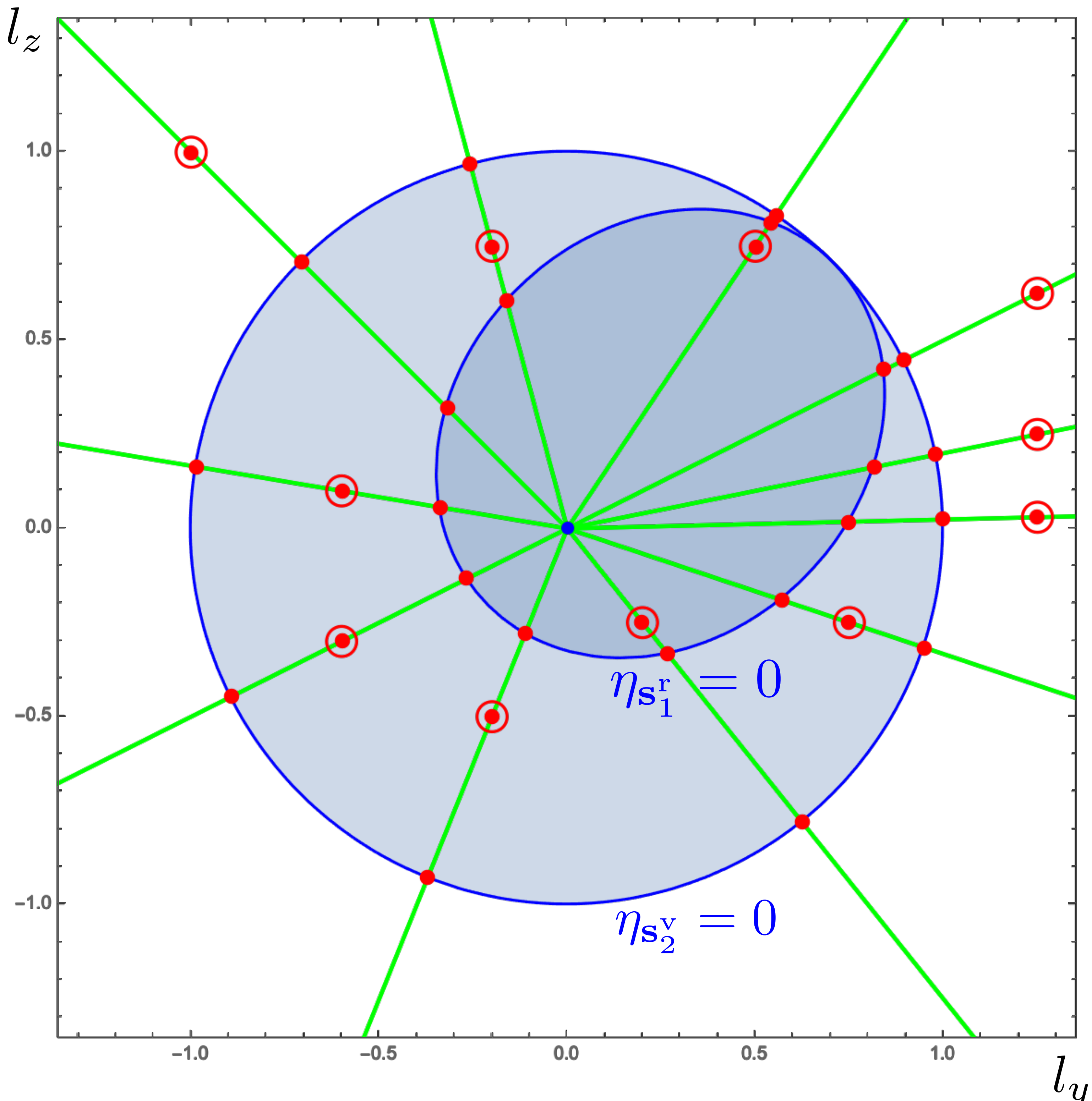

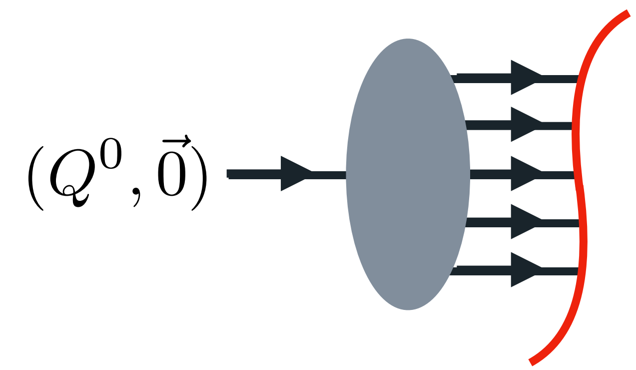

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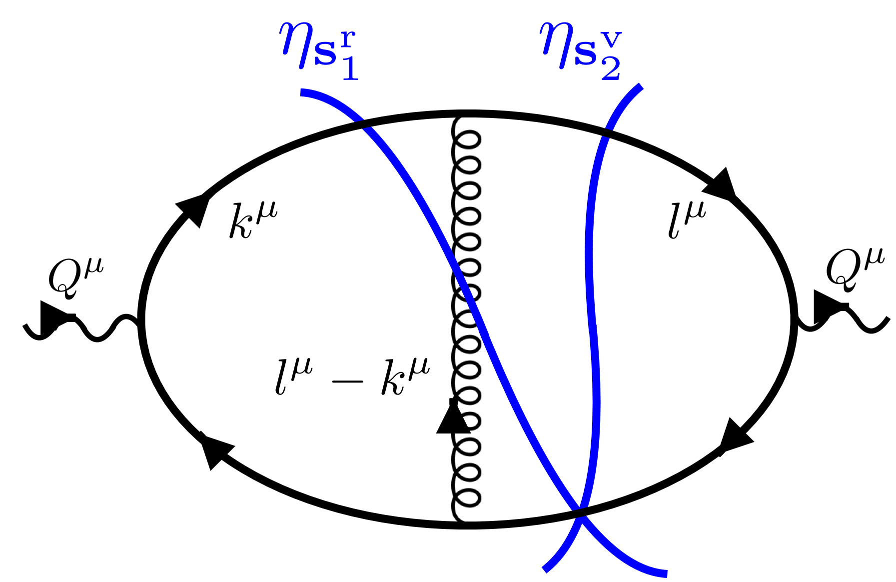

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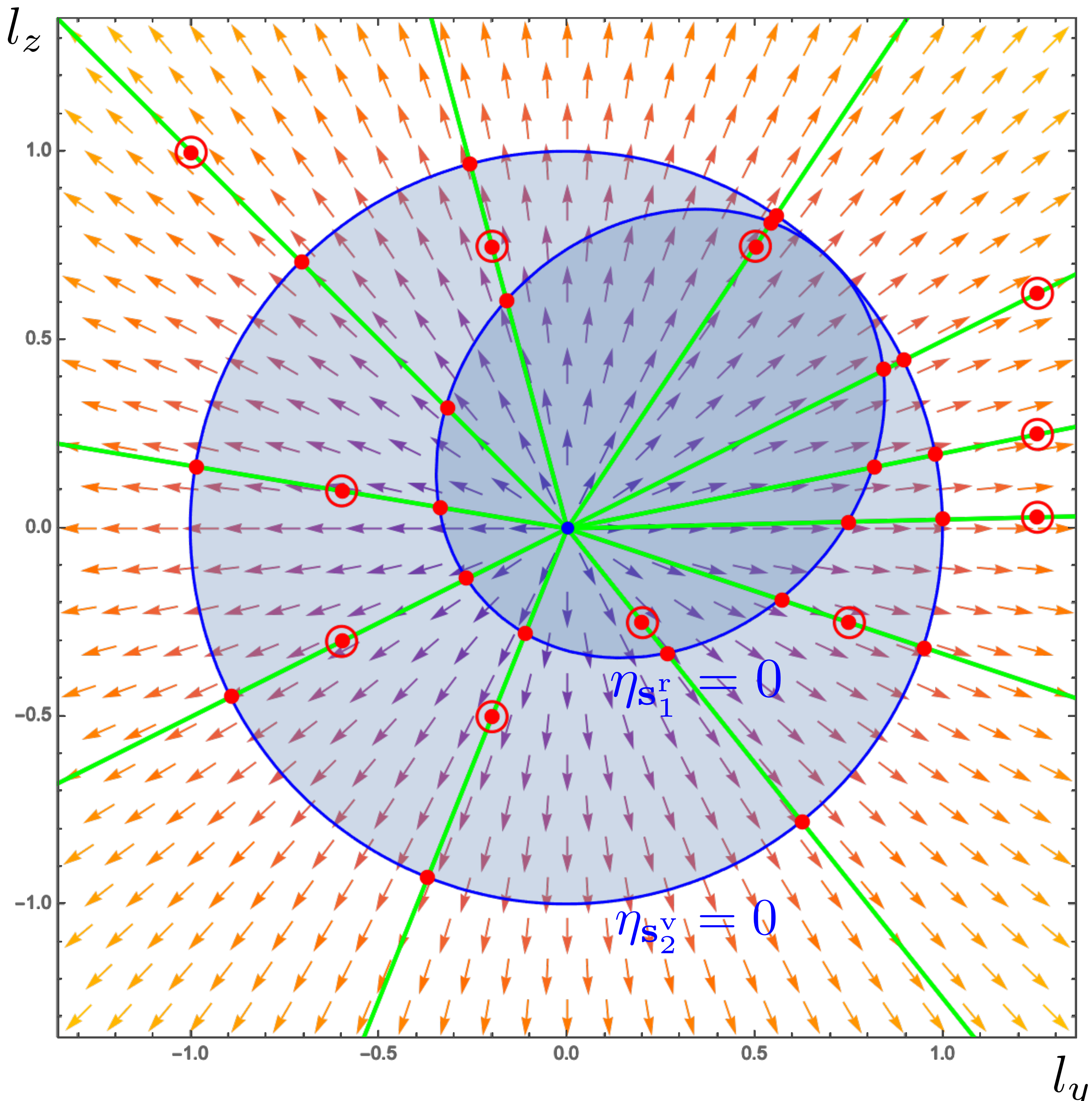

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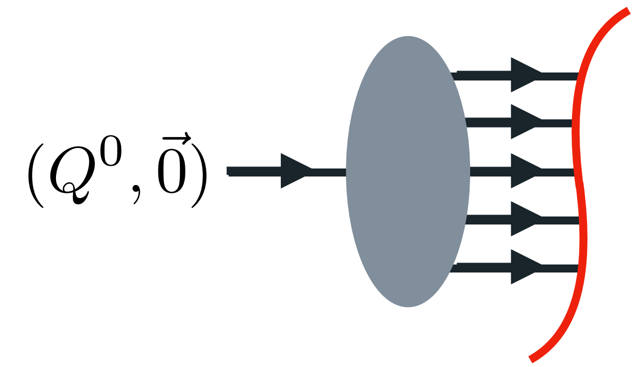

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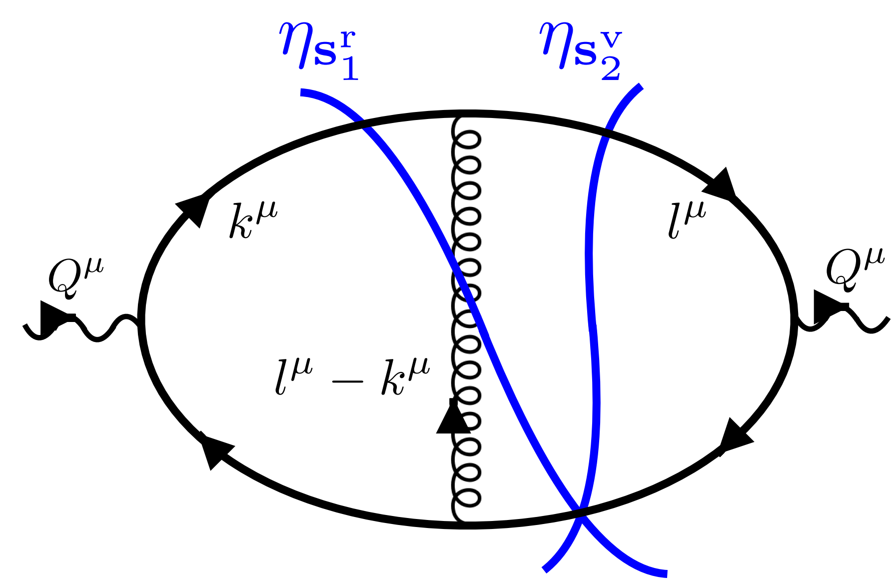

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Locality Unitarity: all-orders proof

[Z. Capatti, VH, A. Pelloni, B. Ruijl, arXiv : 2010.01068]   [Summary in proceedings, arXiv : 2110.15662]

Locality Unitarity: all-orders proof

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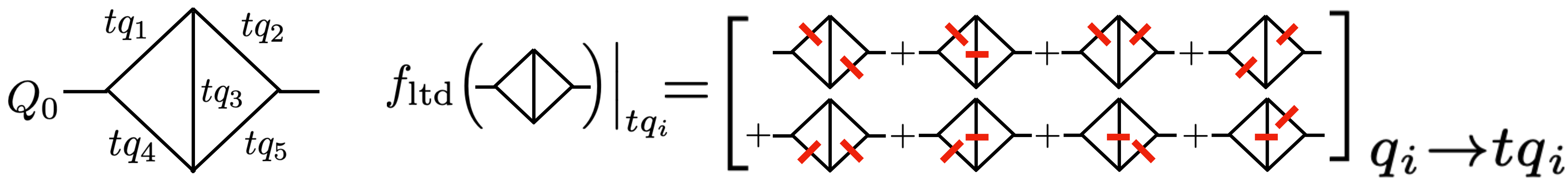
The **LTD representation** of the double triangle with rescaled momenta is



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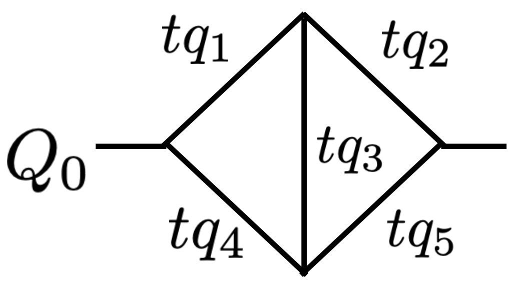
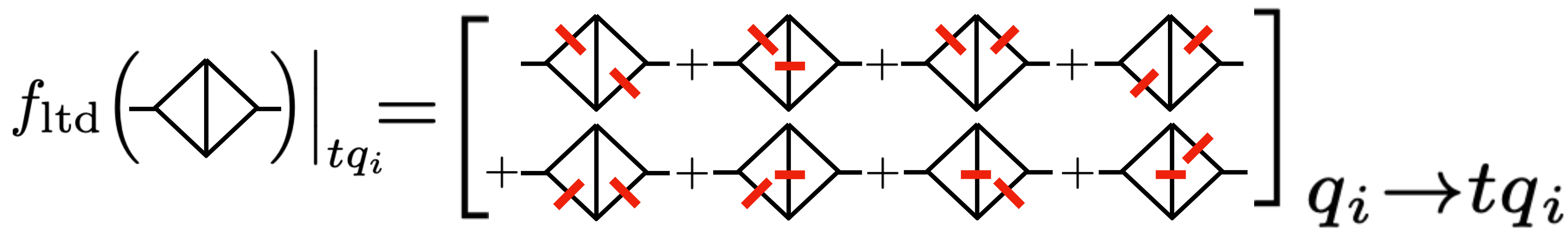
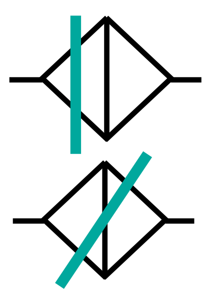
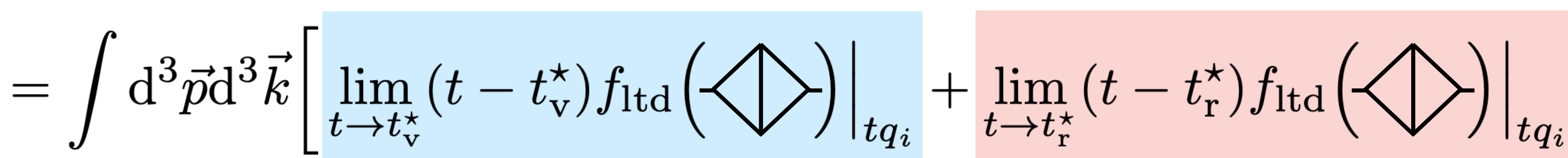




Then one can capture the **thresholds** of this forward-scattering graphs with





g_{v}, g_{r} can be written as different limits of the same function!

Locality Unitarity: all-orders proof

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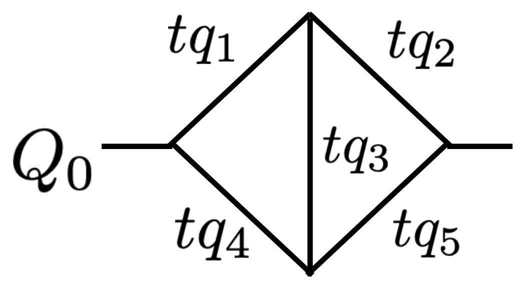
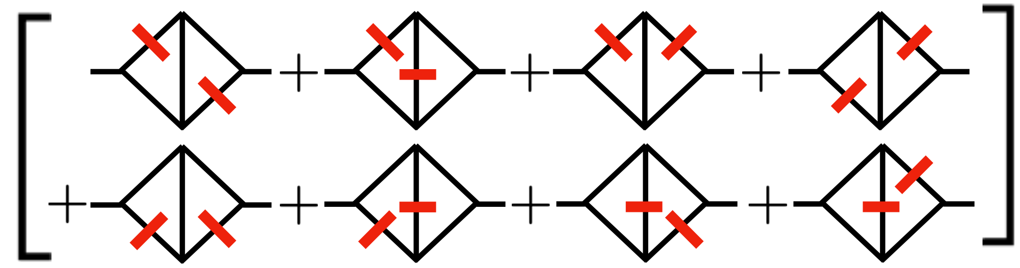


$$f_{\rm ltd}\left(\quad \right)\Big|_{tq_i} =$$



$$q_i \to t q_i$$

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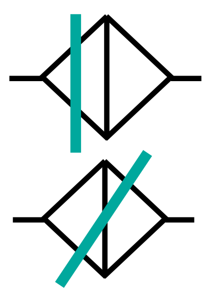
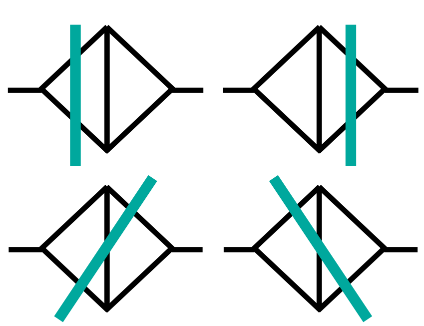


$$= \int \mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{k} \left[\lim_{t\to t^\star_{\rm v}} (t - t^\star_{\rm v}) f_{\rm ltd}\left(\quad \right)\Big|_{tq_i} + \lim_{t\to t^\star_{\rm r}} (t - t^\star_{\rm r}) f_{\rm ltd}\left(\quad \right)\Big|_{tq_i} \right]$$

$g_{\rm v}$, $g_{\rm r}$ can be written as different limits of the same function!

Solving delta in the scaling variable $\Longrightarrow$ **1d residue theorem along the line** $\gamma(t) = (t\vec{k}, t\vec{p})$



$$= \int \mathrm{d}^3\vec{p}\;\mathrm{d}^3\vec{k} \underbrace{\left[\sum_{i=1}^{4} \lim_{t\to t^\star_i} (t - t^\star_i) f_{\rm ltd}\left(\quad \right)\Big|_{tq_i} \right]}_{}$$

$= \sigma_{\rm d}$

LU representation

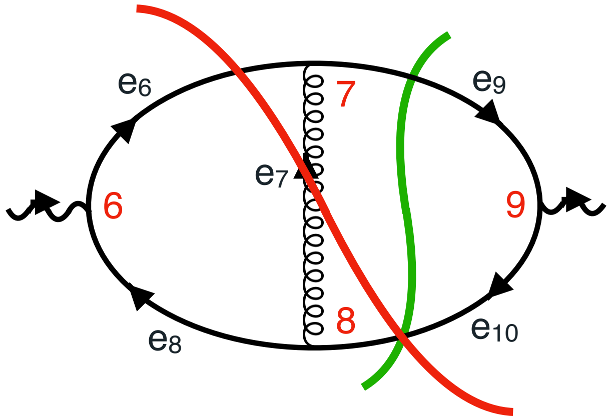
Cutkosky, but at the local level! We prove cancellations by studying the limit $t^\star_r \to t^\star_{\rm v}$

Locality Unitarity

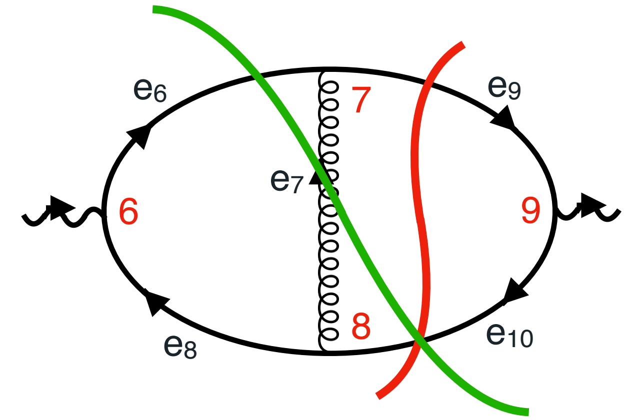
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

= Cutkosky cut   = threshold singularity



cancels



Locality Unitarity

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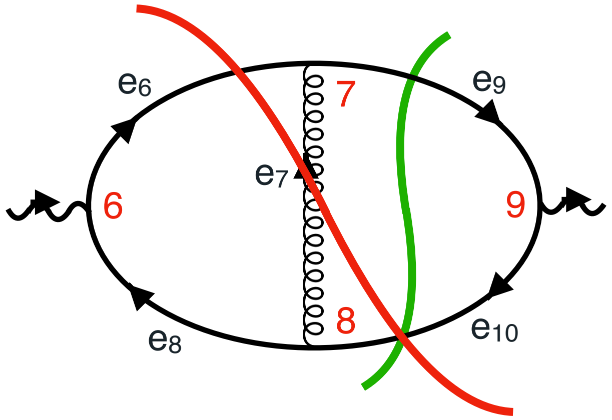

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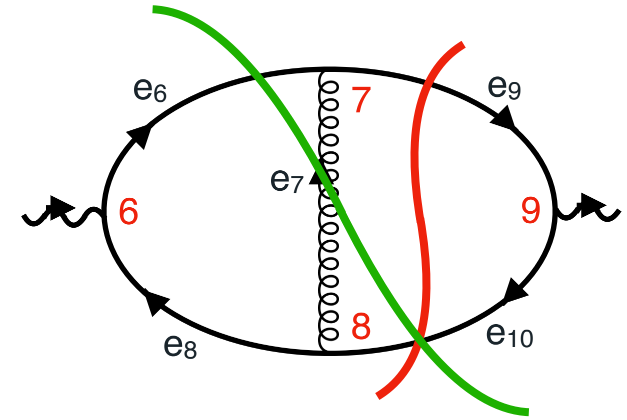

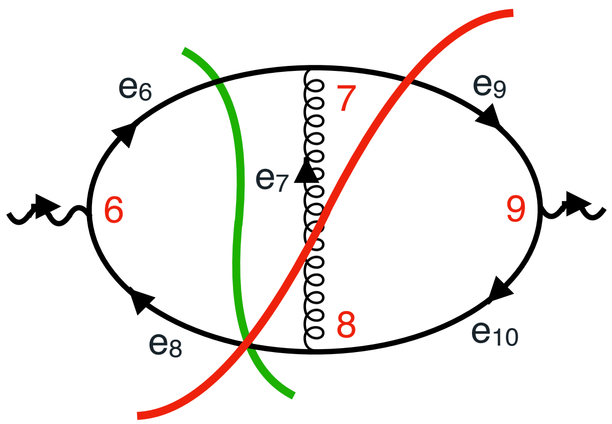

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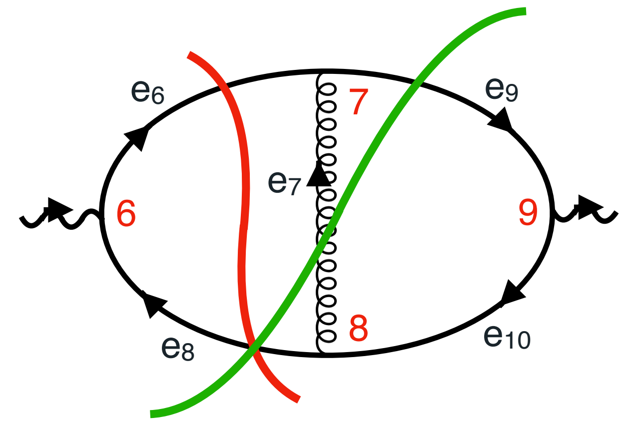
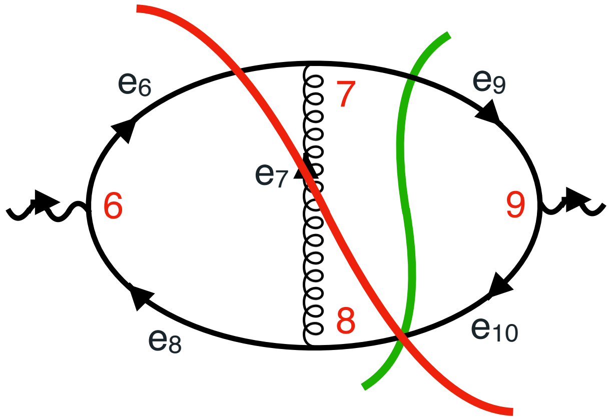


LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

(red line) = Cutkosky cut (green line) = threshold singularity



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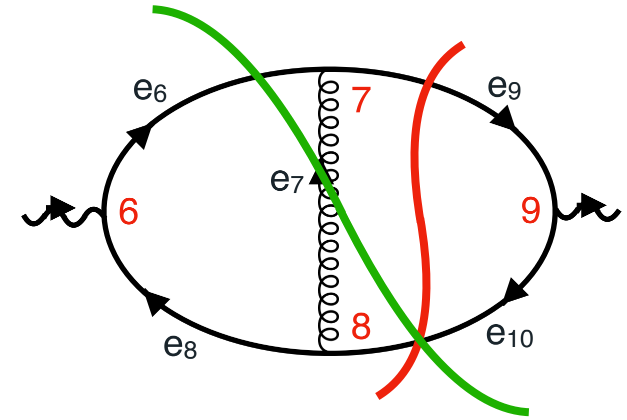

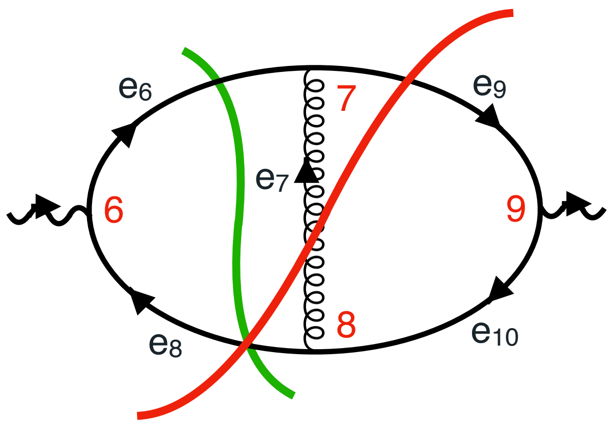

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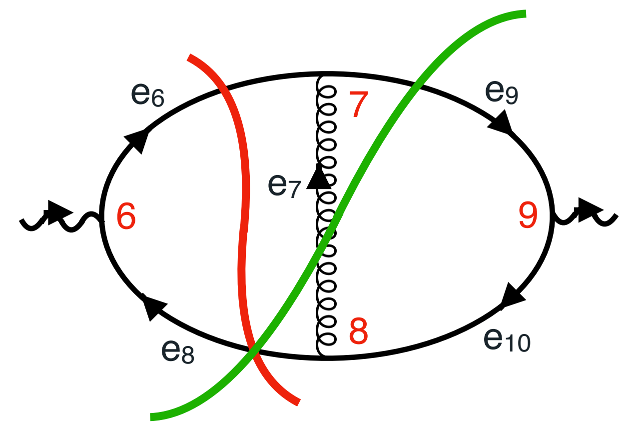
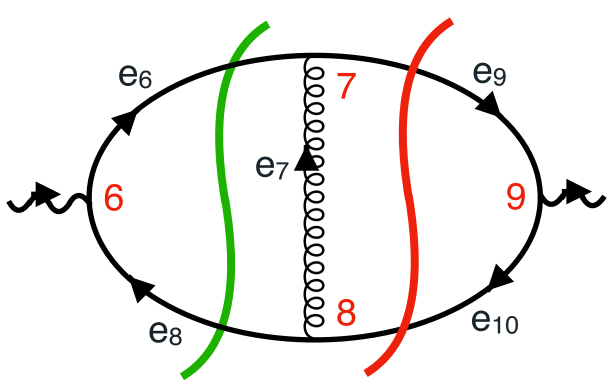
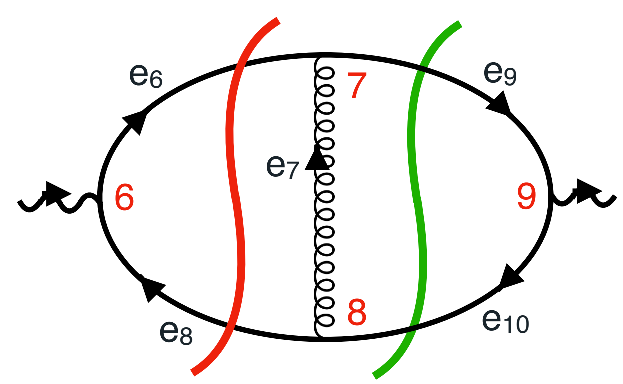


Even for **non-pinched singular threshold** ! (when $\mathcal{O}_{\mathbf{s}} \equiv 1$) :



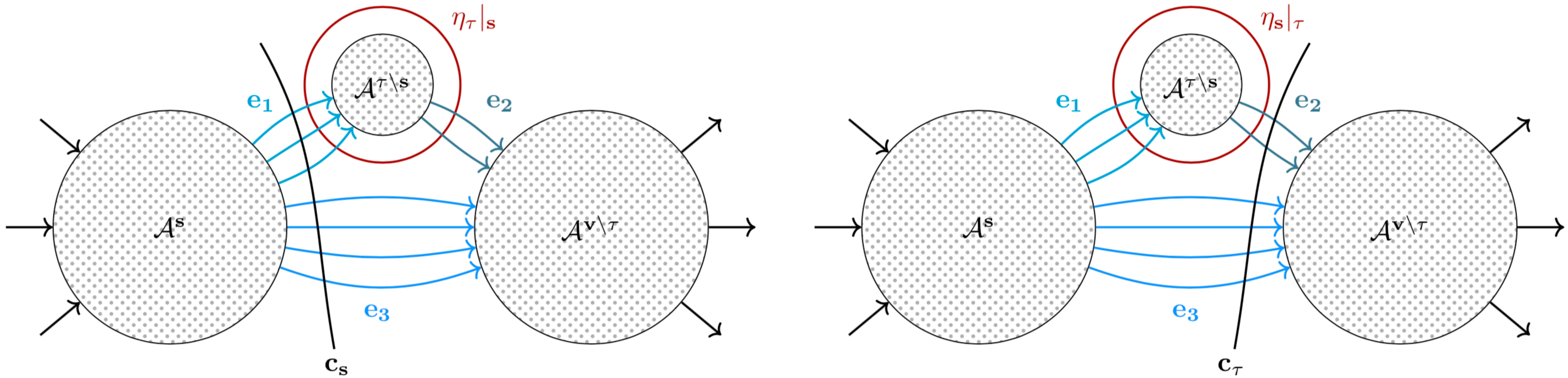
cancels



Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :



Tropical sampling in momentum space

for taming integrable singularities

$$I^{(\mathrm{eucl.})}[f] = \int d^{DL}\mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}}$$

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Tropical sampling of Euclidean Feynman integrals

$$\boxed{I^{\mathrm{eucl.}}[f=1]}$$

E	$\ell(G)$	σ_I/I	samples per second	preprocessing time	RAM
6	3	0.9	$1.1 \cdot 10^6$ / s	$3.0 \cdot 10^{-5}$ s	1 KB
8	4	1.1	$7.5 \cdot 10^5$ / s	$1.3 \cdot 10^{-4}$ s	4 KB
10	5	1.3	$5.1 \cdot 10^5$ / s	$6.0 \cdot 10^{-4}$ s	16 KB
12	6	1.6	$4.1 \cdot 10^5$ / s	$2.7 \cdot 10^{-3}$ s	64 KB
14	7	1.8	$3.2 \cdot 10^5$ / s	$1.2 \cdot 10^{-2}$ s	256 KB
16	8	2.1	$2.6 \cdot 10^5$ / s	$5.3 \cdot 10^{-2}$ s	1 MB
18	9	2.5	$2.1 \cdot 10^5$ / s	$2.3 \cdot 10^{-1}$ s	4 MB
20	10	2.8	$1.4 \cdot 10^5$ / s	$1.1 \cdot 10^0$ s	16 MB
22	11	3.2	$1.0 \cdot 10^5$ / s	$4.7 \cdot 10^0$ s	64 MB
24	12	3.7	$8.6 \cdot 10^4$ / s	$2.1 \cdot 10^1$ s	256 MB
26	13	4.2	$6.9 \cdot 10^4$ / s	$9.5 \cdot 10^1$ s	1 GB
28	14	4.8	$5.9 \cdot 10^4$ / s	$4.4 \cdot 10^2$ s	4 GB
30	15	5.3	$5.1 \cdot 10^4$ / s	$1.9 \cdot 10^3$ s	16 GB
32	16	6.3	$4.3 \cdot 10^4$ / s	$8.7 \cdot 10^3$ s	64 GB
34	17	7.2	$3.6 \cdot 10^4$ / s	$3.9 \cdot 10^4$ s	256 GB

Table 1: Benchmark of Feynman integral evaluations with different numbers of edges.

[M. Borinsky, arXiv : 2008.12310]

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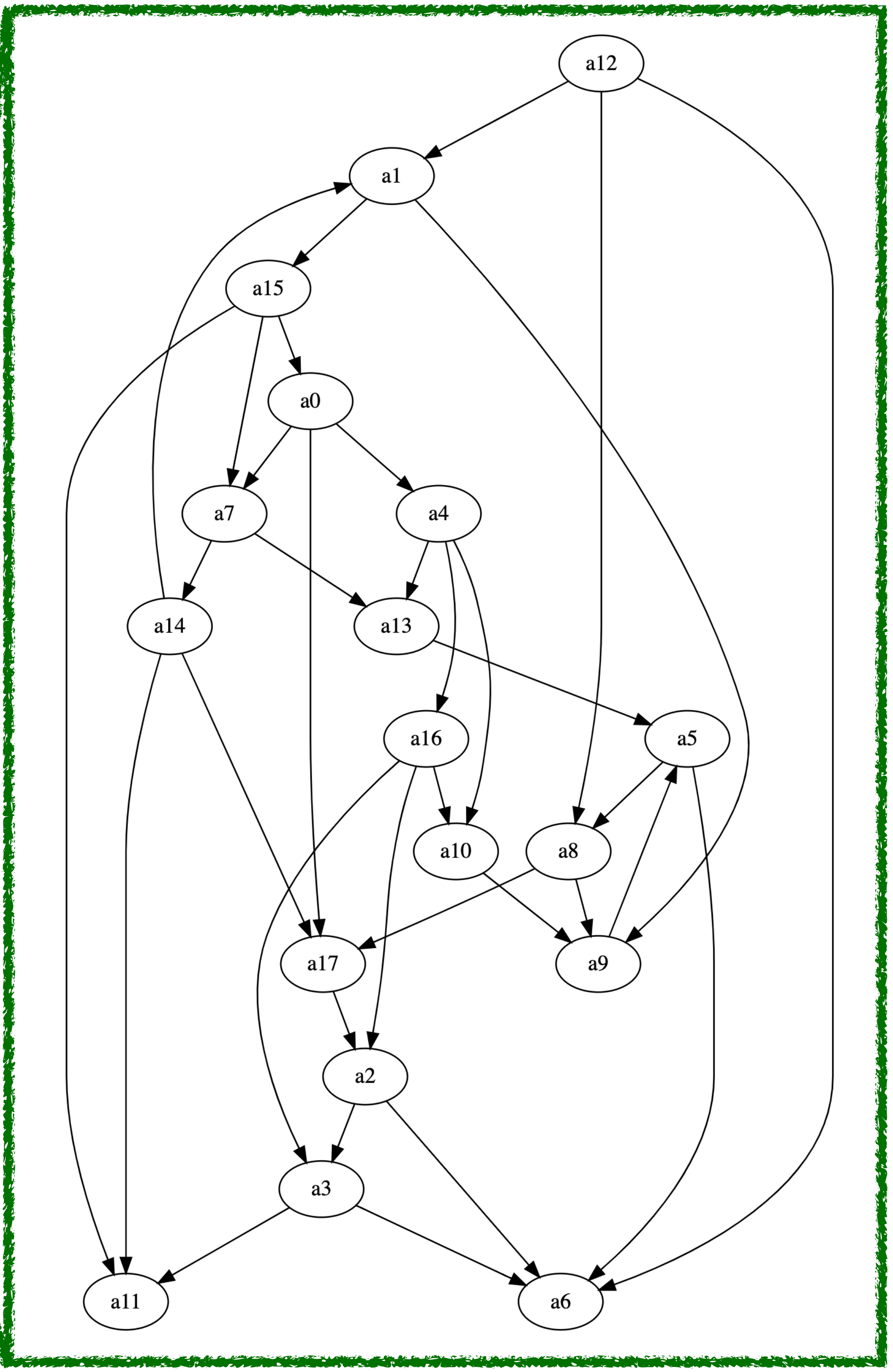
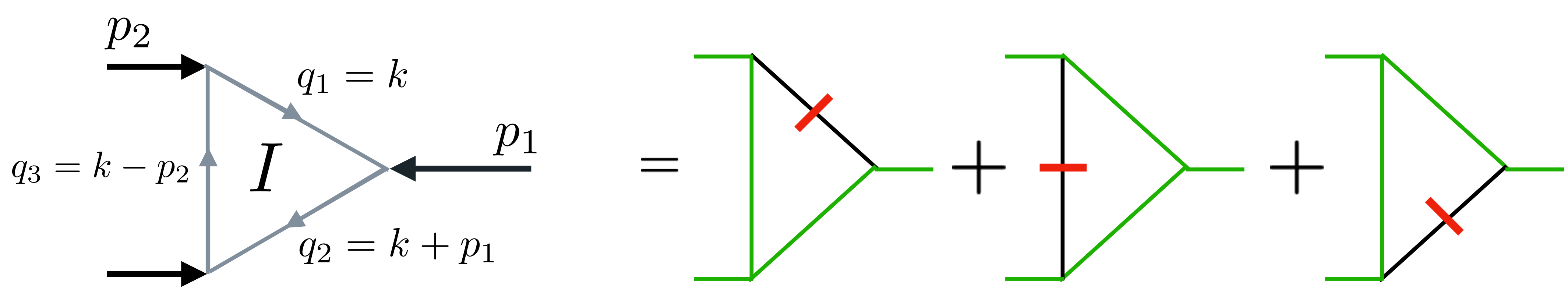


Tropical sampling of Loop-Tree Duality integrands

[Inspired from and in collaboration with M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

[Investigation from **Mathijs Fraije**]

Recall:



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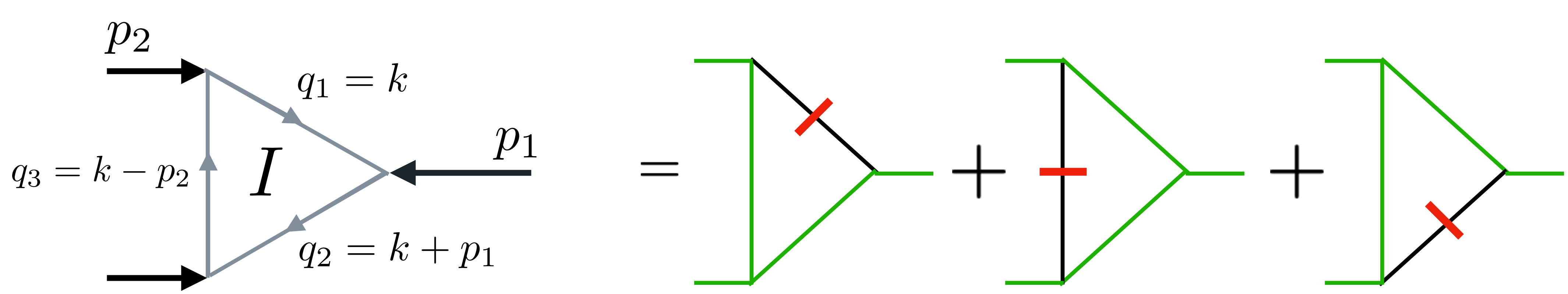

$$= \int \mathrm{d}^4 k \frac{1}{D_1 D_2 D_3} \left(D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3) \right) \supset \int d^3\vec{k} \frac{1}{E_1 E_2 E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \overset{m=0}{=} |\vec{q}_i|$.

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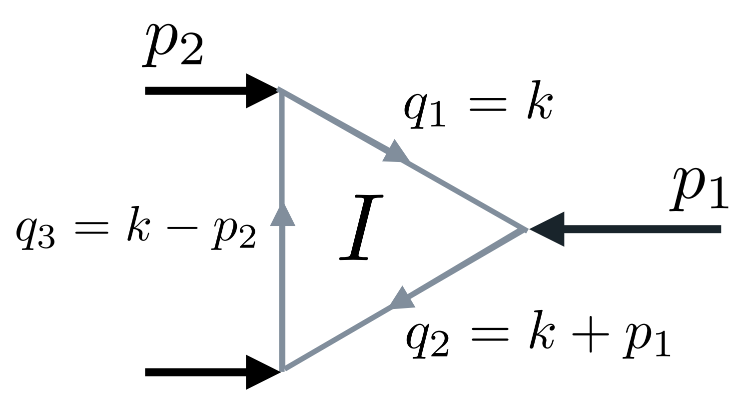
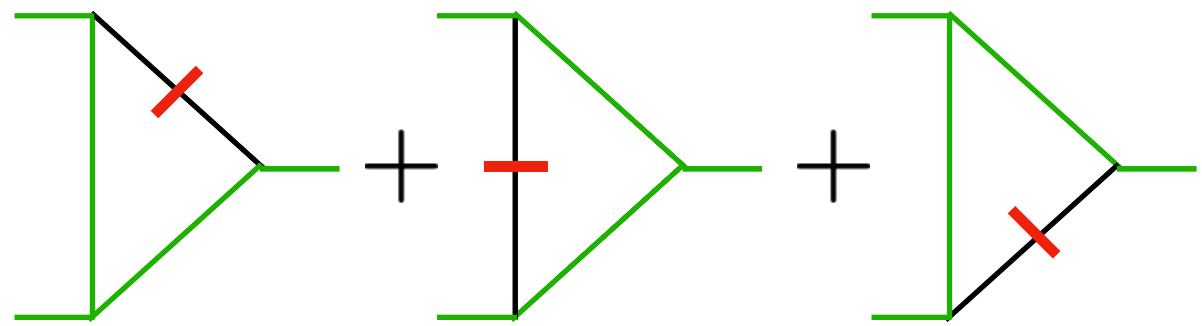


$=$



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With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \overset{m=0}{=} |\vec{q}_i|$. The prefactor $\frac{1}{E_1 E_2 E_3}$ contains point-like integrable singularities :

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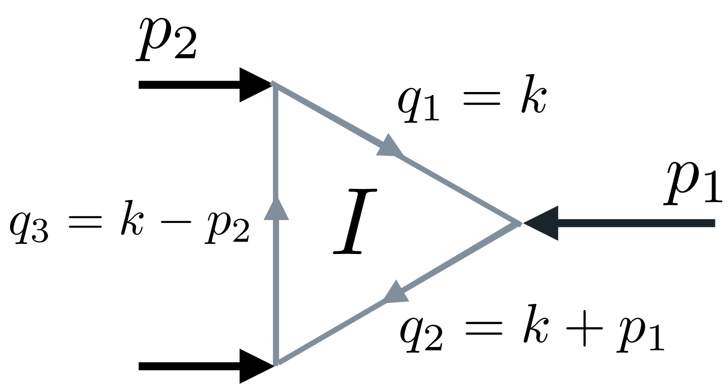

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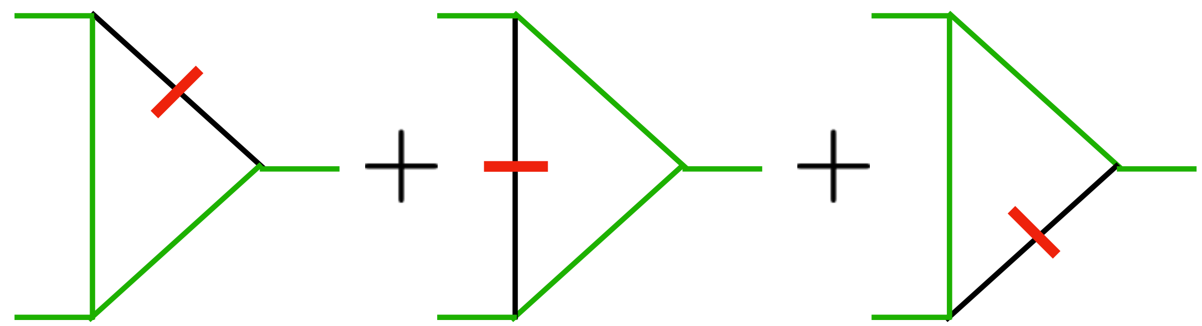

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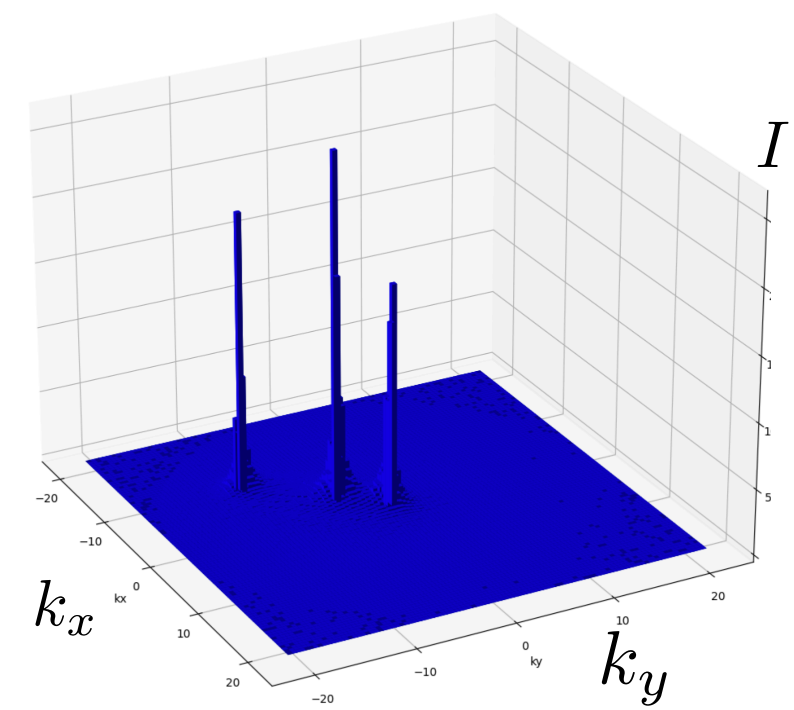
2D example:



Tropical sampling of Loop-Tree Duality integrands

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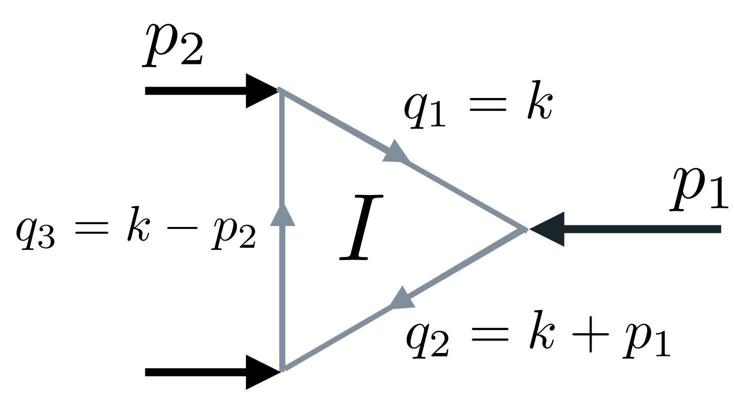

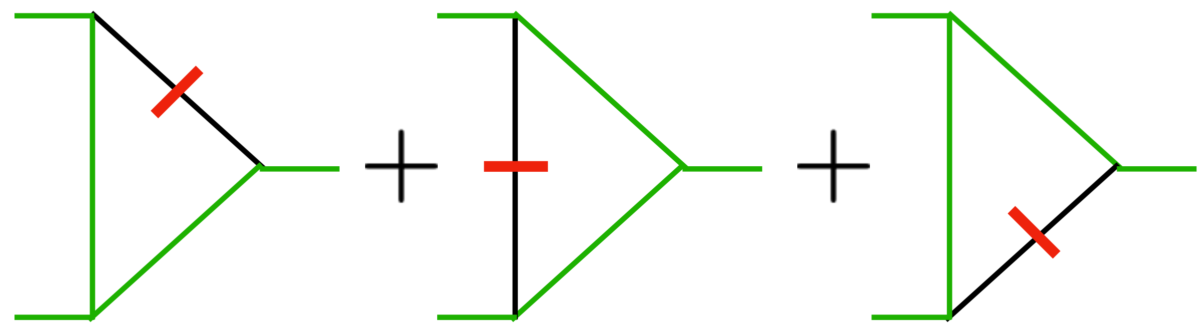

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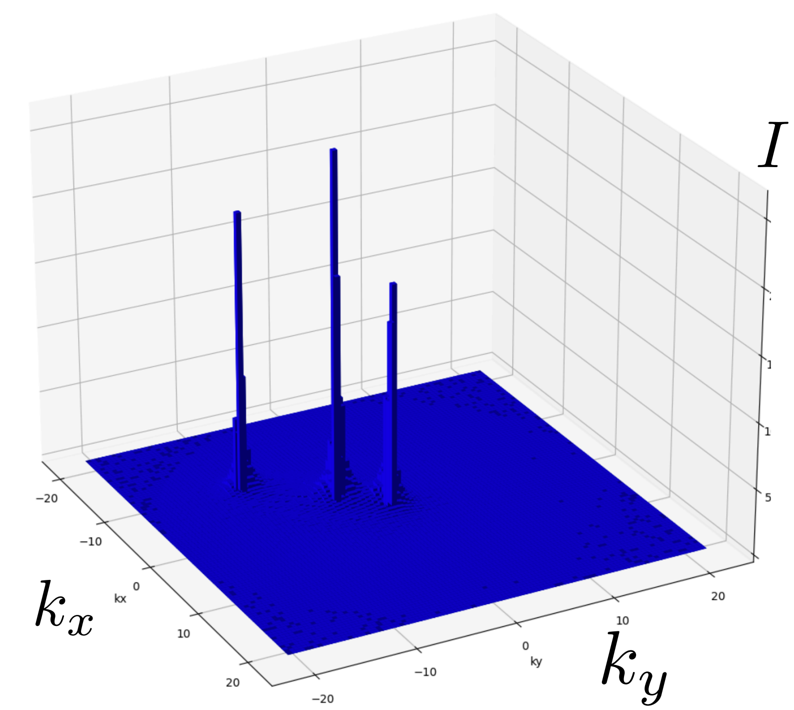

[Originates from $\frac{1}{\left(k^0 - E(\vec{k})\right)\left(k^0 + E(\vec{k})\right)} = \frac{1}{2E(\vec{k})}\left(\frac{1}{E(\vec{k}) - k^0} + \frac{1}{E(\vec{k}) + k^0}\right)$]

Tropical sampling of Loop-Tree Duality integrands

[Inspired from and in collaboration with M. Borinsky, arXiv : 2008.12310]

[Investigation from **Mathijs Fraije**]

$$I^{(\mathrm{LTD})} \supset \int d^3\vec{k} \frac{1}{|\vec{k}||\vec{k}+\vec{p_1}||\vec{k}-\vec{p_2}|} \frac{\mathcal{N}(\vec{k})}{((E_1+E_2-p_1^0)(E_1+E_2+p_1^0)}$$

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Multi-channeling is the canonical approach to flatten these three integrable singularities.

But it would be far better to build a single parameterisation whose Jacobian vanishes simultaneously at *all three points* $\vec{k}=\left\{\vec{0},-\vec{p}_1,\vec{p}_2\right\}$:

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But it would be far better to build a single parameterisation whose Jacobian vanishes simultaneously at *all three points* $\vec{k} = \left\{\vec{0}, -\vec{p}_1, \vec{p}_2\right\}$:

$$= \int d^3\vec{k} \frac{1}{D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} D_3^{\frac{1}{2}}} f(\vec{k})$$

with **euclidean** propagators :

$$D_1 = \vec{k}\cdot\vec{k}$$

$$D_2 = (\vec{k}+\vec{p_1})\cdot(\vec{k}+\vec{p_1})$$

$$D_3 = (\vec{k}-\vec{p_2})\cdot(\vec{k}-\vec{p_2})$$

Ready for being tropical-sampled !

$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

Tropical sampling in momentum space

$$I^{\text{eucl.}}[f] = \int d^{DL}\mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}}$$

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$$I^{\text{eucl.}}[f] = \int d^{DL}\mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}} \propto \int d\mathbf{k} \int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{f(\mathbf{k})}{\left(\sum_e x_e D_e(\mathbf{k})\right)^{\sum_e \nu_e}}$$

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$$\propto \int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{1}{\mathcal{U}(\mathbf{x})^{D/2}\mathcal{V}(\mathbf{x})^{\omega}} \int_0^\infty d\lambda \lambda^{\omega-1} e^{-\lambda} \int \prod_l d^D q_l e^{-q_l^2/2} f\left(\mathbf{k}(\mathbf{x},\mathbf{q},\lambda)\right)$$

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Feynman params.

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Quite a bit of algebra, leading to ~ Schwinger param.

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Easy to sample momenta from this PDF **exactly**

Tropical sampling in momentum space

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$$\propto \underbrace{\int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{1}{\mathcal{U}(\mathbf{x})^{D/2}\mathcal{V}(\mathbf{x})^{\omega}}}_{\text{Tropical sampling [M. Borinsky 2008.12310]}} \underbrace{\int_0^\infty d\lambda \lambda^{\omega-1} e^{-\lambda} \int \prod_l d^D q_l e^{-q_l^2/2}}_{\text{Easy to sample momenta from this PDF exactly}} f\left(\mathbf{k}(\mathbf{x},\mathbf{q},\lambda)\right)$$

$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

$$K(\mathbf{z}) \propto \left(\frac{\mathcal{U}^{\text{tr}}(\mathbf{x})}{\mathcal{U}(\mathbf{x})}\right)^{D/2} \left(\frac{\mathcal{V}^{\text{tr}}(\mathbf{x})}{\mathcal{V}(\mathbf{x})}\right)^{\omega}$$

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Tropical sampling [M. Borinsky 2008.12310]

Easy to sample momenta from this PDF **exactly**

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- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\text{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$

Tropical sampling in momentum space

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Tropical sampling [M. Borinsky 2008.12310]

Easy to sample momenta from this PDF **exactly**

$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z})) \qquad K(\mathbf{z}) \propto \left(\frac{\mathcal{U}^{\text{tr}}(\mathbf{x})}{\mathcal{U}(\mathbf{x})}\right)^{D/2} \left(\frac{\mathcal{V}^{\text{tr}}(\mathbf{x})}{\mathcal{V}(\mathbf{x})}\right)^{\omega}$$

- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\text{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$
- Moreover, a systematic algorithm exists to sample according to p^{tr} (note:. sampling dimensionality increases)

Tropical sampling in momentum space

$$I^{\text{eucl.}}[f] = \int d^{DL}\mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}} \propto \int d\mathbf{k} \underbrace{\int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e}}_{\text{Feynman params.}} \frac{f(\mathbf{k})}{\left(\sum_e x_e D_e(\mathbf{k})\right)^{\sum_e \nu_e}}$$

$\omega = \sum_e \nu_e - DL/2$ $\mathcal{V} = \mathcal{F}/\mathcal{U}$ Quite a bit of algebra, leading to ~ Schwinger param.

$$\propto \underbrace{\int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{1}{\mathcal{U}(\mathbf{x})^{D/2}\mathcal{V}(\mathbf{x})^{\omega}}}_{\text{Tropical sampling [M. Borinsky 2008.12310]}} \underbrace{\int_0^\infty d\lambda \lambda^{\omega-1} e^{-\lambda} \int \prod_l d^D q_l e^{-q_l^2/2}}_{\text{Easy to sample momenta from this PDF exactly}} f\left(\mathbf{k}(\mathbf{x},\mathbf{q},\lambda)\right)$$

$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z})) \qquad K(\mathbf{z}) \propto \left(\frac{\mathcal{U}^{\text{tr}}(\mathbf{x})}{\mathcal{U}(\mathbf{x})}\right)^{D/2} \left(\frac{\mathcal{V}^{\text{tr}}(\mathbf{x})}{\mathcal{V}(\mathbf{x})}\right)^{\omega}$$

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- Moreover, a systematic algorithm exists to sample according to p^{tr} (note:. sampling dimensionality increases)
- One can show that the ratio K = $\frac{p^{\text{tr}}}{p}$ is bounded from below and above when $p \equiv \mathcal{U}$ or $\mathcal{V}$

Tropical sampling in momentum space

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$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

$$K(\mathbf{z}) \propto \left(\frac{\mathcal{U}^{\text{tr}}(\mathbf{x})}{\mathcal{U}(\mathbf{x})}\right)^{D/2} \left(\frac{\mathcal{V}^{\text{tr}}(\mathbf{x})}{\mathcal{V}(\mathbf{x})}\right)^{\omega}$$

- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\text{tr}}(x,y) = \max\{x^2, y^3, xy\}$
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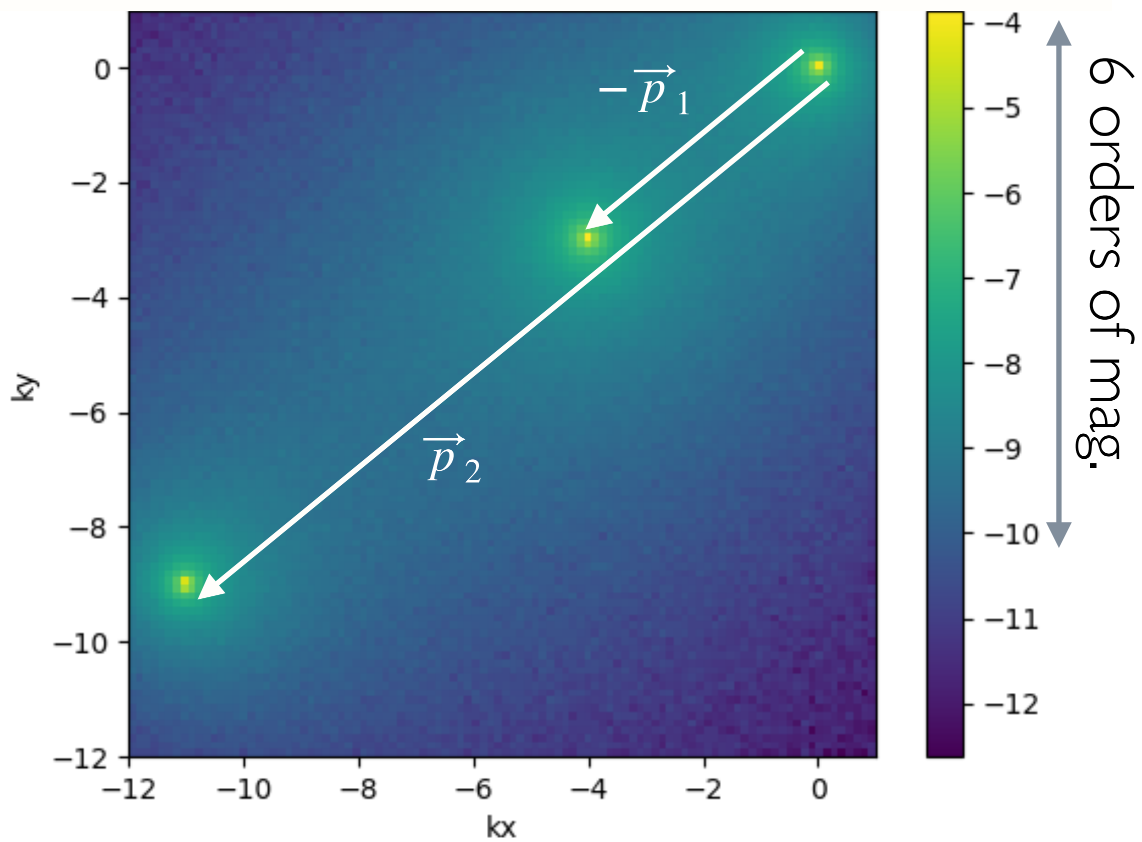
- One can show that the ratio K = $\frac{p^{\text{tr}}}{p}$ is bounded from below and above when $p \equiv \mathcal{U}$ or $\mathcal{V}$
- M. Fraije and M. Borinsky will soon publish a **Rust** implementation of the map: $\phi^{\text{tr}}[\Gamma](\mathbf{z}) \to (K(\mathbf{z}), \mathbf{k}(\mathbf{z}))$

Tropical sampling of THE LTD triangle integral

Let's apply this approach to our example.

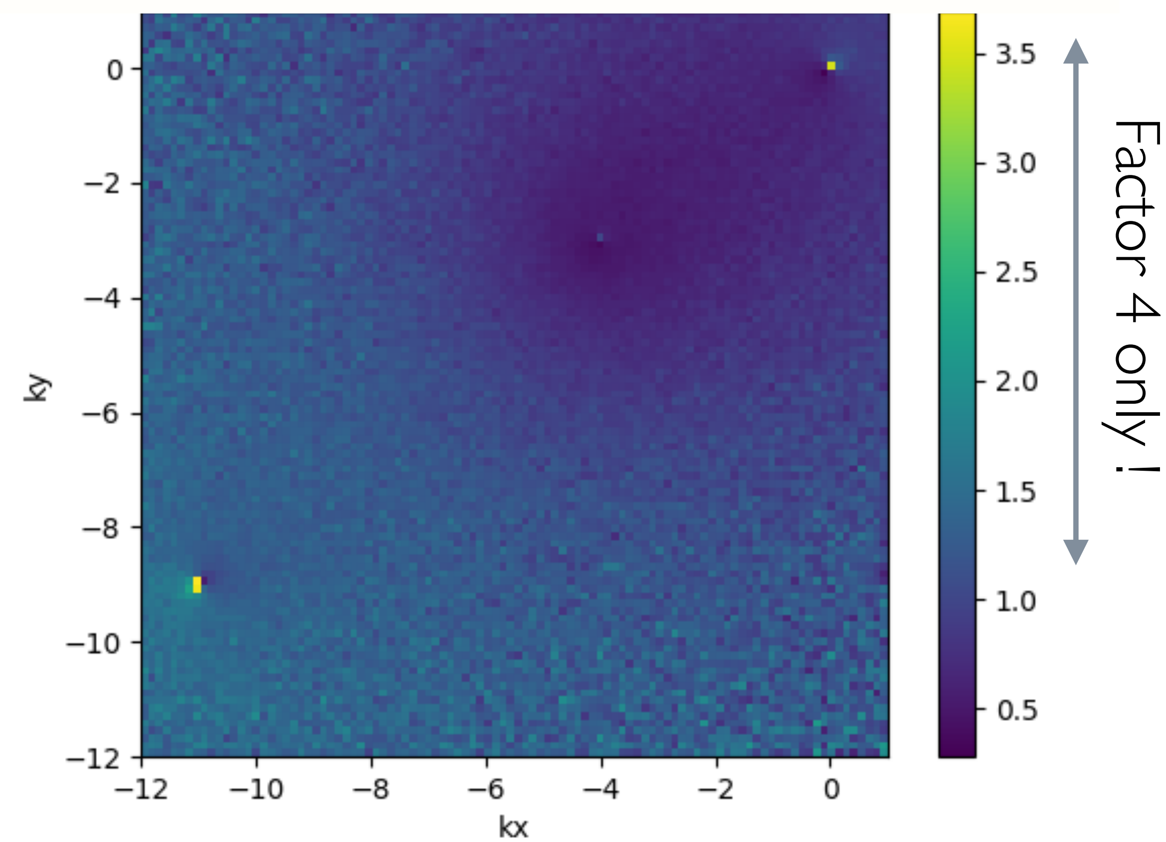
In 2D with tropical sampling for removing blow ups from $p := \dfrac{1}{|\vec{k}||\vec{k}+\vec{p}_1||\vec{k}-\vec{p}_2|}$:

$\log(\,\mathbb{J}^{\mathrm{tr}}(\,k_x, k_y\,)\,)$



Log of tropical ~sampling density

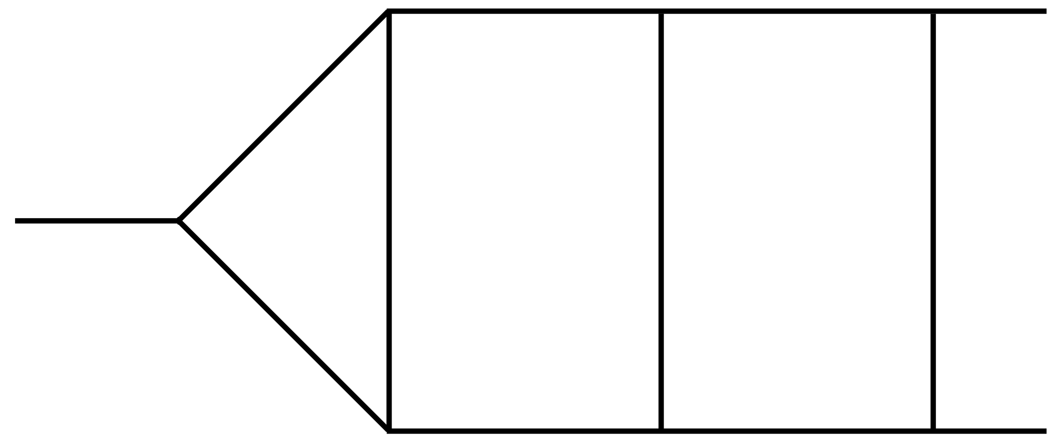
$\mathbb{J}^{\mathrm{tr}}(\,k_x, k_y\,)\,/\,p$



Integrable singularities conquered !

- <u>Many details</u> of this approach <u>omitted</u> here. In our implementation: **arbitrary numerators** supported!

Tropical sampling results

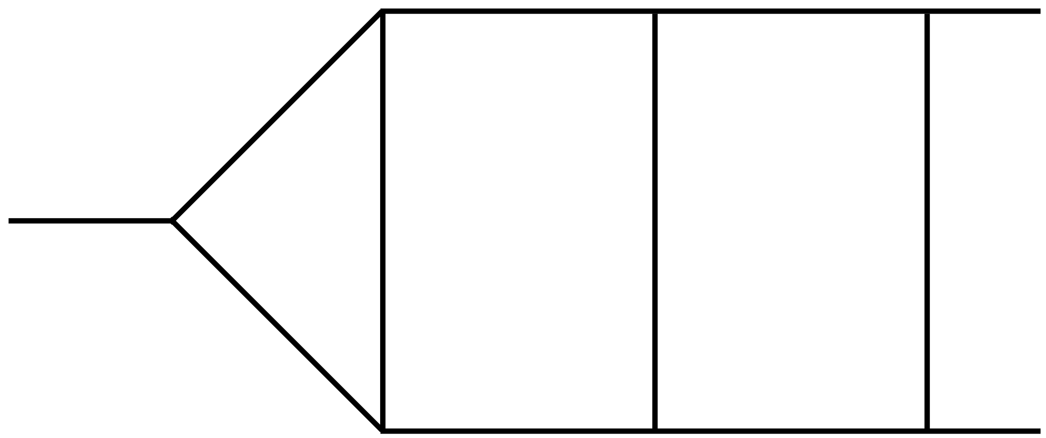


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/18$)

N Samples	Naive	Tropical
0.01M 30 ms	3.51(63)e-8 **18%**	
0.1M 0.3 s	3.78(24)e-8 **6%**	
1M 3 s	3.99(11)e-8 **2.7%**	
10M 30 s	4.045(36)e-8 **0.9%**	

Credits: Mathijs Fraaije

Tropical sampling results

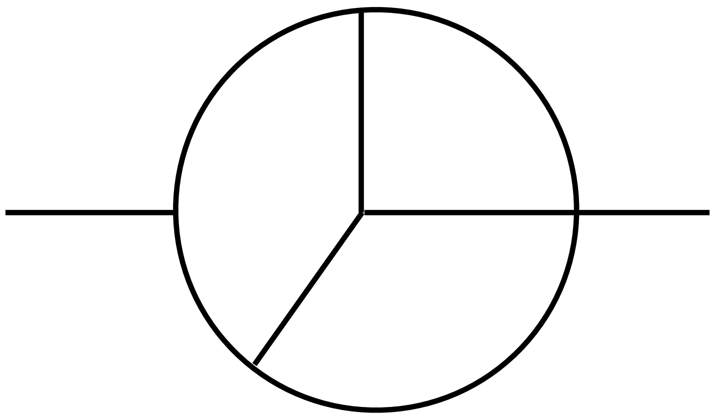


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/18$)

N Samples	Naive	Tropical
0.01M 30 ms	3.51(63)e-8 **18%**	4.050(35)e-8 **0.9%**
0.1M 0.3 s	3.78(24)e-8 **6%**	4.030(11)e-8 **0.3%**
1M 3 s	3.99(11)e-8 **2.7%**	4.0379(35)e-8 **0.09%**
10M 30 s	4.045(36)e-8 **0.9%**	4.0358(11)e-8 **0.03%**

Credits: Mathijs Fraaije

Tropical sampling results

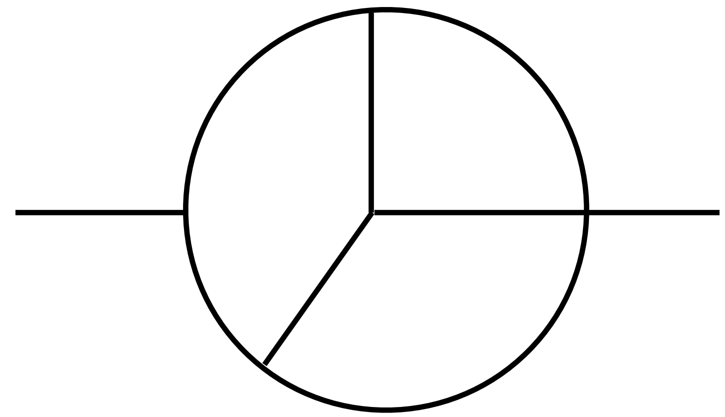


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/14$)

N Samples	Naive	Tropical
0.01M 30 ms	8.9(2.9)e-7 33%	
0.1M 0.3 s	3.5(1.0)e-6 29%	
1M 3 s	5.6(1.2)e-6 22%	
10M 30 s	1.23(41)e-5 34%	

Credits: Mathijs Fraaije

Tropical sampling results

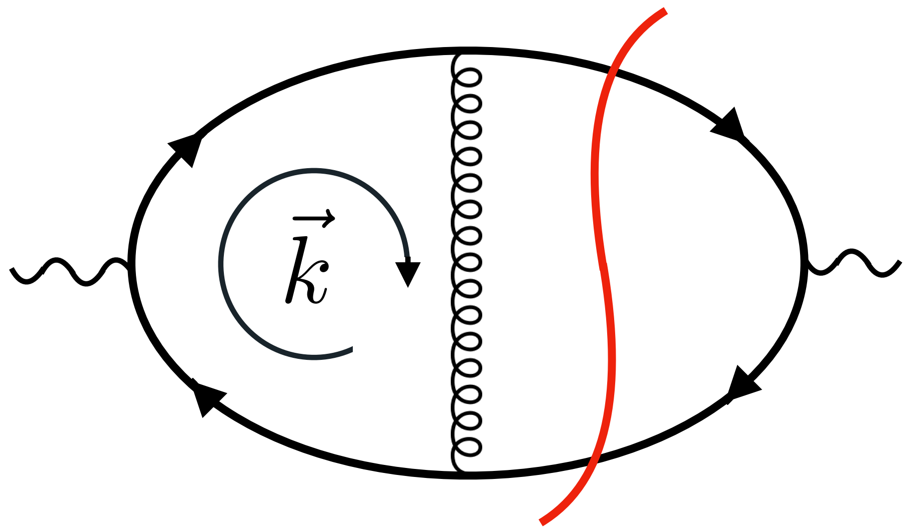


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/14$)

N Samples	Naive	Tropical
0.01M 30 ms	8.9(2.9)e-7 33%	9.403(60)e-6 0.6%
0.1M 0.3 s	3.5(1.0)e-6 29%	9.518(19)e-6 0.2%
1M 3 s	5.6(1.2)e-6 22%	9.4984(60)e-6 0.06%
10M 30 s	1.23(41)e-5 34%	9.4986(19)e-6 0.02%

Credits: Mathijs Fraaije

Localised Renormalisation: BPHZ



$$\lim_{|\vec{k}|\to\infty} I^{(\text{Local Unitarity})} \to \infty$$

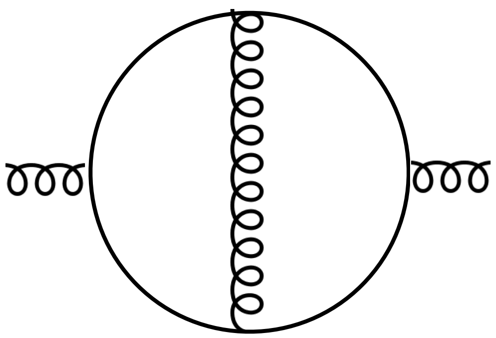
Localised Renormalisation

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

$$\mathrm{R}\left(\Gamma\right) = \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$$

Localised Renormalisation

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

R (Γ = ) $= \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$

Localised Renormalisation

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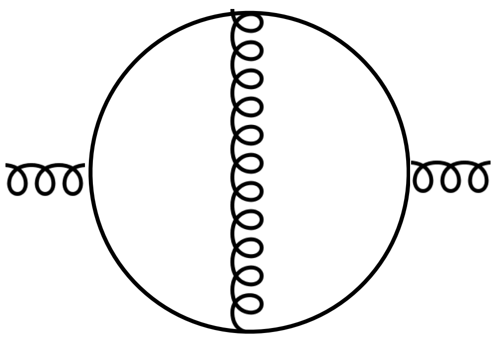
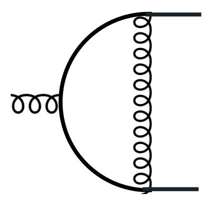
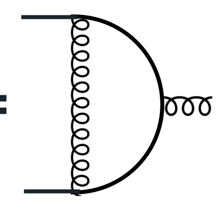
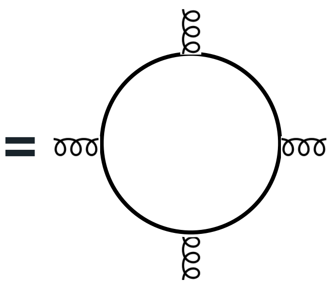
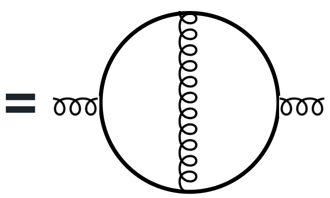
$$R\Bigg(\Gamma =$$  $$\Bigg) = \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma\setminus S * \prod_{\gamma'\in S} Z(\gamma')\right)$$

UV subgraphs :

$\mathrm{dod}\left(\gamma_{\{1,2,3\}}\right) = 0$

$\mathrm{dod}\left(\gamma_4\right) = 2$

$\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

Localised Renormalisation

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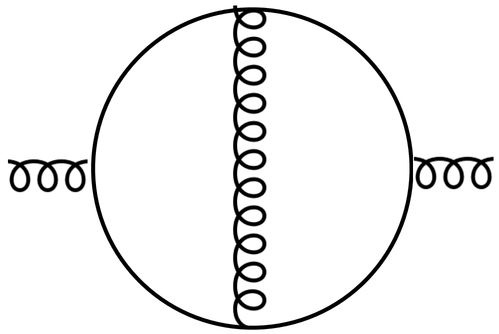
R (Γ = ) $= \sum_{S\in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma\in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S\in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma'\in S} Z(\gamma') \right)$

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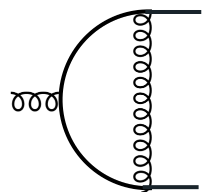
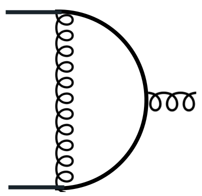
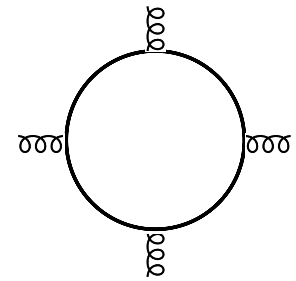
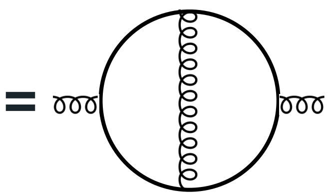
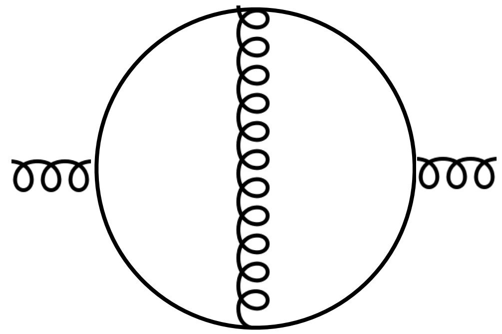
$\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$R(\Gamma) = \Gamma - K(\gamma_1)*\Gamma\setminus\gamma_1 - K(\gamma_2)*\Gamma\setminus\gamma_2 - K(\gamma_3)*\Gamma\setminus\gamma_3 - K(\gamma_4)*\Gamma\setminus\gamma_4$$

$$+ K(K(\gamma_1)*\Gamma\setminus\gamma_1) + K(K(\gamma_2)*\Gamma\setminus\gamma_2) + K(K(\gamma_3)*\Gamma\setminus\gamma_3)$$

Localised Renormalisation

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

R (Γ = ) $= \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$

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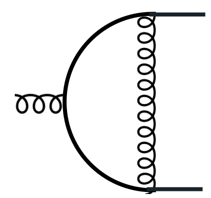
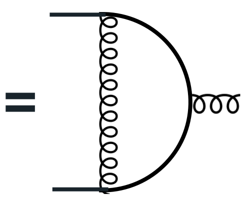
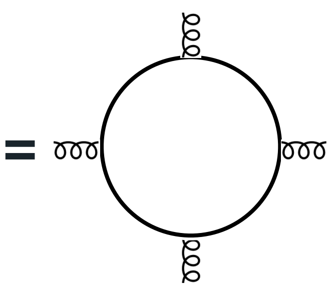
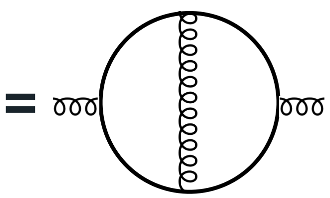
γ_1 =  γ_2 =  γ_3 =  γ_4 = 

$$R(\Gamma) = \Gamma - K(\gamma_1) * \Gamma \setminus \gamma_1 - K(\gamma_2) * \Gamma \setminus \gamma_2 - K(\gamma_3) * \Gamma \setminus \gamma_3 - K(\gamma_4) * \Gamma \setminus \gamma_4$$
$$+ K(K(\gamma_1) * \Gamma \setminus \gamma_1) + K(K(\gamma_2) * \Gamma \setminus \gamma_2) + K(K(\gamma_3) * \Gamma \setminus \gamma_3)$$

What is the operator K(γ) ? Anything we want ! so long as it:

- Locally cancels UV divergences of γ, even in the presence of nestings
- Yields results immediately renormalised in the chosen scheme ($\overline{\text{MS}} + \text{OS}$)
- Minimal analytics: at most single-scale all-massive vacuum integrals

Local renormalisation operator K

Our solution: $\mathsf{K}(\gamma) := \mathsf{T}(\gamma)$

$\mathsf{T}(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

Local renormalisation operator K

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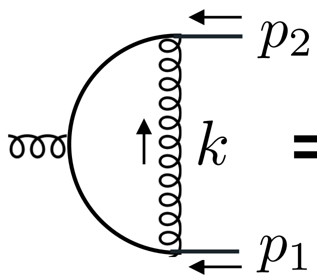
$\gamma_1 =$



$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

Local renormalisation operator K

Our solution: $\mathrm{K}(\gamma) := \mathrm{T}(\gamma)$

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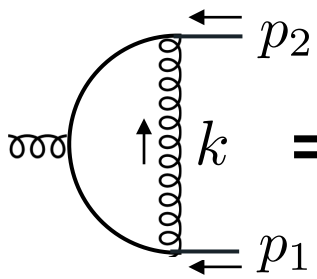
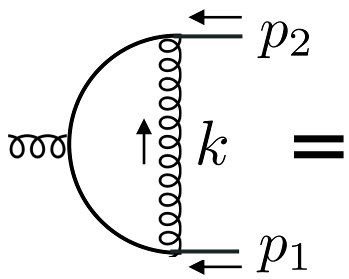
$\gamma_1 =$  $= \dfrac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$

$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k-\lambda p_1)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2))(k^2 - m_{\mathrm{UV}}^2)((k+\lambda p_2)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2)\ )}$$

Local renormalisation operator K

Our solution: $K(\gamma) := T(\gamma)$

$T(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

$\gamma_1 =$ 

$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

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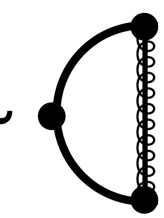
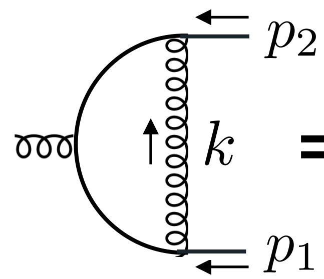
$$T(\gamma) = T_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma^{\lambda}\bigg|_{\lambda=0},$$

$$T_0(\gamma_1) = \frac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\mathrm{UV}}^2)^3} \sim$$ 

Local renormalisation operator K

Our solution: $\mathsf{K}(\gamma) := \mathsf{T}(\gamma) - [\mathsf{T}](\gamma)$

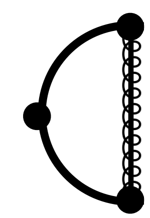
$\mathsf{T}(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

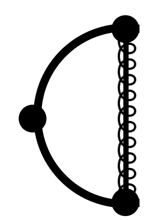
$\gamma_1 =$ 

$= \dfrac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$

$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k-\lambda p_1)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2))(k^2 - m_{\mathrm{UV}}^2)((k+\lambda p_2)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2)\ )}$$

$$\mathsf{T}(\gamma) = \mathsf{T}_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma^{\lambda}\Big|_{\lambda=0},$$

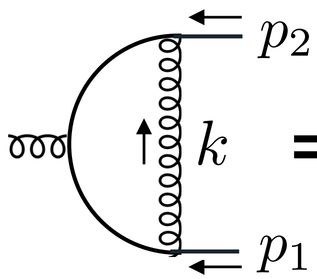
$\mathsf{T}_0(\gamma_1) = \dfrac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\mathrm{UV}}^2)^3} \sim$ 

$[\mathsf{T}](\gamma) :=$ **Integrated CT** , $[\mathsf{T}](\gamma_1) = \left(\dfrac{\mu_r^2}{4\pi e^{-\gamma_{\mathrm{E}}}}\right)^{\epsilon} \displaystyle\int d^{4-2\epsilon}k$  $= \displaystyle\sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$

Local renormalisation operator K

Our solution: $\mathrm{K}(\gamma) := \mathrm{T}(\gamma) - [\mathrm{T}](\gamma) + \delta^{\overline{\mathrm{MS}}+\mathrm{OS}}(\gamma)$

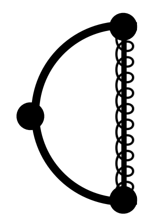
$\mathrm{T}(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

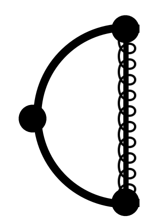
$\gamma_1 =$ 

$= \dfrac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$

$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k-\lambda p_1)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2))(k^2 - m_{\mathrm{UV}}^2)((k+\lambda p_2)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2)\ )}$$

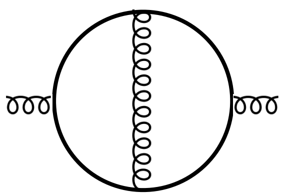
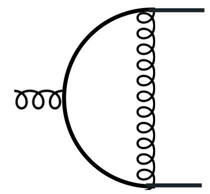
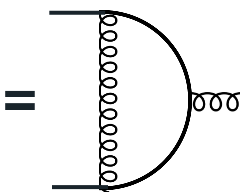
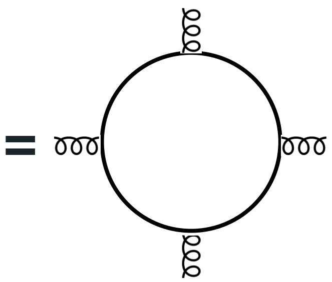
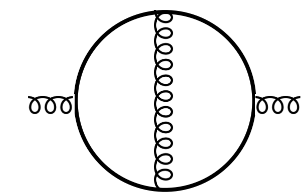
$$\mathrm{T}(\gamma) = \mathrm{T}_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!} \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} \gamma^{\lambda} \Big|_{\lambda=0},$$

$\mathrm{T}_0(\gamma_1) = \dfrac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\mathrm{UV}}^2)^3} \sim$ 

$[\mathrm{T}](\gamma) :=$ **Integrated CT** , $[\mathrm{T}](\gamma_1) = \left(\dfrac{\mu_r^2}{4\pi e^{-\gamma_{\mathrm{E}}}}\right)^{\epsilon} \displaystyle\int d^{4-2\epsilon}k$  $= \displaystyle\sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$

$\delta^{\mathrm{X}}(\gamma) :=$ **Renormalisation CT** in scheme X , $(-[\mathrm{T}] + \delta^{\overline{\mathrm{MS}}}) := \bar{\mathrm{K}}$, $\bar{\mathrm{K}}(\gamma_1) = \displaystyle\sum_{k=0}^{+\infty} \alpha_k \epsilon^k$

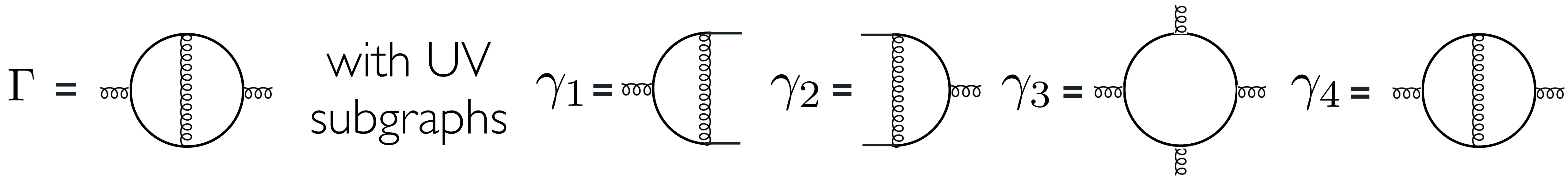
R-operator unfolding

Γ =  with UV subgraphs γ_1 =  γ_2 =  γ_3 =  γ_4 = 

$$R(\Gamma) = \Gamma - T_0(\gamma_1)*\Gamma\backslash\gamma_1 - T_0(\gamma_2)*\Gamma\backslash\gamma_2 - T_0(\gamma_3)*\Gamma\backslash\gamma_3 - T_2(\gamma_4)*\Gamma\backslash\gamma_4$$

$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K}\text{ terms}$$

R-operator unfolding



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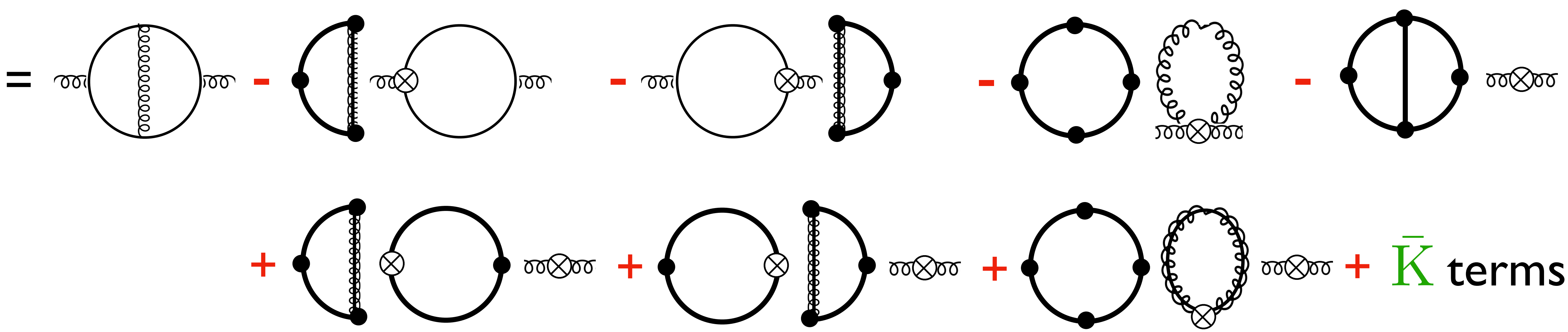
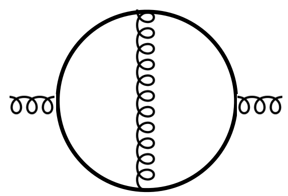
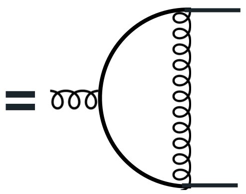
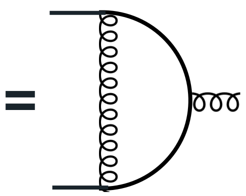
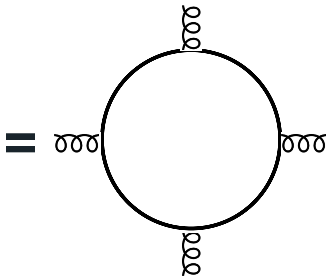
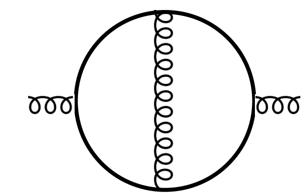


R-operator unfolding

$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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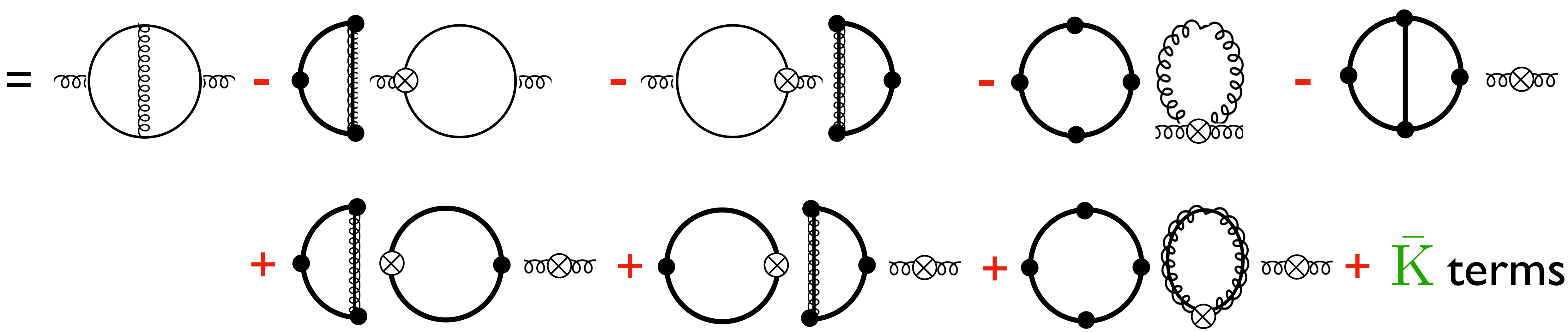

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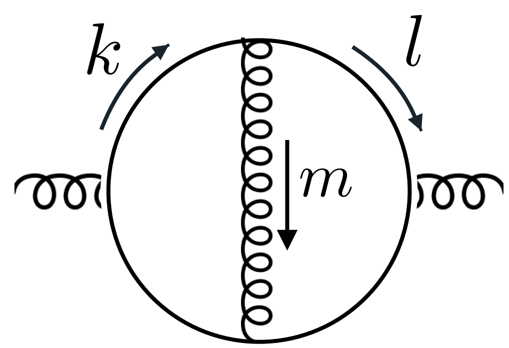
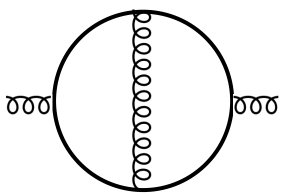
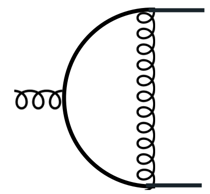
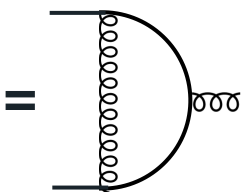
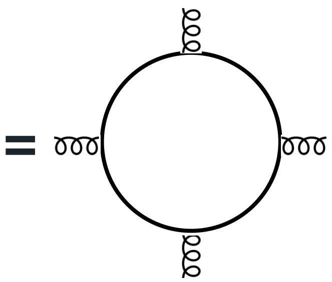
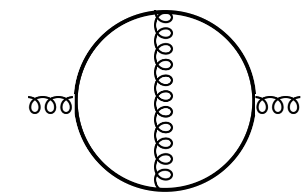


$$\begin{array}{lll} k, m & \to \infty & , l \text{ finite} \\ l, m & \to \infty & , k \text{ finite} \\ k, l & \to \infty & , m \text{ finite} \\ k, l, m & \to \infty & \end{array}$$

R-OPERATOR UNFOLDING

$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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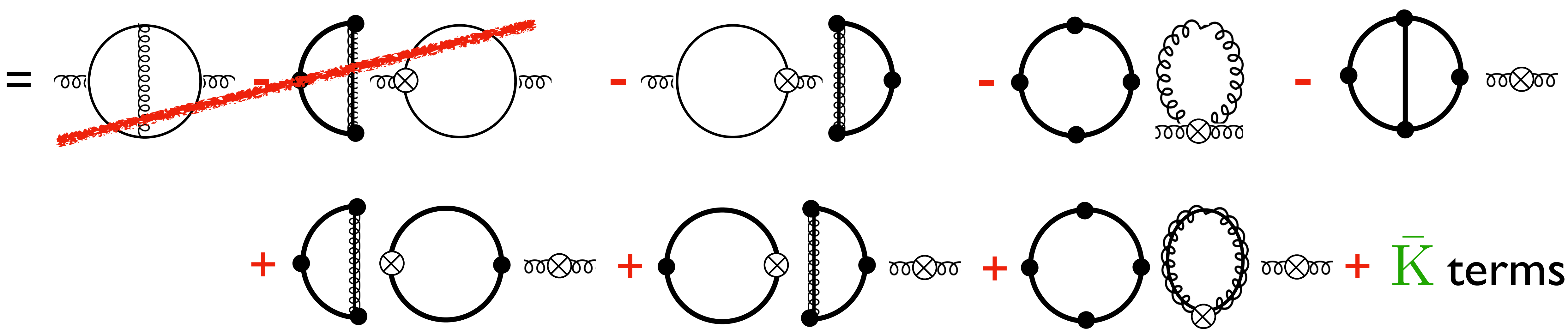

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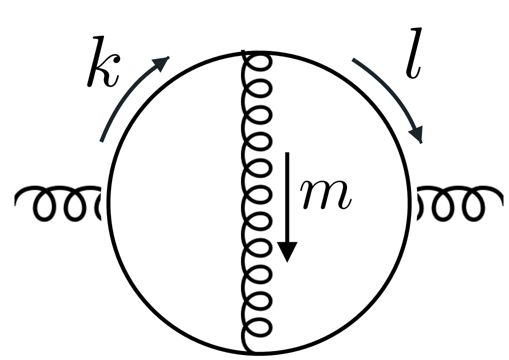

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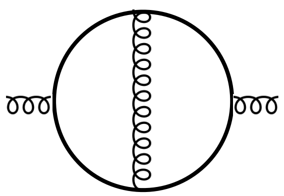
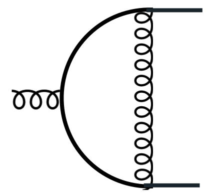
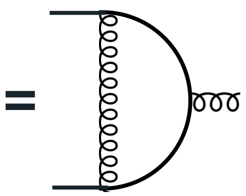
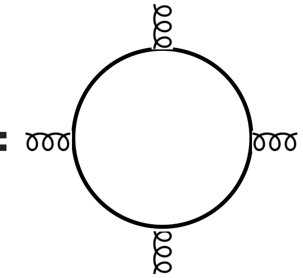
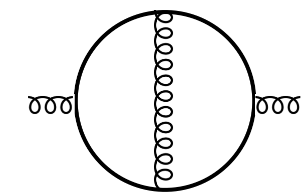
$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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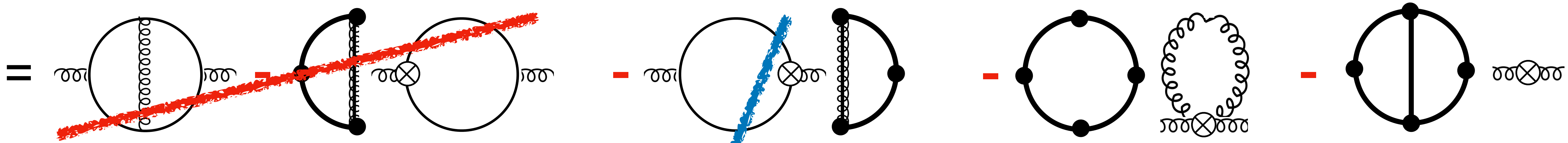

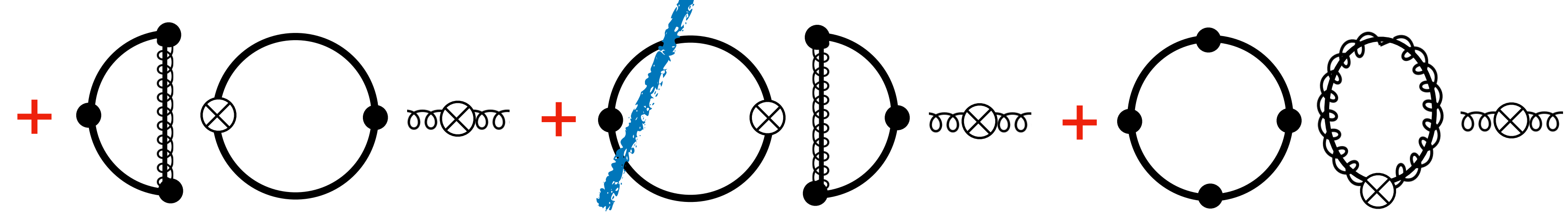 + $\bar{K}$ terms

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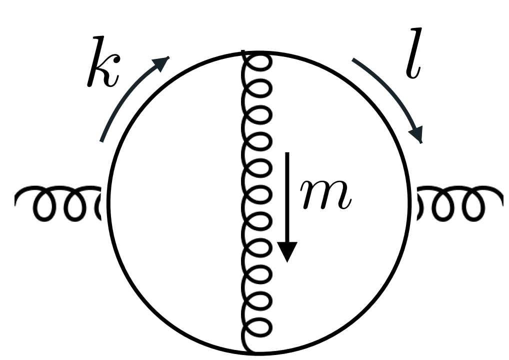

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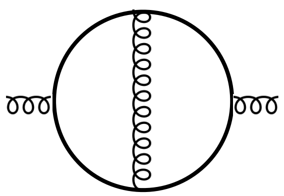
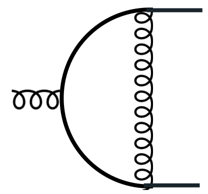
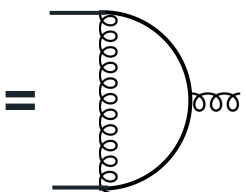
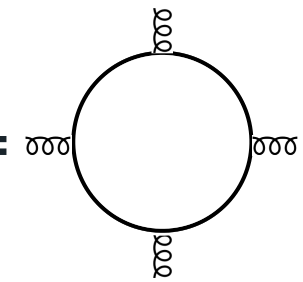
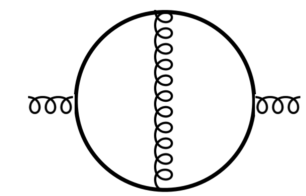
$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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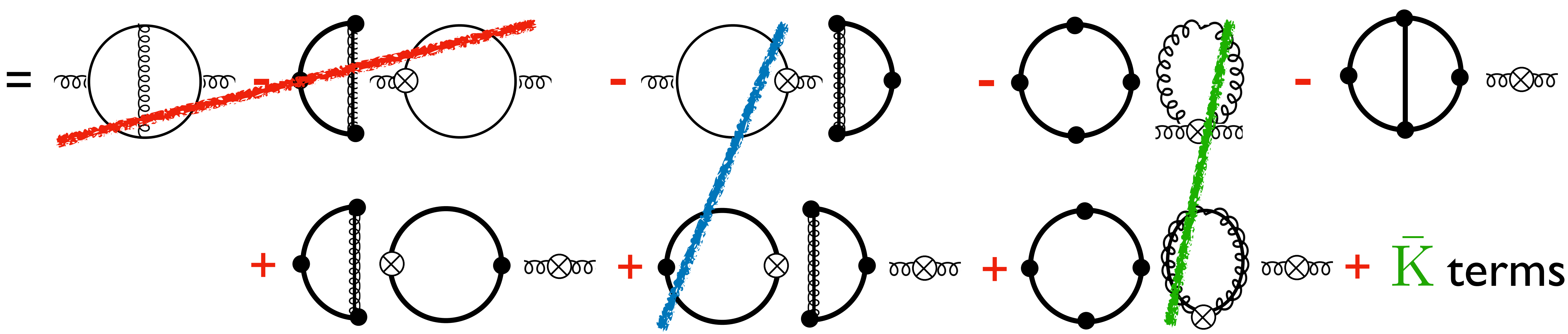

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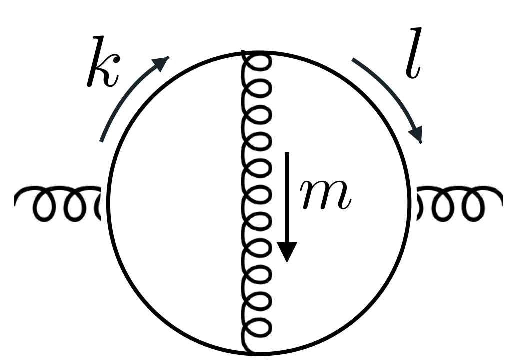

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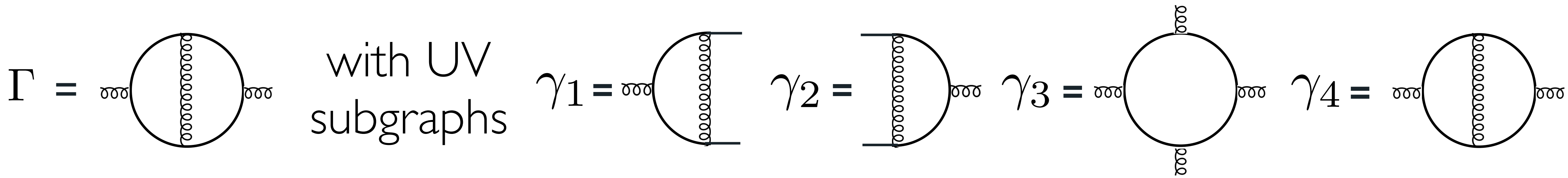

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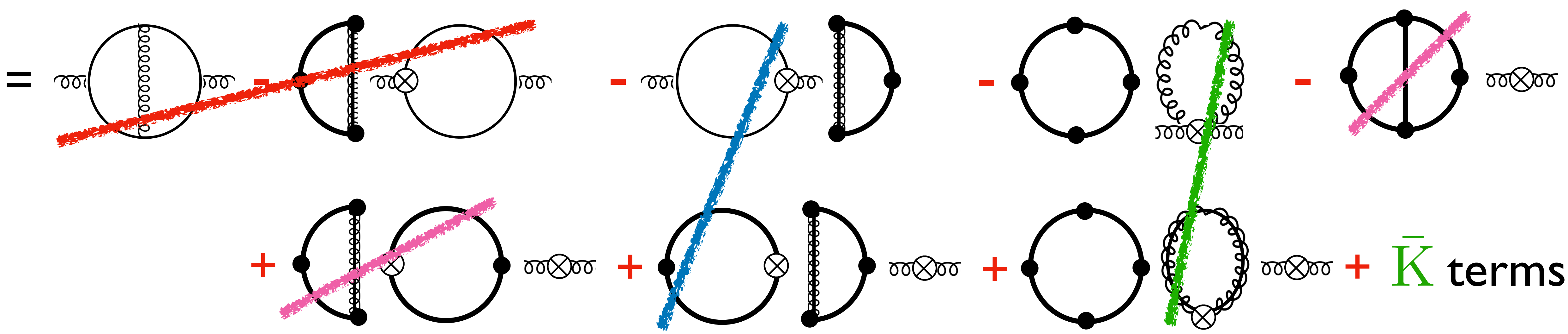

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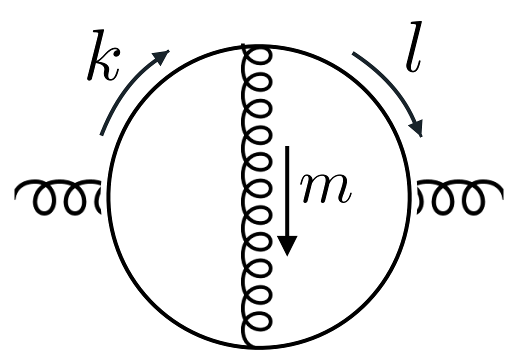

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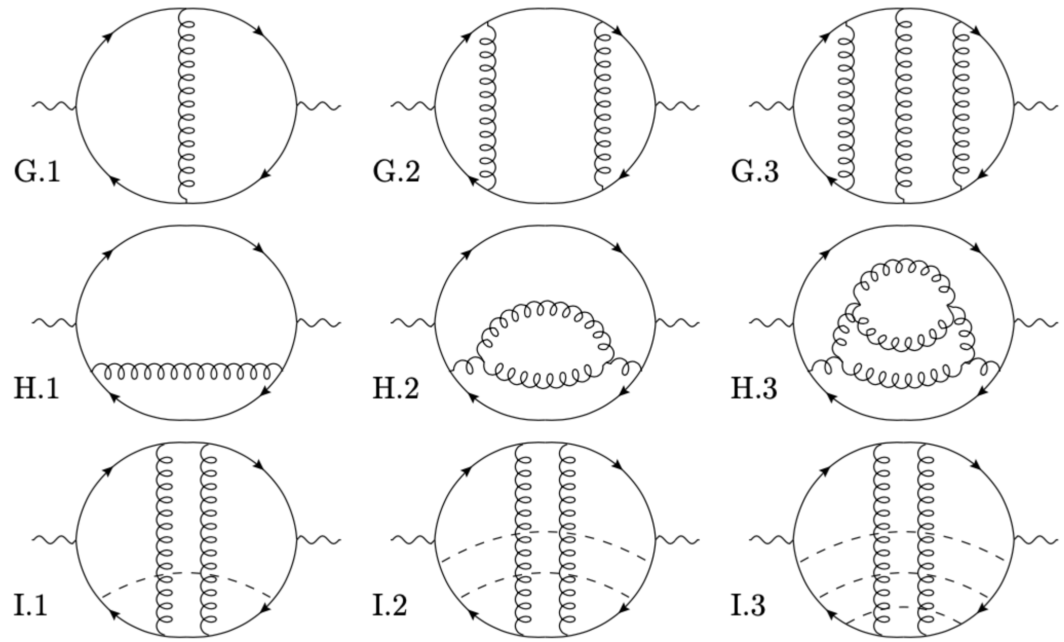
Efficiency in LU implementation

αLoop → γLoop



SG	proc.	order	$t_{\rm gen}$ [s]	$M_{\rm disk}$ [MB]	$N_{\rm sg}$ [-]	$N_{\rm cuts}$ [-]	$t_{\rm eval}$ [ms]	$t_{\rm eval}^{\rm (f128)}$ [ms]
G.1	$1 \to 2$	NLO	0.1	0.13	2	4	0.004	0.13
G.2	$1 \to 2$	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	$1 \to 2$	N3LO	36K	509	220	16	17.6	281
H.1	$1 \to 2$	NLO	0.07	0.12	2	2	0.006	0.14
H.2	$1 \to 2$	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	$1 \to 2$	N3LO	255	43	220	4	2.35	56
I.1	$1 \to 3$	NNLO	126	22	266	9	0.32	12.4
I.2	$1 \to 4$	NNLO	1.9K	120	4492	9	4.4	67
I.3	$1 \to 5$	NNLO	36K	20K	$\mathcal{O}$(100K)	9	3.6K	17.3K

NB: most recent version of αLoop does better, but scaling is similar

Numerical gamma chains with spenso.rs

spenso.rs

[Lucien Huber : crates.io/crates/spenso]

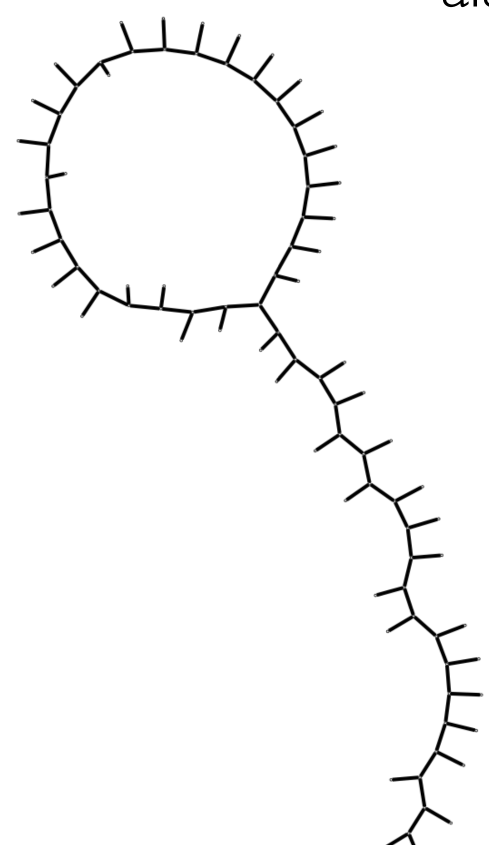
$$p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3}\,\mathbf{Tr}(\gamma_\mu \gamma_{\mu_1} \gamma_\nu \gamma_{\mu_2} \gamma_\rho \gamma_{\mu_3})$$

γ -algebra

$$\begin{aligned}
&-4(p_1 \cdot p_2)p_{3,\mu}\eta_{\nu\rho}\\
&+4(p_1 \cdot p_2)p_{3,\nu}\eta_{\mu\rho}\\
&-4(p_1 \cdot p_2)p_{3,\rho}\eta_{\mu\nu}\\
&+4(p_1 \cdot p_3)p_{2,\mu}\eta_{\nu\rho}\\
&-4(p_1 \cdot p_3)p_{2,\nu}\eta_{\mu\rho}\\
&-4(p_1 \cdot p_3)p_{2,\rho}\eta_{\mu\nu}\\
&-4p_{1,\mu}(p_2 \cdot p_3)\eta_{\nu\rho}\\
&+4p_{1,\mu}p_{2,\nu}p_{3,\rho}\\
&+4p_{1,\mu}p_{2,\rho}p_{3,\nu}
\end{aligned}$$

necessary for d-dimensions
but scales badly for $d = 4$

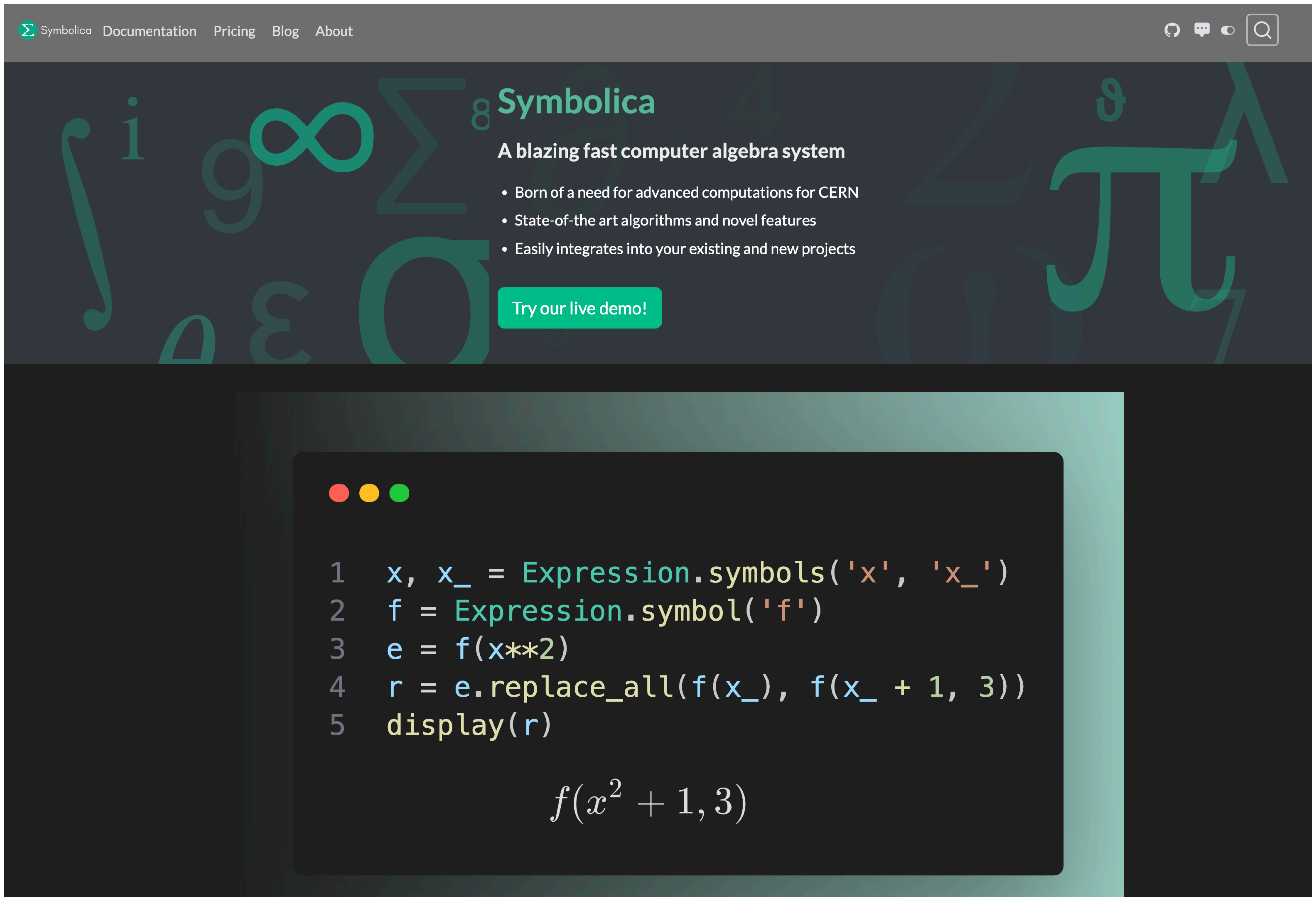
numerical tensors
"ala MadGraph"



Spenso	Spenso Compiled	Hardcode Fortran
104 μs	16 μs	31μs

Symbolica

Ben Ruijl's successor to FORM



Try it for yourself in this colab notebook !

Summary - Possible overlap with LSS

Formalism :

- **Local** cancellation of all **final-state IR singularities**
- Completely **generic** (masses, kinematics, topologies, observables…)
- New **theoretical perspectives** on perturbative expansions
- **Automated renormalisation** with minimal analytical computations
- **Falsehood :** "Analytical = solved && Numerical = black box"

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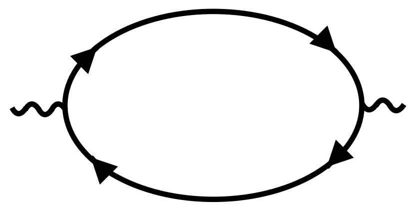
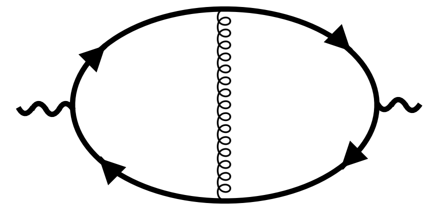
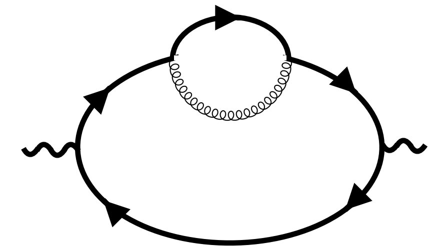
Outlook :

- Rewrite **LU** impl. **cleaner and more efficient**: $\alpha\text{Loop} \rightarrow \gamma\text{Loop}$
- Apply **LU** to **new unknown corrections** : e.g. $b\bar{b}$ N^3LO AFB or decays
- Applications in finite **T** and finite μ pQFT : see [Navarrete & al.: 2403.02180]
- Generalise **LU** so as to apply of **initial-state singularities** as well
- Match **LU** to some form of **numerical resummation** (PSMC).
- Method likely well-suited for deployment on GPU.

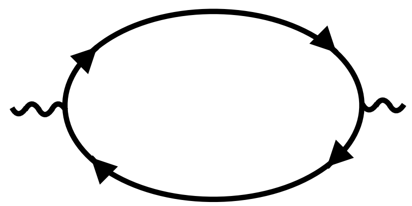
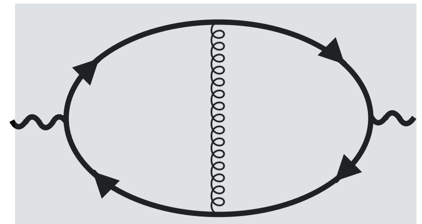
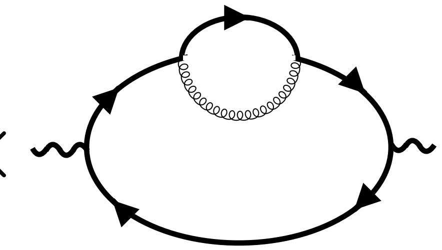
Back-up slides

Local Unitarity: x-sec results

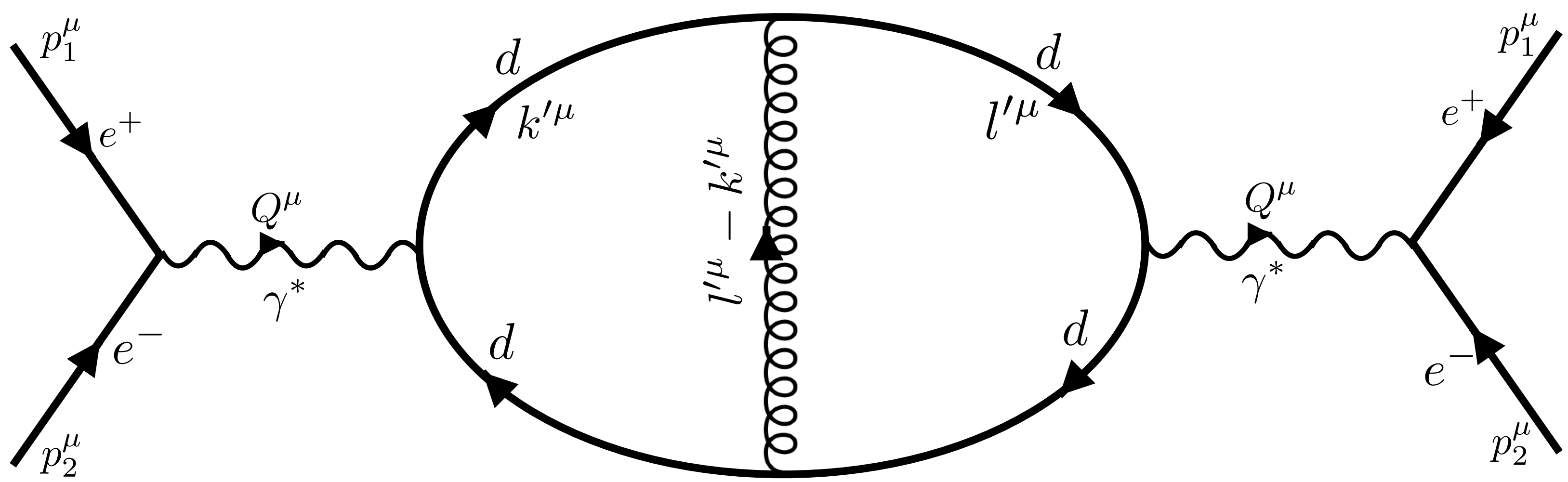
Numerical results for $e^+e^- \rightarrow \gamma^\star \rightarrow d\bar{d}$ @ NLO

$\sigma^{(\mathrm{LU})}_{\gamma^\star \rightarrow d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+\ 2\times$  $\Big]$

NUMERICAL RESULTS FOR $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+ 2\times$  $\Big]$

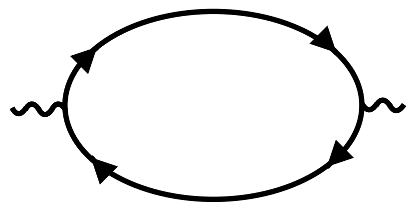
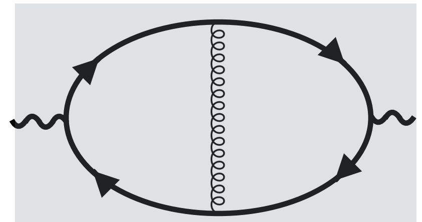
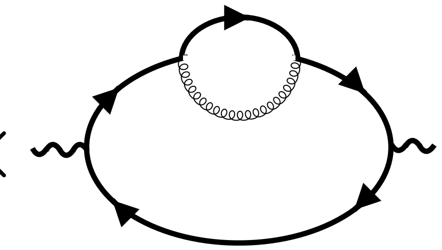
Visualisation of the **LU integrand** for the **Double-Triangle** supergraph and :



$$p_1^\mu = (1, 0, 0, 1) \qquad 0.4 < p_{t,j_1} < 0.8$$

$$p_2^\mu = (1, 0, 0, -1) \qquad (\,\vec{k}, \vec{l}\,) = (\,(0, k_y, \frac{1}{\sqrt{2}})\,, (0, \frac{1}{\sqrt{2}}, l_z)\,)$$

Numerical results for $e^+e^- \rightarrow \gamma^\star \rightarrow d\bar{d}$ @ NLO

$\sigma^{(\mathrm{LU})}_{\gamma^\star \rightarrow d\bar{d}} = \mathrm{LU}\,\Big[$  $+$  $+ 2\times$  $\Big]$

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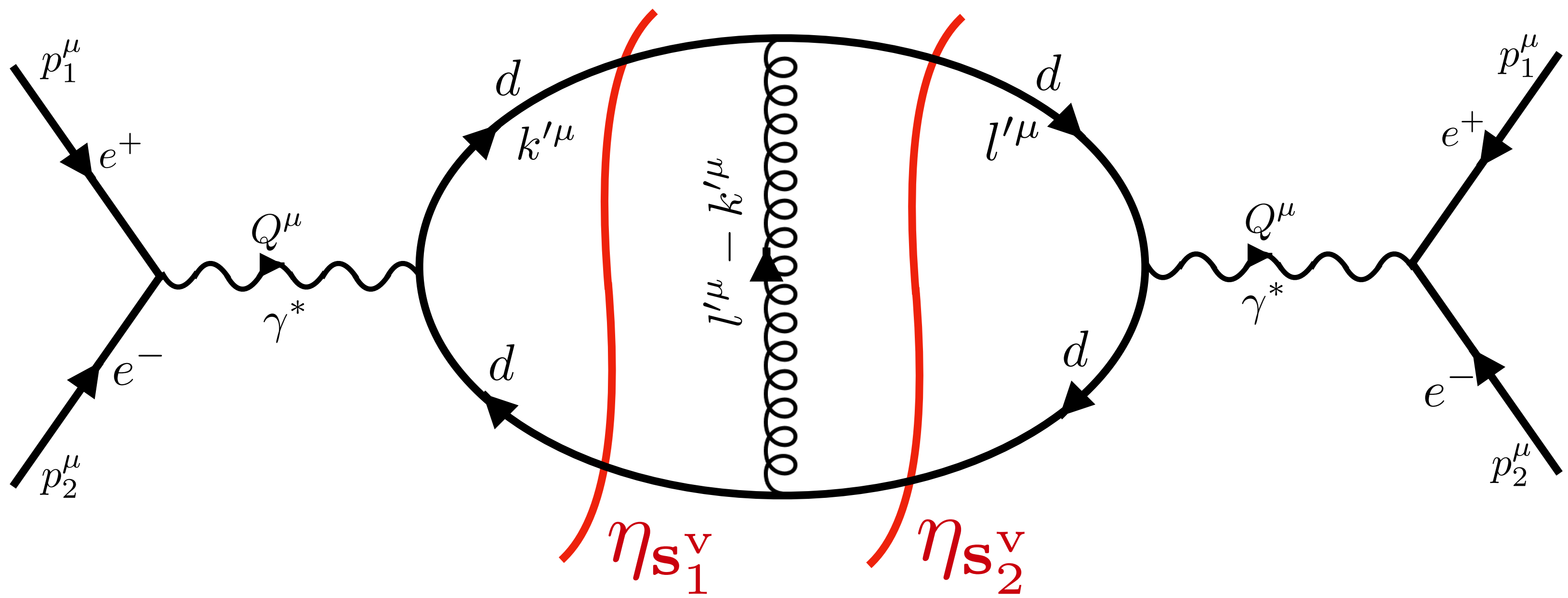

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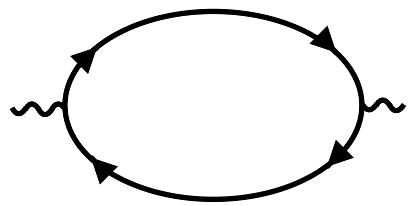
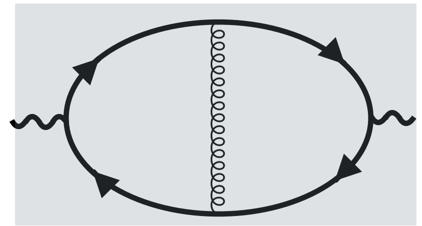
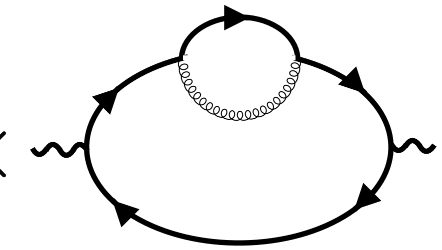
Cancellation of **non-pinched E-surfaces** for : $\eta_{\mathbf{s}_1^{\mathrm{v}}} = \eta_{\mathbf{s}_2^{\mathrm{v}}} \rightarrow k'_y = l'_z$

Numerical results for $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+\, 2\times$  $\Big]$

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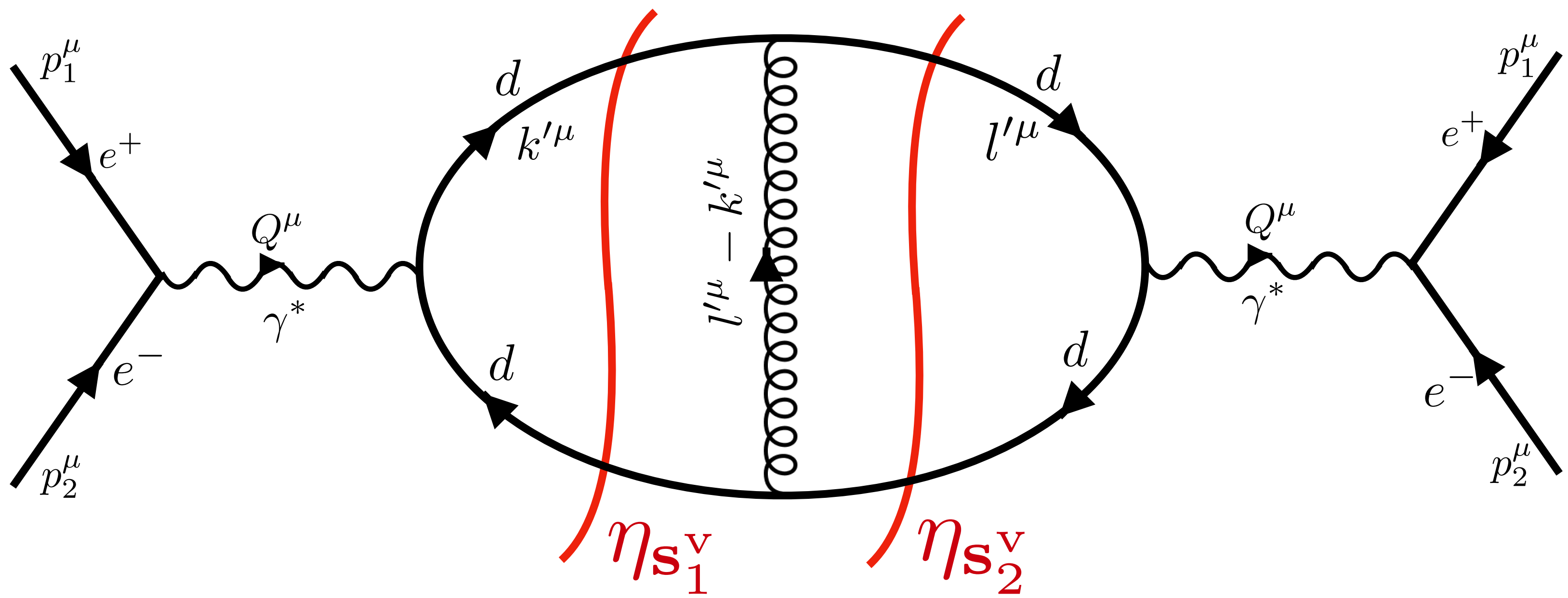

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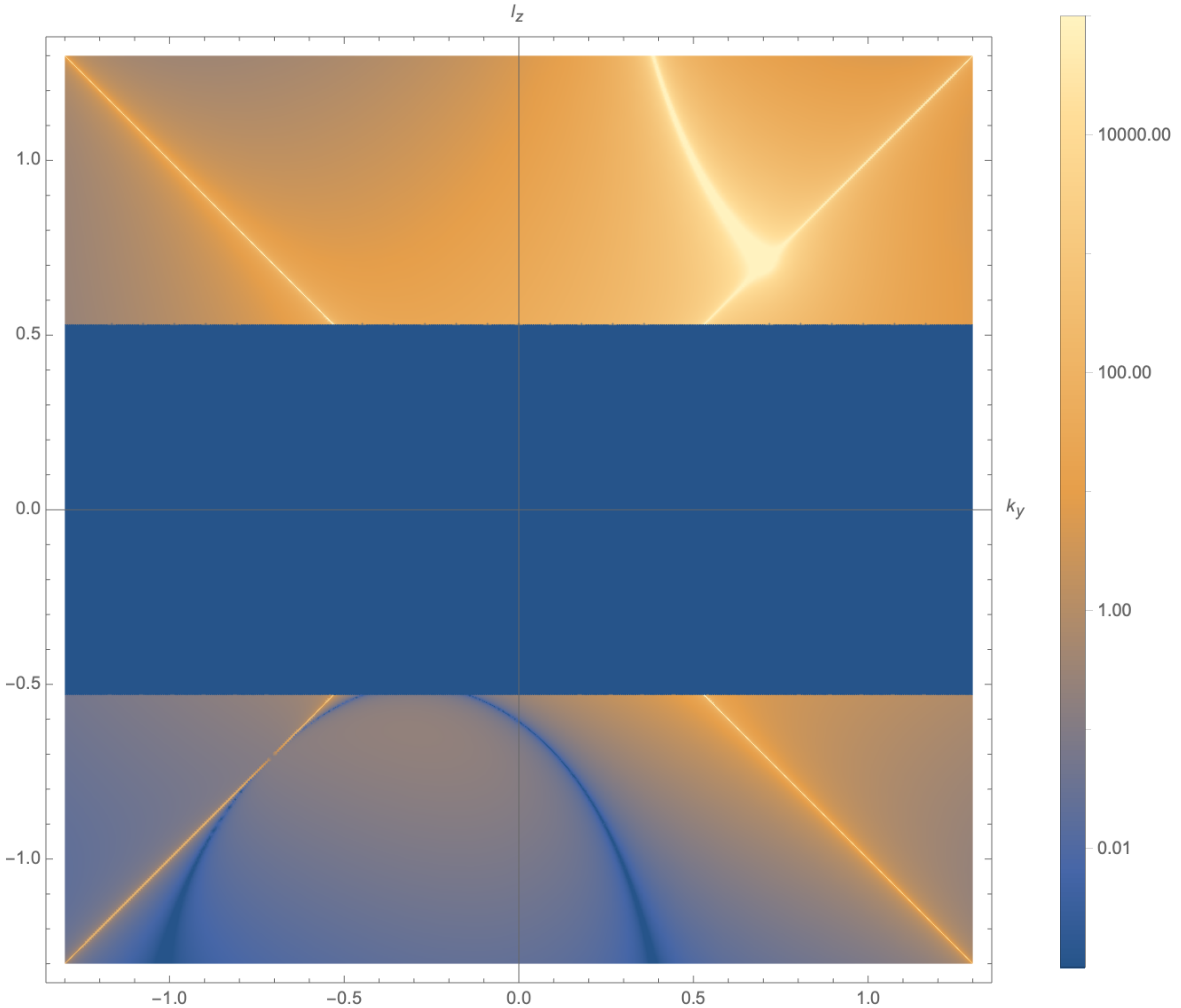
Cancellation of **non-pinched E-surfaces** for : $\eta_{\mathbf{s}_1^{\mathrm{v}}} = \eta_{\mathbf{s}_2^{\mathrm{v}}} \;\to\; k'_y = l'_z$

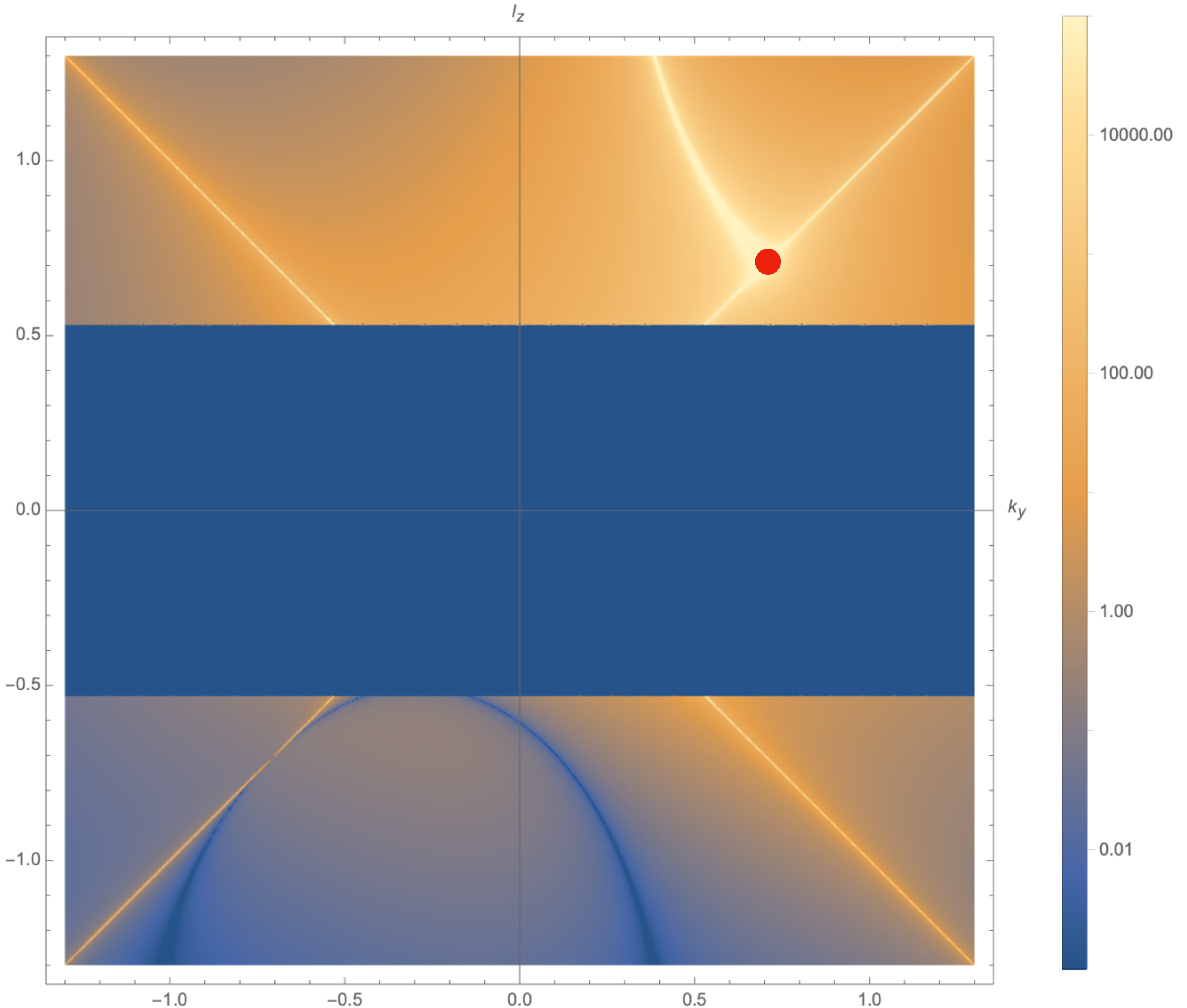
Soft configuration for : $|\vec{l}\,' - \vec{k}| = 0 \;\to\; k'_y = l'_z = \frac{1}{\sqrt{2}}$

$I_{\mathbf{s}_1^{\mathrm{v}}}$



$I_{\mathbf{s}_1^{\mathrm{v}}}$

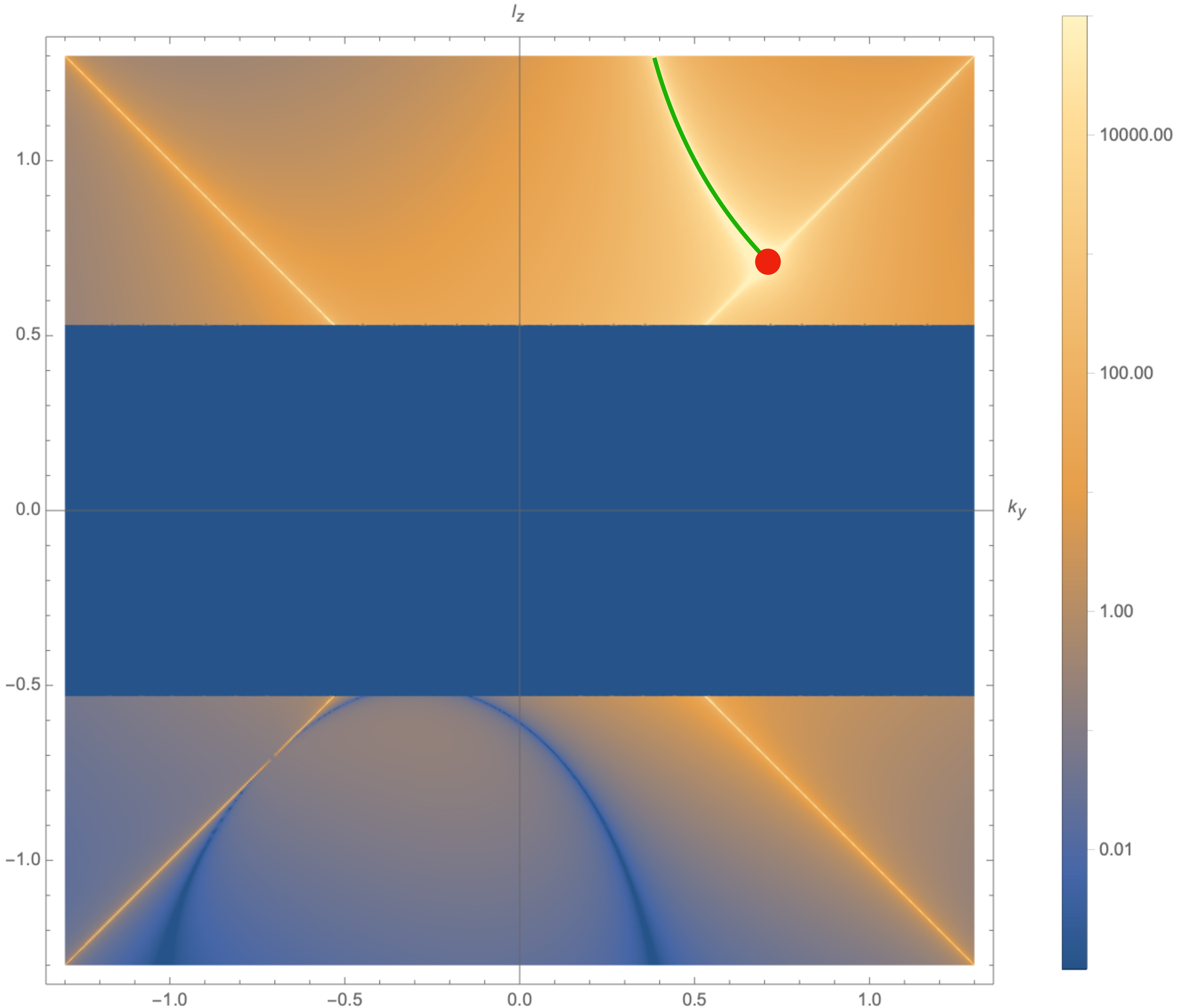
soft



$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

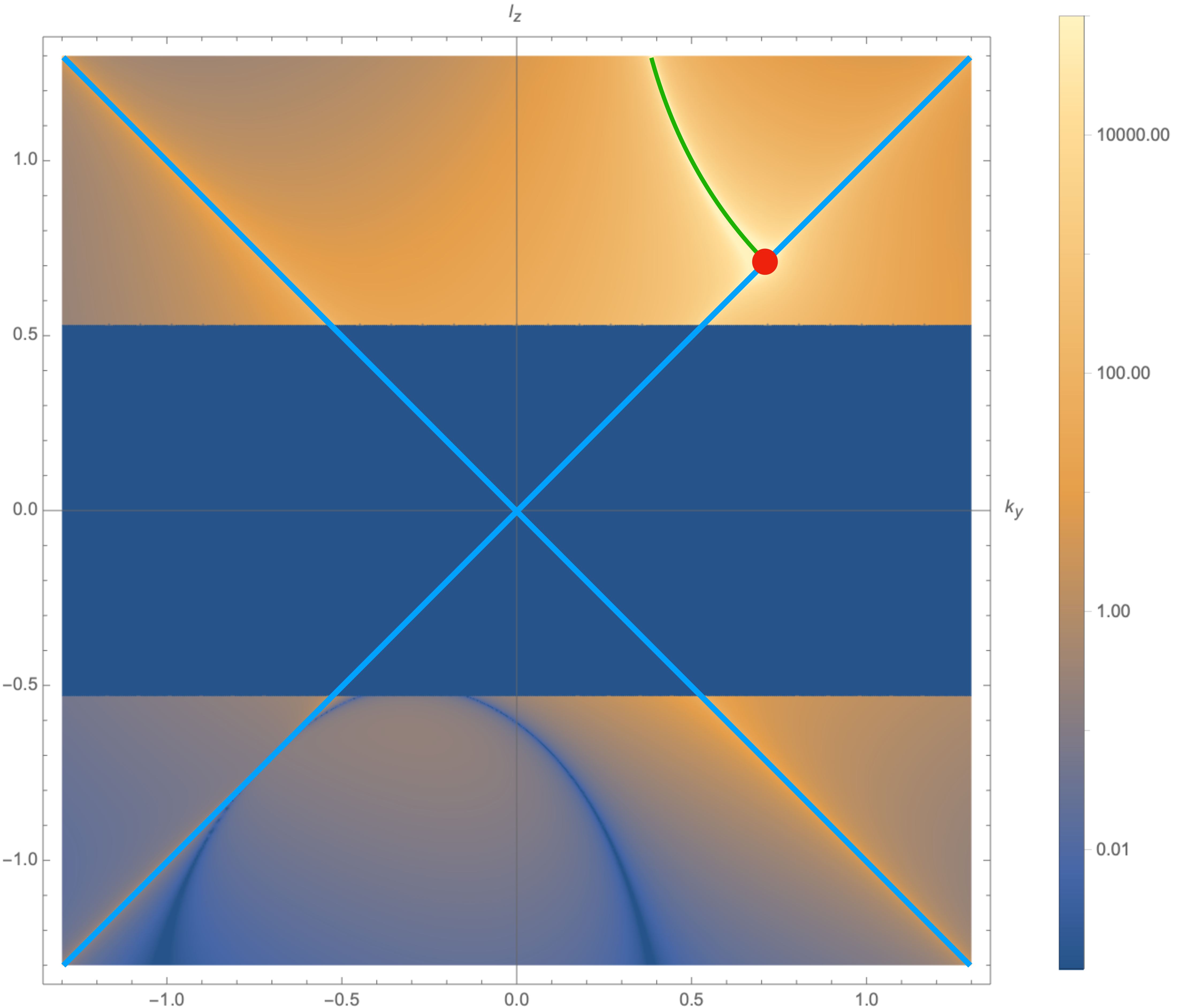


$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

non-pinch cancellation



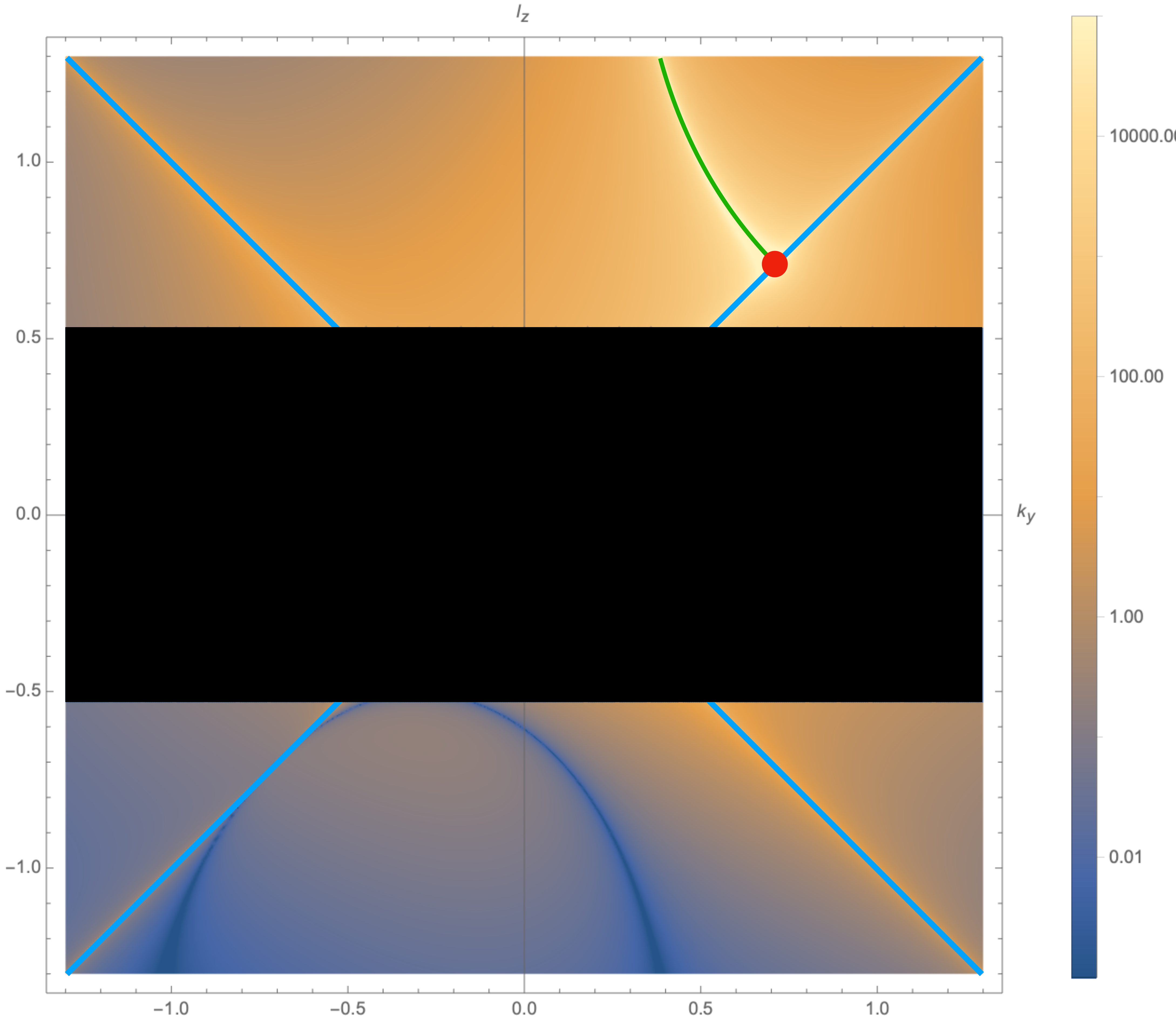
$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

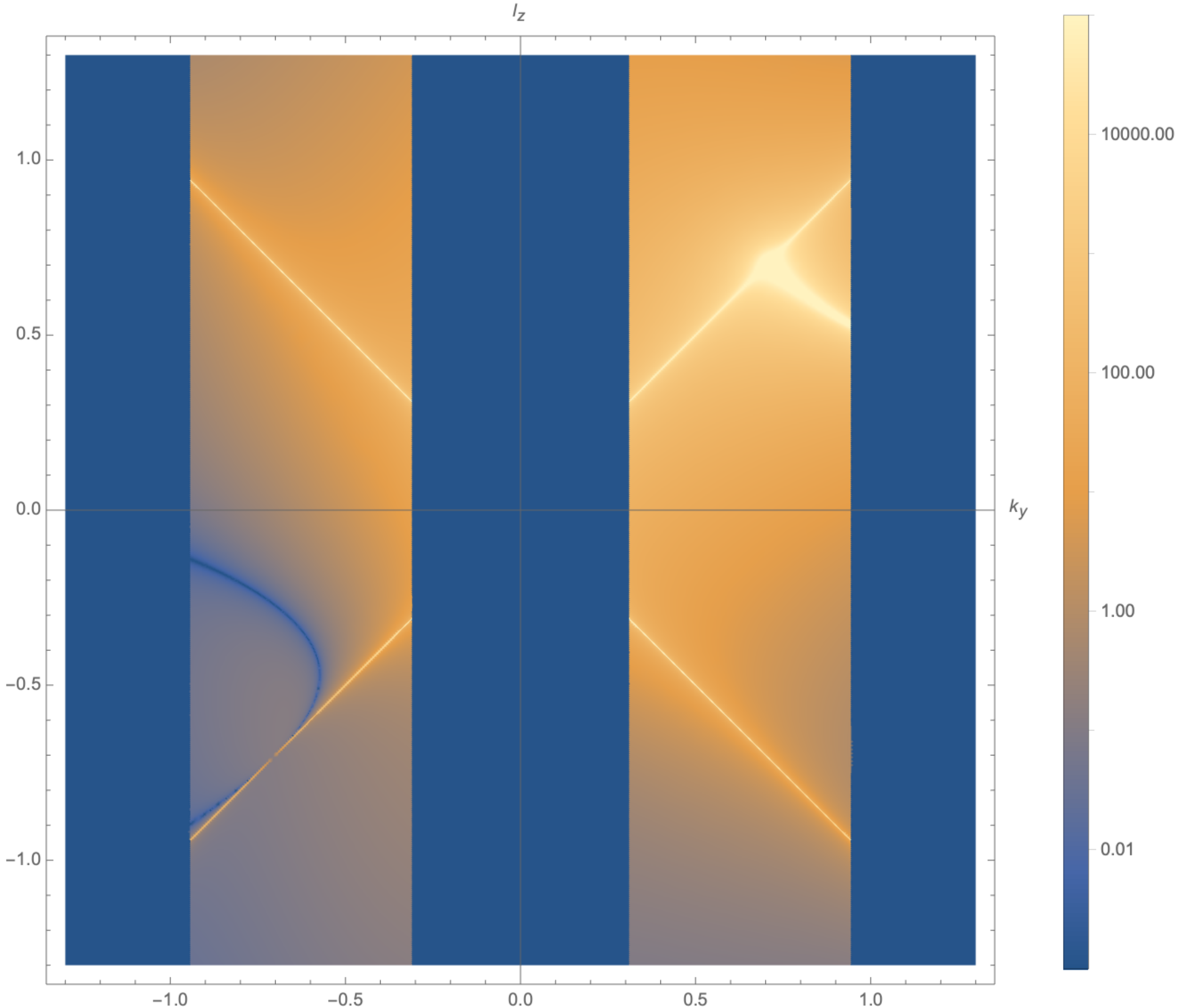
collinear

non-pinch cancellation

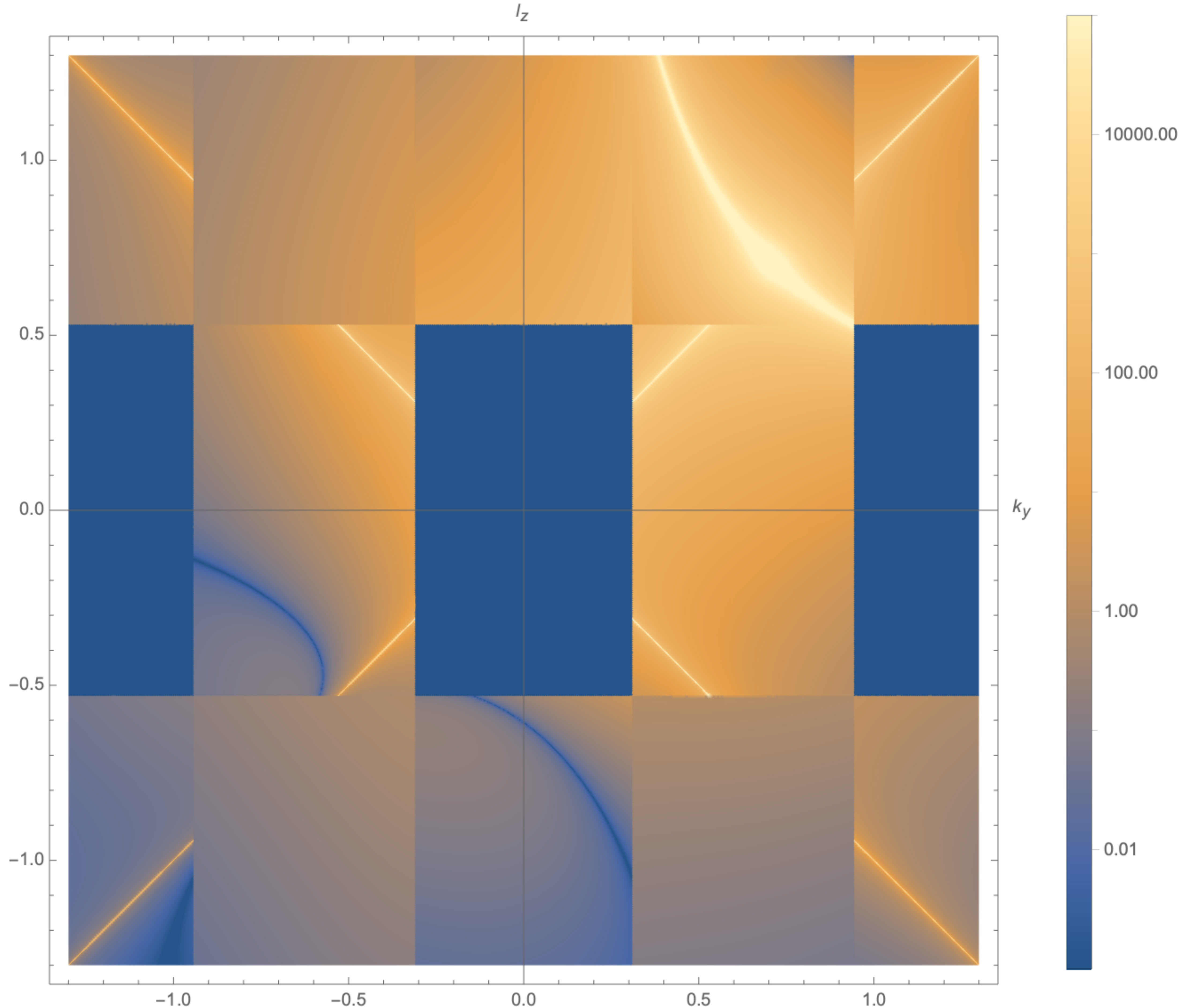
observable cuts



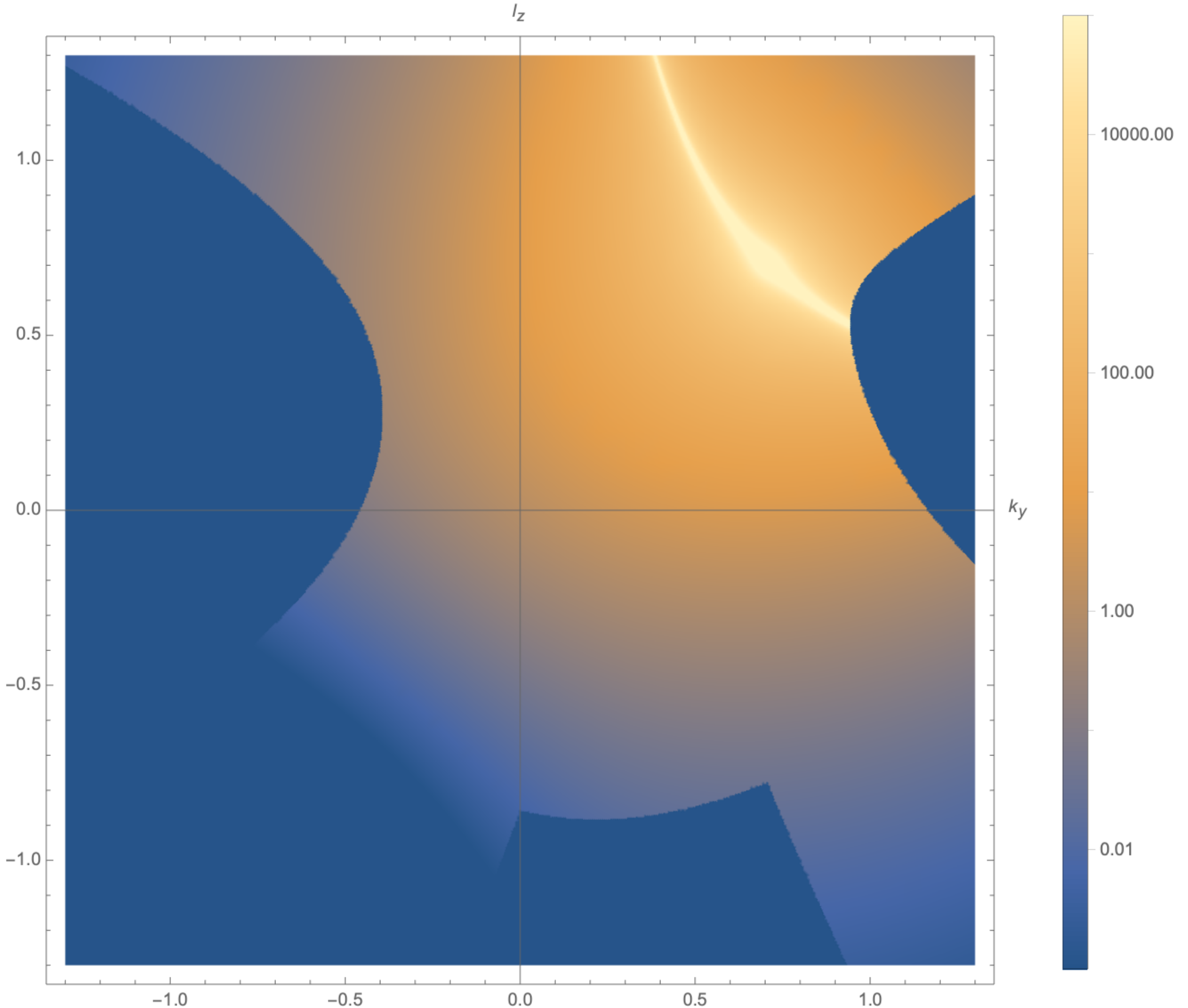
$I_{\mathbf{s}_2^{\mathrm{v}}}$



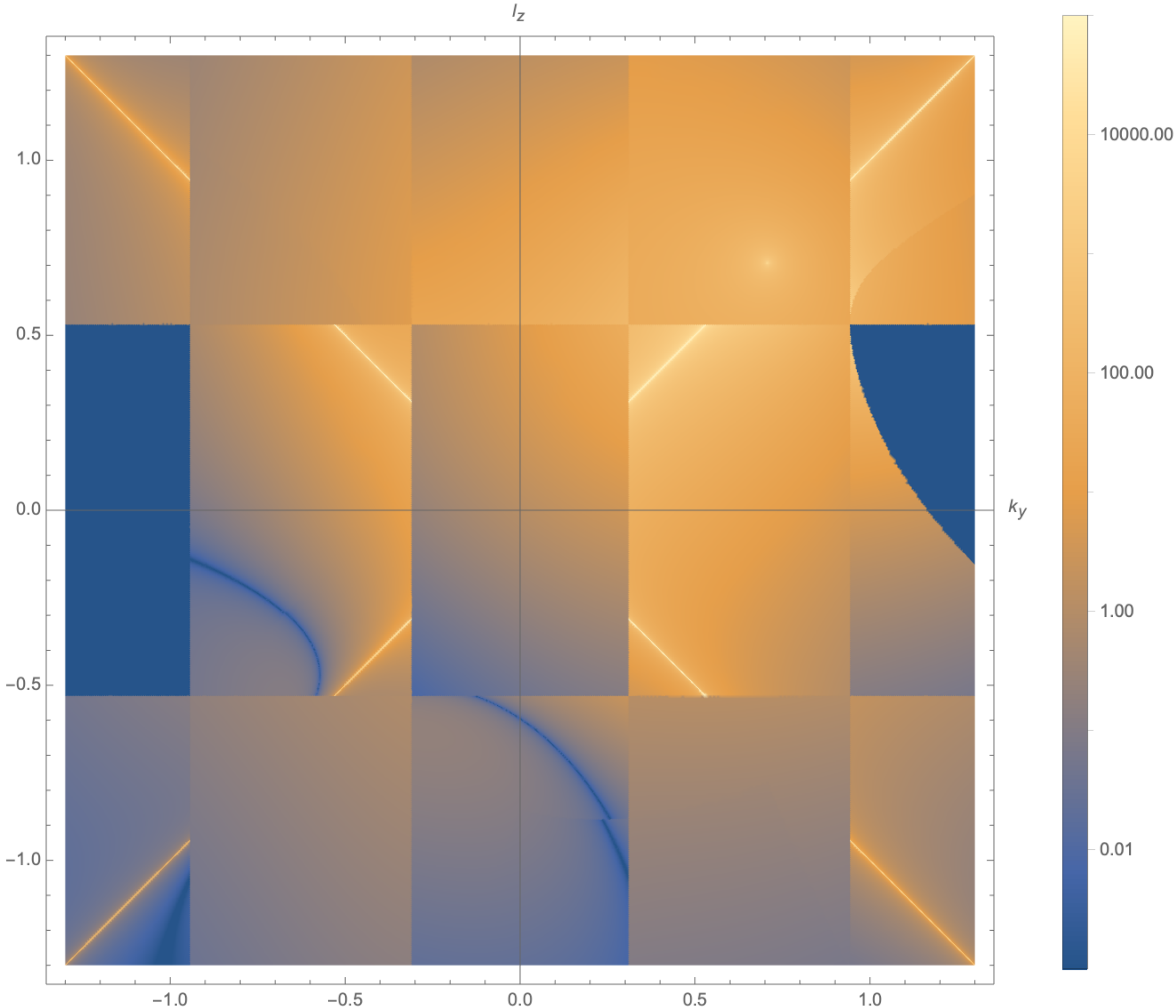
$I_{\mathbf{s}_1^{\mathrm{v}}} + I_{\mathbf{s}_2^{\mathrm{v}}}$



$$I_{\mathbf{s}_1^{\mathrm{r}}} + I_{\mathbf{s}_2^{\mathrm{r}}}$$

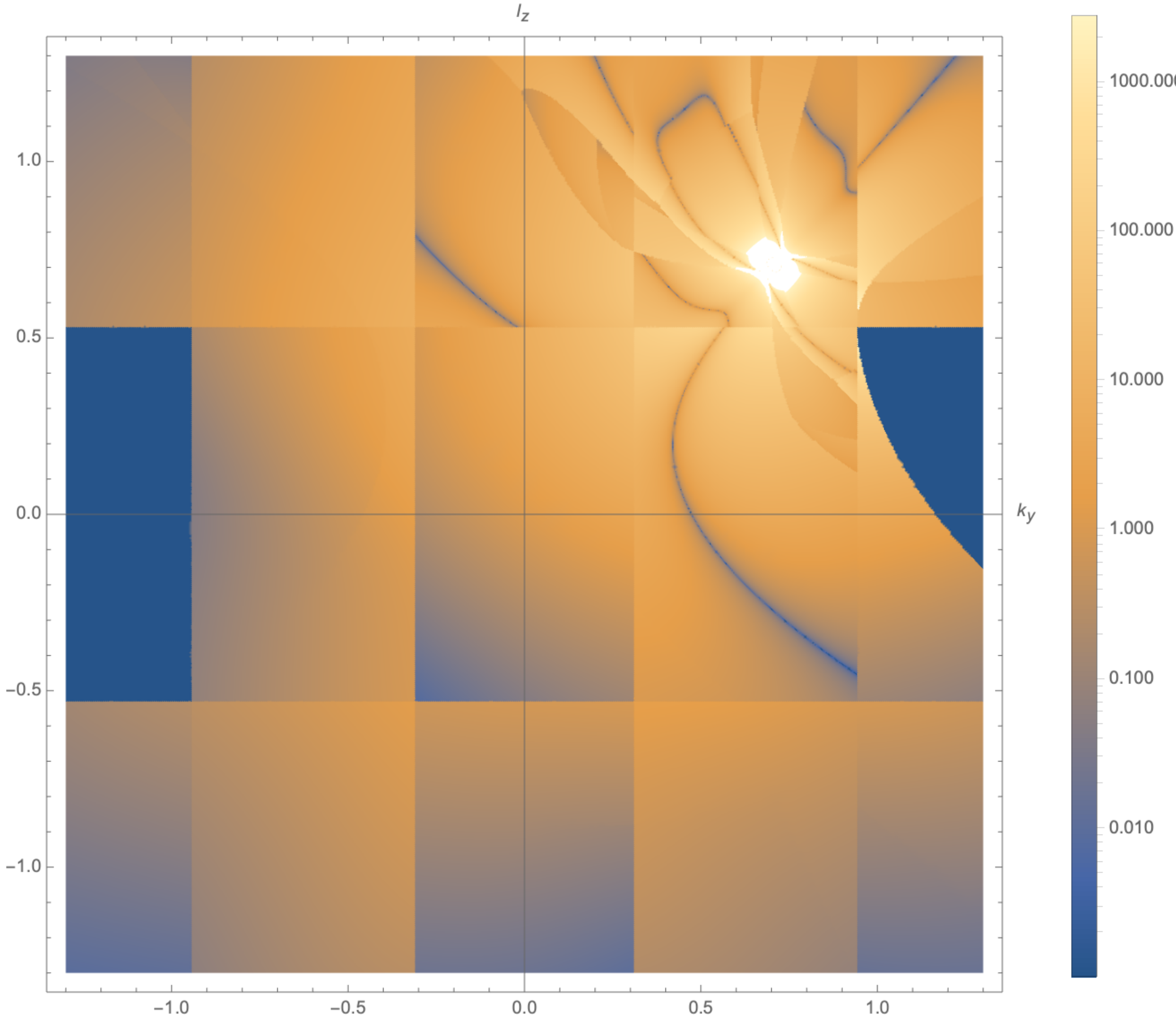


$I_{\sum}$

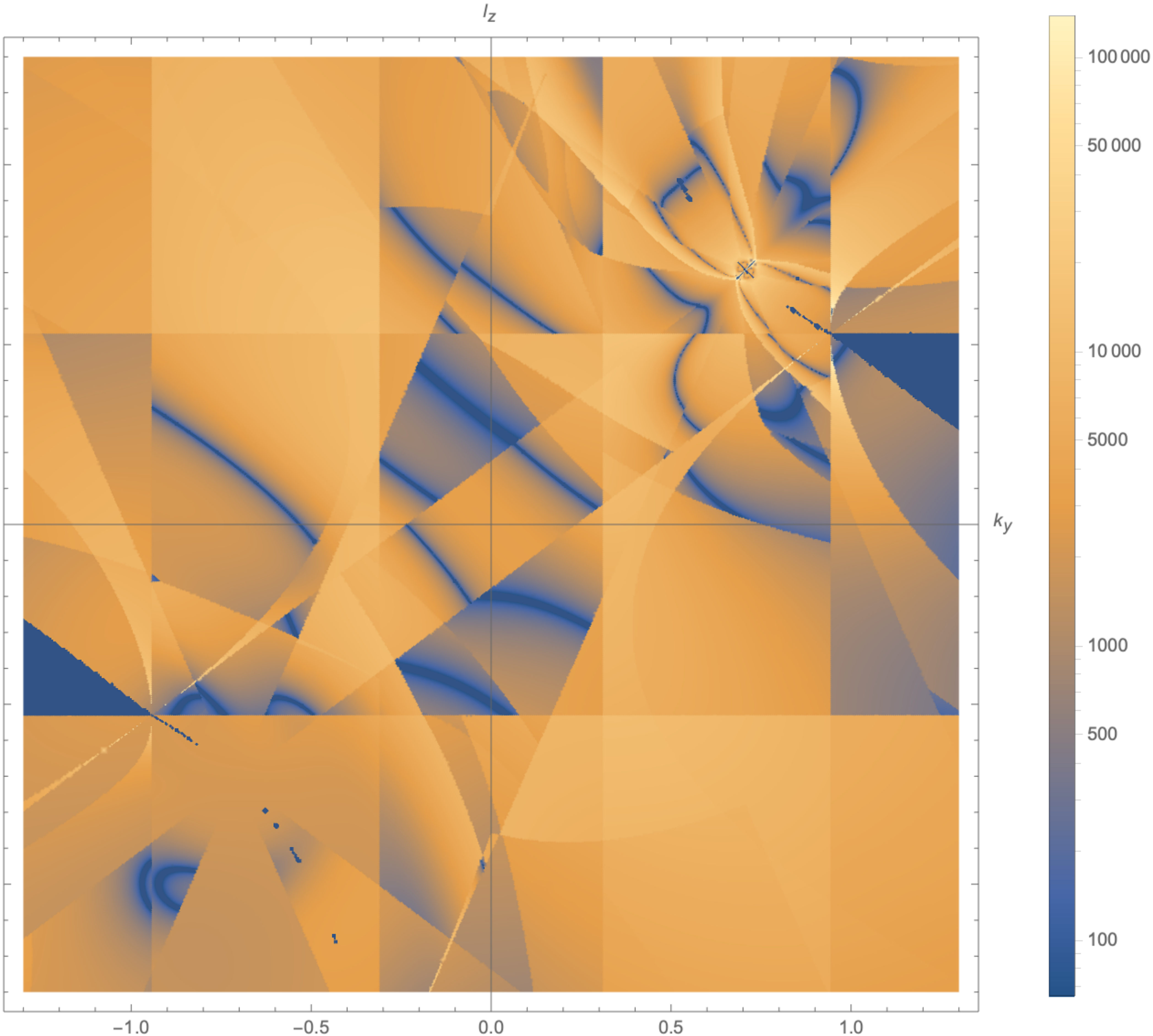


$\mathrm{Re}\left[I_{\sum}\right]$

with deformation



$\mathrm{Re}\left[I_{\sum}\right]$
with
deformation
and
multichanneling



Numerical results

We have computed the following fixed-order processes with **Local Unitarity**:

NLO

$e^+e^- \to \gamma \to jj$ $p_t(j_1)$ distribution

$e^+e^- \to \gamma \to jjj$ semi-inclusive

$e^+e^- \to \gamma \to t\bar{t}h$ (semi-)inclusive

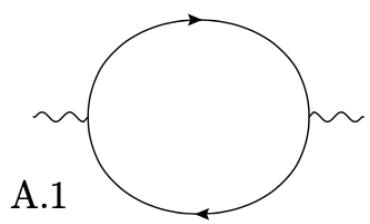
NNLO

$\gamma^\star \to jj$ inclusive

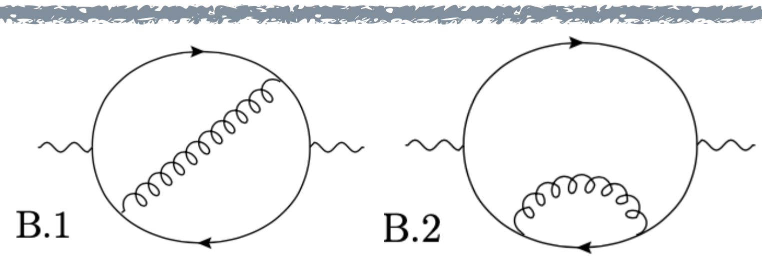
$\gamma^\star \to t\bar{t}$ inclusive

First NNLO cross-sections computed **fully numerically in momentum space**.

LO



NLO



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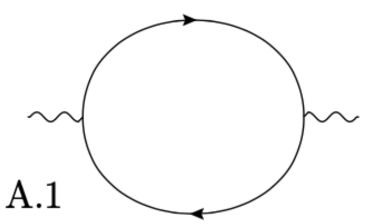
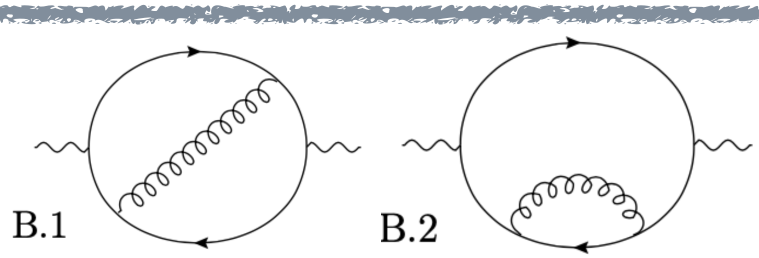
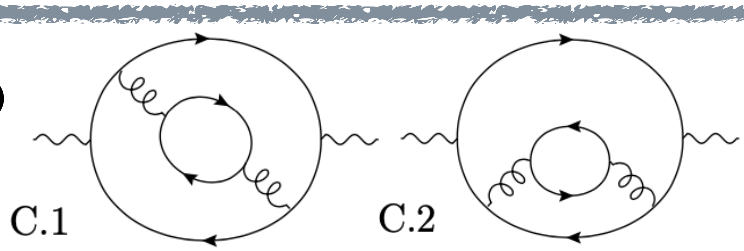


NLO



NNLO
$\mathcal{O}(n_f)$



Numerical results

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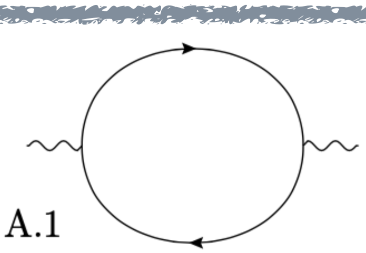
- $e^+e^- \to \gamma \to jj$ $\quad p_t(j_1)$ distribution
- $e^+e^- \to \gamma \to jjj$ $\quad$ semi-inclusive
- $e^+e^- \to \gamma \to t\bar{t}h$ $\quad$ (semi-)inclusive

NNLO

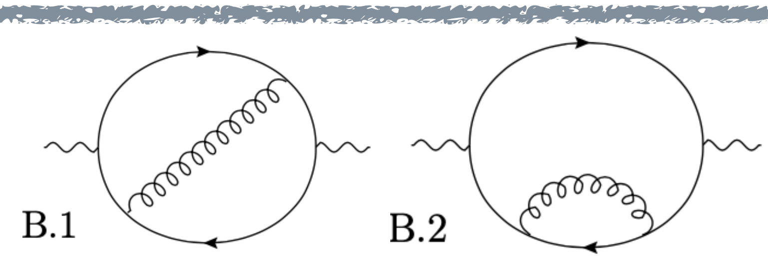
- $\gamma^\star \to jj$ $\quad$ inclusive
- $\gamma^\star \to t\bar{t}$ $\quad$ inclusive

First NNLO cross-sections computed **fully numerically in momentum space**.

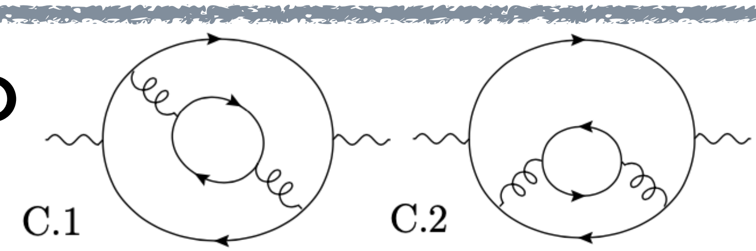
LO

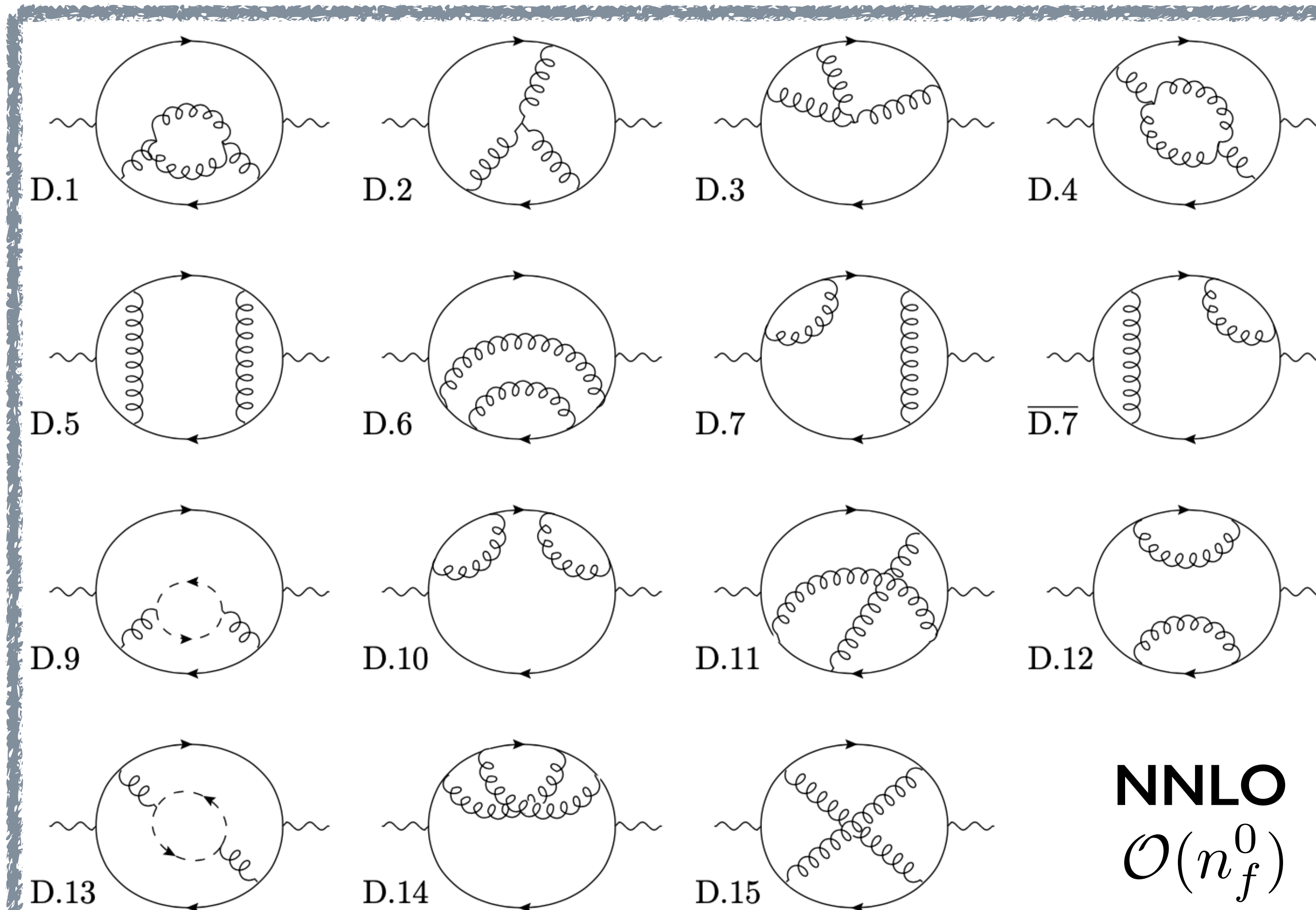


NLO



NNLO $\mathcal{O}(n_f)$





NNLO $\mathcal{O}(n_f^0)$

Numerical results

SG id	Ξ	$\sigma^{(\overline{\text{MS}})}_{\gamma^\star\to jj}$ [GeV^{-2}] $p^2_{\gamma^\star}=\mu_r^2=(400\text{ GeV})^2$	Δ [%]	$\sigma^{\left[\alpha_s^{(\overline{\text{MS}})},m_t^{(\text{OS})}\right]}_{\gamma^\star\to t\bar{t}}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p^2_{\gamma^\star}=(400\text{ GeV})^2$	Δ [%]
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	$5.031049\cdot10^{-01}$	0.0018	$1.387586\cdot10^{+00}$	0.0011
Total		$5.031049\cdot10^{-01}$	0.0018	$1.387586\cdot10^{+00}$	0.0011
NLO		$\mathcal{O}(\alpha_s^1)$			
B.1	1	$5.03926\cdot10^{-02}$	0.0075	$2.52705\cdot10^{-01}$	0.034
B.2	2	$-3.14956\cdot10^{-02}$	0.018	$1.80050\cdot10^{-01}$	0.049
Total		$1.88970\cdot10^{-02}$	0.036	$4.3276\cdot10^{-01}$	0.028
Benchmark		$1.889690\cdot10^{-02}$	0.00053	$4.32831\cdot10^{-01}$	-0.018
NNLO		$\mathcal{O}(\alpha_s^2 n_f)$			
C.1	1	$-4.66342\cdot10^{-04}$	0.019	$-1.0022\cdot10^{-03}$	0.17
C.2	2	$3.8448\cdot10^{-04}$	0.036	$-4.6982\cdot10^{-03}$	0.081
Total		$-8.186\cdot10^{-05}$	0.20	$-5.7004\cdot10^{-03}$	0.073
Benchmark		$-8.1834\cdot10^{-05}$	0.036	$-5.6982\cdot10^{-03}$	0.038
NNLO		$\mathcal{O}(\alpha_s^2)$			
D.1	2	$-2.30886\cdot10^{-03}$	0.017	$3.8886\cdot10^{-02}$	0.031
D.2	2	$6.42018\cdot10^{-03}$	0.0055	$5.6351\cdot10^{-03}$	0.14
D.3	2	$-6.91254\cdot10^{-03}$	0.0046	$1.76075\cdot10^{-02}$	0.055
D.4	1	$3.20278\cdot10^{-03}$	0.0084	$8.8163\cdot10^{-03}$	0.078
D.5	1	$1.68148\cdot10^{-03}$	0.013	$9.200\cdot10^{-04}$	0.79
D.6	2	$6.6698\cdot10^{-04}$	0.027	$5.1058\cdot10^{-03}$	0.15
D.7	2	$-1.30381\cdot10^{-03}$	0.013	$6.7284\cdot10^{-03}$	0.10
$\overline{\text{D.7}}$	2	$-1.30395\cdot10^{-03}$	0.013	$6.7300\cdot10^{-03}$	0.10
D.9	2	$-1.6661\cdot10^{-04}$	0.064	$2.3361\cdot10^{-03}$	0.12
D.10	2	$6.64155\cdot10^{-04}$	0.012	$3.7418\cdot10^{-03}$	0.14
D.11	2	$2.34300\cdot10^{-04}$	0.031	$2.0845\cdot10^{-03}$	0.083
D.12	1	$4.11063\cdot10^{-04}$	0.017	$3.5114\cdot10^{-03}$	0.12
D.13	1	$2.41514\cdot10^{-04}$	0.026	$8.222\cdot10^{-04}$	0.19
D.14	2	$5.8386\cdot10^{-05}$	0.088	$1.76075\cdot10^{-02}$	0.055
D.15	1	$-1.75957\cdot10^{-04}$	0.022	$-7.242\cdot10^{-04}$	0.14
Total		$1.40910\cdot10^{-03}$	0.056	$1.04214\cdot10^{-01}$	0.024
Benchmark		$1.40941\cdot10^{-03}$	-0.022	$1.0386\cdot10^{-01}$	0.34

Analytic benchmarks :

$$\sigma^{\overline{\text{MS}}}_{\gamma^\star\to X}(p^2,\mu_r^2=p^2)\supset K_X n_f\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^{(\text{LO})}_{\gamma^\star\to X}(p^2)$$

Numerical results

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$$\gamma^\star\to jj$$

$$\boxed{K_{jj}}=-C_F\left[11-8\zeta_3\right]$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

Numerical results

SG id	Ξ	$\sigma^{(\overline{\text{MS}})}_{\gamma^\star\to jj}$ [GeV^{-2}] $p^2_{\gamma^\star}=\mu_r^2=(400\text{ GeV})^2$	Δ [%]	$\sigma^{[\alpha_s^{(\overline{\text{MS}})},m_t^{(\text{OS})}]}_{\gamma^\star\to t\bar{t}}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p^2_{\gamma^\star}=(400\text{ GeV})^2$	Δ [%]
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	$5.031049\cdot 10^{-01}$	0.0018	$1.387586\cdot 10^{+00}$	0.0011
Total		$5.031049\cdot 10^{-01}$	0.0018	$1.387586\cdot 10^{+00}$	0.0011
NLO		$\mathcal{O}(\alpha_s^1)$			
B.1	1	$5.03926\cdot 10^{-02}$	0.0075	$2.52705\cdot 10^{-01}$	0.034
B.2	2	$-3.14956\cdot 10^{-02}$	0.018	$1.80050\cdot 10^{-01}$	0.049
Total		$1.88970\cdot 10^{-02}$	0.036	$4.3276\cdot 10^{-01}$	0.028
Benchmark		$1.889690\cdot 10^{-02}$	0.00053	$4.32831\cdot 10^{-01}$	-0.018
NNLO		$\mathcal{O}(\alpha_s^2 n_f)$			
C.1	1	$-4.66342\cdot 10^{-04}$	0.019	$-1.0022\cdot 10^{-03}$	0.17
C.2	2	$3.8448\cdot 10^{-04}$	0.036	$-4.6982\cdot 10^{-03}$	0.081
Total		$-8.186\cdot 10^{-05}$	0.20	$-5.7004\cdot 10^{-03}$	0.073
Benchmark		$-8.1834\cdot 10^{-05}$	0.036	$-5.6982\cdot 10^{-03}$	0.038
NNLO		$\mathcal{O}(\alpha_s^2)$			
D.1	2	$-2.30886\cdot 10^{-03}$	0.017	$3.8886\cdot 10^{-02}$	0.031
D.2	2	$6.42018\cdot 10^{-03}$	0.0055	$5.6351\cdot 10^{-03}$	0.14
D.3	2	$-6.91254\cdot 10^{-03}$	0.0046	$1.76075\cdot 10^{-02}$	0.055
D.4	1	$3.20278\cdot 10^{-03}$	0.0084	$8.8163\cdot 10^{-03}$	0.078
D.5	1	$1.68148\cdot 10^{-03}$	0.013	$9.200\cdot 10^{-04}$	0.79
D.6	2	$6.6698\cdot 10^{-04}$	0.027	$5.1058\cdot 10^{-03}$	0.15
D.7	2	$-1.30381\cdot 10^{-03}$	0.013	$6.7284\cdot 10^{-03}$	0.10
$\overline{\text{D.7}}$	2	$-1.30395\cdot 10^{-03}$	0.013	$6.7300\cdot 10^{-03}$	0.10
D.9	2	$-1.6661\cdot 10^{-04}$	0.064	$2.3361\cdot 10^{-03}$	0.12
D.10	2	$6.64155\cdot 10^{-04}$	0.012	$3.7418\cdot 10^{-03}$	0.14
D.11	2	$2.34300\cdot 10^{-04}$	0.031	$2.0845\cdot 10^{-03}$	0.083
D.12	1	$4.11063\cdot 10^{-04}$	0.017	$3.5114\cdot 10^{-03}$	0.12
D.13	1	$2.41514\cdot 10^{-04}$	0.026	$8.222\cdot 10^{-04}$	0.19
D.14	2	$5.8386\cdot 10^{-05}$	0.088	$1.76075\cdot 10^{-02}$	0.055
D.15	1	$-1.75957\cdot 10^{-04}$	0.022	$-7.242\cdot 10^{-04}$	0.14
Total		$1.40910\cdot 10^{-03}$	0.056	$1.04214\cdot 10^{-01}$	0.024
Benchmark		$1.40941\cdot 10^{-03}$	-0.022	$1.0386\cdot 10^{-01}$	0.34

Analytic benchmarks :

$$\sigma^{\overline{\text{MS}}}_{\gamma^\star\to X}(p^2,\mu_r^2=p^2)\supset K_X n_f\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^{(\text{LO})}_{\gamma^\star\to X}(p^2)$$

$$\gamma^\star\to jj$$

$$\boxed{K_{jj}}=-C_F\left[11-8\zeta_3\right]$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

$$\gamma^\star\to t\bar{t}$$

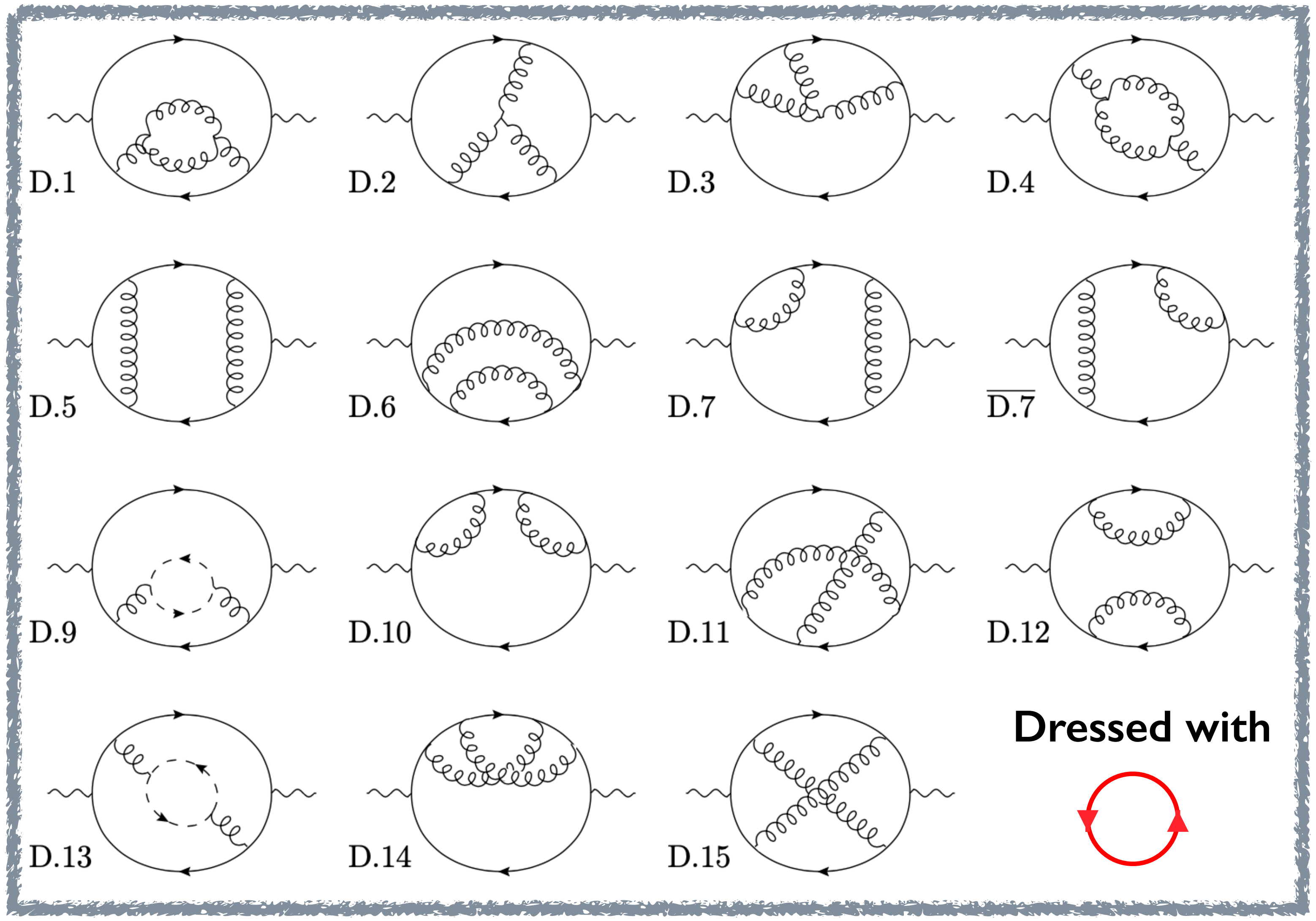
$$\boxed{K_{t\bar{t}}}=$$

$$\begin{aligned}
\delta^{(2)} = & -\frac{(3-v^2)\,(1+v^2)}{6}\times \\
& \Big\{ \mathrm{Li}_3(p) - 2\,\mathrm{Li}_3(1-p) - 3\,\mathrm{Li}_3(p^2) - 4\,\mathrm{Li}_3\Big(\frac{p}{1+p}\Big) - 5\,\mathrm{Li}_3(1-p^2) + \frac{11}{2}\zeta(3) \\
& + \mathrm{Li}_2(p)\,\ln\Big(\frac{4\,(1-v^2)}{v^4}\Big) + 2\,\mathrm{Li}_2(p^2)\,\ln\Big(\frac{1-v^2}{2\,v^2}\Big) + 2\,\zeta(2)\Big[\ln p - \ln\Big(\frac{1-v^2}{4\,v}\Big)\Big] \\
& - \frac{1}{6}\ln\Big(\frac{1+v}{2}\Big)\Big[36\ln 2\ln p - 44\ln^2 p + 49\ln p\ln\Big(\frac{1-v^2}{4}\Big) + \ln^2\Big(\frac{1-v^2}{4}\Big)\Big] \\
& - \frac{1}{2}\ln p\ln v\Big[36\ln 2 + 21\ln p + 16\ln v - 22\ln(1-v^2)\Big]\Big\} \\
& + \frac{1}{24}\Big\{(15-6\,v^2-v^4)\Big(\mathrm{Li}_2(p)+\mathrm{Li}_2(p^2)\Big) + 3\,(7-22\,v^2+7\,v^4)\,\mathrm{Li}_2(p) \\
& - (1-v)\,(51-45\,v-27\,v^2+5\,v^3)\,\zeta(2) \\
& + \frac{(1+v)\,(-9+33\,v-9\,v^2-15\,v^3+4\,v^4)}{v}\ln^2 p \\
& + \Big[(33+22\,v^2-7\,v^4)\ln 2 - 10\,(3-v^2)\,(1+v^2)\ln v \\
& \qquad - (15-22\,v^2+3\,v^4)\ln\Big(\frac{1-v^2}{4\,v^2}\Big)\Big]\ln p \\
& + 2\,v\,(3-v^2)\ln\Big(\frac{4\,(1-v^2)}{v^4}\Big)\Big[\ln v - 3\ln\Big(\frac{1-v^2}{4\,v}\Big)\Big] \\
& + \frac{237-96\,v+62\,v^2+32\,v^3-59\,v^4}{4}\ln p - 16\,v\,(3-v^2)\ln\Big(\frac{1+v}{4}\Big) \\
& - 2\,v\,(39-17\,v^2)\ln\Big(\frac{1-v^2}{2\,v^2}\Big) - \frac{v\,(75-29\,v^2)}{2}\Big\}.\ \ldots
\end{aligned}\tag{B.3}$$

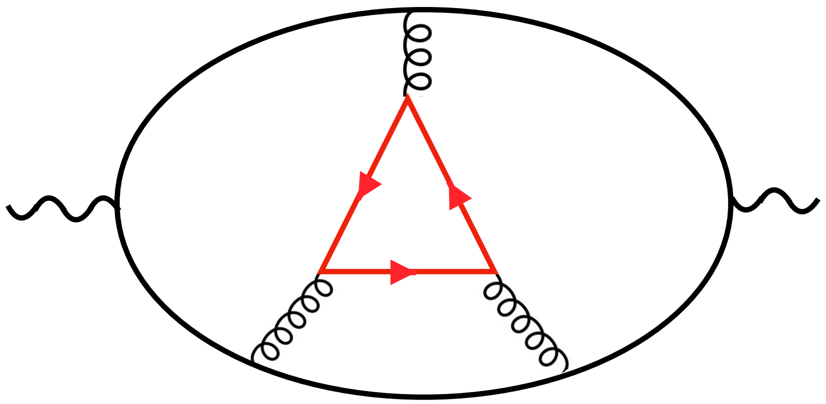
[Chetyrkin, Kuehn, Steinhauser, arxiv : 9606230]

Preliminary N3LO results

n_f contributions :



+ new topologies, such as:



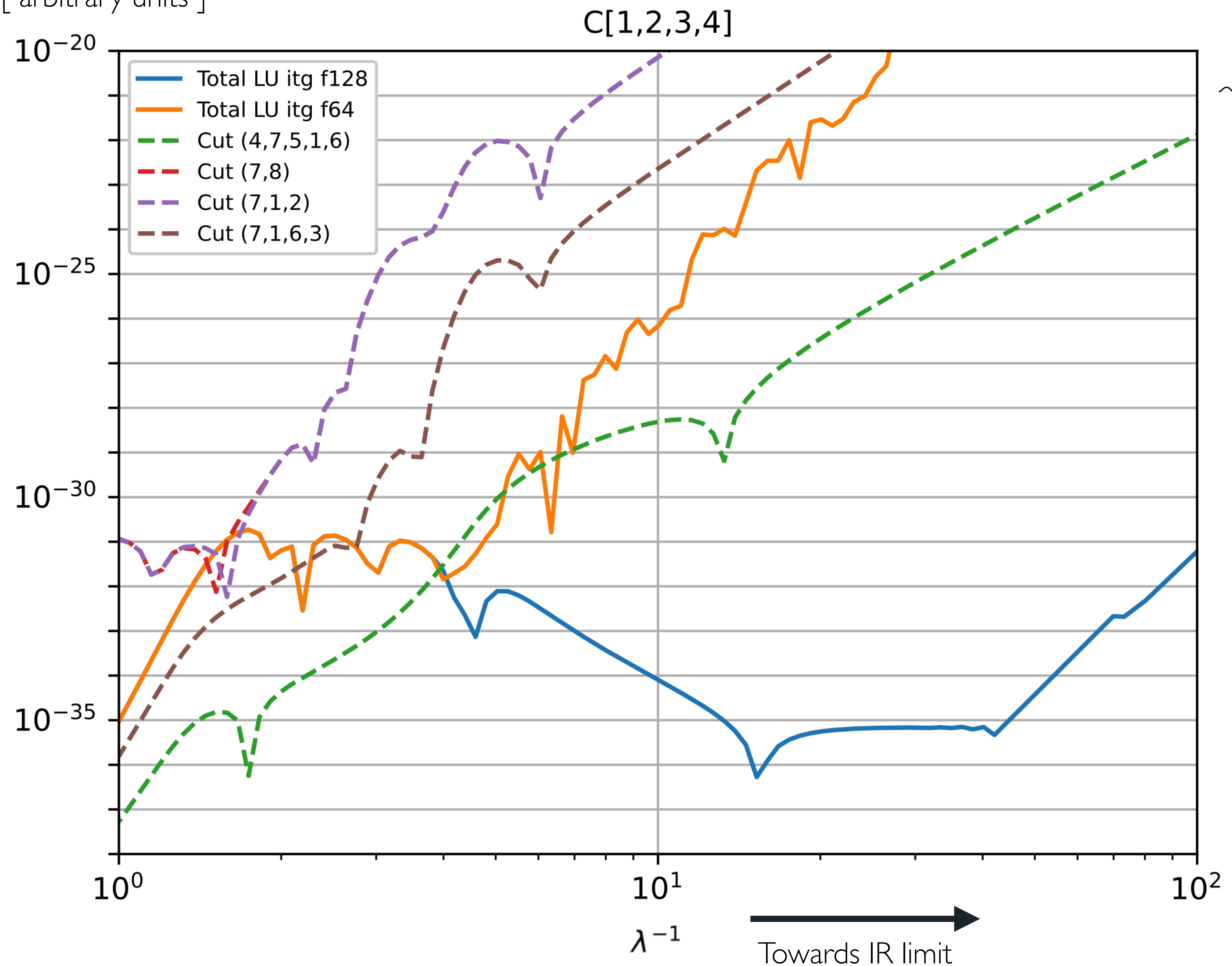
$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f),(\text{MC LU})} = -77.1(1.7)$$

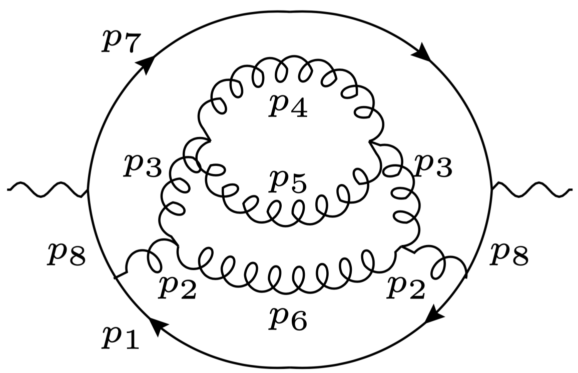
$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f)} = -C_F^2\left(\frac{29}{2} - 152\zeta_3 + 160\zeta_5\right) - C_F C_A\left(\frac{15520}{27} - \frac{88}{3}\zeta_2 - \frac{3584}{9}\zeta_3 - \frac{80}{3}\zeta_5\right) = -76.8086$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

Testing IR quadruple collinear Limits

[arbitrary units]





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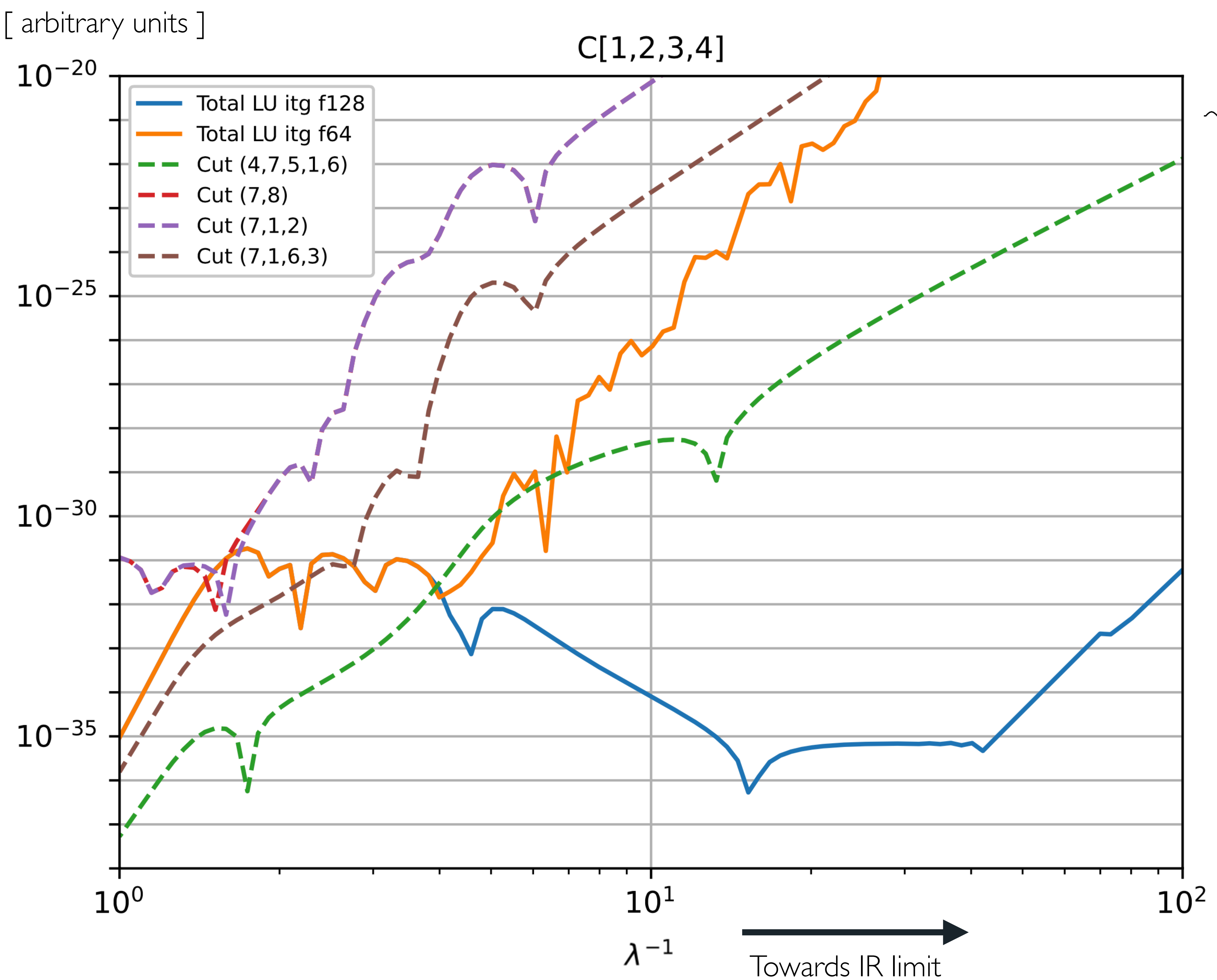
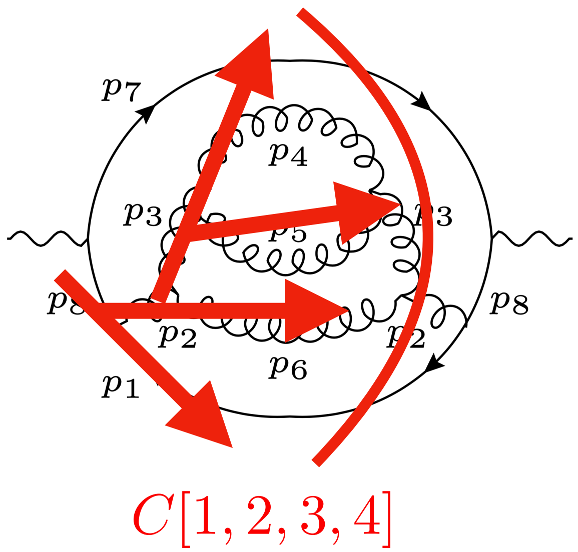
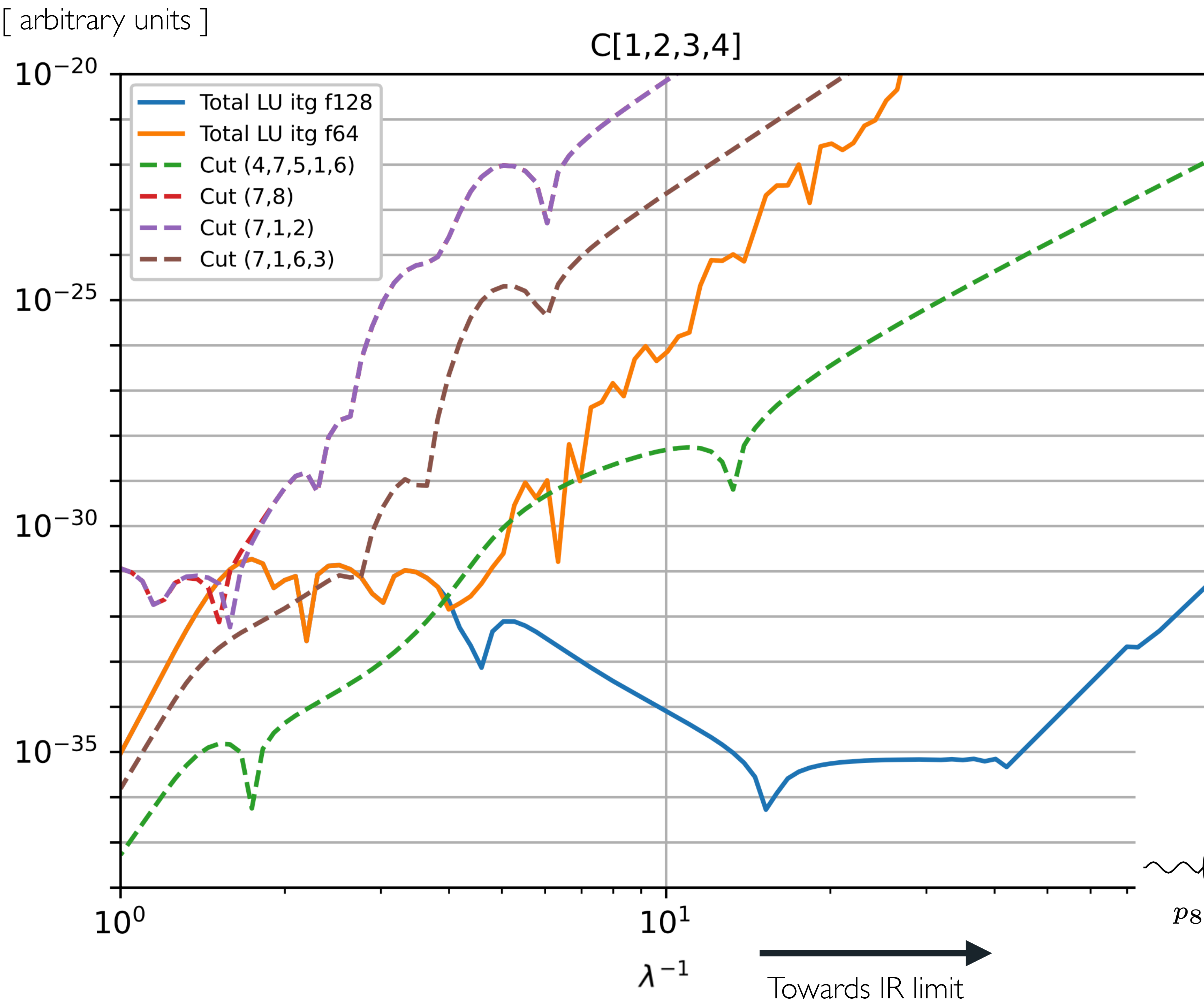
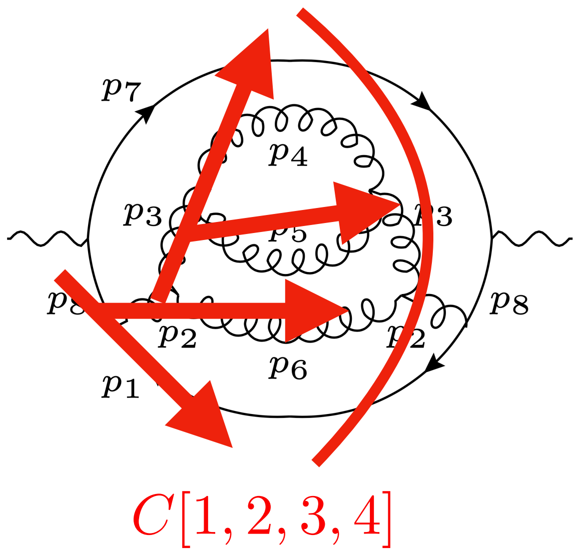
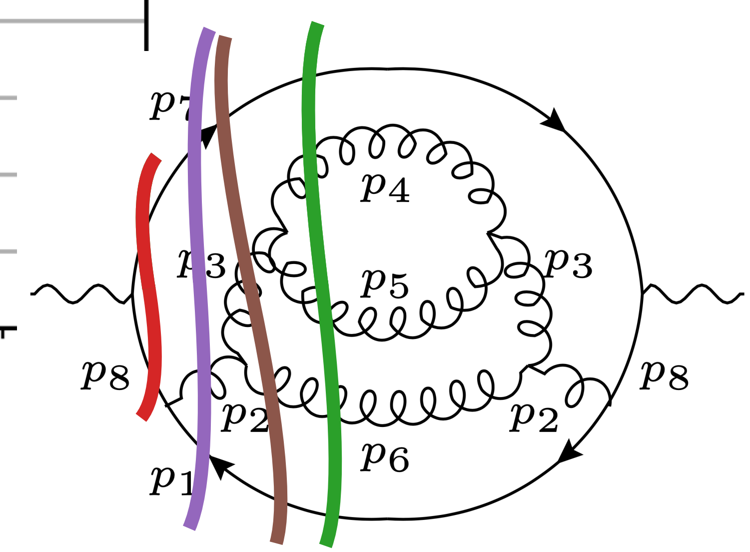




Testing IR quadruple collinear Limits







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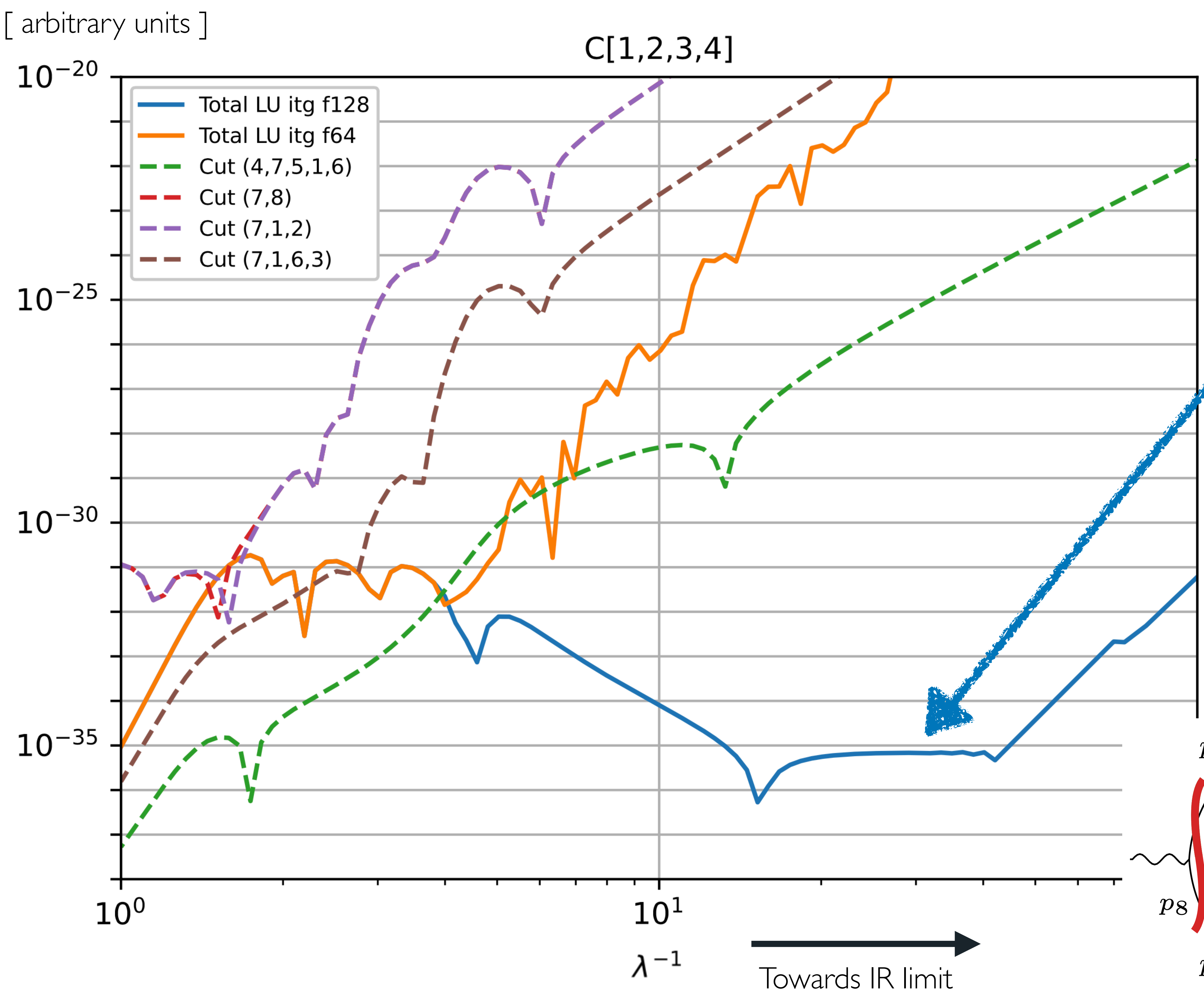
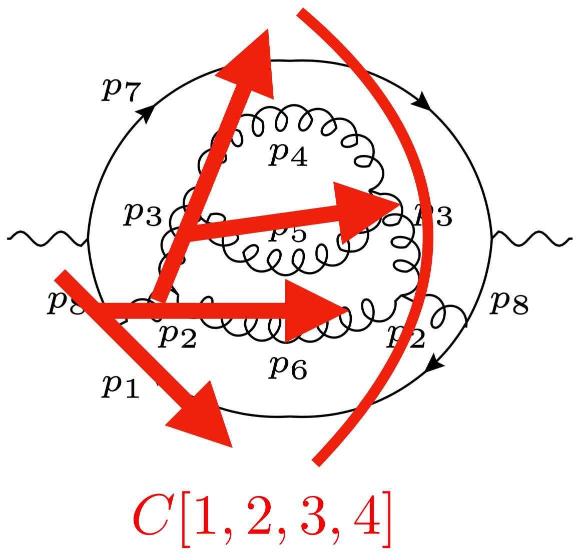
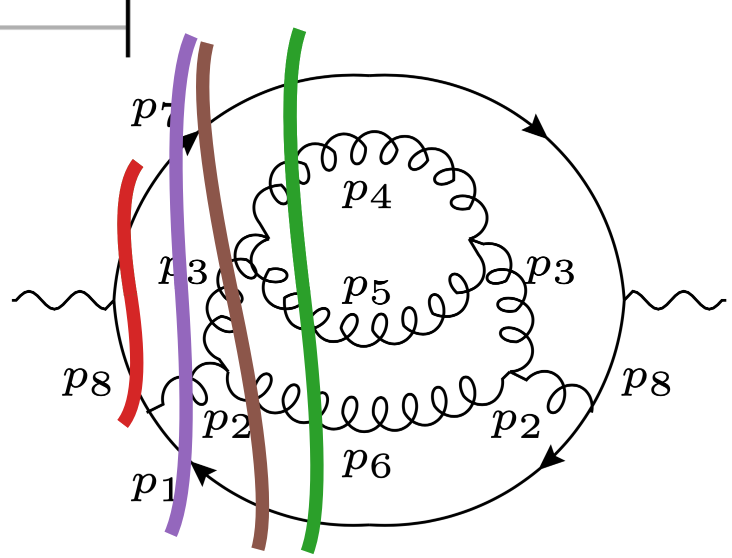




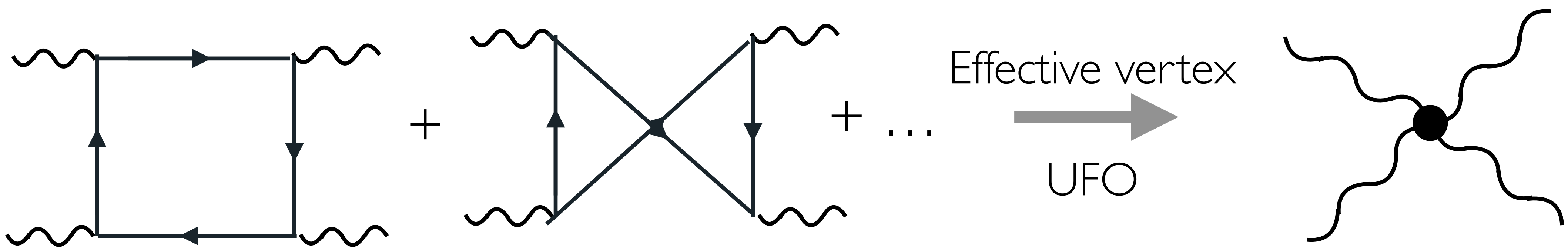
LU integrand always **goes to a constant** on **collinear limits** without incl. *any* scaling of the measure ! No residual integrable singularity.



Photon-Photon scattering in HI UPC

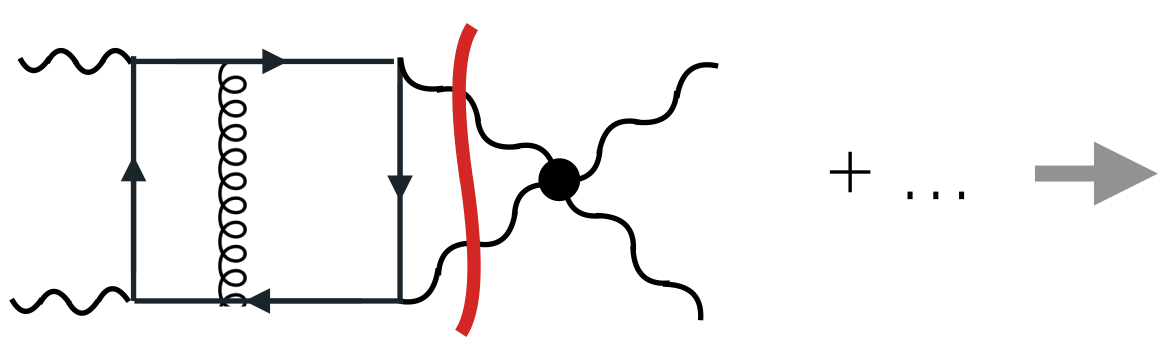
[In collaboration with H.S. -Shao, M. Fraaije, E. Chaubet]

- Too early to present / show much, but alphaLoop delivered a first NLO unknown x-sec!



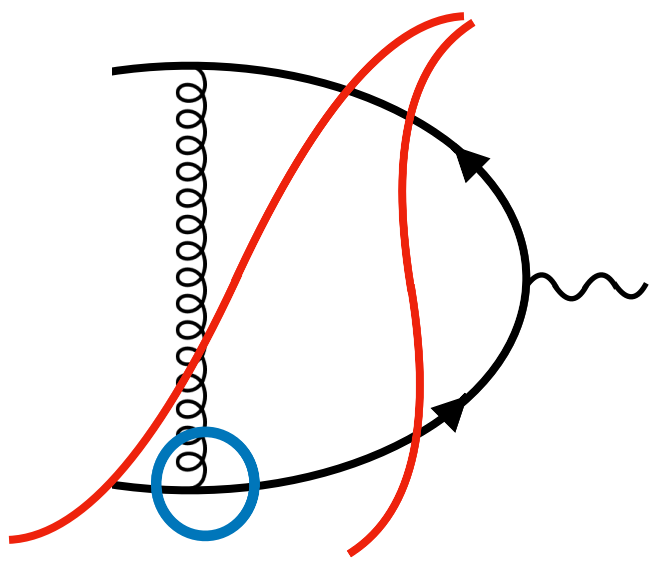
$$\sim \sum_{i=1}^{3} C_i(s,t,u,m_f^2)T_i^{\mu_1\mu_2\mu_3\mu_4}(\{p_i\})$$

- In alphaloop we then do



Yielding our first "prediction" for a piece of a yet unknown cross-section: NLO photon scattering. Successful validation vs analytics!

Initial-state Singularities



KLN CAN WORK FOR INITIAL-STATE !

Initial-state singularities: Idea

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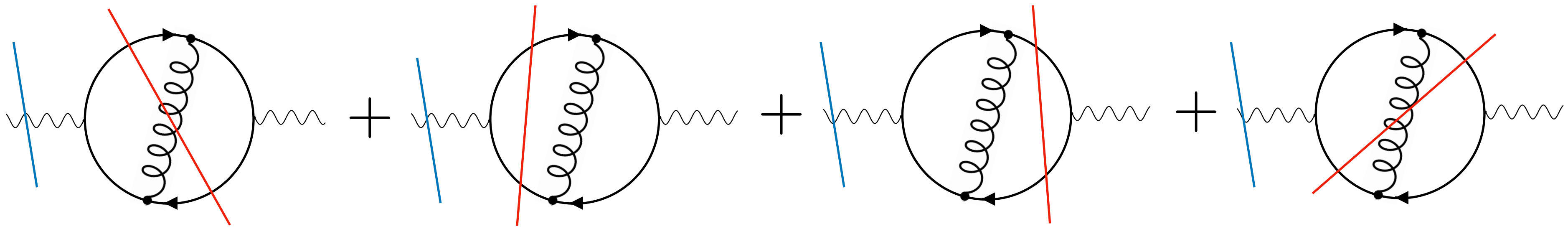
For $e^+e^- \to 2j$ @ NLO we used



(Including all degenerate configurations, higher final-state multiplicities)

Initial-state singularities: Idea

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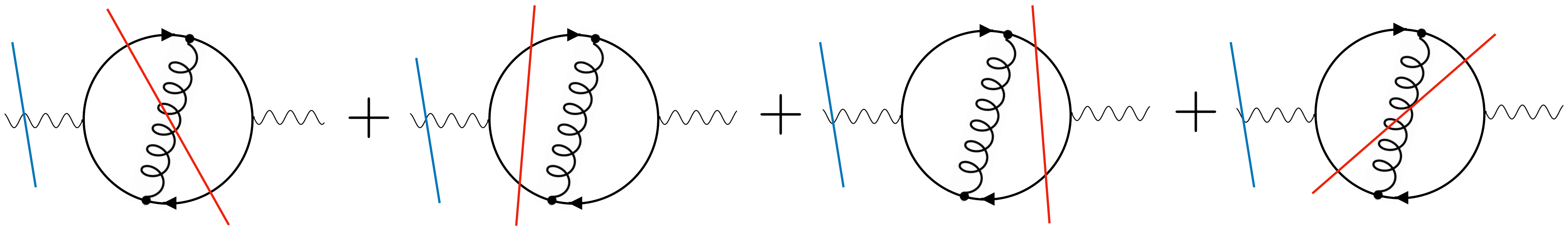

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Flip it, and obtain the answer for Drell-Yan, $2\,\mathrm{j} \to e^+e^-$

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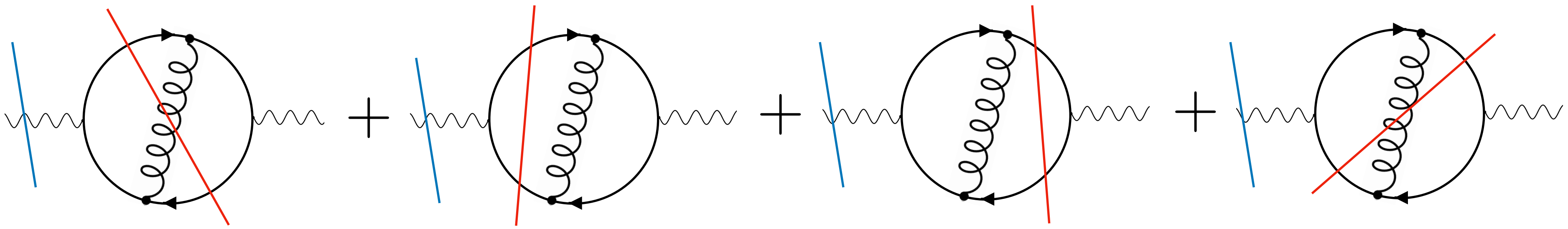

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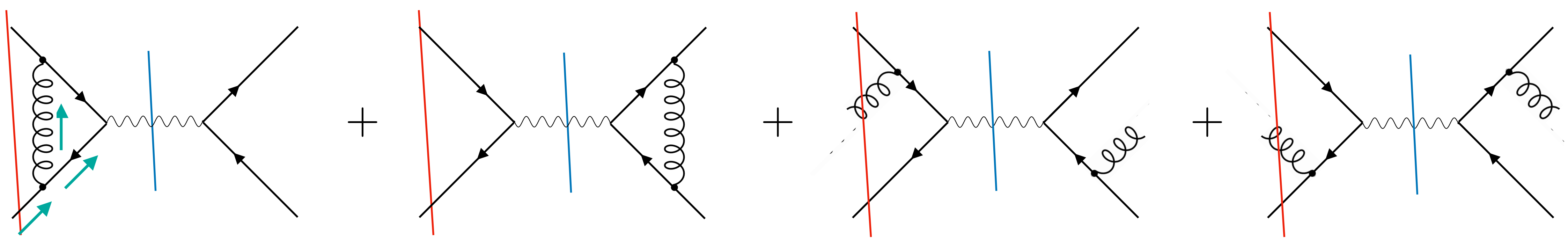

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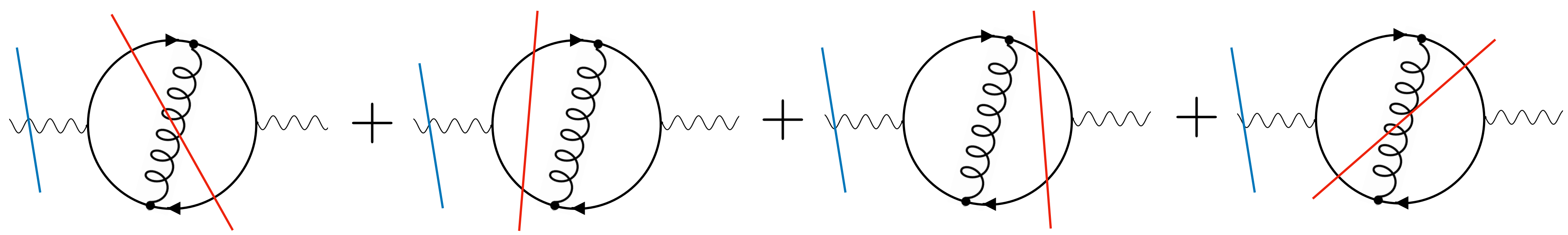

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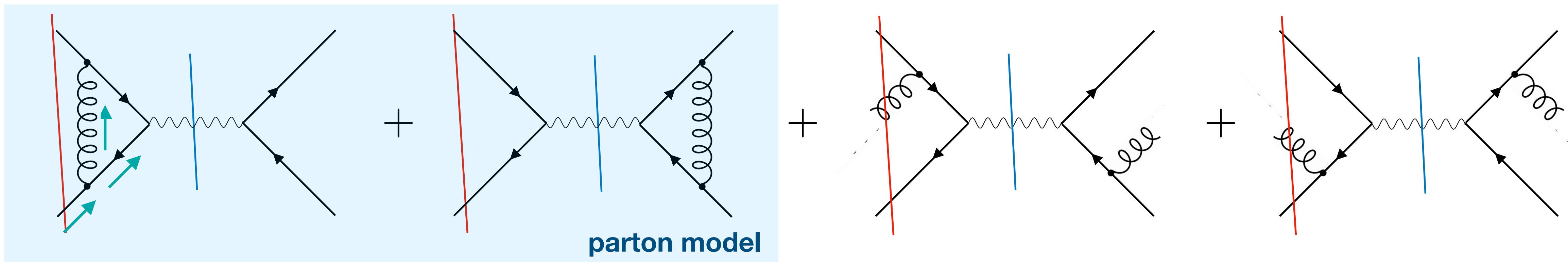

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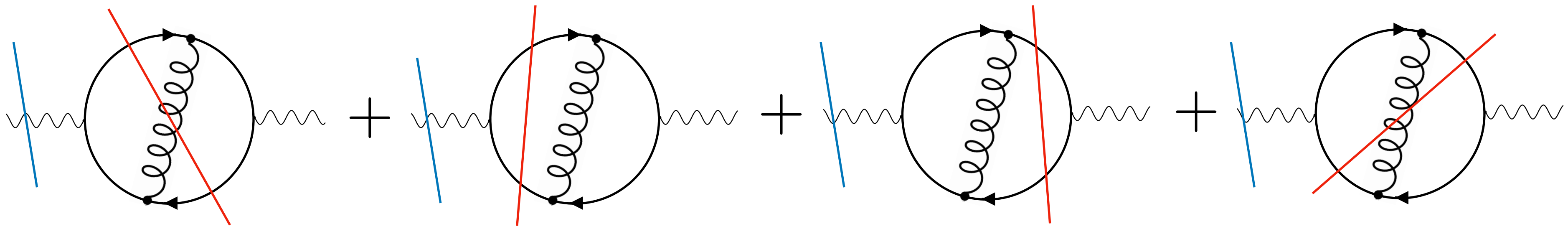

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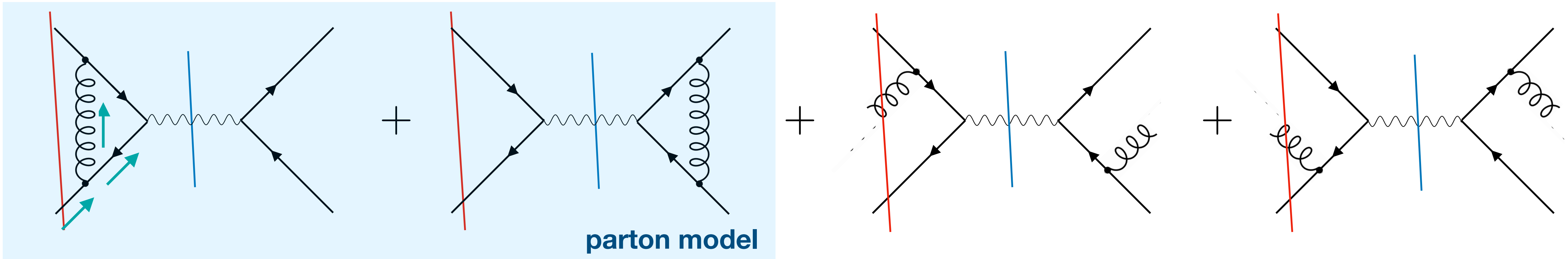

The initial state singularity is now absent!

Initial-state singularities: Idea

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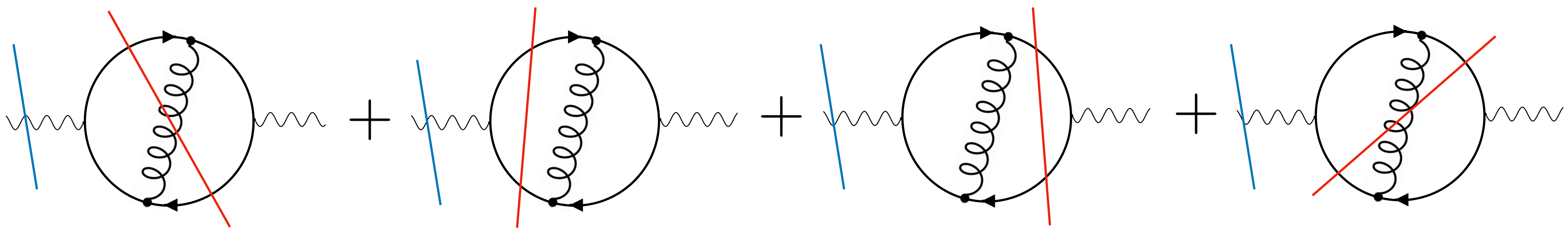

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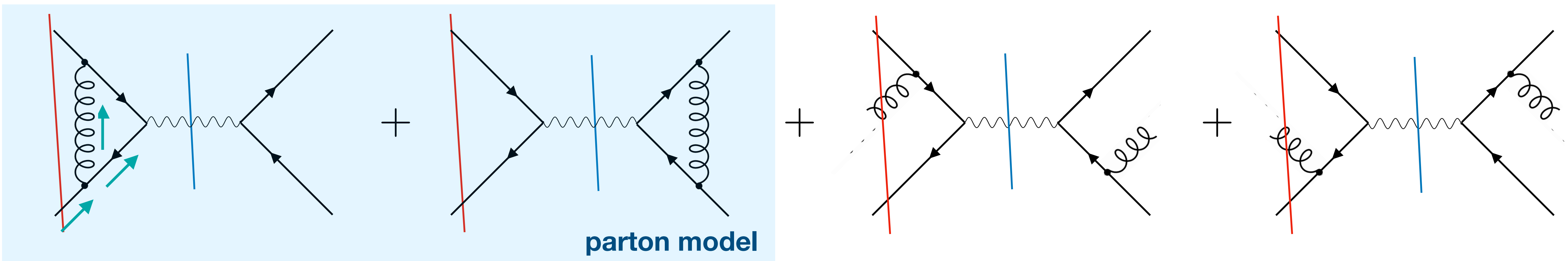

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Include degenerate initial states $\longrightarrow$ **Higher multiplicity initial states**

Initial-state singularities: Idea

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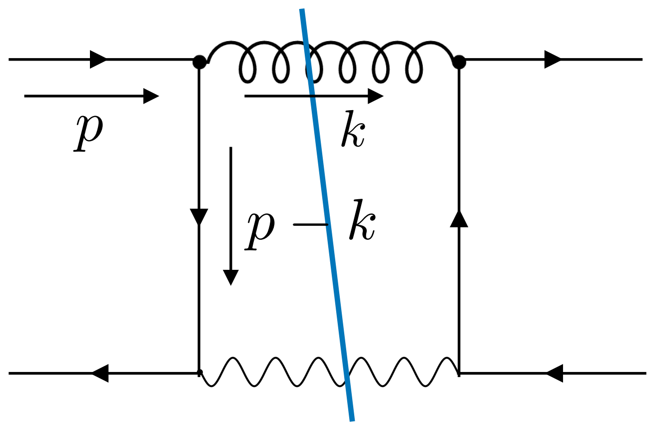
What about this diagram?



Also has collinear singularity at $k = xp$

Initial-state singularities: Idea

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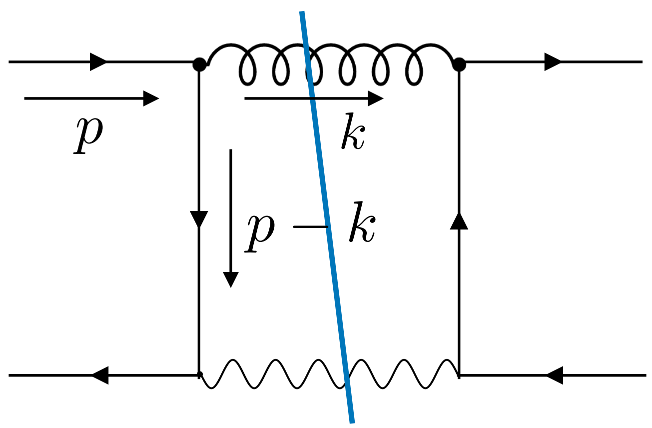
this diagram?



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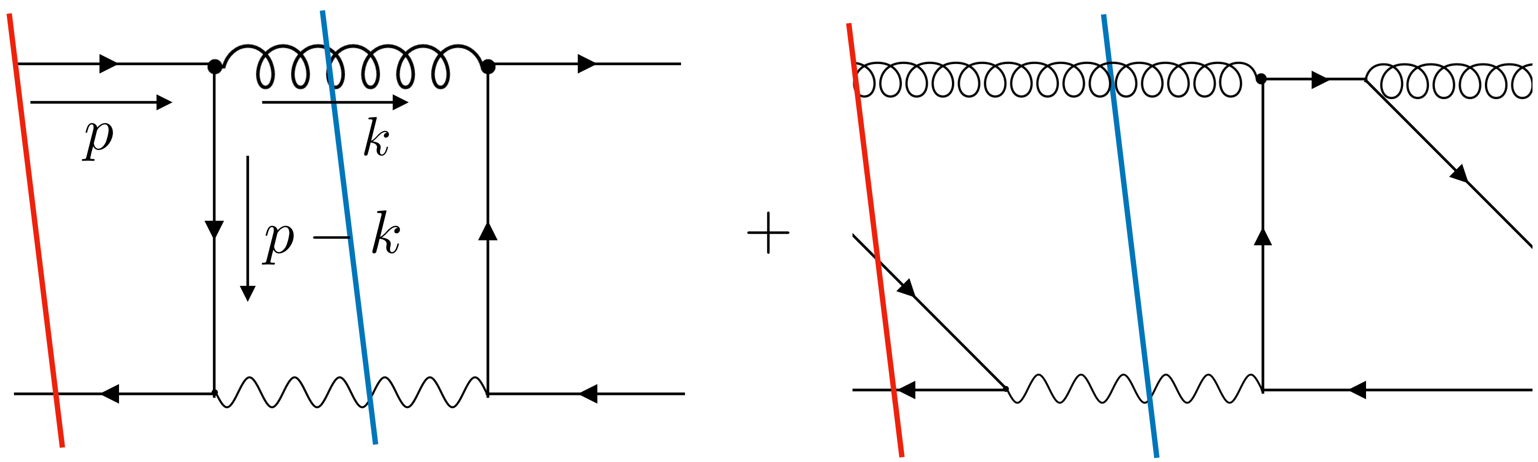
In this case, the cancelling partner is



Higher multiplicity initial states, but also **disconnected**! **Free travelling gluon!**

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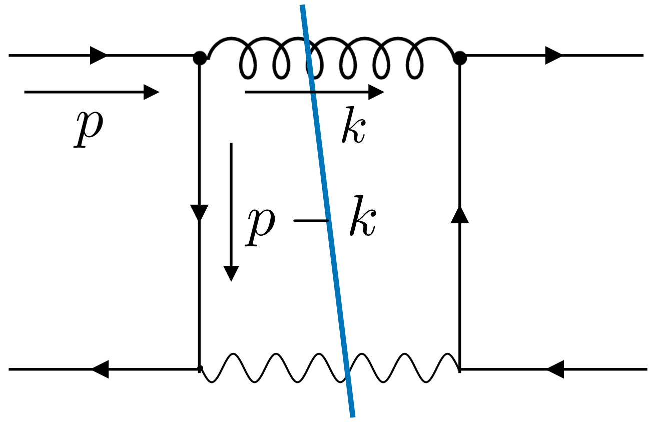

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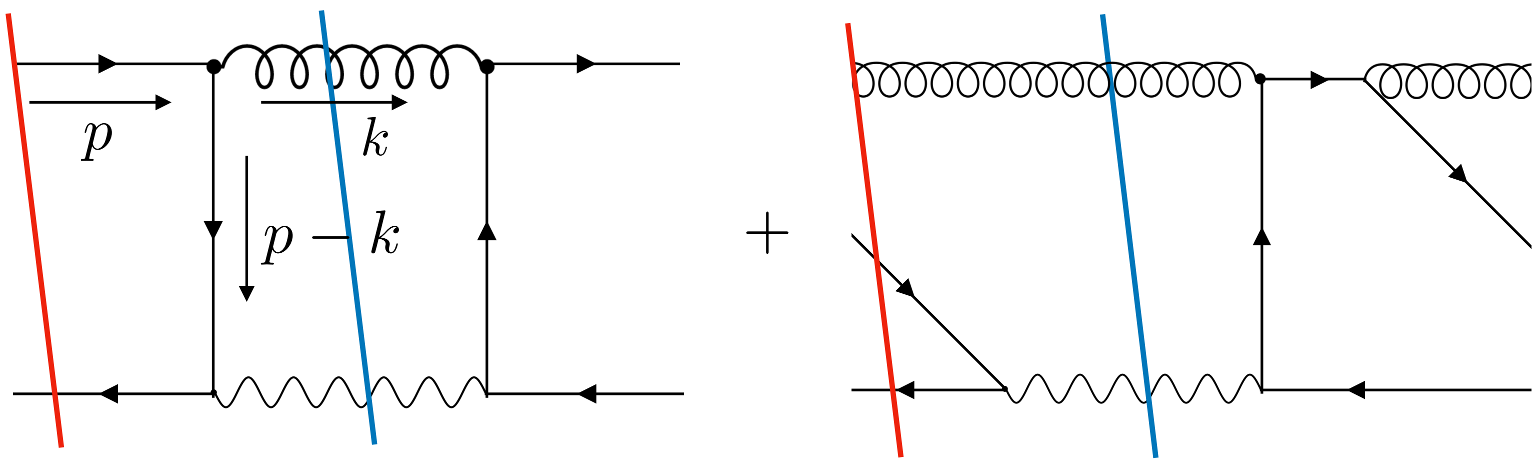

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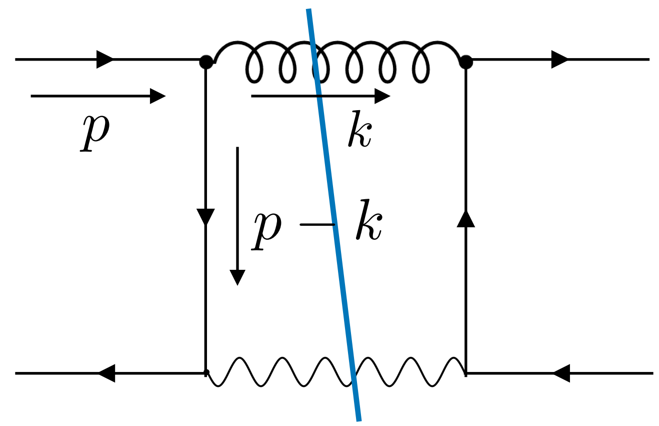

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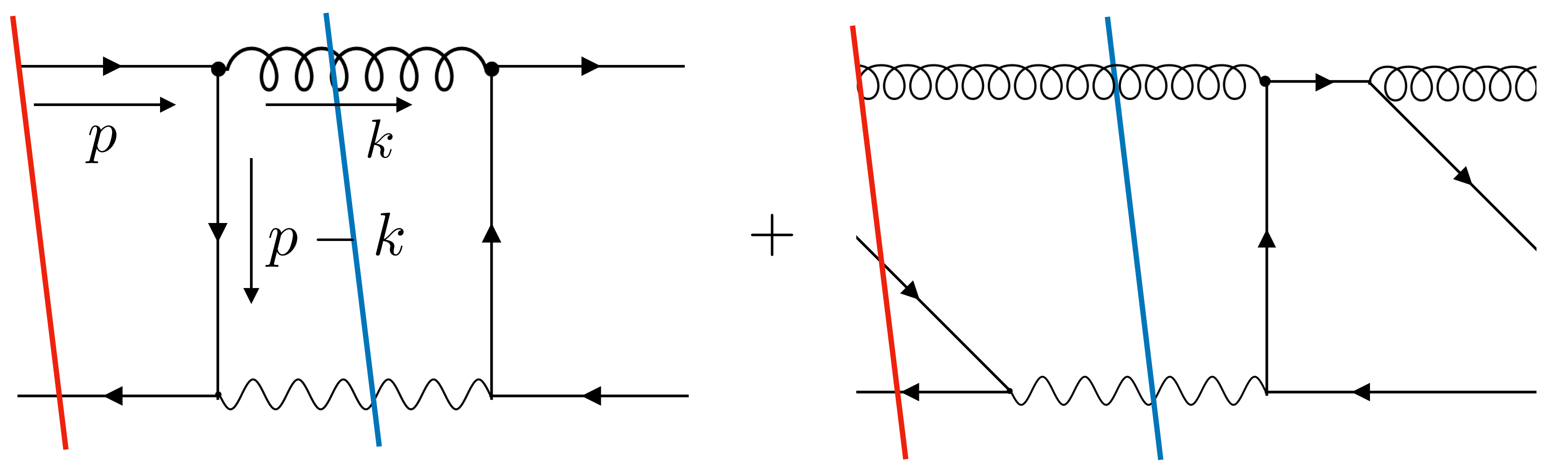

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Initial-state singularities: Idea

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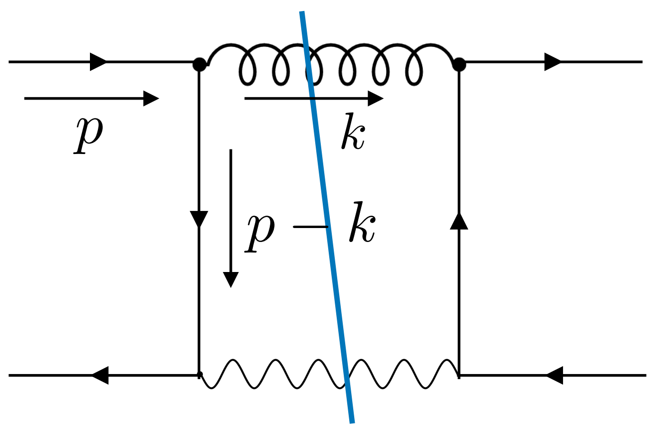

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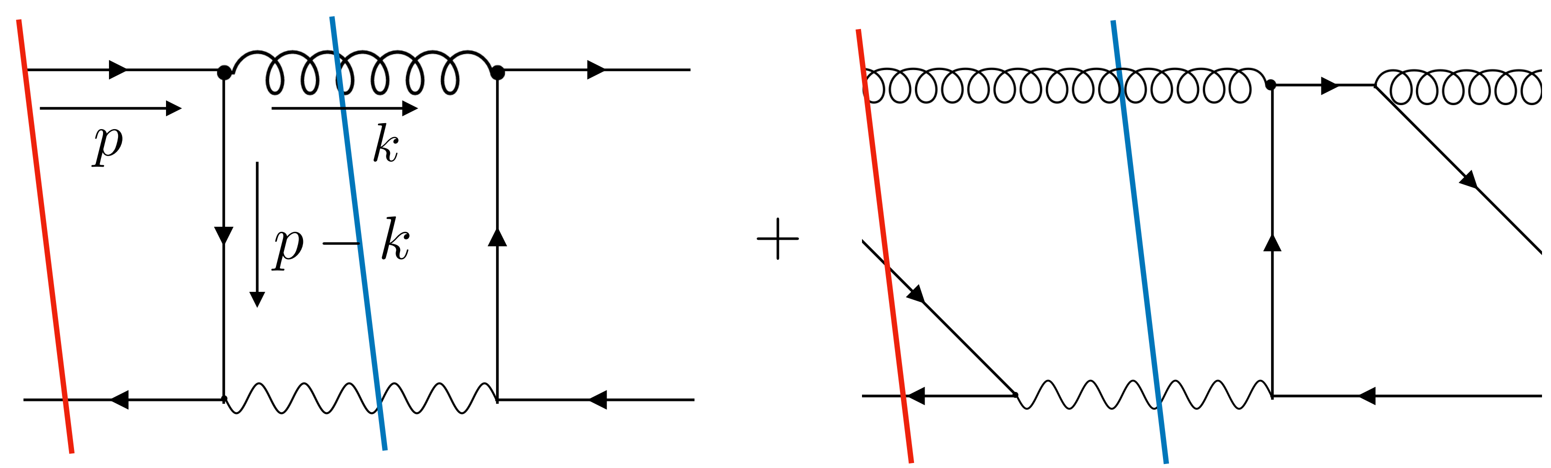

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But also more recently, they were studied in:

Frye, Hannesdottir, Paul, Schwartz, Yan arXiv:1810.10022 (2019)

Initial-state singularities: preliminary tests

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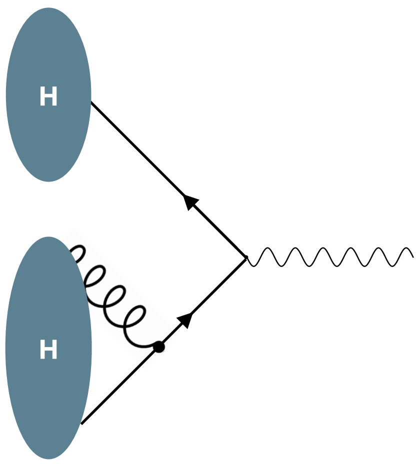
This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons



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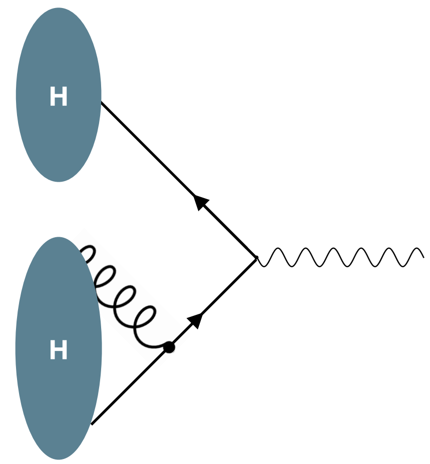
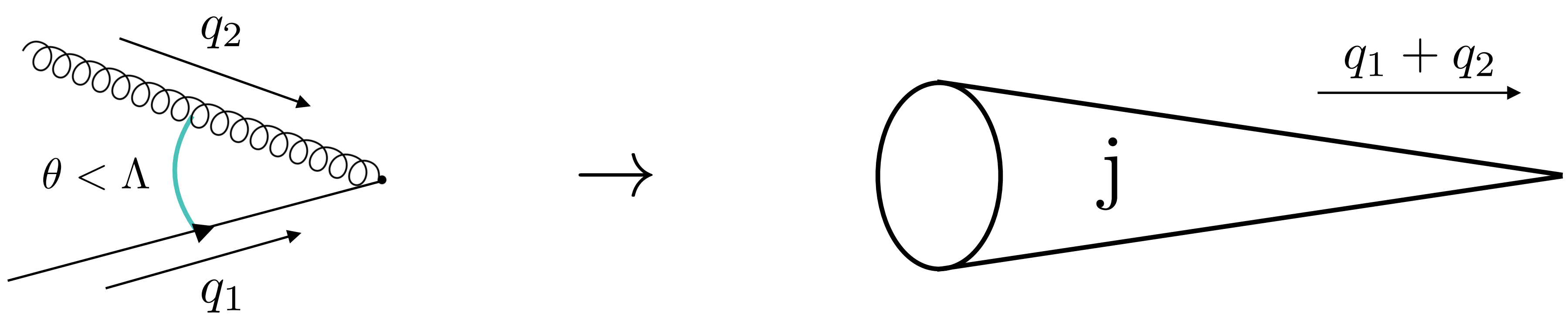


and that the multiple partons should be clustered into **two jet-like objects** that resemble boosted hadrons



After clustering, we get two jets with momenta

$$P_1^{\mathrm{j}} \qquad\qquad P_2^{\mathrm{j}}$$

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

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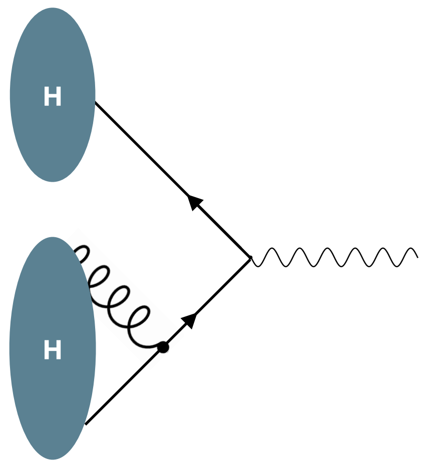

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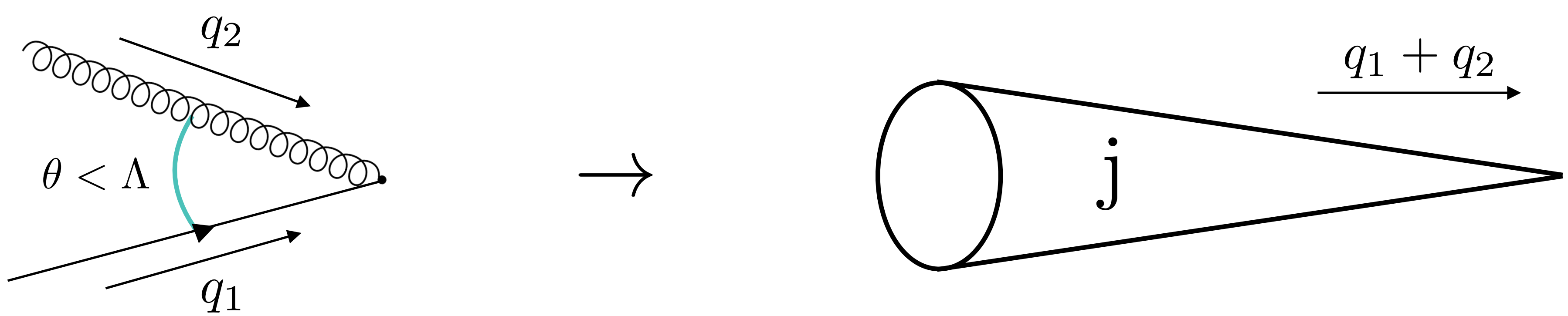

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$$P_1^{\mathrm{j}} \qquad\qquad P_2^{\mathrm{j}}$$

Cluster initial states analogously to final states: symmetry initial-final state

Initial-state singularities: preliminary tests

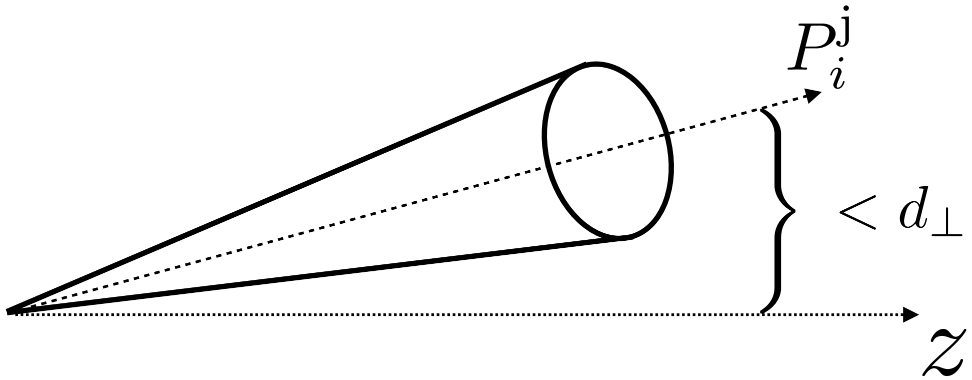
There are two relevant scales for the two initial state jets reconstructed:

Initial-state singularities: preliminary tests

There are two relevant scales for the two initial state jets reconstructed:

- One measuring the allowed phase space for the **total momentum** of the jet

$$(P_i^{\mathrm{j}})_\perp < d_\perp$$



If the scale is zero, the jet lies **exactly** on the z axis

If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

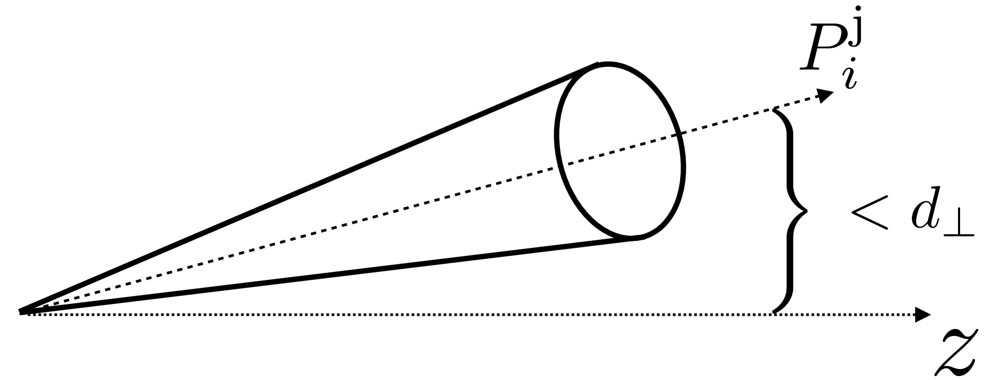
$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

Initial-state singularities: preliminary tests

There are two relevant scales for the two initial state jets reconstructed:

- One measuring the allowed phase space for the **total momentum** of the jet

$$(P_i^{\mathrm{j}})_\perp < d_\perp$$



If the scale is zero, the jet lies **exactly** on the z axis

If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

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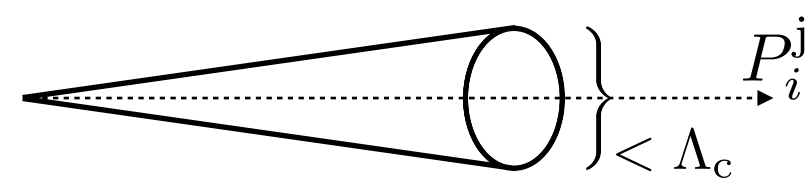

The smaller this scale, the more **collinear** the partons are

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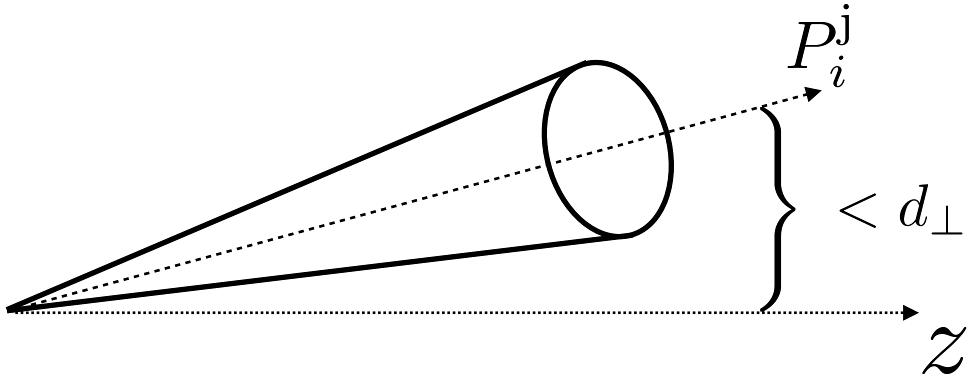

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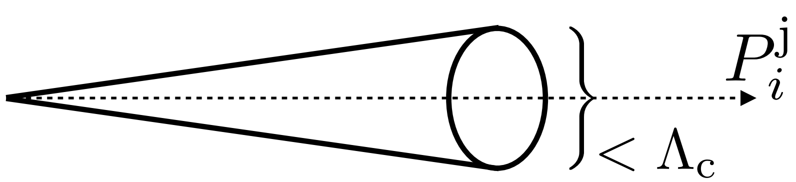

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Λ_{c} **is the equivalent of the factorisation scale!** $\approx \log(\Lambda_{\mathrm{c}})$

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This allows us to define **Bjorken variables**

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(not in MS bar)

- Vary the factorisation scale Λ_c and interpolate the dependence on the factorisation scale **Numerical resummation?** Banfi, Salam, Zanderighi, arXiv:0407286 (2004)

Initial-state singularities: "PDFs"

We started with a very generic formalism for scattering

$$\sigma(HH \to X + n\,\mathrm{j}) = \sum_m \int \left[\prod_{i=1}^{m} \mathrm{d}^3\vec{p}_i\right] f(p_1, ..., p_m) \;\; \frac{\mathrm{d}^m \sigma}{\mathrm{d}p_1 ... \mathrm{d}p_m}(p_1, ..., p_m \to X + n\mathrm{j})$$

Sum over number of initial state partons

Integration over initial state partons momenta

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Cross-sections for m initial state partons

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And we "forced" the initial-state observable to reproduce the usual factorised structure:

$$\sigma(HH \to X + n\,\mathrm{j}) = \int \mathrm{d}x_1\mathrm{d}x_2\ f(x_1,\Lambda_\mathrm{c})\ f(x_2,\Lambda_\mathrm{c})\ \frac{\mathrm{d}^2\sigma_p}{\mathrm{d}x_1\mathrm{d}x_2}(2\,\mathrm{j} \to X + n\,\mathrm{j},\Lambda_\mathrm{c})$$

But we did not need to start from this factorised ansatz!

Initial-state singularities: preliminary tests

Numerical example result for this finite sum of two interference diagrams:

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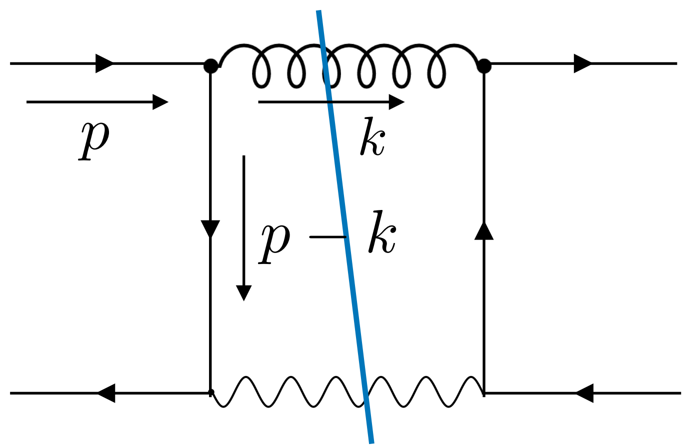

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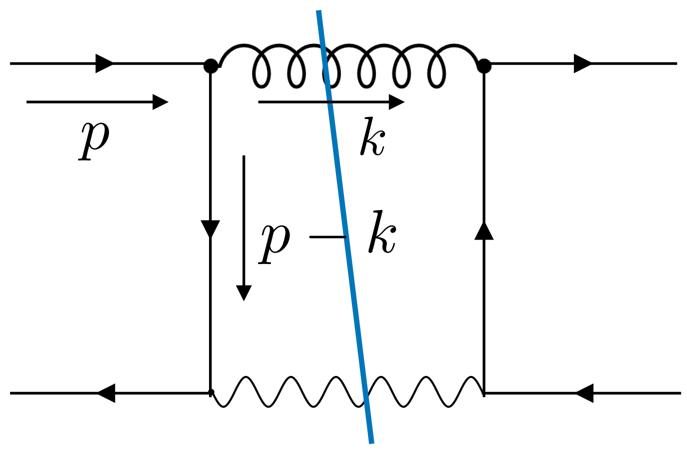

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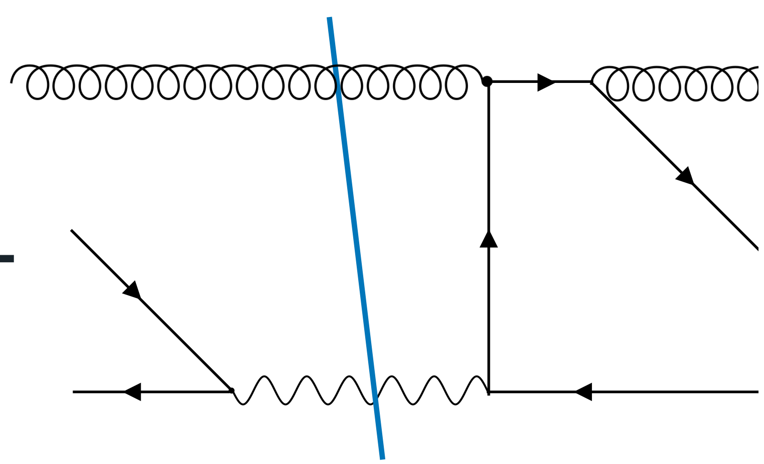

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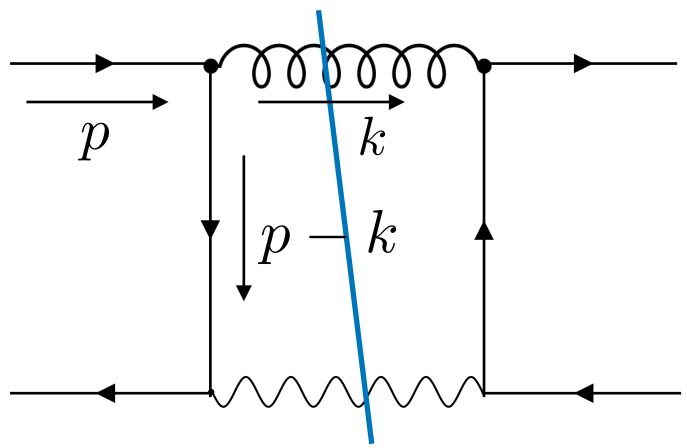

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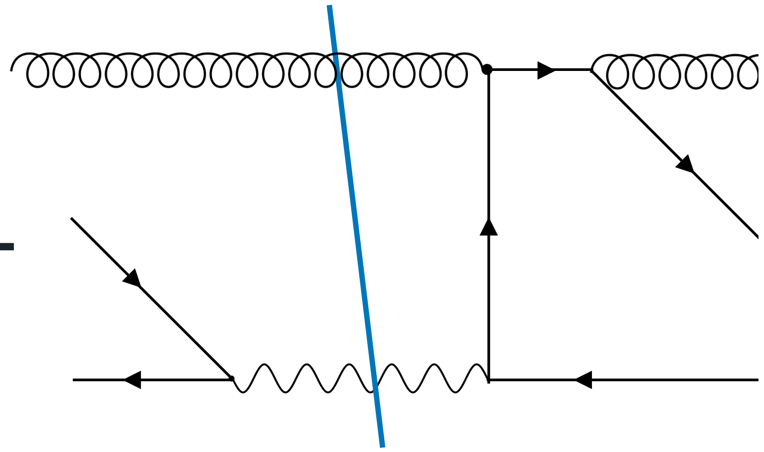

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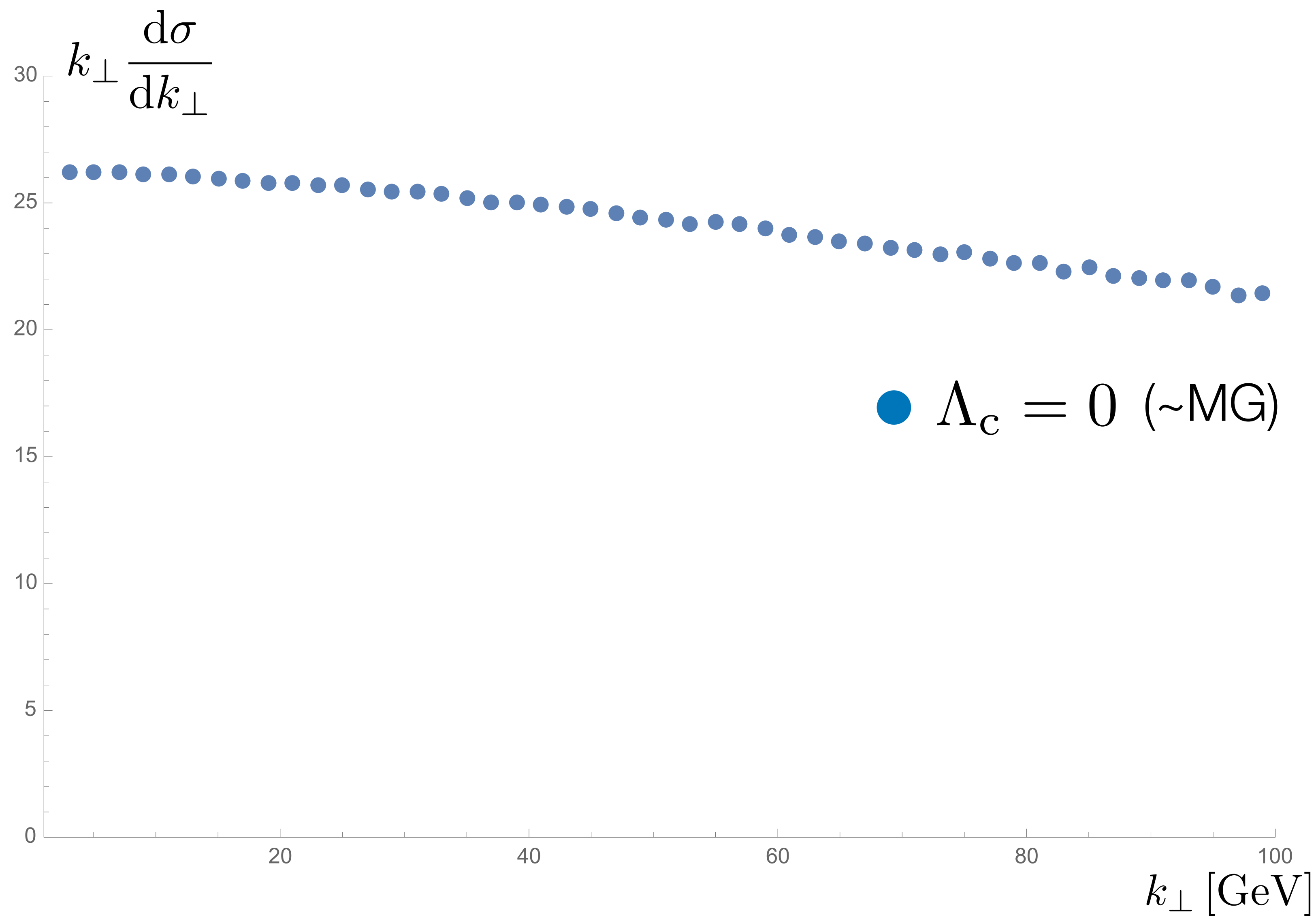

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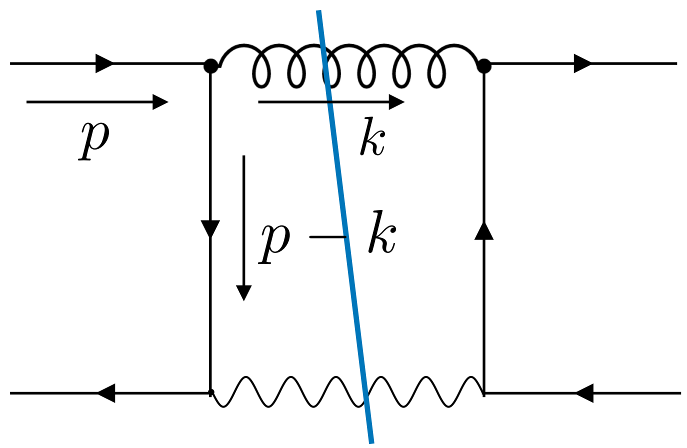

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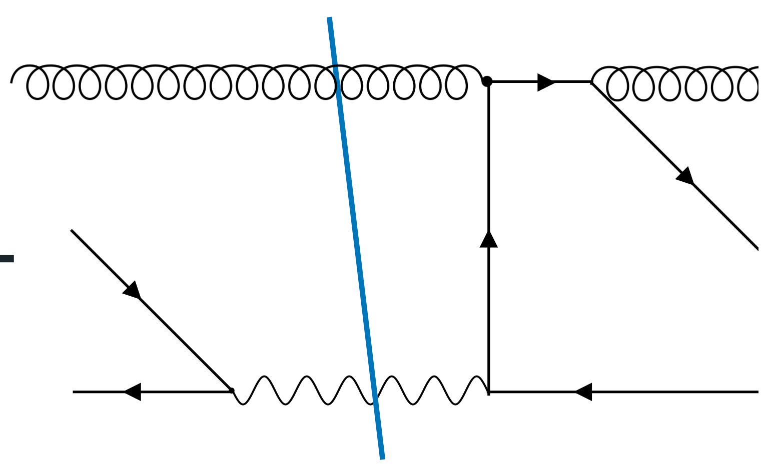

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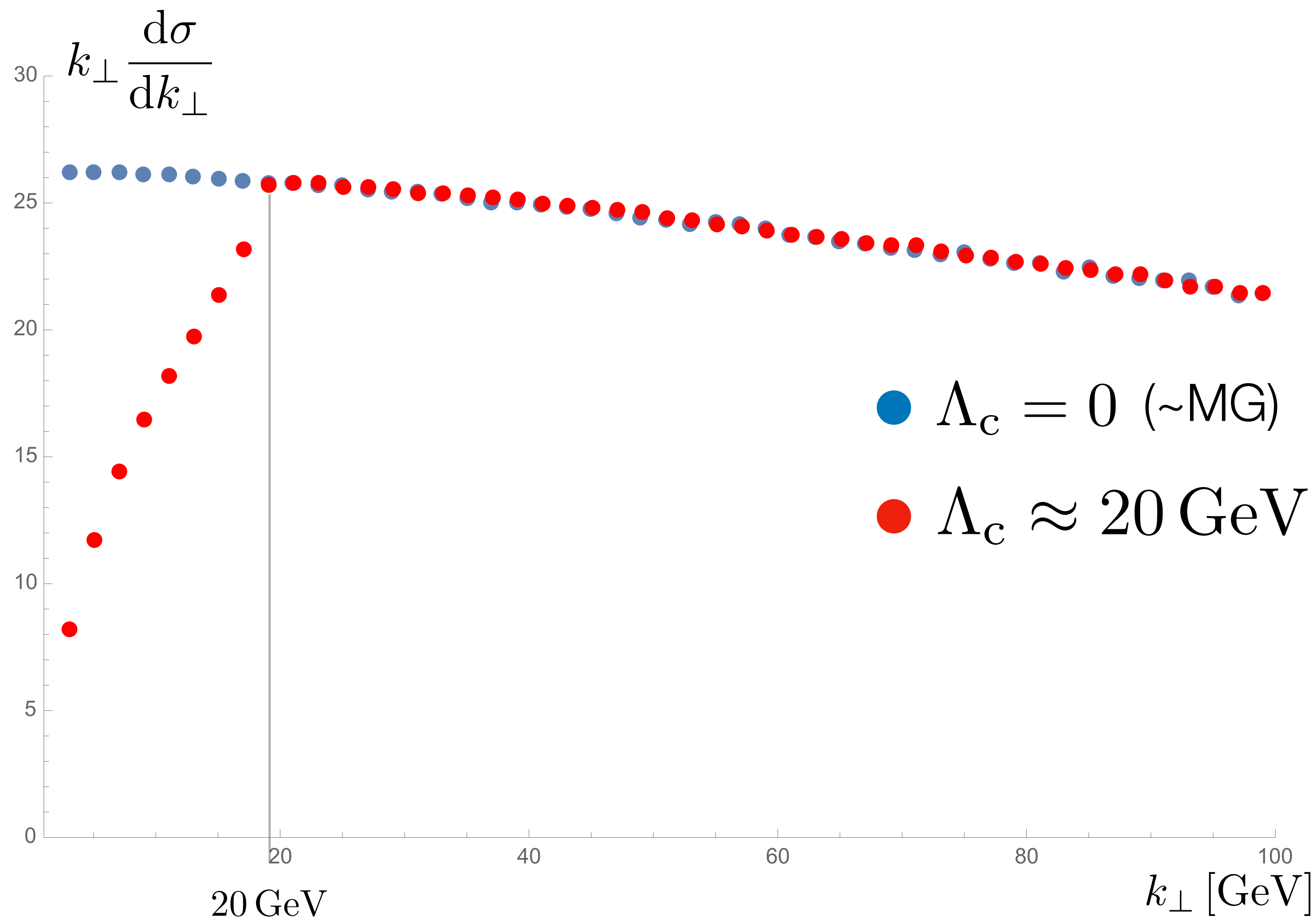

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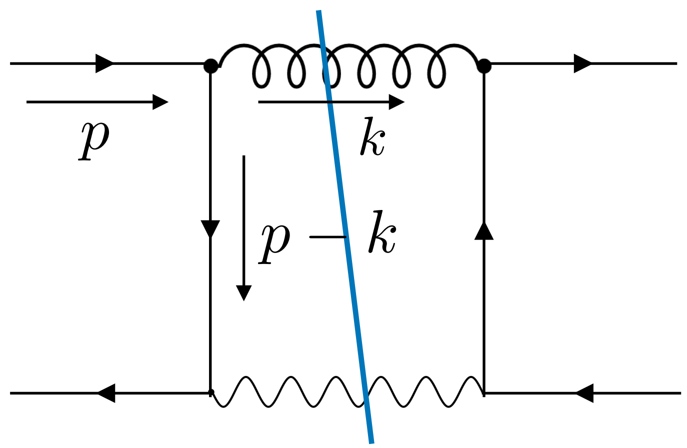

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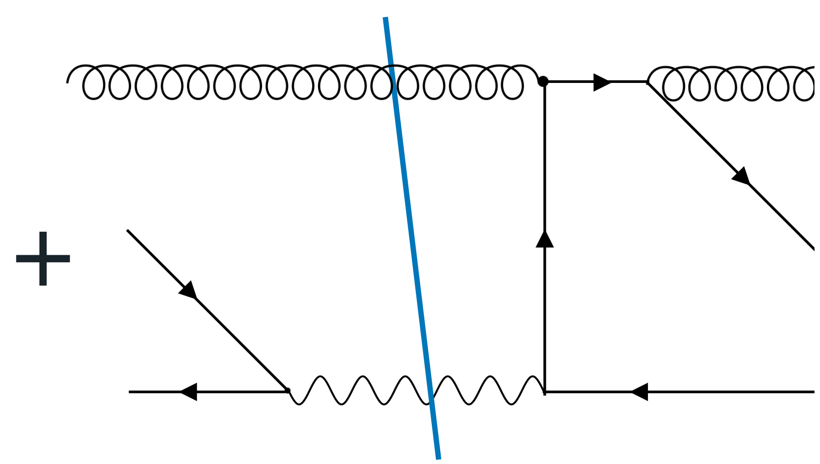

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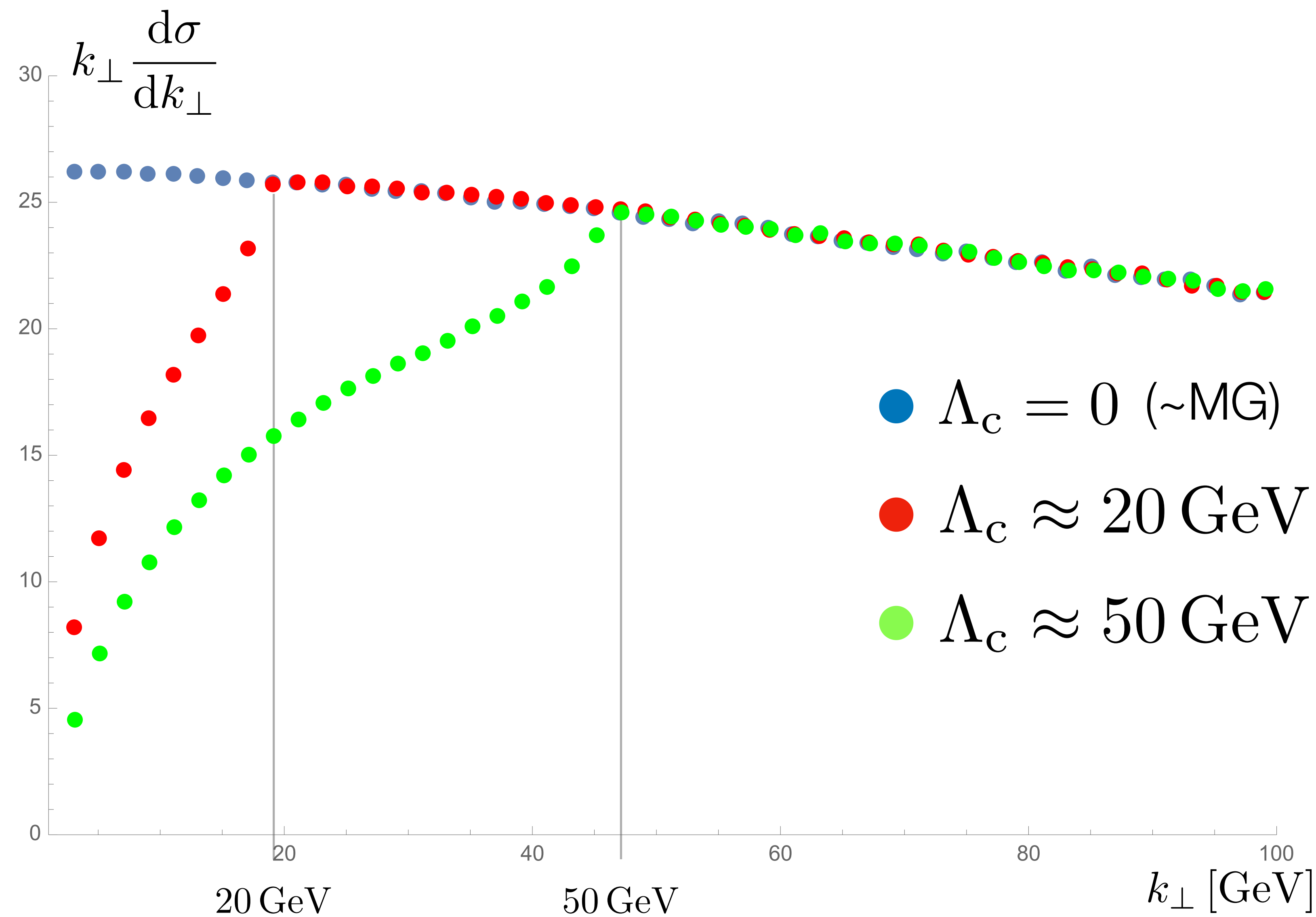

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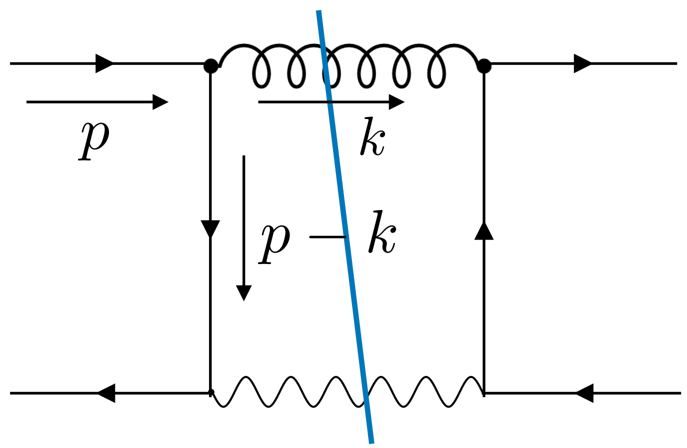

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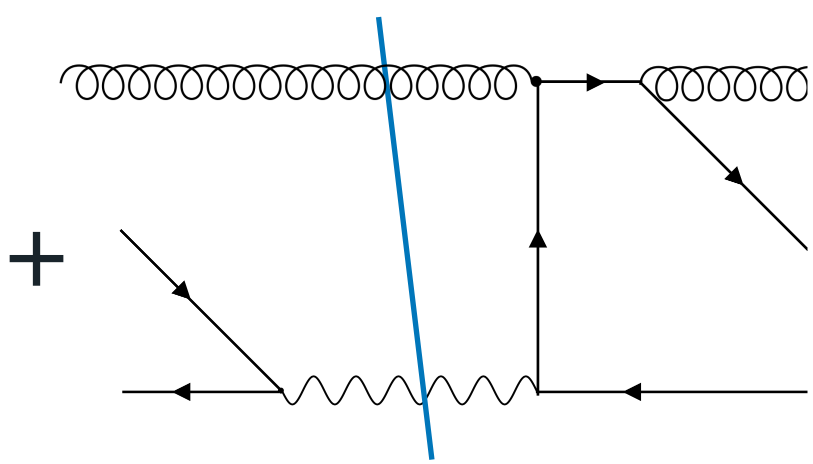

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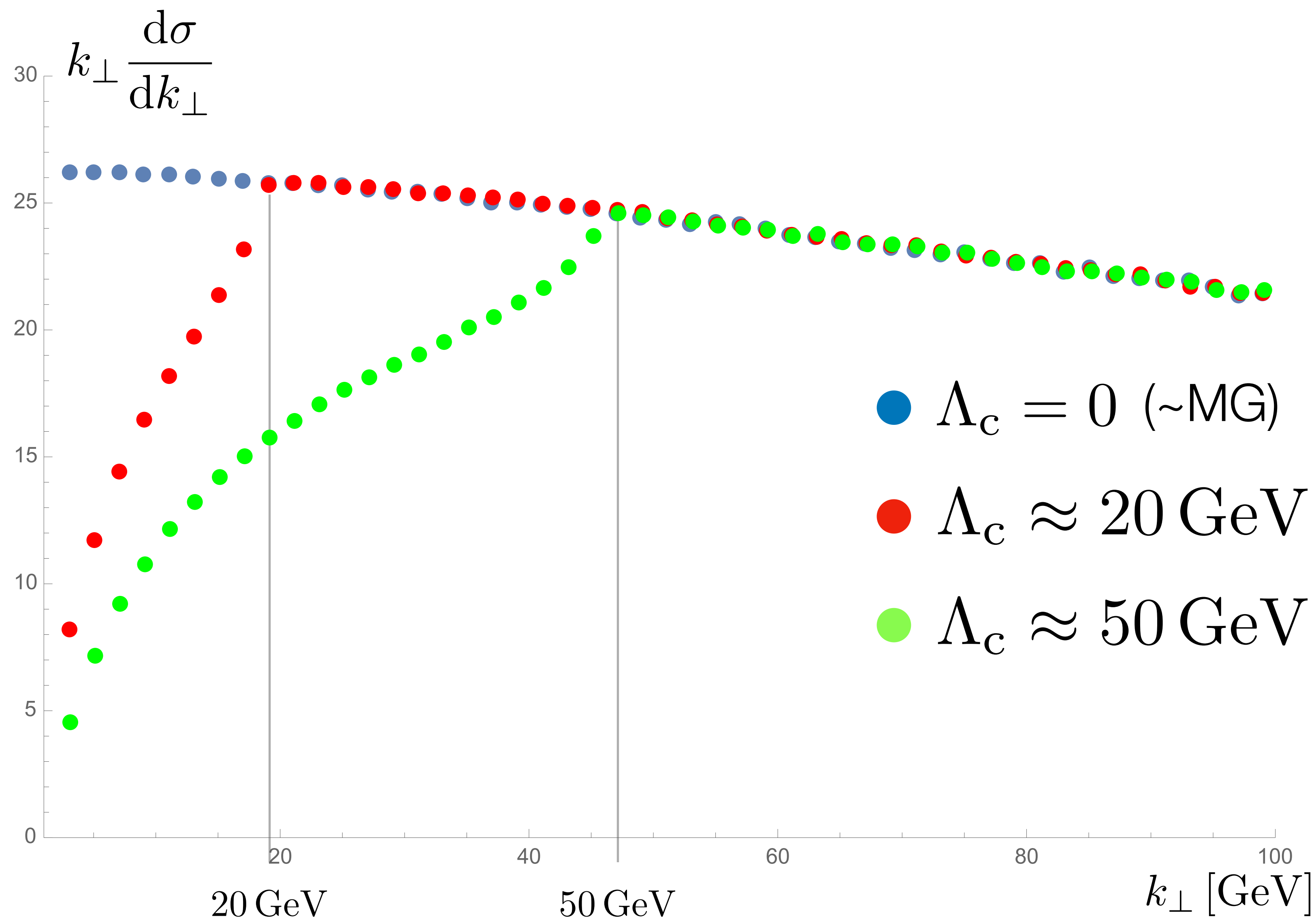

Note that : for $\Lambda_c > 50\,\mathrm{GeV}$
the distribution does not change anymore
because highest separation of two partons
in a jet is of **order of Z mass**

Initial-state singularities: scale dependence

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Consider now the inclusive integral

$$I(\Lambda_{\rm c}) = \int_0^{100} \mathrm{d}k_\perp \frac{\mathrm{d}\sigma}{\mathrm{d}k_\perp}(\Lambda_{\rm c})$$

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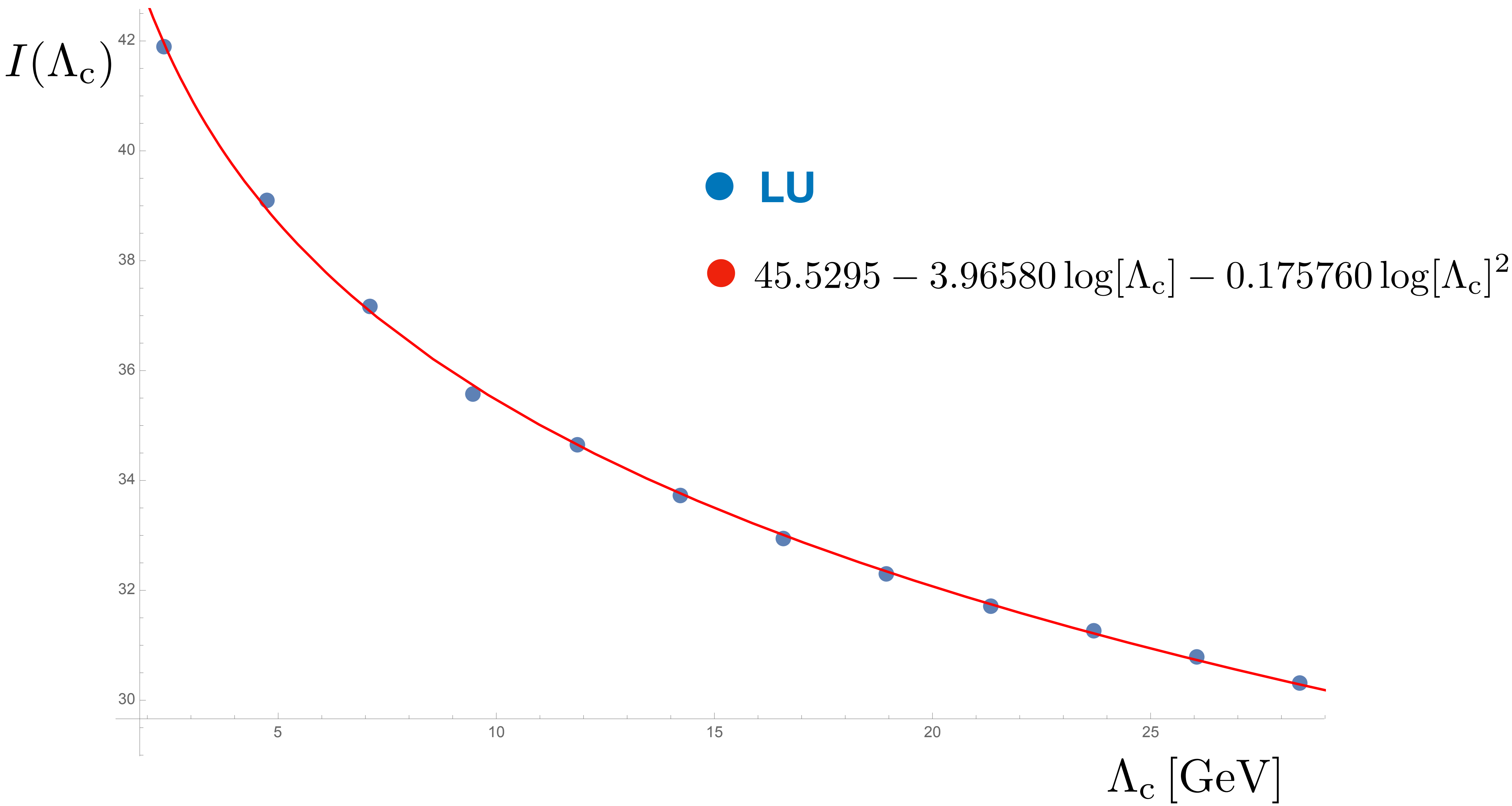

Subtraction vs Local Unitarity

- One can introduce the following local (in x) **counterterm**:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

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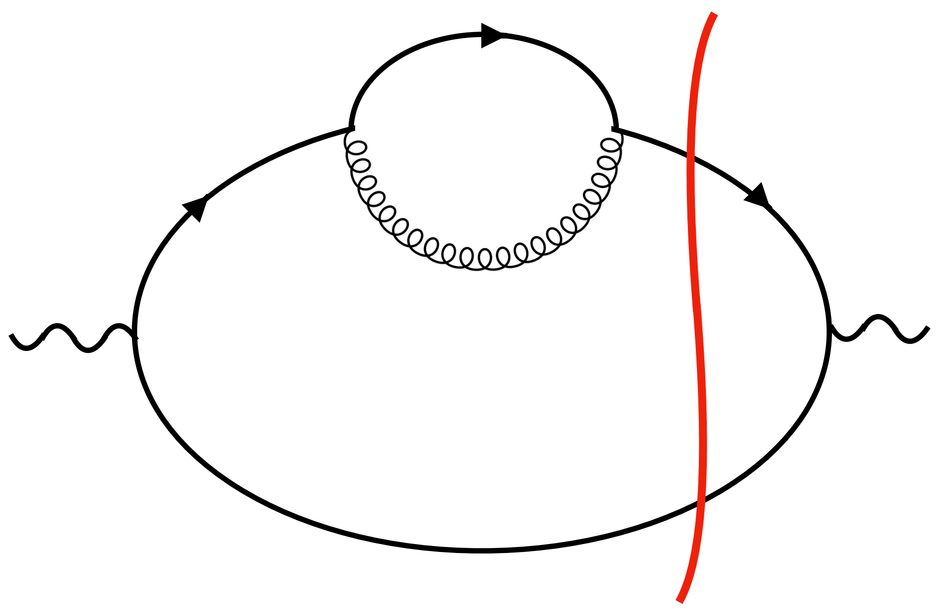
- **Local Unitarity** aligns the measure and combines “real and virtual”:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) + \frac{-e^{-x}}{x}\mathcal{J}(0)\Theta(1-x)\right]$$

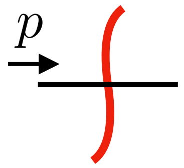
Locality Unitarity: raised propagators

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider ***truncated*** amplitudes only :



Traditional Cutkosky rule



$$= -2\pi \mathrm{i} \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

would not apply here !

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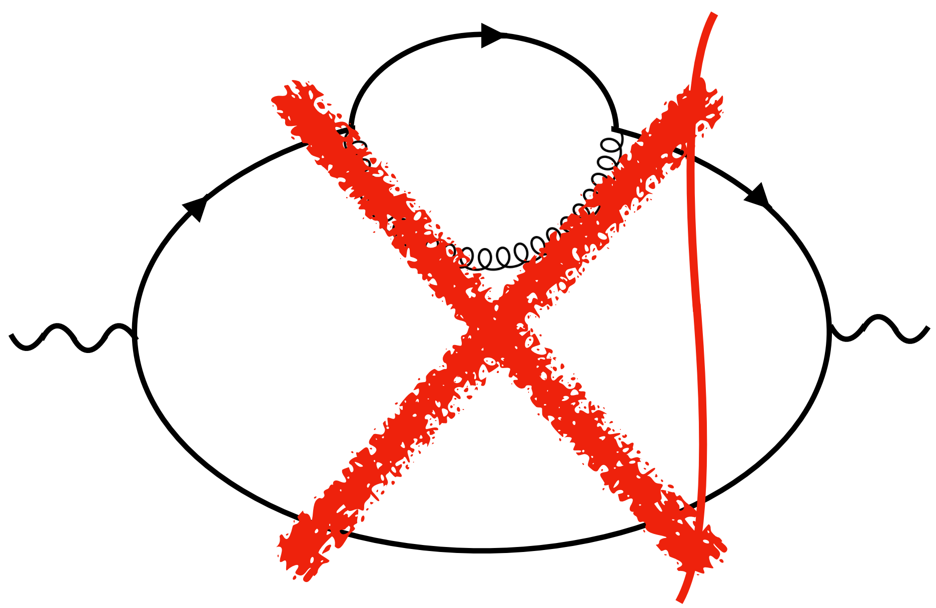

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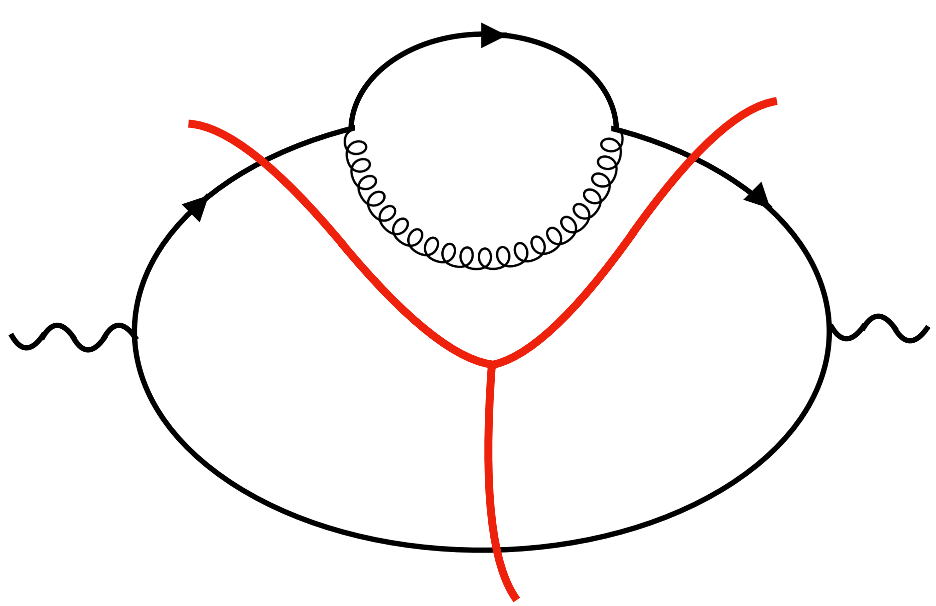
So consider this Cutkosky cut as a **higher-order residue** ➔ **Generalised cutting rule**



$$\cdots \bullet \bullet \bullet \cdots \quad = -2\pi\mathrm{i}\frac{\delta^{(n)}\left[p^0 - E(\vec{p})\right]}{(p^0 + E(\vec{p}))^2}$$

$n - \mathrm{times}$

$$\int \mathrm{d}x\, \delta^{(n+1)}[x] f(x) = \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\bigg|_{x=0}$$

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[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
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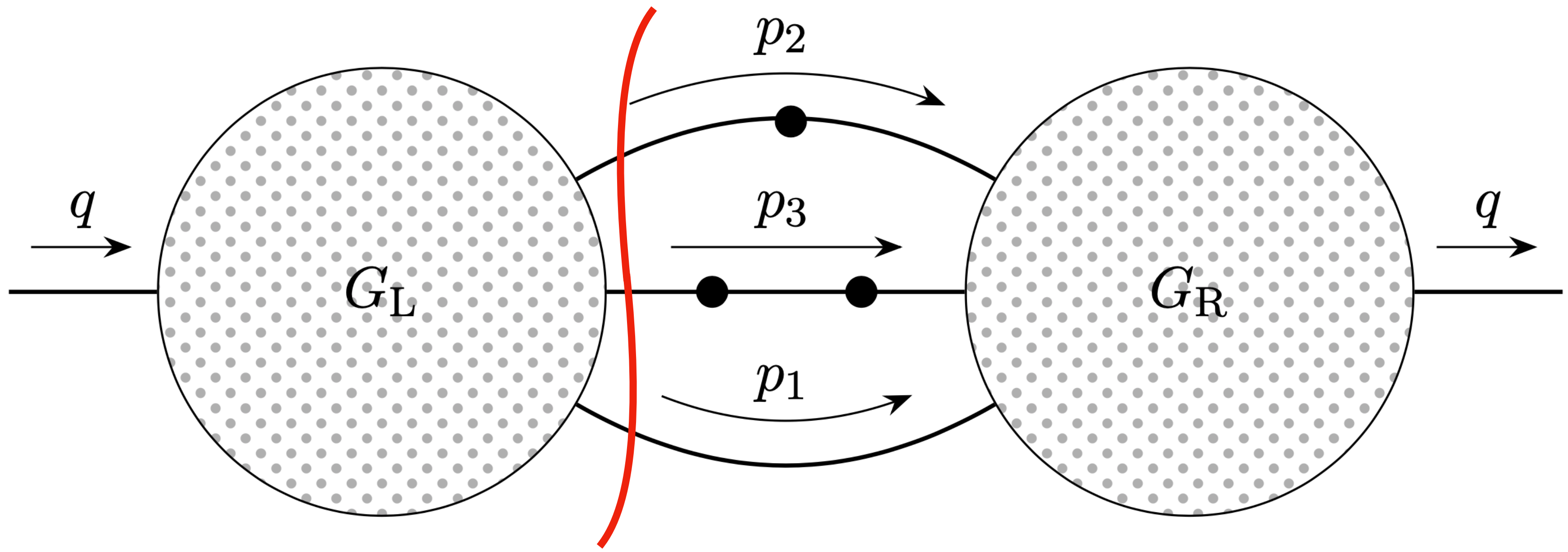

$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1}\ )]\delta^{(2)}[p_2^0 - E(\vec{p_2}\ )]\delta^{(3)}[p_3^0 - E(\vec{p_3}\ )]$$

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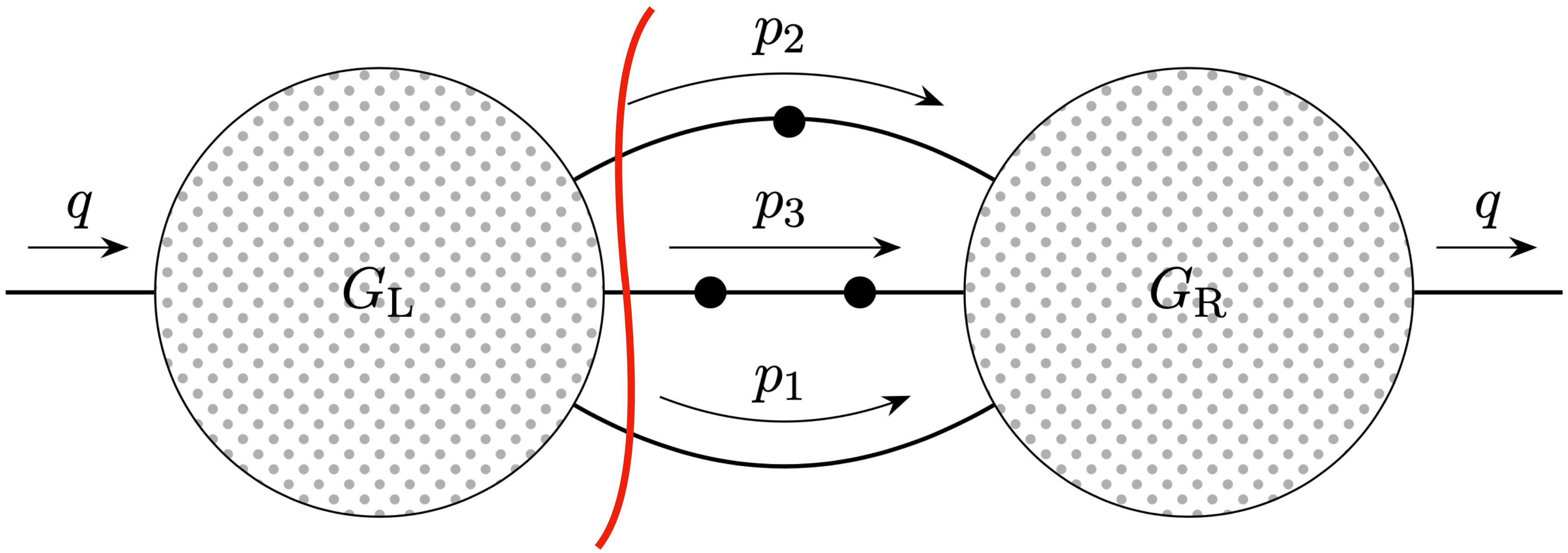

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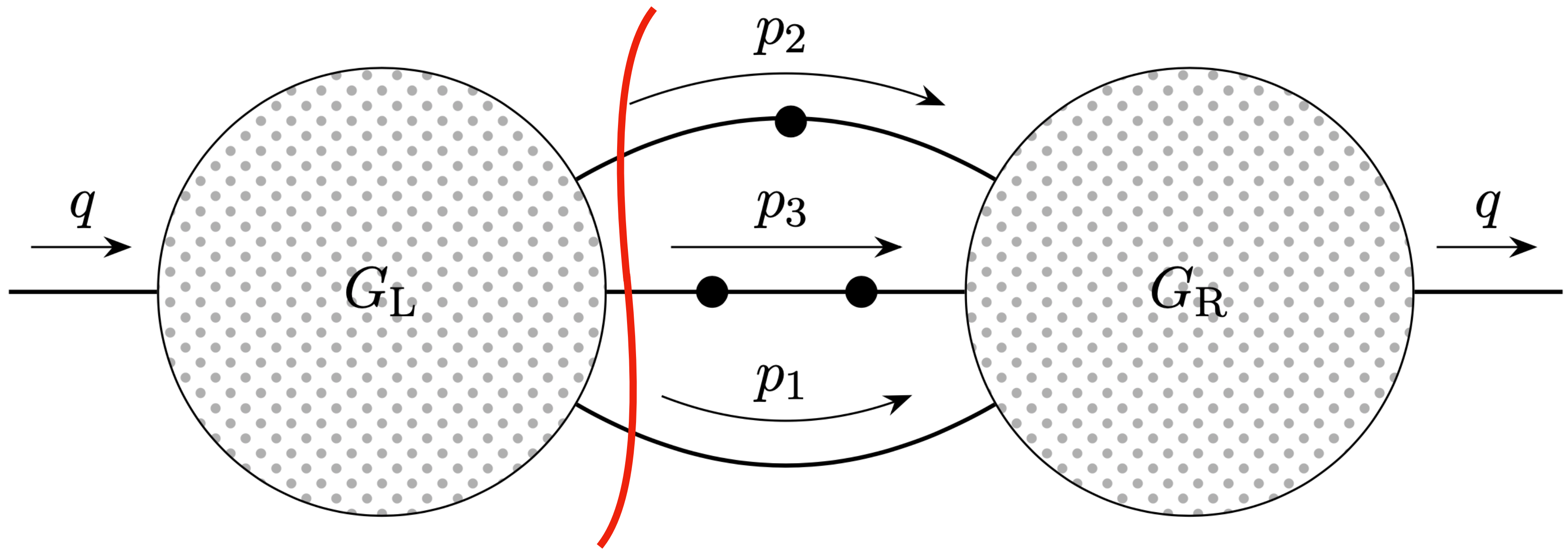

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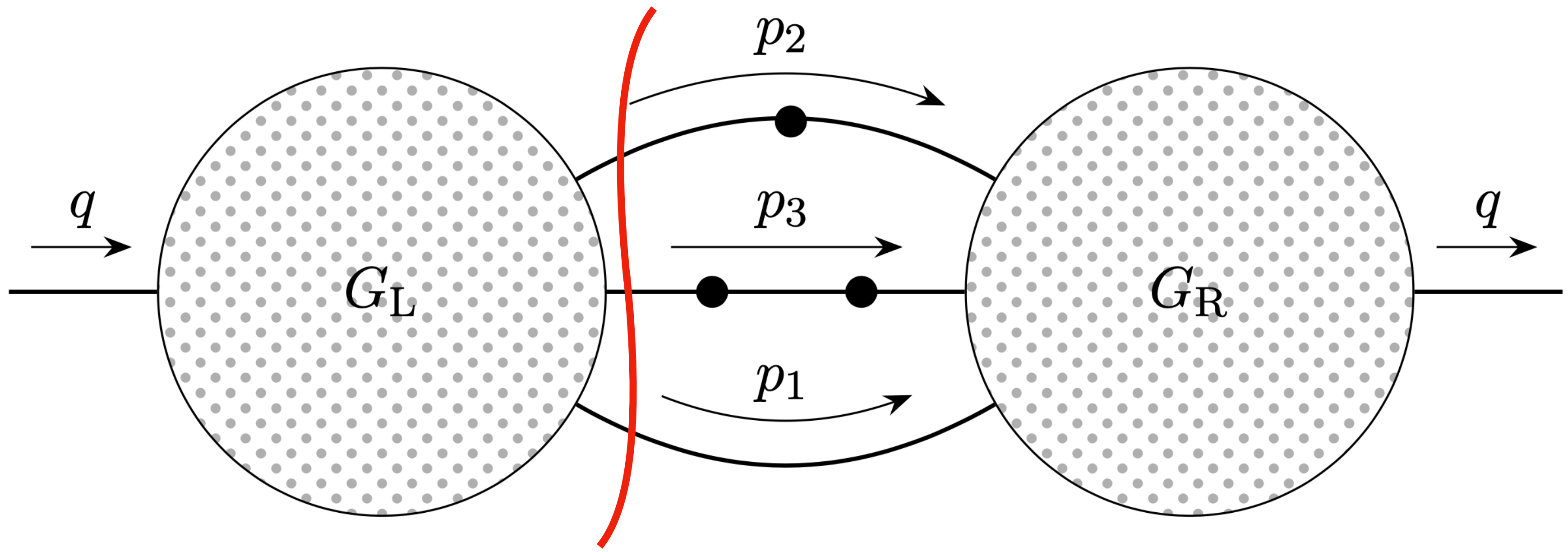

$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1}\,)]\delta^{(2)}[p_2^0 - E(\vec{p_2}\,)]\delta^{(3)}[p_3^0 - E(\vec{p_3}\,)]$$

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Locality Unitarity: raised propagators

[Capatti, VH, Ruijl, arxiv : 2203.11038]

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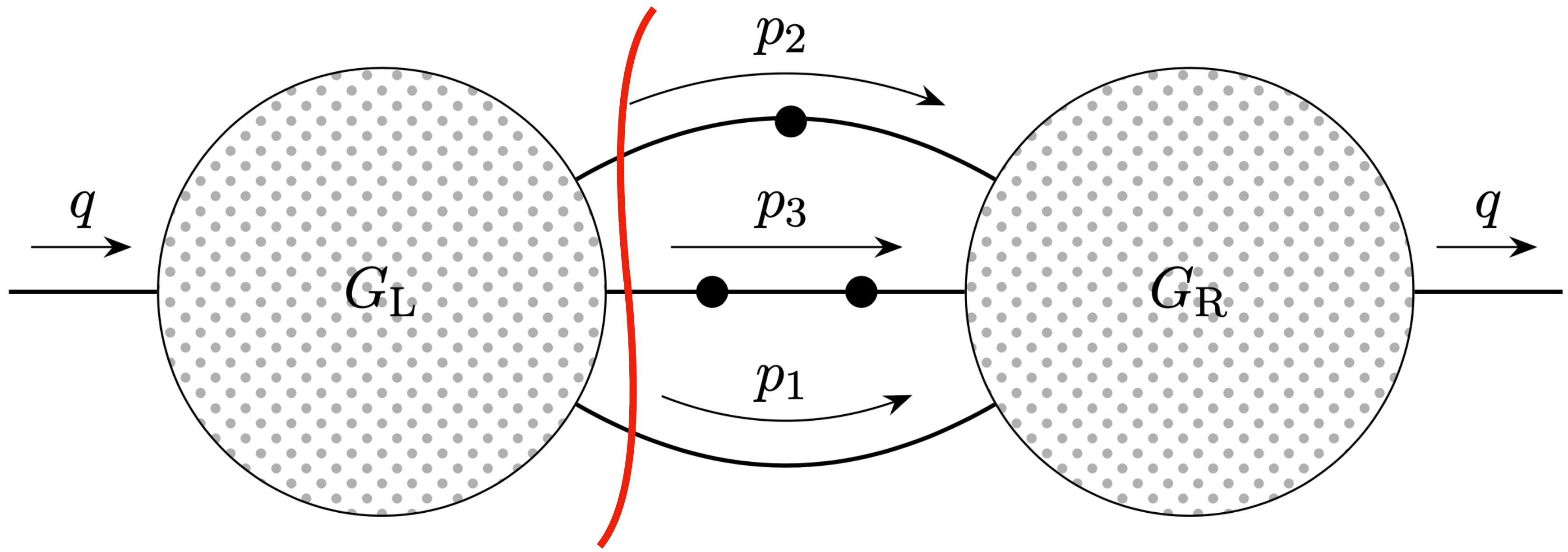

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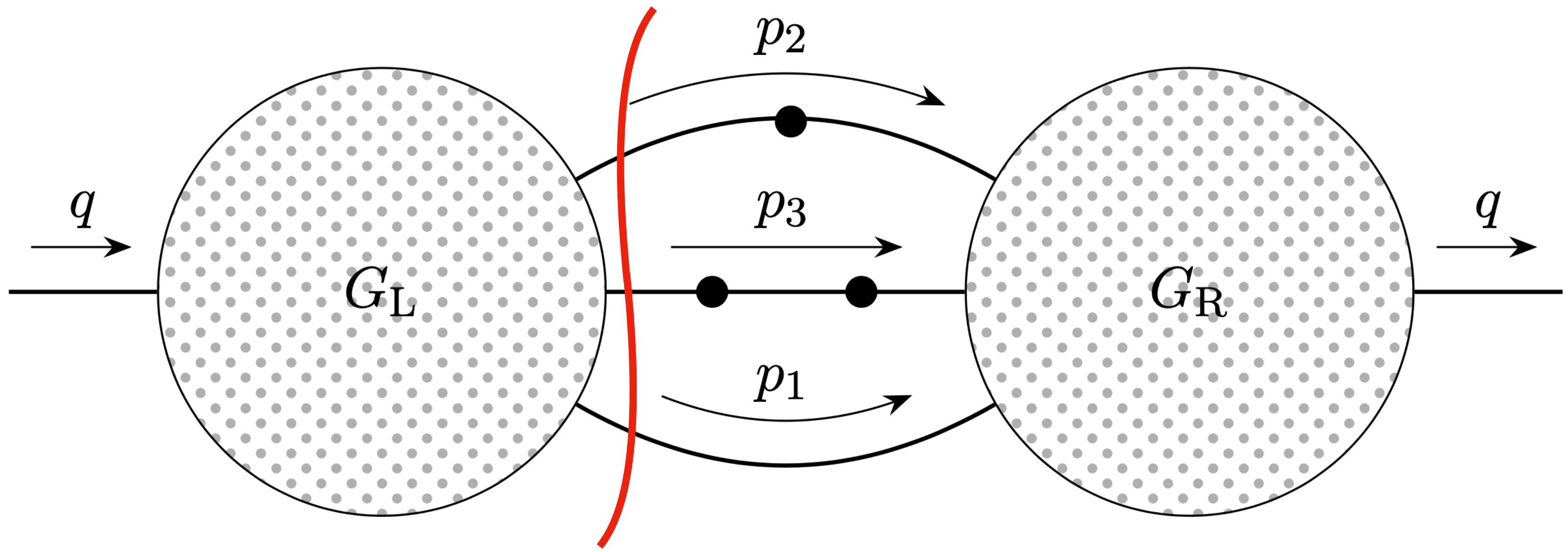

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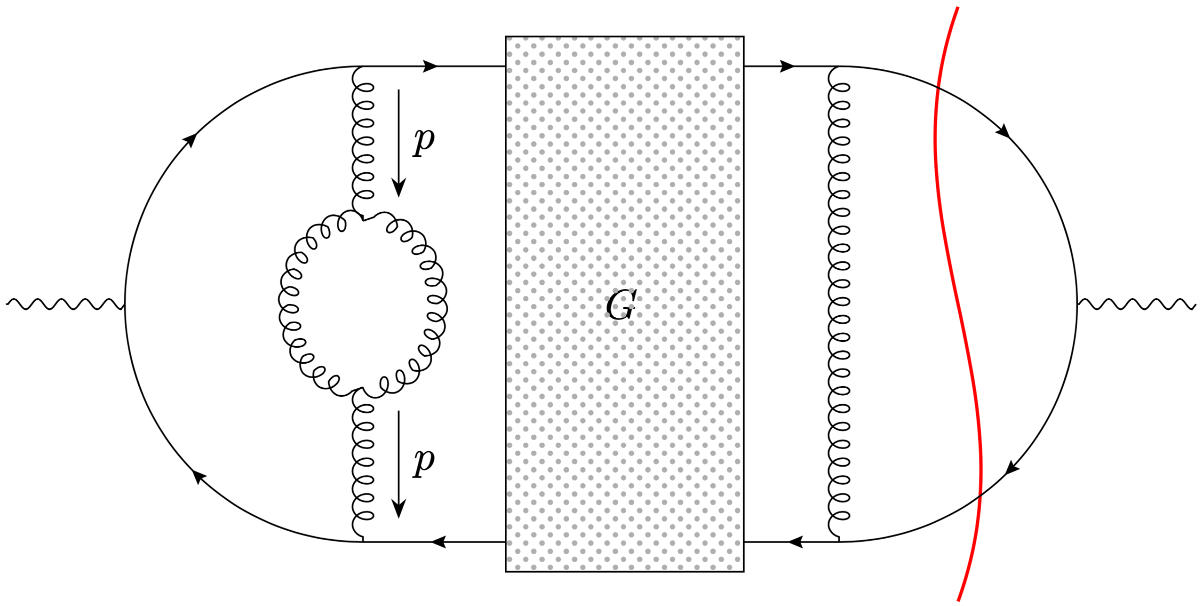
Use **multivariate dual numbers** (auto-differentiation) in order to **efficiently compute amplitude derivatives** of G_L and G_R in p_2^0 and t (in this example)

Spurious soft singularities

[Capatti, VH, Ruijl, arxiv : 2203.11038]

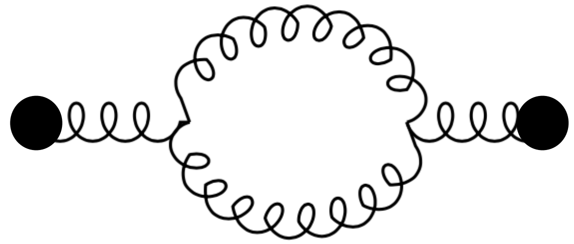


$$\propto \frac{1}{(p^2)^2}$$

For $p = 0$

this induces a spurious soft divergence whose cancellation has nothing to do with KLN!

Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

At the integrated level, we have  $\propto \frac{1}{p^2}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\frac{1}{p^2}$

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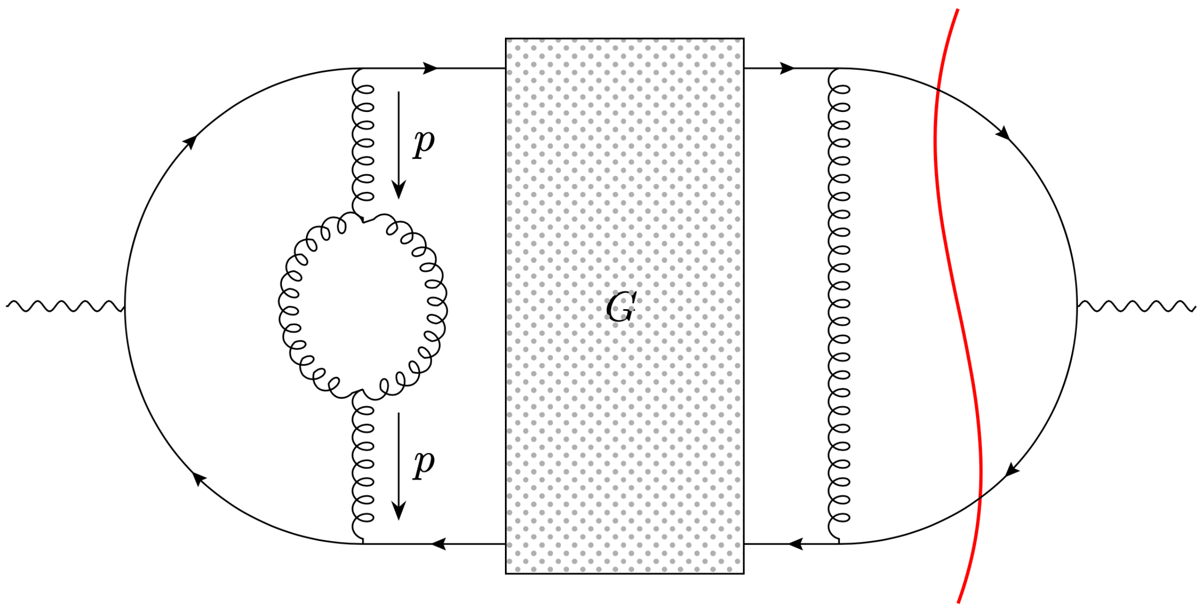

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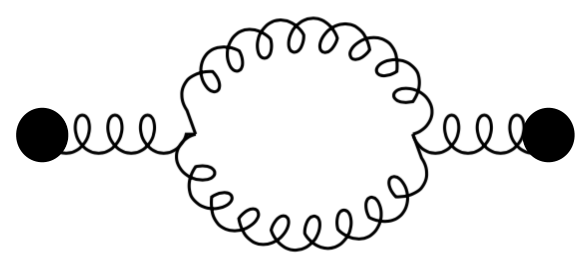
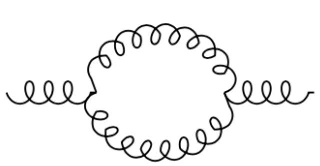
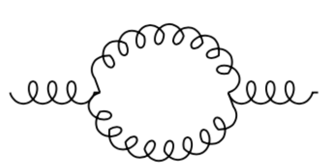
At the integrated level, we have  $\propto \frac{1}{p^2}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\frac{1}{p^2}$

but not at the local level; we must introduce **spurious soft counterterms** :

 $-\tilde{T}_1\Big($  $\Big)$, $\tilde{T}_{\mathrm{soft\_dod}}(\gamma) = \sum_{j=0}^{\mathrm{soft\_dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma(\lambda\, p)\Big|_{\lambda=0}$, $[\tilde{T}] = 0$

Combined UV and Spurious IR forest

Eurekâ moment:

- Remarkably, we always have : $\text{soft\_dod} = \text{UV\_dod} - 1$
- Spurious soft expansion also valid as UV counter term.
- Spurious soft IR forest similar to the one produced by the R-operation

so that we can combine the UV and spurious soft subtraction as one !

$$\boxed{\hat{T}_{\text{dod}} = T_{\text{dod}} + \tilde{T}_{\text{dod}-1} - T_{\text{dod}}\tilde{T}_{\text{dod}-1}}$$

until we realised that we had just re-invented the wheel: [J. H. Lowenstein, 1976]

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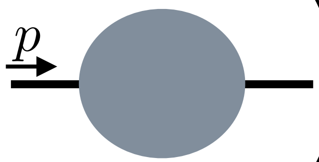
Novelty though: **automatic renormalisation** of fermion **masses in the OS scheme:**

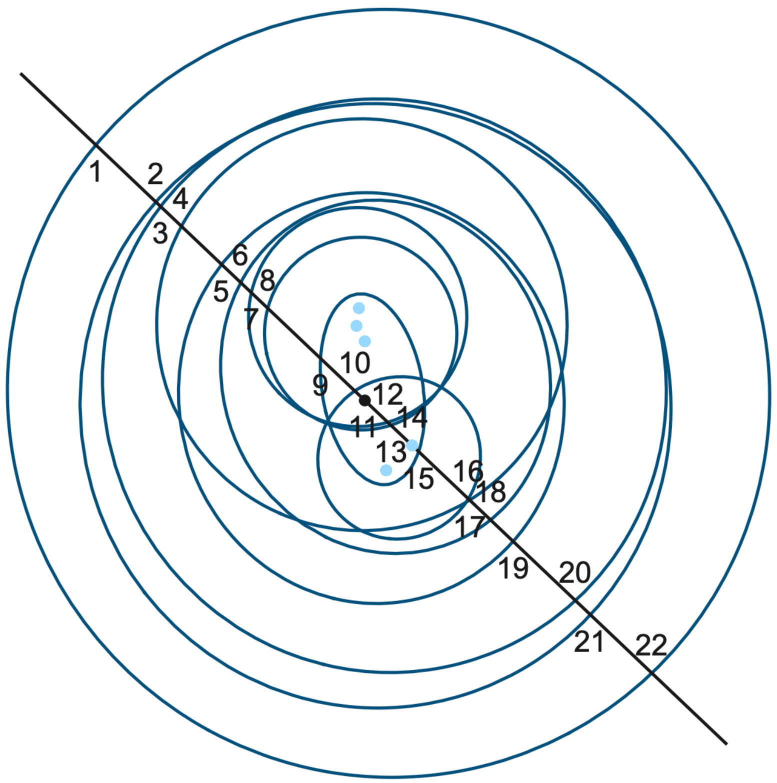
$T^{\text{os}\pm}\Big(\Sigma =$  $\Big) = (1 \pm \gamma^0)\Sigma(p = \pm p^{\text{os}})$, $\quad p^{\text{os}} = (m, 0, 0, 0)$

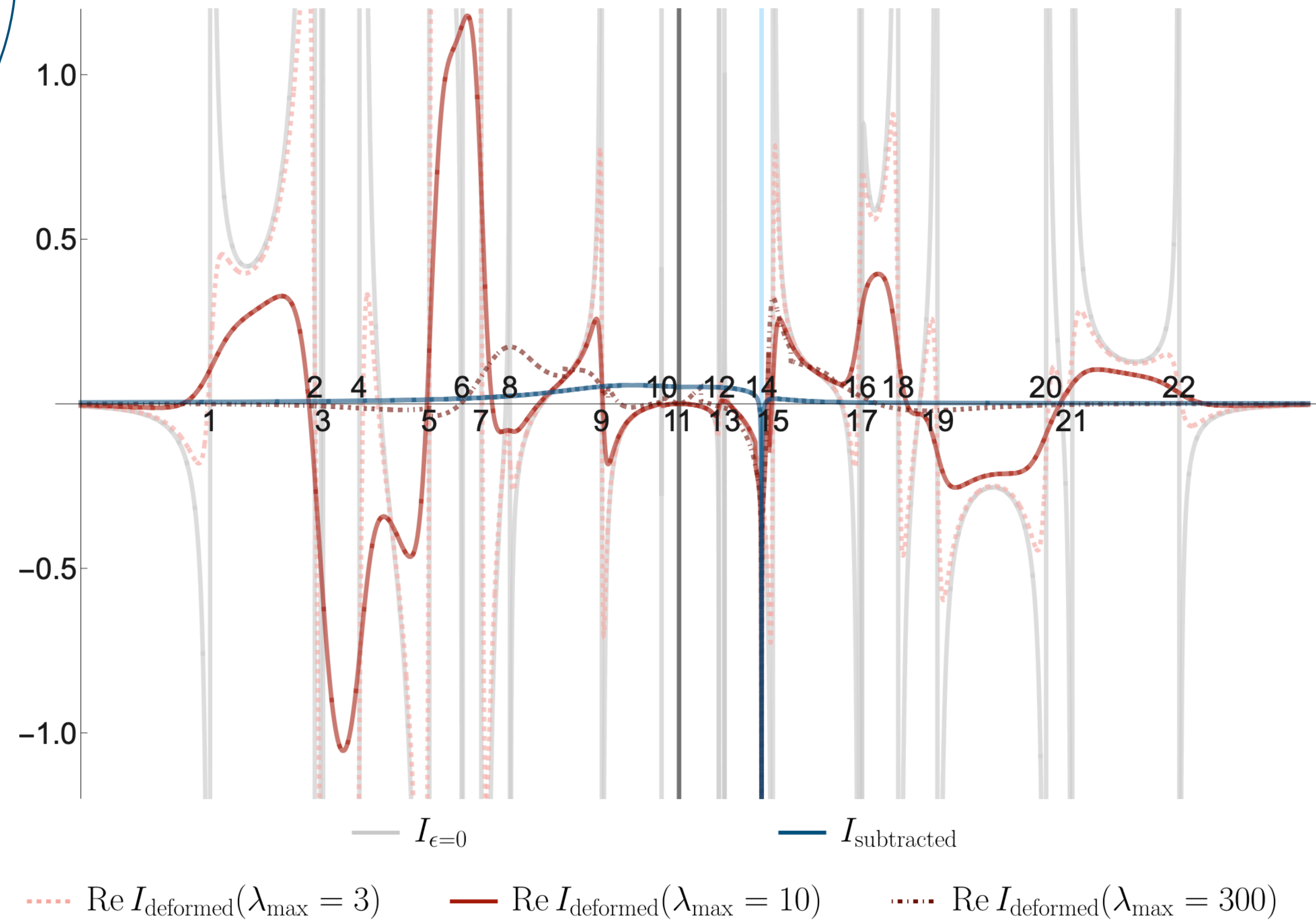
$$\frac{1}{2}\left([T^{\text{os}+}(\Sigma)] + [T^{\text{os}-}(\Sigma)]\right) = \delta m^{\text{os}}$$

Implying that our local UV counterterm T^{os} automatically generates the OS mass renormalisation counterterm !

THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

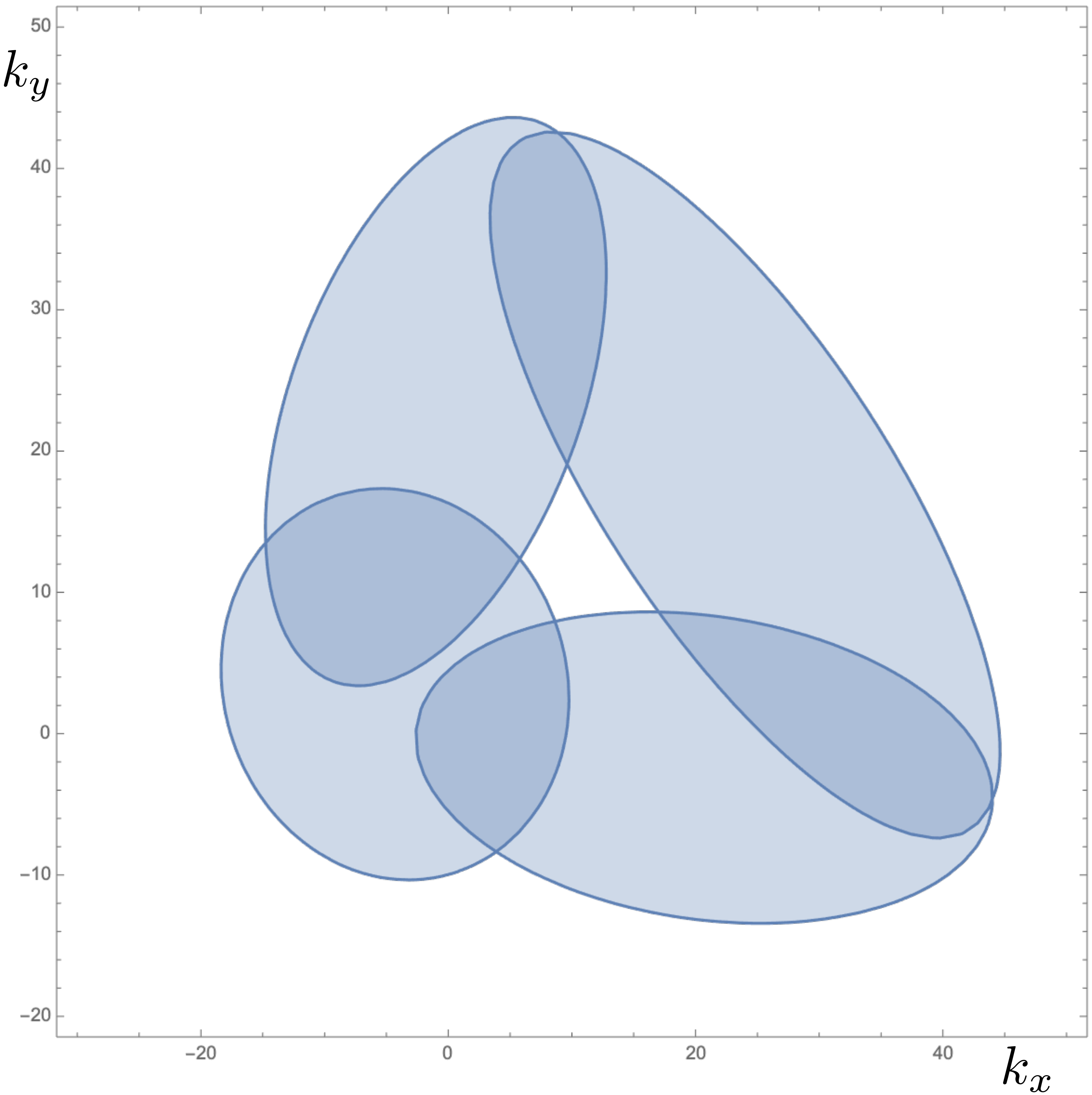
[D. Kermanschah, arXiv : 2110.06869]

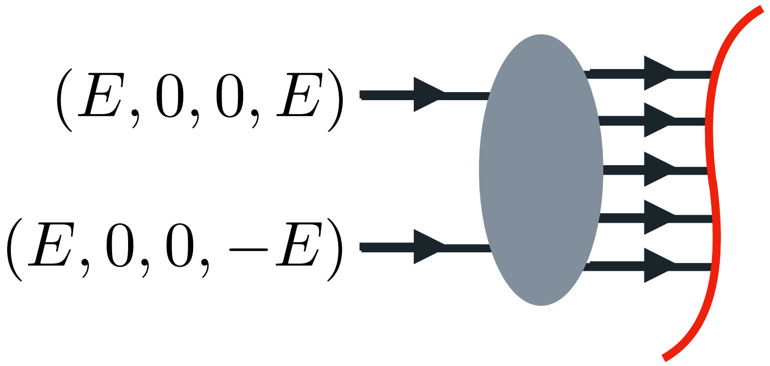




Locality Unitarity: Causal Flow

The **rescaling** change of variables is however **not general** : (**<u>but always sufficient in practice</u>**!)





Ex: Box_4E from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

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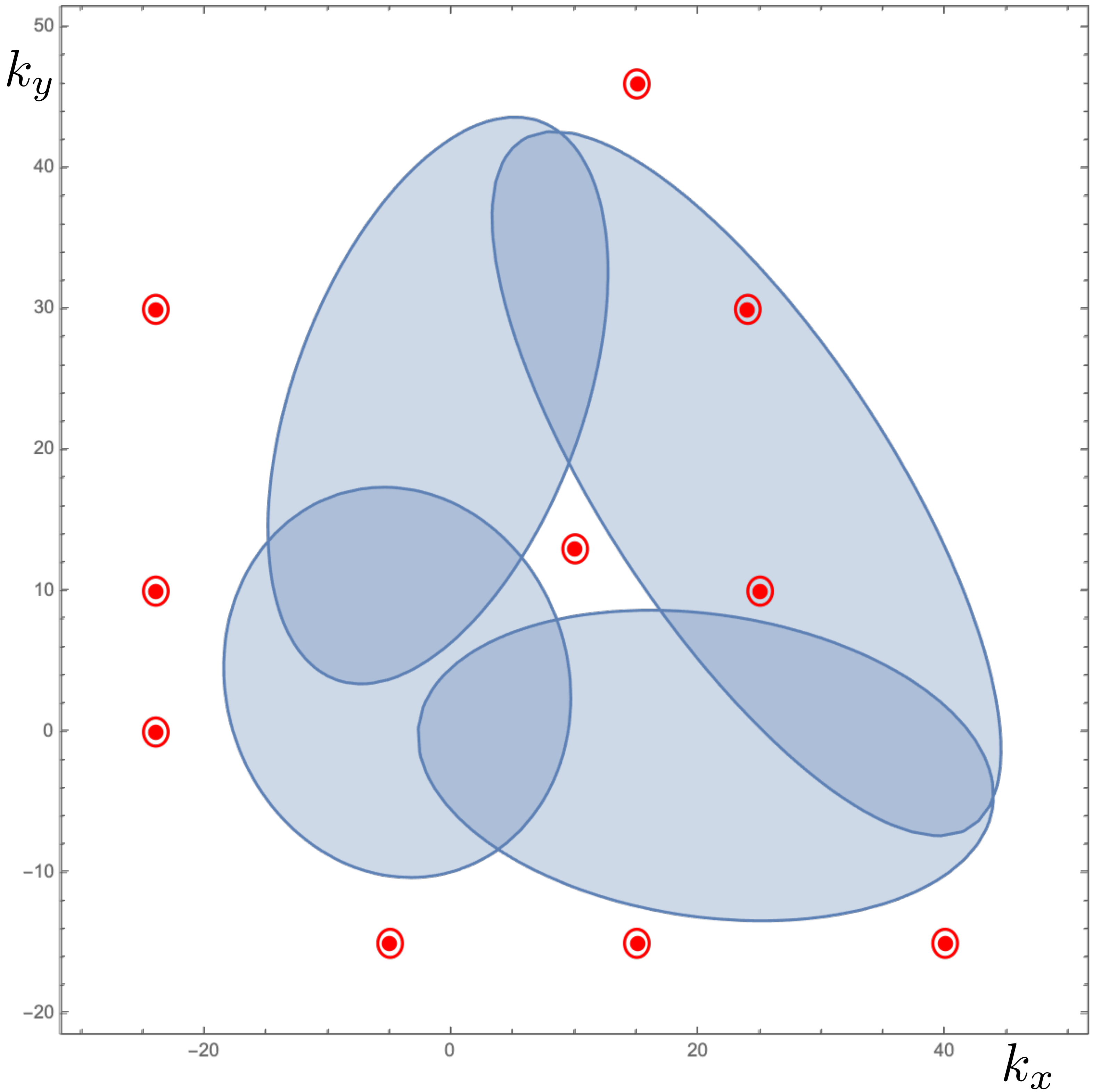

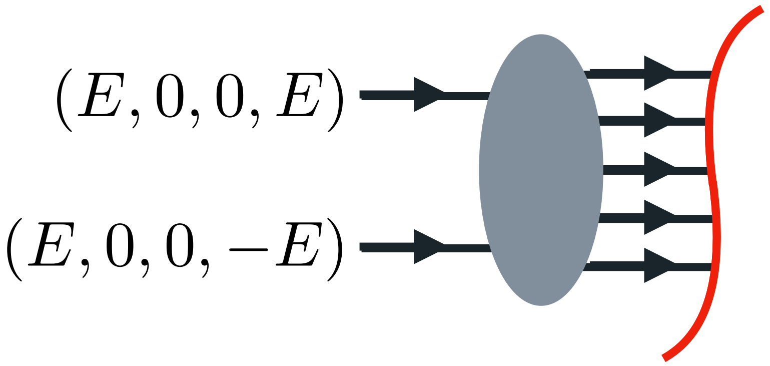

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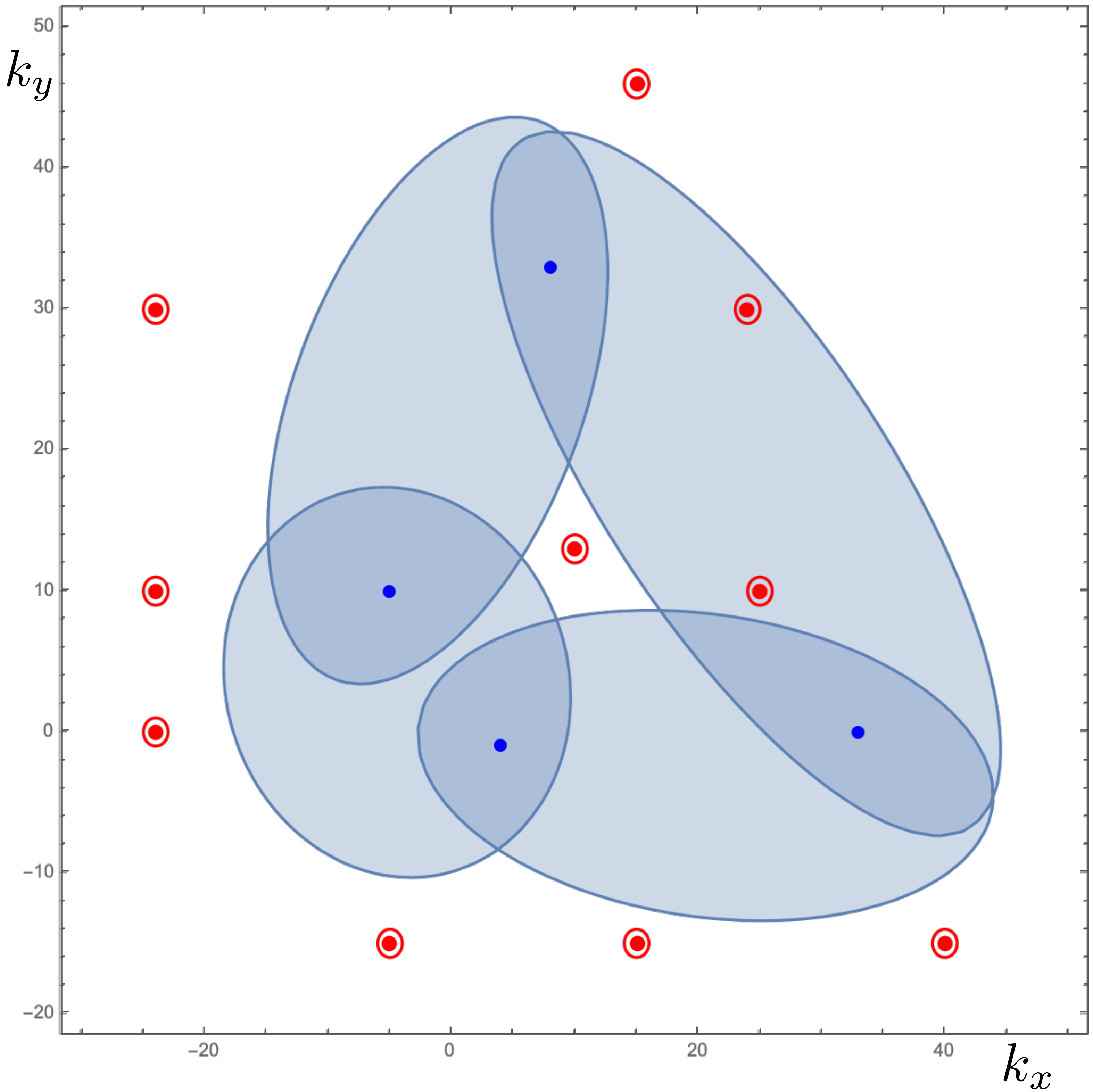

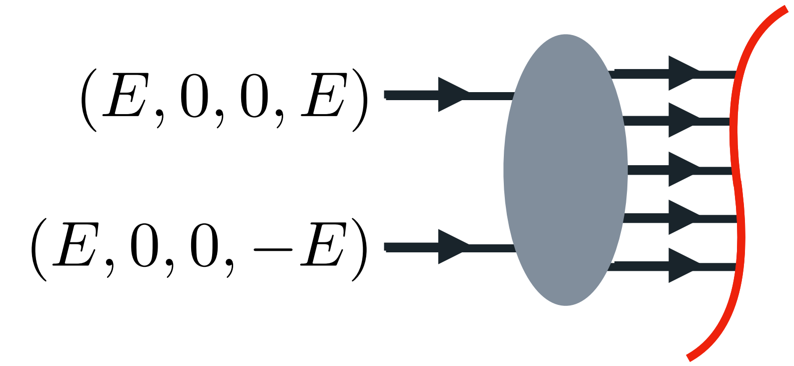

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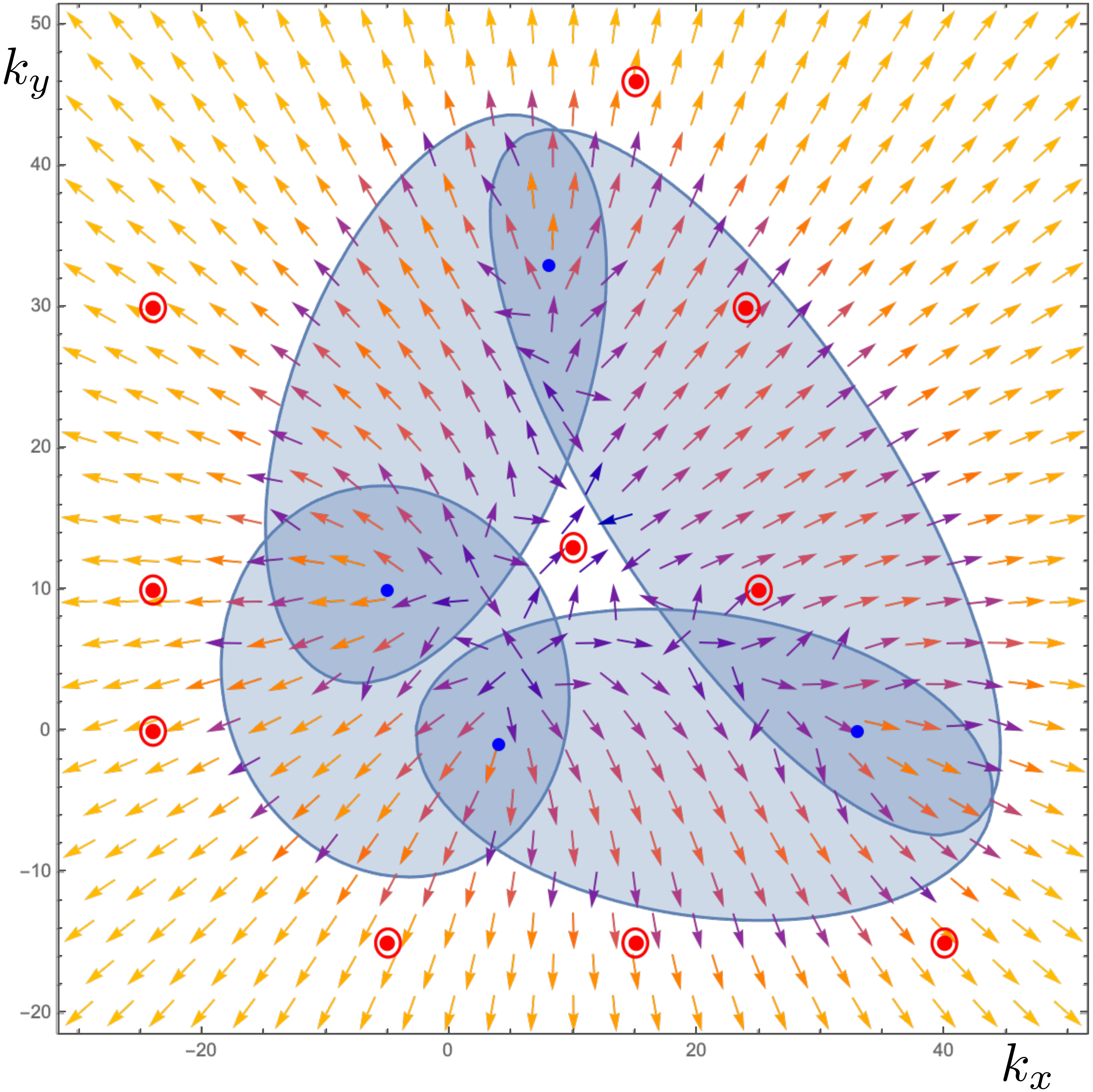

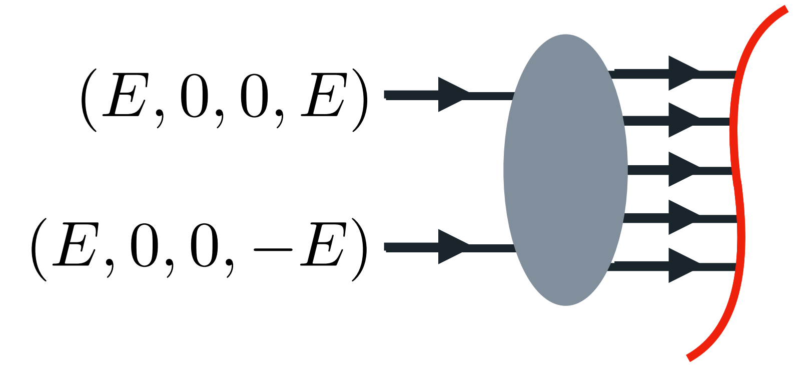

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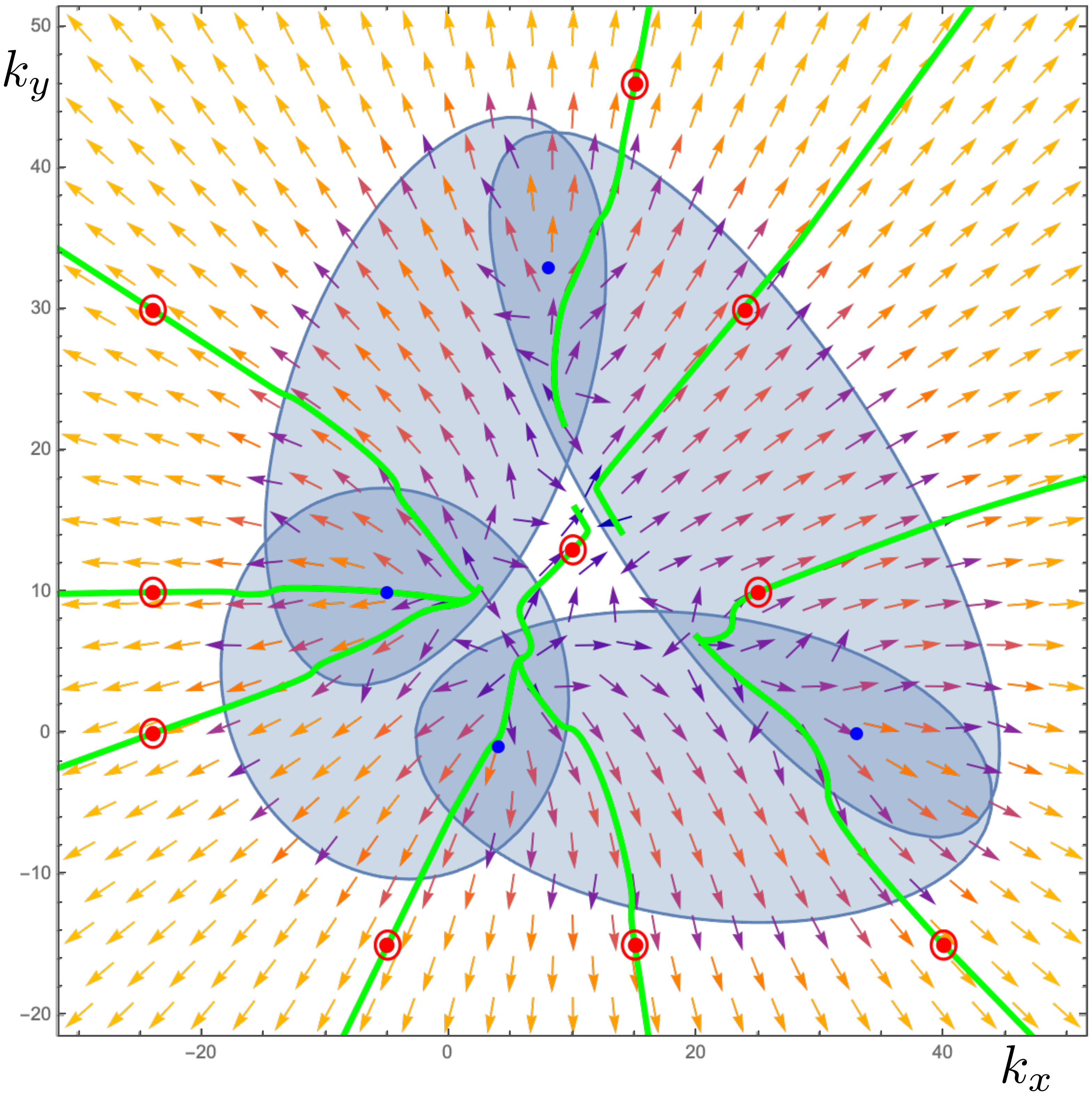

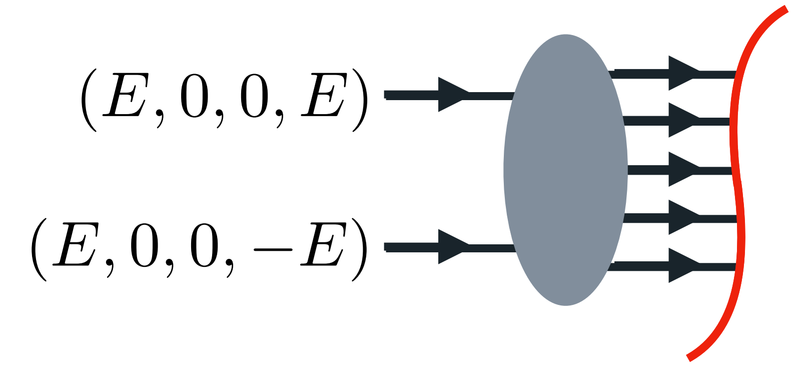

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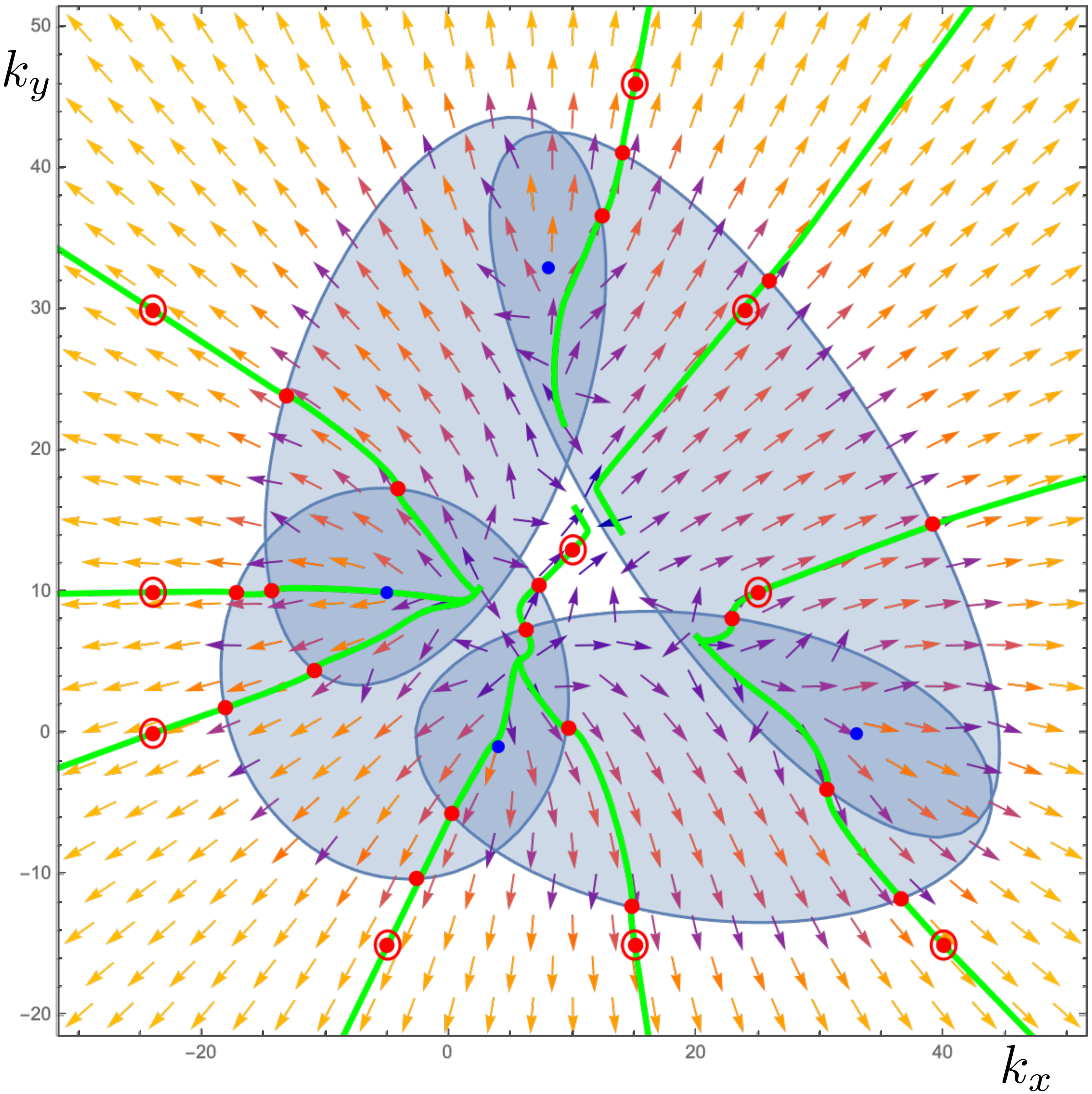

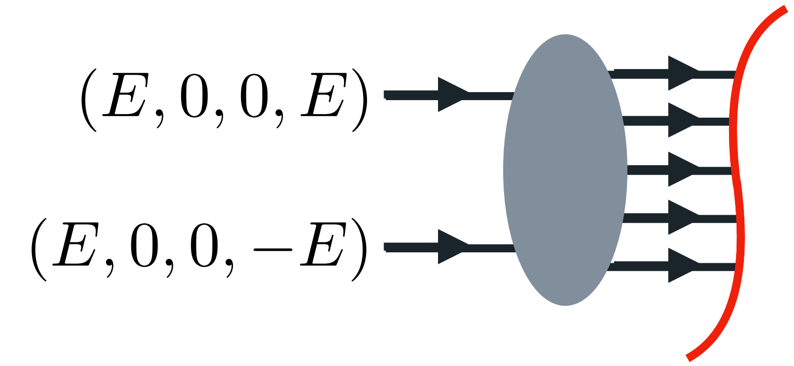

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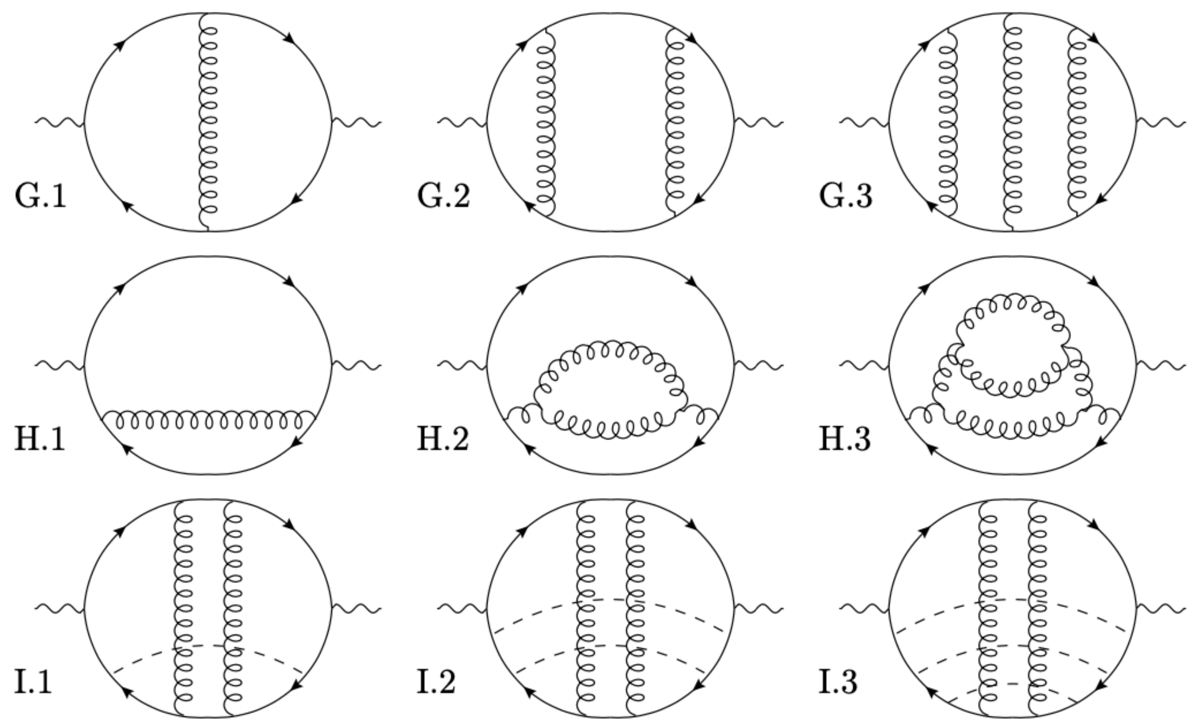
Implementation run-time performance



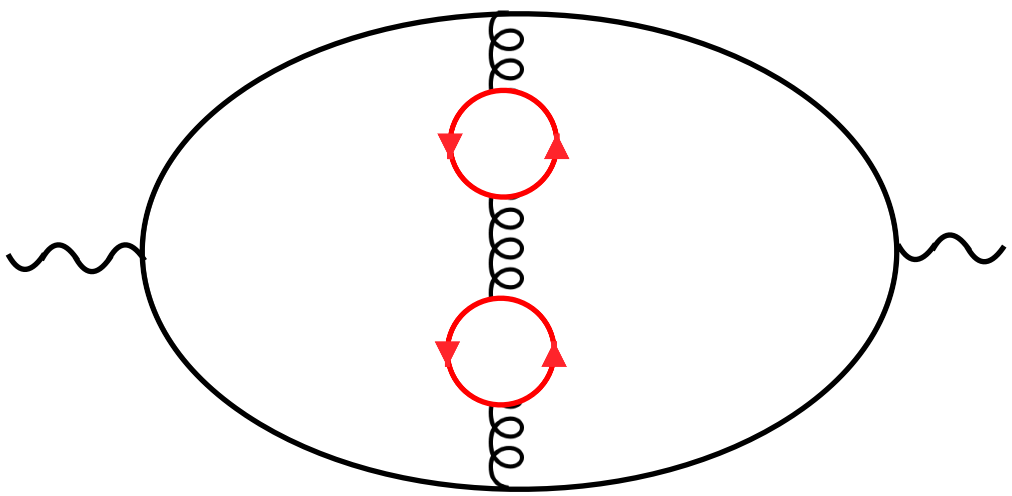
SG	proc.	order	t_{gen} [s]	M_{disk} [MB]	N_{sg} [-]	N_{cuts} [-]	t_{eval} [ms]	$t^{(\text{f128})}_{\text{eval}}$ [ms]
G.1	$1 \to 2$	NLO	0.1	0.13	2	4	0.004	0.13
G.2	$1 \to 2$	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	$1 \to 2$	N3LO	36K	509	220	16	17.6	281
H.1	$1 \to 2$	NLO	0.07	0.12	2	2	0.006	0.14
H.2	$1 \to 2$	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	$1 \to 2$	N3LO	255	43	220	4	2.35	56
I.1	$1 \to 3$	NNLO	126	22	266	9	0.32	12.4
I.2	$1 \to 4$	NNLO	1.9K	120	4492	9	4.4	67
I.3	$1 \to 5$	NNLO	36K	20K	$\mathcal{O}$(100K)	9	3.6K	17.3K

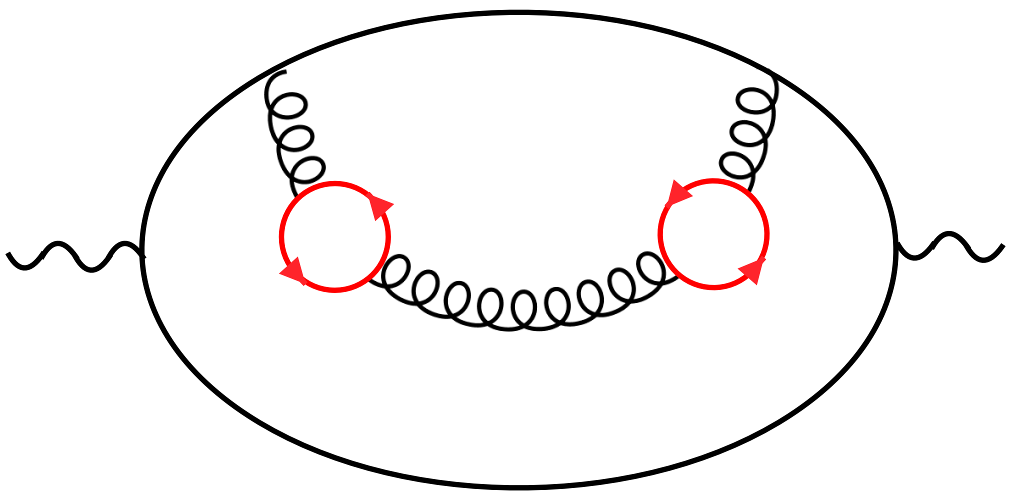
NB: these are **integrand** performance.
(Note: we recently found massive speedup w.r.t the above)

Integration (sampling) not optimised yet.
so we do not report quantitatively on it yet.

Partial N3LO results

n_f^2 contributions :





$$K_{jj}^{\mathrm{(MC\ LU)\ I}} = 24.45(10)$$

$$K_{jj}^{\mathrm{(MC\ LU)\ II}} = -24.80(22)$$

(Large accidental cancellation between the two graphs, but validation otherwise successful)

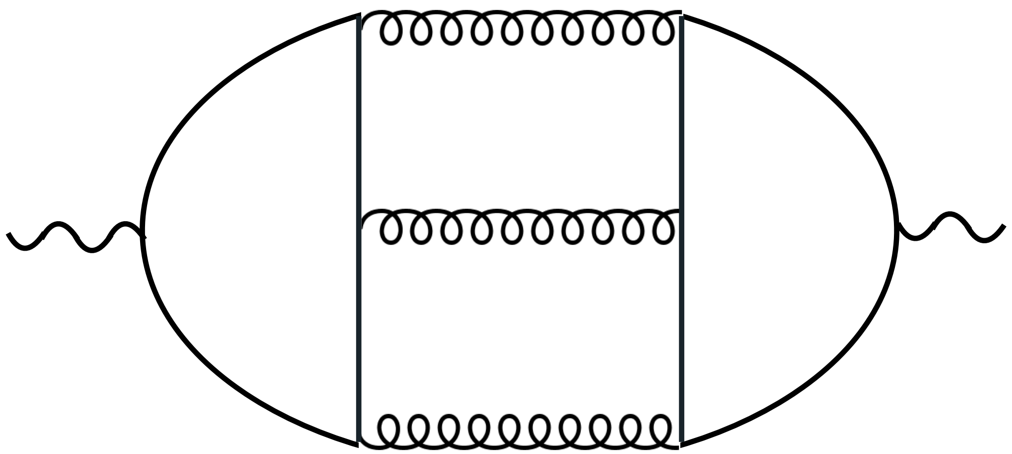
$$K_{jj}^{\mathrm{(MC\ LU)\ I+II}} = -0.35(24)$$

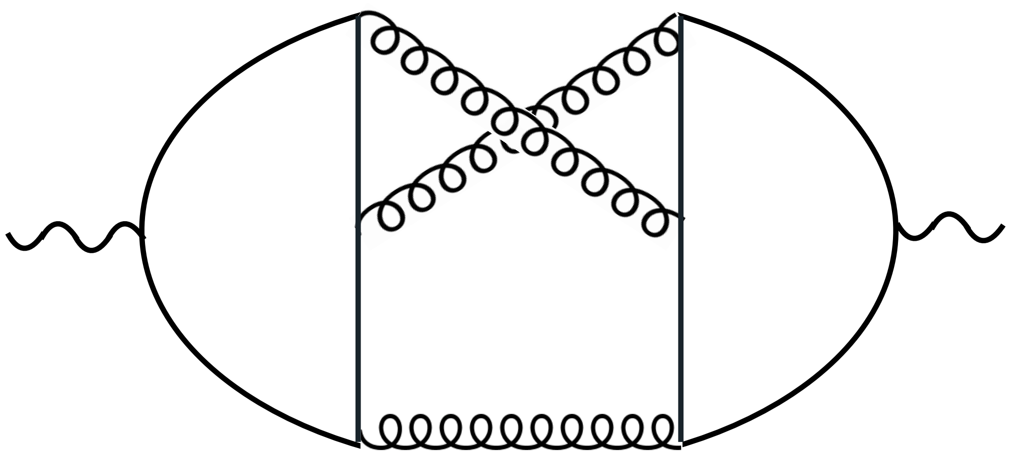
$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f^2)} = C_F\left(\frac{1208}{27} - \frac{8}{3}\zeta_2 - \frac{304}{9}\zeta_3\right) = -0.331415$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

Partial N3LO results

Singlet contributions : (Results for low Monte-Carlo statistics here)





$$K_{jj}^{(\text{MC LU})\text{I}} = 48.4(1.0)$$

$$K_{jj}^{(\text{MC LU})\text{II}} = -74.0(1.1)$$

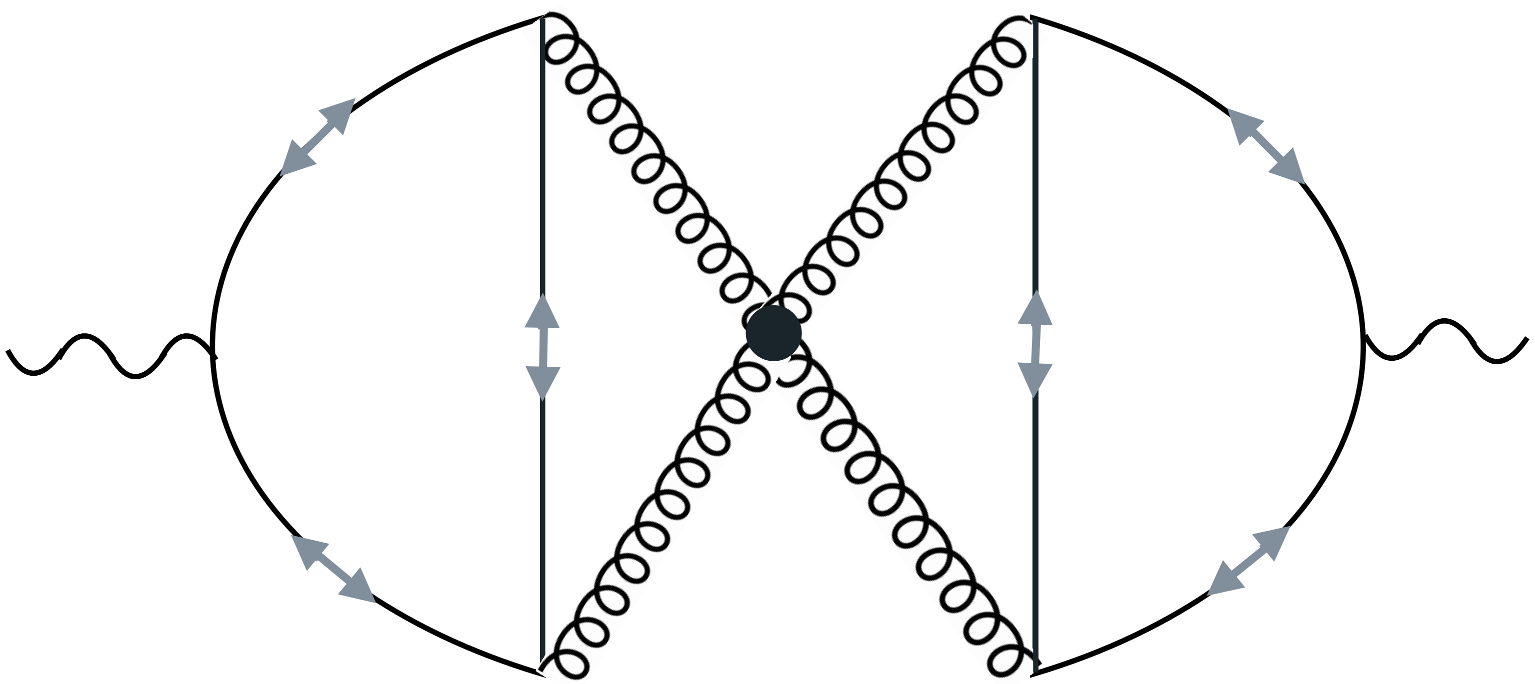
$$K_{jj}^{(\text{MC LU})\text{I+II}} = -25.6(1.5)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3),\,\text{singlet}} = \frac{d_F^{abc} d_F^{abc}}{N_R}\left(\frac{176}{3} - 128\zeta_3\right) = -26.4435$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

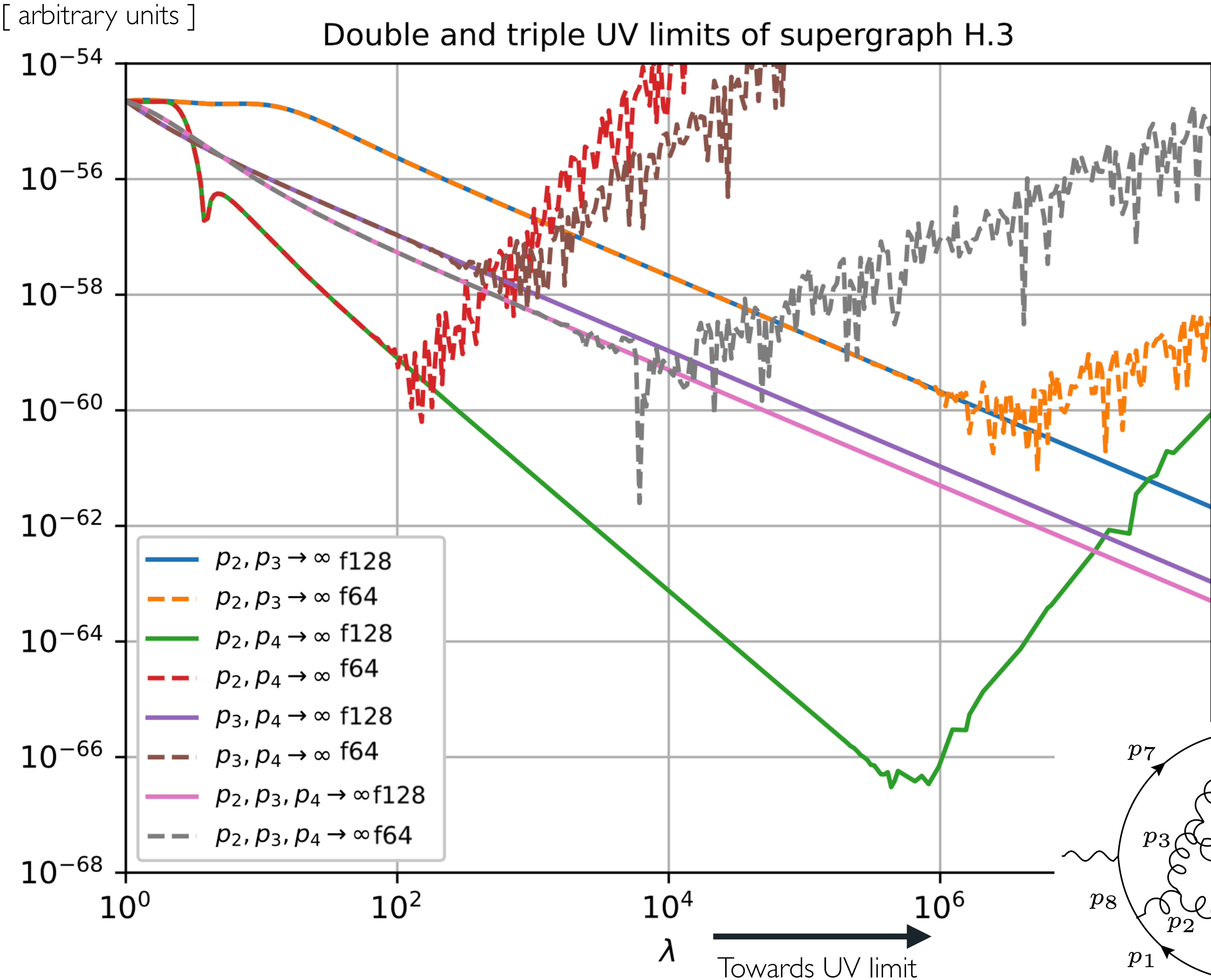
Curiosity

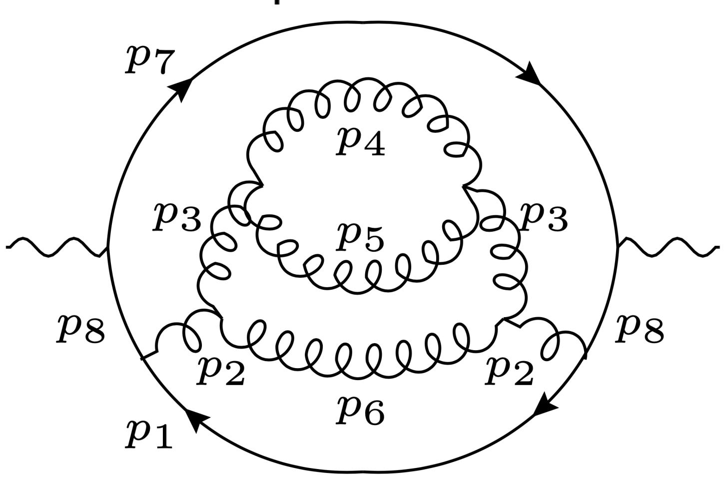
Singlet contributions : A non-obvious zero contribution…



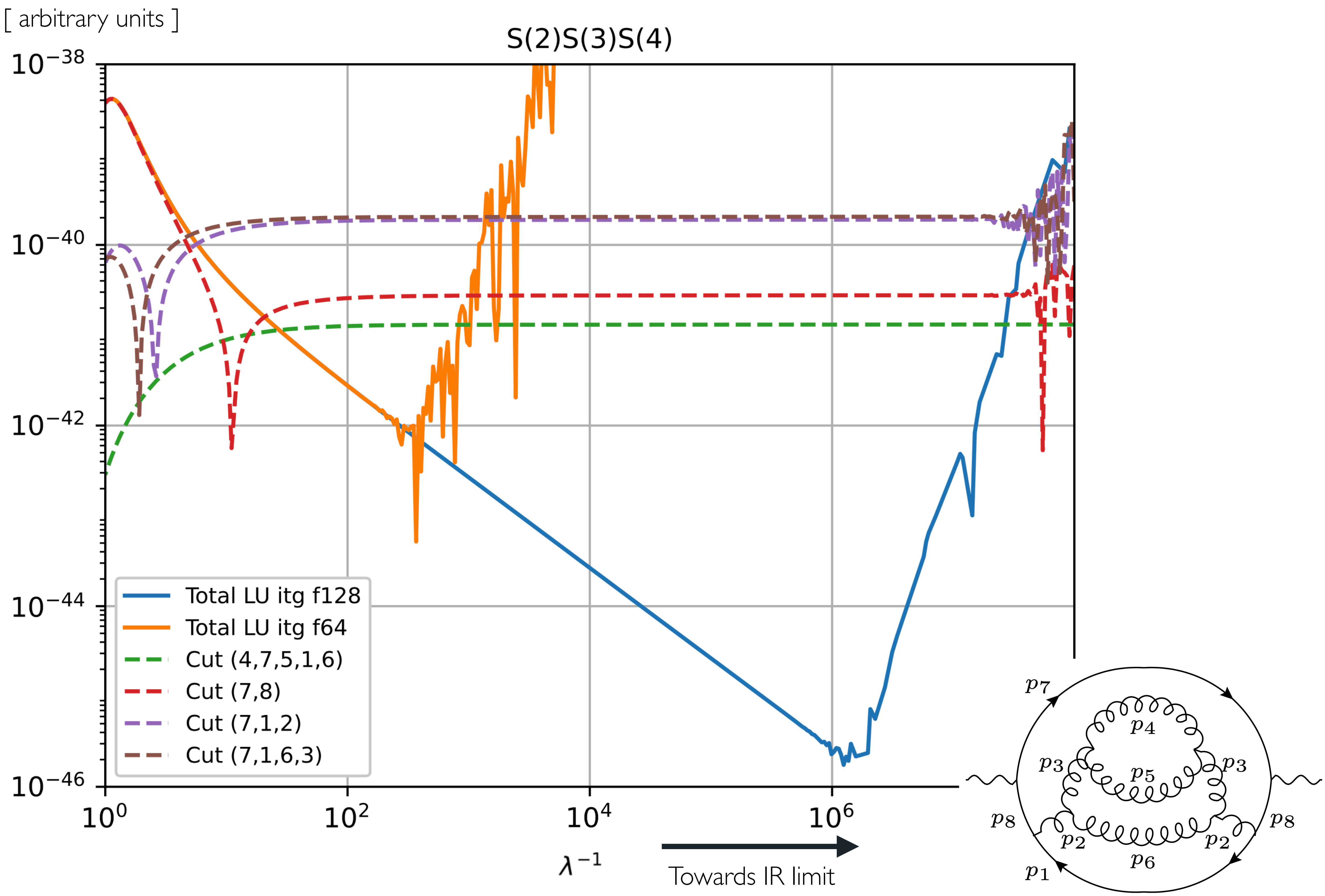
This graph gives no contribution inclusively.
I have not looked into it with any depth, but I don't see
an obvious really as to why it should be zero…

Testing N3LO UV Limits

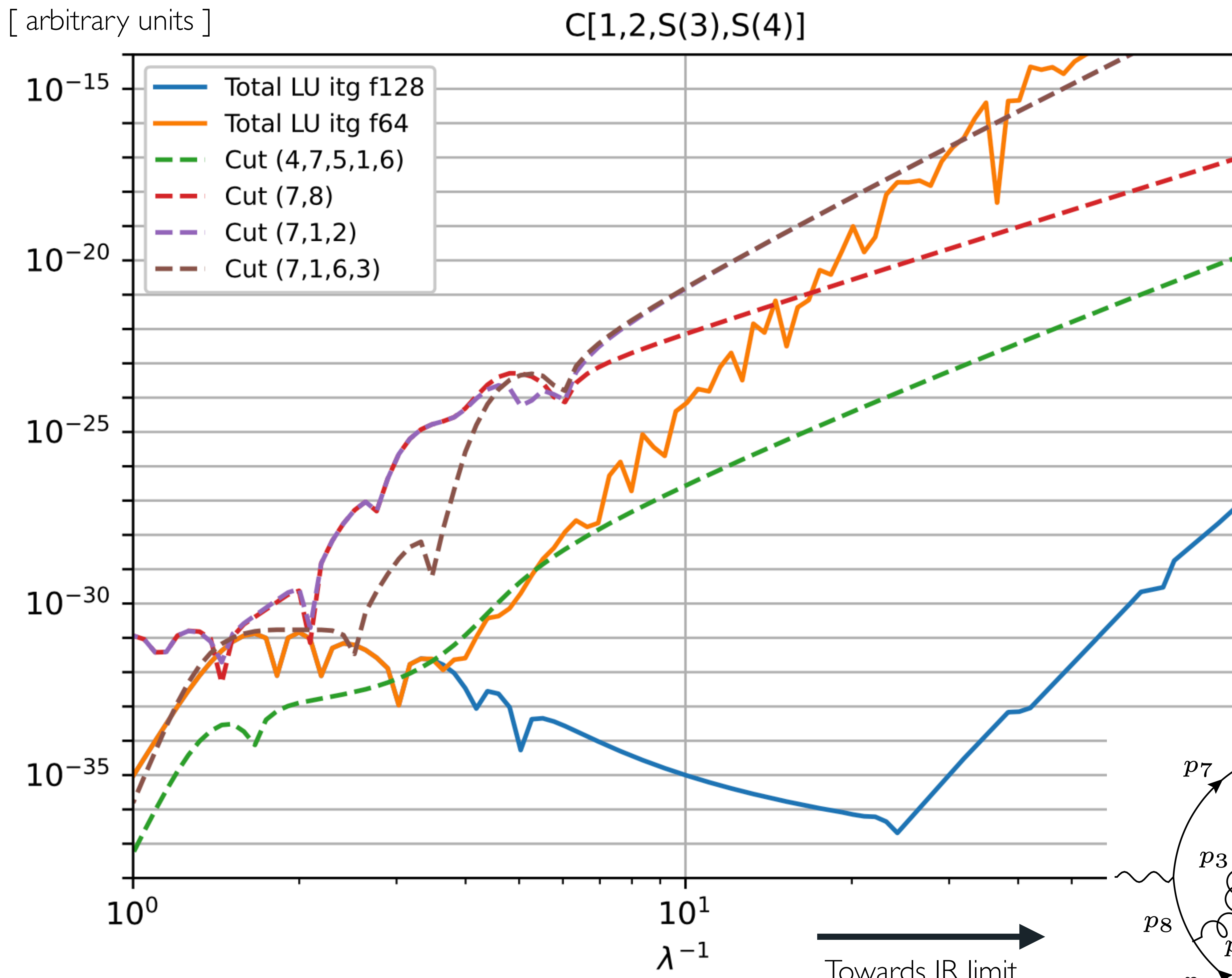


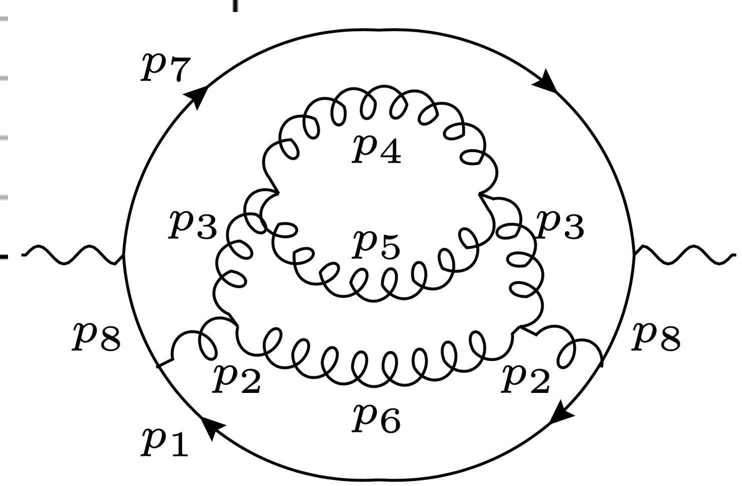


Testing IR soft Limits

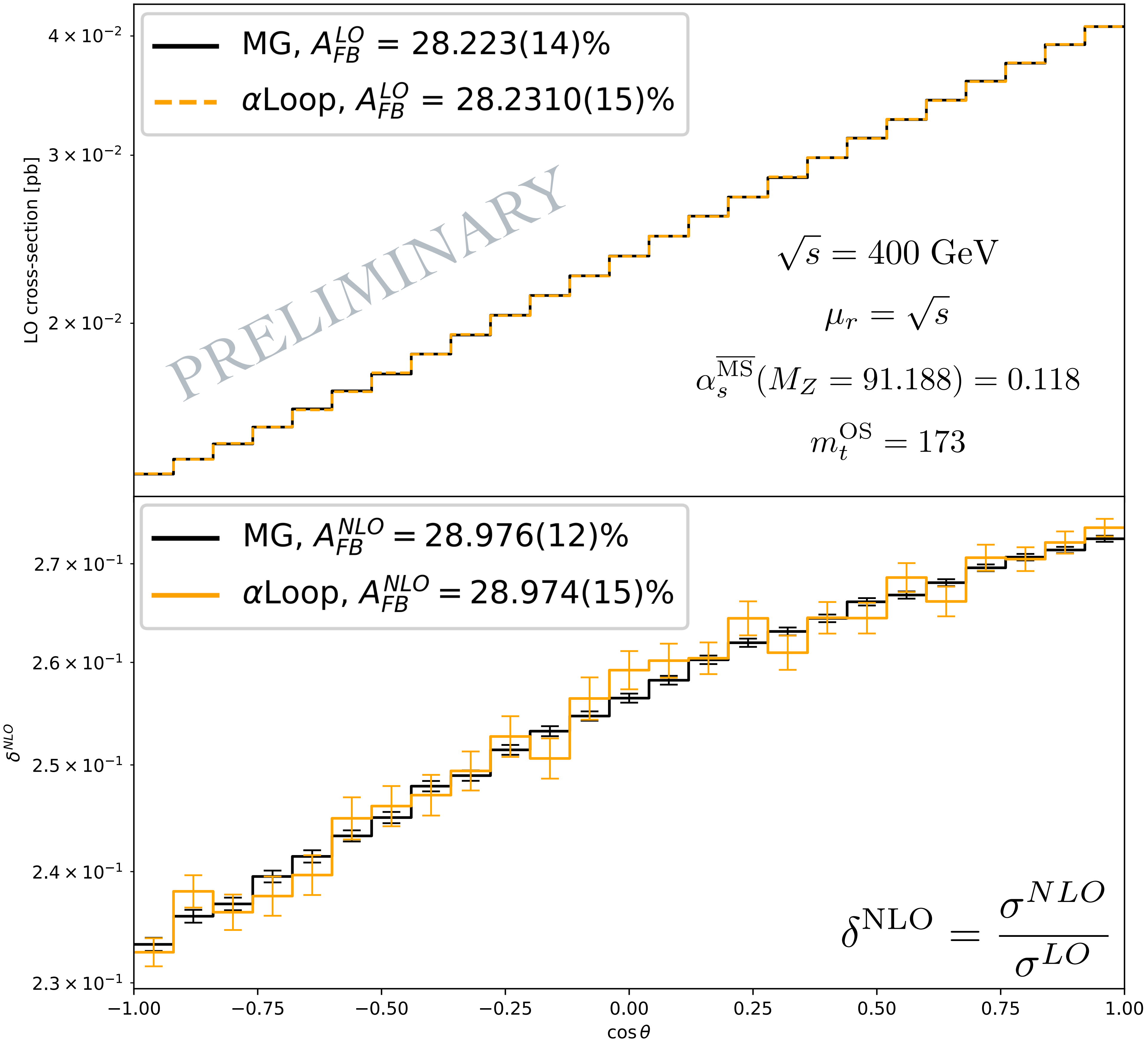


Testing IR soft-collinear Limits





EXAMPLE II : NLO AFB FOR $e^+e^- \to \gamma^\star/Z \to t\bar{t}$



First result in LU with γ^5 and EW-boson

Contour deformation well-behaved in this case

Credits to ETHZ student

Max Hofer