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TUM

LOCAL UNITARITY

BUILDING A NUMERICAL COLLIDER

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TUM SEMINAR - MUNICH

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{ ANARICAL }

$\log(2)$

EXPRESSION

$$\int_1^2 dx \frac{1}{x}$$

{ ANARICAL }

$\log(2)$

ANALYTICAL ?

EXPRESSION

LAY MAN
CLASSIFICATION

$$\int_1^2 dx \frac{1}{x}$$

NUMERICAL ?

{ ANARICAL }

$$\log(2)$$

ANALYTICAL ?

$$\sim \sum_{i=1}^N \frac{(1-x)^i}{-i} \Big|_{x=2}$$

EXPRESSION

**LAY MAN
CLASSIFICATION**

IMPLEMENTATION

$$\int_1^2 dx \frac{1}{x}$$

NUMERICAL ?

$$\sim \frac{1}{N} \sum \frac{1}{(1 + \text{rdm}())}$$

{ ANARICAL }

$$\log(2)$$

ANALYTICAL ?

$$\sim \sum_{i=1}^N \frac{(1-x)^i}{-i} \Big|_{x=2}$$

$$t \propto D \log(D)^2$$

EXPRESSION

**LAY MAN
CLASSIFICATION**

IMPLEMENTATION

**COMPLEXITY FOR
“D” ACCURATE DIGITS**

$$\int_1^2 dx \frac{1}{x}$$

NUMERICAL ?

$$\sim \frac{1}{N} \sum \frac{1}{(1 + \text{rdm}())}$$

$$t \propto D^2$$

OVERVIEW OF LOCAL UNITARITY



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with \mathcal{J} a measurement function, over $x \in [0, 10]$

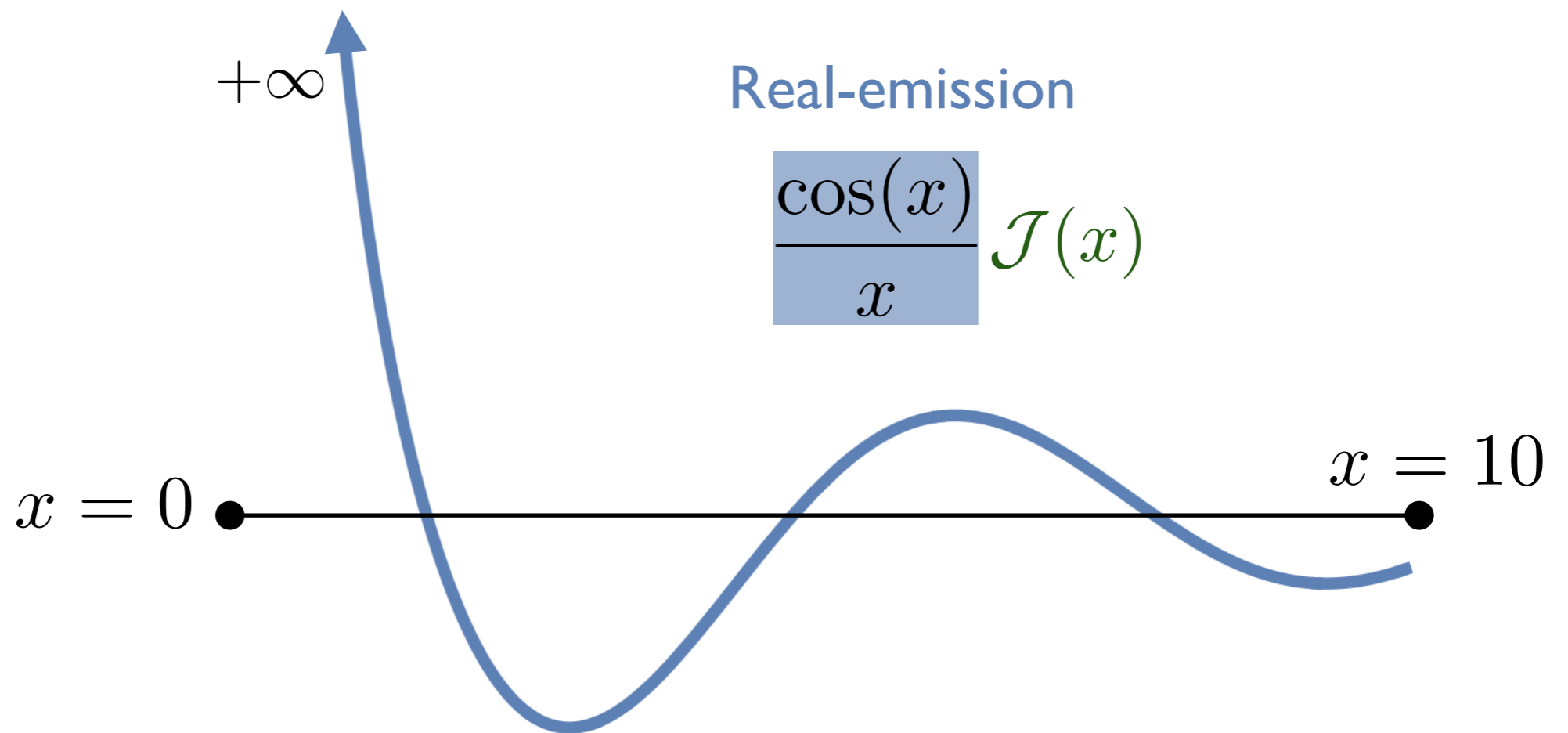
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with \mathcal{J} a measurement function, over $x \in [0, 10]$

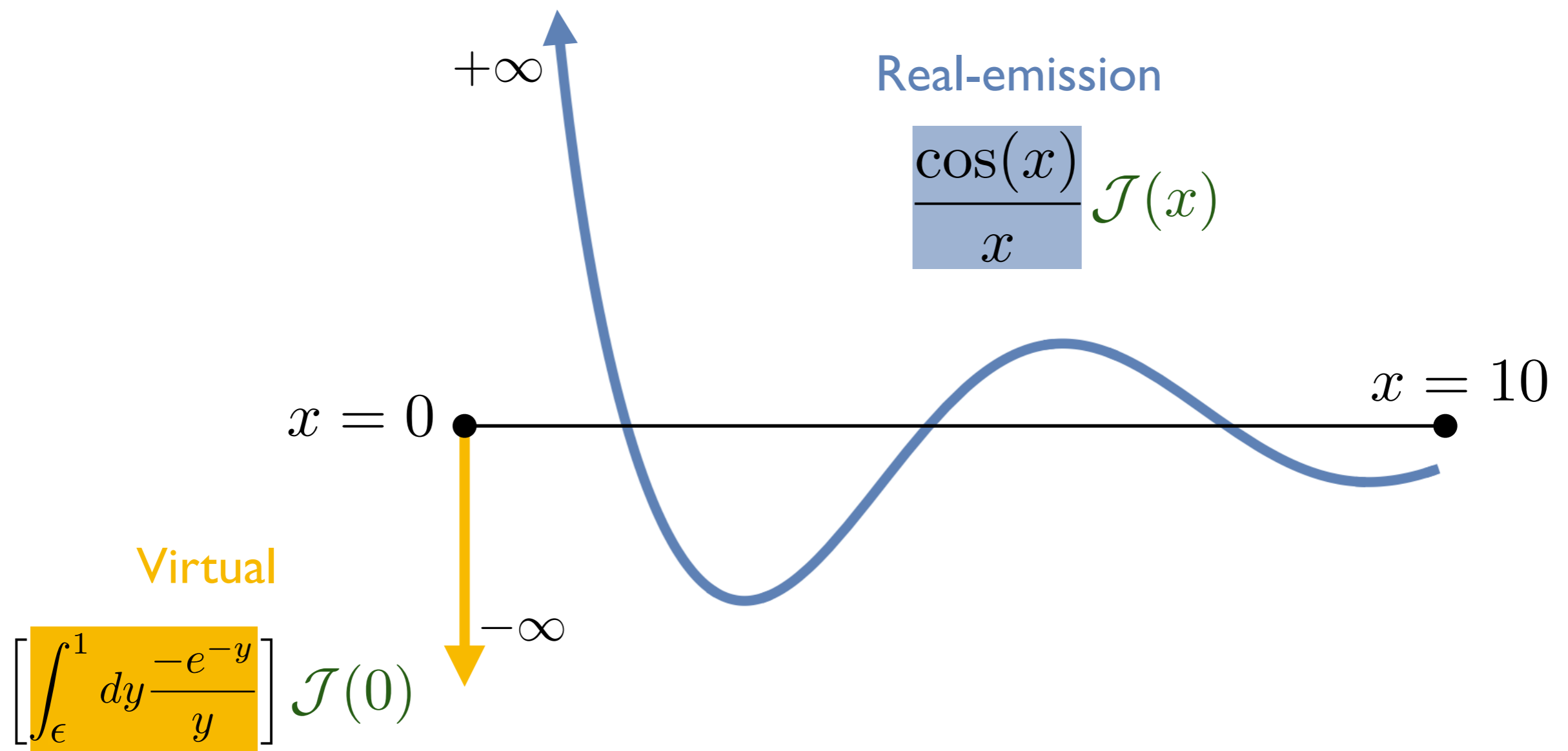
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



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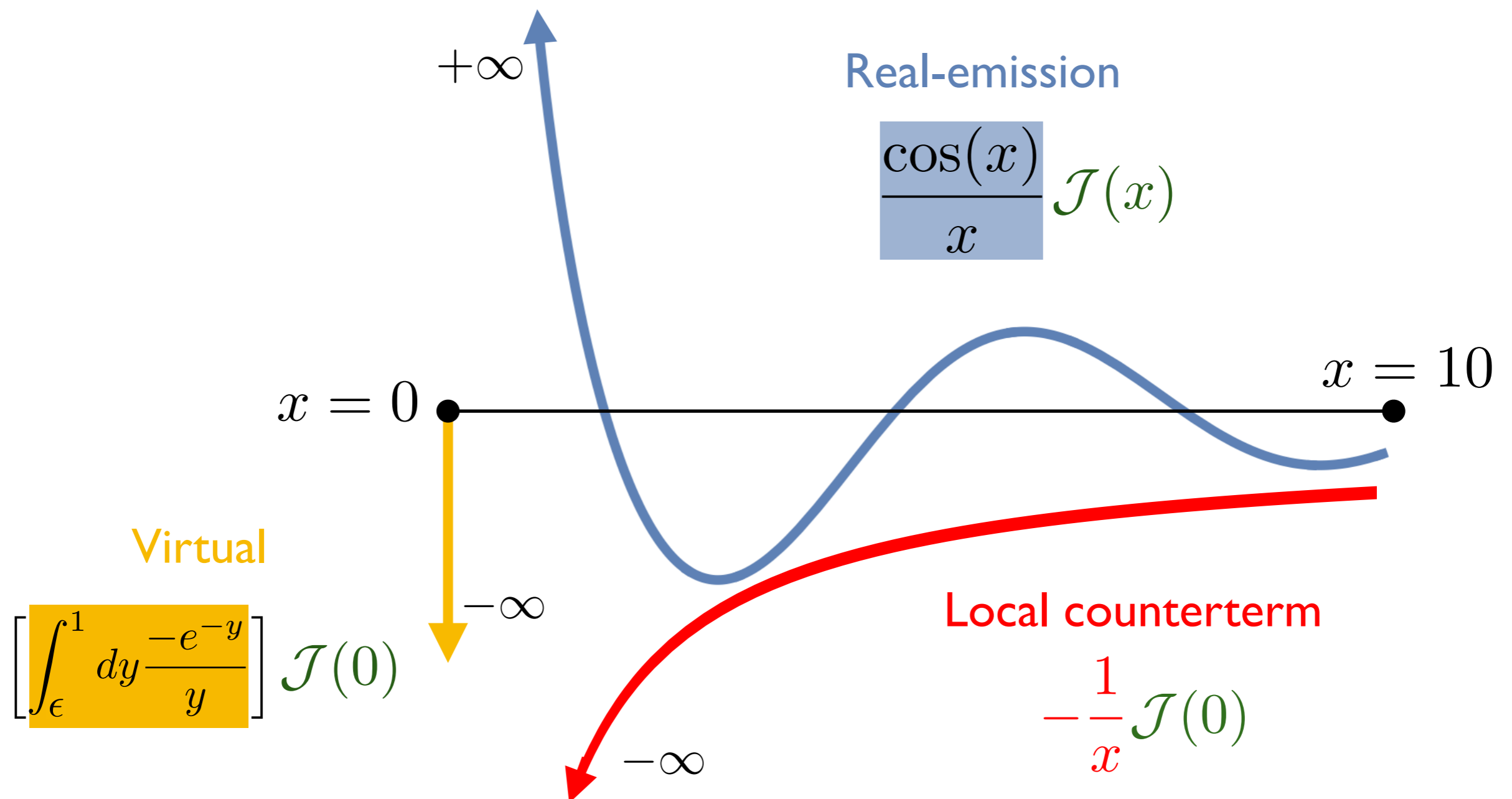
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with \mathcal{J} a measurement function, over $x \in [0, 10]$

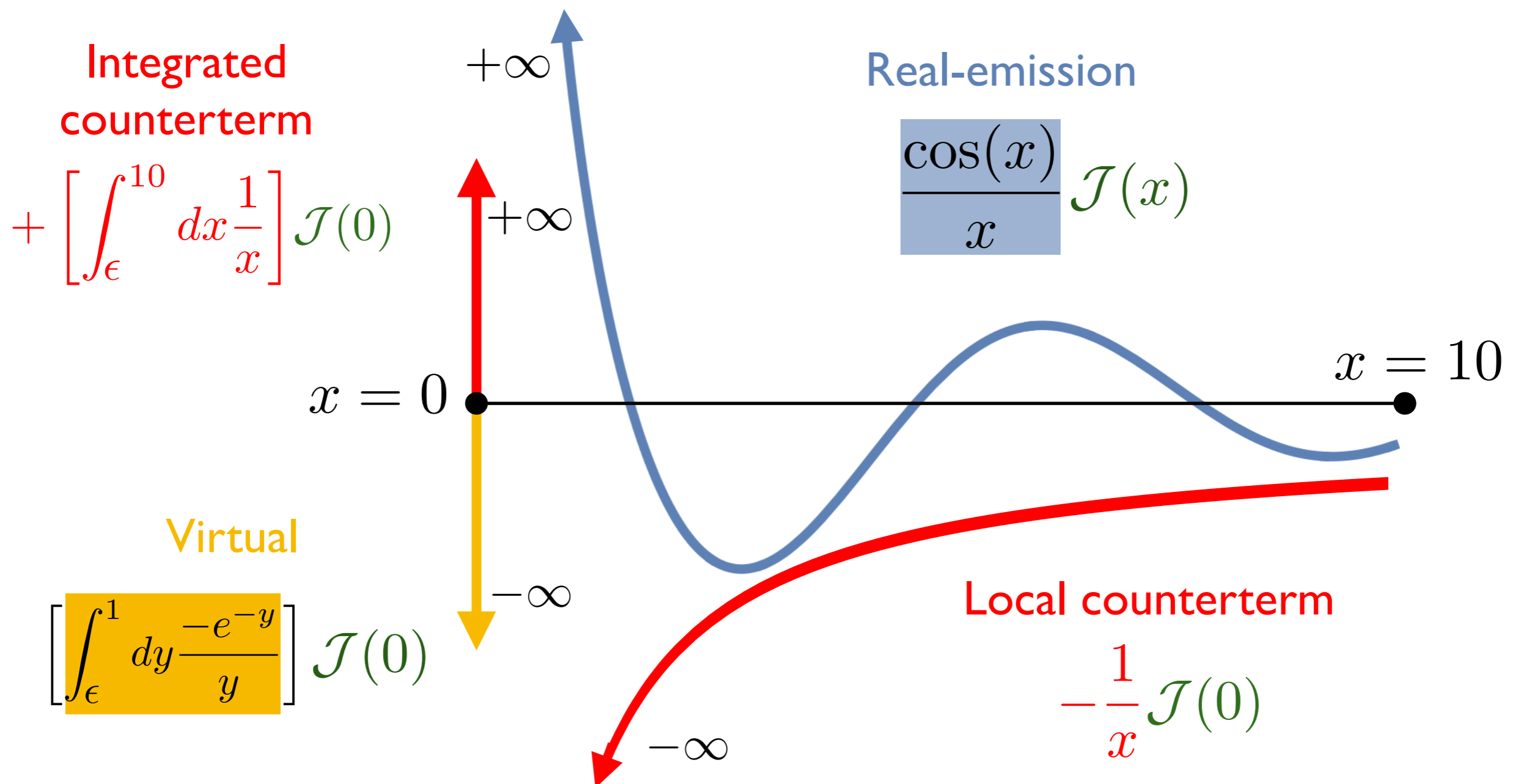
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with \mathcal{J} a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0) + \left[\int_0^{10} dx \frac{1}{x} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \quad \sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \right]$$



ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \qquad \sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{e^{-x/10}}{x} \mathcal{J}(0) \right]$$

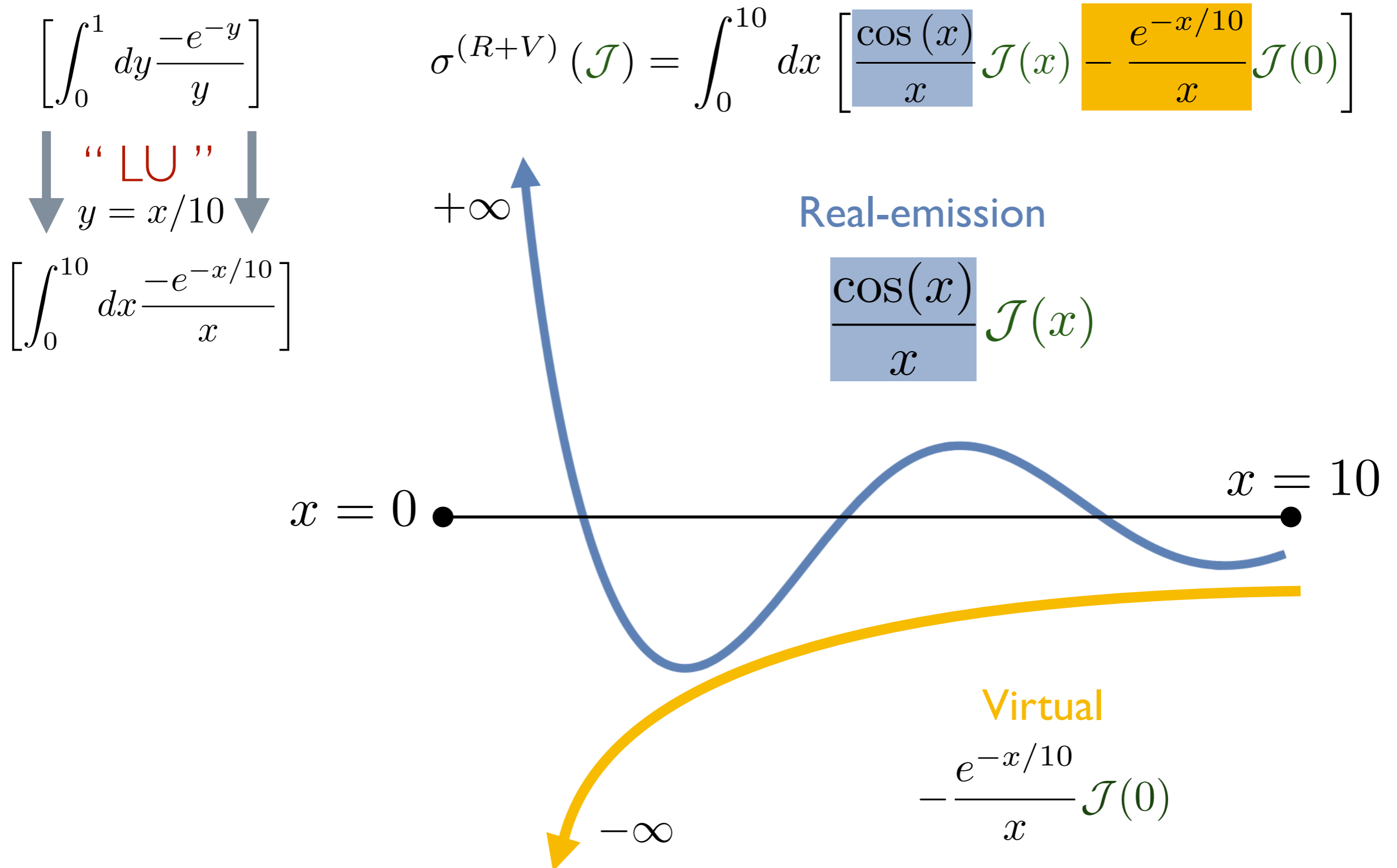
“LU”
↓ $y = x/10$ ↓

$$\left[\int_0^{10} dx \frac{-e^{-x/10}}{x} \right]$$



ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual



REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The diagrams are Feynman-like graphs for forward scattering. Each graph consists of a wavy line (photon) interacting with a fermion loop (quark).
 - Diagram 1: Photon couples to the left vertex of the loop.
 - Diagram 2: Photon couples to the right vertex of the loop.
 - Diagram 3: Photon couples to the top vertex of the loop.
 - Diagram 4: Photon couples to the bottom vertex of the loop.
 The asterisk (*) denotes the complex conjugate of the corresponding diagram.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} =$$

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9}
 \end{aligned}$$

The diagrams are as follows:

- Diagram 1: A blue square on the left, a wavy line, and a semi-circle with a vertical dashed line.
- Diagram 2: A wavy line and a semi-circle with a vertical dashed line.
- Diagram 3: A green square on the left, a wavy line, and a semi-circle with a vertical dashed line.
- Diagram 4: A wavy line and a semi-circle with a vertical dashed line.
- Diagram 5: A wavy line and a semi-circle with a diagonal dashed line.
- Diagram 6: A wavy line and a semi-circle with a diagonal dashed line.
- Diagram 7: A wavy line and a semi-circle with a diagonal dashed line.
- Diagram 8: A wavy line and a semi-circle with a diagonal dashed line.
- Diagram 9: A circle with a wavy line on the left and a wavy line on the right, split by a vertical red line. The left half is blue and the right half is green.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9} + \text{Diagram 10}
 \end{aligned}$$

The diagrams are Feynman-like graphs for the process $\gamma^* \rightarrow d\bar{d}$.

 - Diagram 1: A wavy line (photon) enters from the left and splits into a quark and an antiquark.

 - Diagram 2: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a blue square.

 - Diagram 3: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 4: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square, and the entire diagram is marked with a star.

 - Diagram 5: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 6: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 7: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square, and the entire diagram is marked with a star.

 - Diagram 8: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square, and the entire diagram is marked with a star.

 - Diagram 9: A circular loop diagram with a wavy line entering from the left and a quark-antiquark pair exiting from the right. A red vertical line is drawn through the loop.

 - Diagram 10: Similar to Diagram 9, but the loop is highlighted in a blue square on the left and a green square on the right, with a red vertical line through it.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11}
 \end{aligned}$$

The diagrams are Feynman-like graphs for the process $\gamma^* \rightarrow d\bar{d}$.

 - Diagram 1: A wavy line (photon) enters from the left and hits a blue shaded semi-circular region.

 - Diagram 2: A wavy line enters from the left and hits a semi-circular region with a vertical wavy line inside.

 - Diagram 3: A wavy line enters from the left and hits a semi-circular region.

 - Diagram 4: A wavy line enters from the left and hits a green shaded semi-circular region with a vertical wavy line inside.

 - Diagram 5: A wavy line enters from the left and hits a semi-circular region with a diagonal wavy line inside.

 - Diagram 6: A wavy line enters from the left and hits a semi-circular region with a diagonal wavy line inside.

 - Diagram 7: A wavy line enters from the left and hits a semi-circular region with a diagonal wavy line inside.

 - Diagram 8: A wavy line enters from the left and hits a semi-circular region with a diagonal wavy line inside.

 - Diagram 9: A wavy line enters from the left, passes through a circle with a vertical red line, and exits to the right.

 - Diagram 10: A wavy line enters from the left, passes through a circle with a vertical wavy line and a vertical red line, and exits to the right.

 - Diagram 11: A wavy line enters from the left, passes through a circle with a vertical wavy line and a vertical red line, and exits to the right. The left half of the circle is blue shaded and the right half is green shaded.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^* \\
 &+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 1} + \text{Diagram 5} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} \\
 &+ \text{Diagram 9}
 \end{aligned}$$

The diagrams are:

- Diagram 1: A wavy line entering from the left, hitting a semi-circular arc on the right. A vertical wavy line is inside the arc.
- Diagram 2: A wavy line entering from the left, hitting a semi-circular arc on the right. A diagonal wavy line is inside the arc.
- Diagram 3: A wavy line entering from the left, hitting a semi-circular arc on the right. A diagonal wavy line is inside the arc. The left part of the arc is shaded blue.
- Diagram 4: A wavy line entering from the left, hitting a semi-circular arc on the right. A diagonal wavy line is inside the arc. The right part of the arc is shaded green.
- Diagram 5: A wavy line entering from the left, hitting a semi-circular arc on the right. A diagonal wavy line is inside the arc. The right part of the arc is shaded green.
- Diagram 6: A circle with a wavy line entering from the left and exiting to the right. A vertical red line is on the left side of the circle.
- Diagram 7: A circle with a wavy line entering from the left and exiting to the right. A vertical wavy line is inside the circle. A vertical red line is on the left side of the circle.
- Diagram 8: A circle with a wavy line entering from the left and exiting to the right. A vertical wavy line is inside the circle. A vertical red line is on the left side of the circle.
- Diagram 9: A circle with a wavy line entering from the left and exiting to the right. A diagonal wavy line is inside the circle. A vertical red line is on the left side of the circle. The left part of the circle is shaded blue, and the right part is shaded green.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^* \\
 &+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 &+ \text{Diagram 8} + \text{Diagram 9}
 \end{aligned}$$

The diagrams are Feynman-like graphs for the process $\gamma^* \rightarrow d\bar{d}$.

 - Diagram 1: A wavy line (photon) enters from the left and splits into a quark and an antiquark.

 - Diagram 2: Similar to Diagram 1, but with a vertical gluon loop between the quark and antiquark lines.

 - Diagram 3: Similar to Diagram 1, but with a diagonal gluon loop.

 - Diagram 4: Similar to Diagram 3, but with a blue shaded square highlighting the loop region.

 - Diagram 5: A circular loop with a red vertical line through it, representing a ghost loop.

 - Diagram 6: Similar to Diagram 5, but with a vertical gluon loop inside the circle.

 - Diagram 7: Similar to Diagram 6, but with a diagonal gluon loop inside the circle.

 - Diagram 8: Similar to Diagram 7, but with a blue shaded square on the left and a green shaded square on the right.

 - Diagram 9: Similar to Diagram 8, but with a diagonal gluon loop inside the circle.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^* \\
 &+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 &+ \text{Diagram 8} + \text{Diagram 9} \\
 &+ \text{Diagram 10}
 \end{aligned}$$

The diagrams are Feynman-like graphs for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagrams 1 and 2: A wavy line (photon) enters from the left and splits into a quark and an antiquark. Diagram 2 includes a vertical gluon loop on the quark line.
 - Diagrams 3 and 4: Similar to 1 and 2, but with a blue and green shaded region respectively, and a diagonal gluon loop.
 - Diagrams 5-9: Circular graphs with a red vertical line through them. Diagrams 5-7 have vertical gluon loops, and diagrams 8-9 have diagonal gluon loops.
 - Diagram 10: A circular graph with a red vertical line, split into a blue shaded left half and a green shaded right half, with a diagonal gluon loop.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The first row shows two terms in large parentheses, each containing a sum of two diagrams. The first diagram in each sum is a wavy line entering from the left and exiting to the right, with a semi-circular arc on the right side. The second diagram in each sum is similar but has a vertical wavy line inside the semi-circular arc. The second row shows two terms in large parentheses, each containing a sum of two diagrams. The first diagram in each sum is a wavy line entering from the left and exiting to the right, with a diagonal wavy line crossing the semi-circular arc. The second diagram in each sum is similar but has a vertical wavy line inside the semi-circular arc. The second row diagrams are highlighted with blue and green squares.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7}$$

$$+ \text{Diagram 8} + \text{Diagram 9}$$

$$+ \text{Diagram 10} + \text{Diagram 11}$$

The equation shows a sum of eleven diagrams. The first three diagrams are circles with a wavy line entering from the left and exiting to the right, and a red vertical line passing through the circle. The second and third diagrams also have a vertical wavy line inside the circle. The next two diagrams are similar to the first two but have a diagonal wavy line crossing the circle. The final two diagrams are similar to the first two but have a diagonal wavy line crossing the circle. The last diagram is highlighted with blue and green squares.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The diagrams in the first row show a wavy line entering from the left and exiting to the right, with a semi-circular arc on the right side. The second diagram in the first row has a vertical wavy line inside the arc. The diagrams in the second row are similar but with a diagonal wavy line inside the arc.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

The diagrams in this equation are circular loops with a wavy line entering from the left and exiting to the right. A red vertical line is drawn through each loop. The first diagram (Diagram 5) is enclosed in a green rectangular box. The other diagrams (6-10) have different internal wavy line configurations: Diagram 6 has a vertical wavy line, Diagram 7 has a vertical wavy line with a red vertical line, Diagram 8 has a diagonal wavy line, Diagram 9 has a diagonal wavy line with a red vertical line, and Diagram 10 has a diagonal wavy line with a red vertical line and a dashed diagonal line.

 LO

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

█ LO
█ NLO, Double-Triangle (DT)

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

- █ LO
- █ NLO, Double-Triangle (DT)
- █ NLO, Self-Energy (SE)

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{tree} + \text{triangle} \right) \times \left(\text{tree} + \text{triangle} \right)^*$$

$$+ \left(\text{tree} + \text{triangle} \right) \times \left(\text{tree} + \text{triangle} \right)^*$$

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LO} + \text{NLO, DT} + \text{NLO, SE}$$

█ LO
█ NLO, Double-Triangle (DT)
█ NLO, Self-Energy (SE)

$\text{red arrow} \equiv \frac{p^2}{2p^0} \delta(p^2) \Theta(p^0)$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi(\text{phase-space}) \left| \text{C} + \text{C} + \text{C} + \text{C} \right|^2$$


LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi(\text{phase-space}) \left| \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right] + \text{LU} \left[\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right] + \text{LU} \left[2 \times \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right]$$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi^{(\text{phase-space})} \left| \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right] + \text{LU} \left[\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right] + \text{LU} \left[2 \times \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right]$$

$$\sum_{c \in \{RRR, RRV, RVV, \dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|_{\text{truncated}}^2$$

IR-subtraction numerical $d = 4$

analytic $d = 4 - 2\epsilon$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi^{(\text{phase-space})} \left| \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{---} \text{---} \text{---} \right] + \text{LU} \left[\text{---} \text{---} \text{---} \right] + \text{LU} \left[2 \times \text{---} \text{---} \text{---} \right]$$

$$\sum_{c \in \{RRR, RRV, RVV, \dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|_{\text{truncated}}^2$$

IR-subtraction numerical $d = 4$

analytic $d = 4 - 2\epsilon$

↓

$$\sum_{j=1}^{n_{\text{supergraphs}}} \int \Pi g_j^{(\text{LU})}$$

numerical $d = 4$
NO IR-subtraction

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Diagram 1} \right] + \text{LU} \left[\text{Diagram 2} \right] + \text{LU} \left[2 \times \text{Diagram 3} \right]$$

The equation shows the Local Unitarity (LU) contribution to the cross-section for the process $\gamma^* \rightarrow d\bar{d}$. It is expressed as a sum of three terms, each enclosed in red square brackets and preceded by the letters "LU" in red. The first term is a circle with two wavy external lines and two arrows on the circle. The second term is a circle with two wavy external lines, two arrows, and a vertical dashed line through the center. The third term is a circle with two wavy external lines, two arrows, and a small loop on top.

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Diagram 1} \right] + \text{LU} \left[\text{Diagram 2} \right] + \text{LU} \left[2 \times \text{Diagram 3} \right]$$

- At the **integral** level, the **Optical Theorem** simply gives: $\text{LU}[\cdot] \propto \text{Im}[\cdot]$

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

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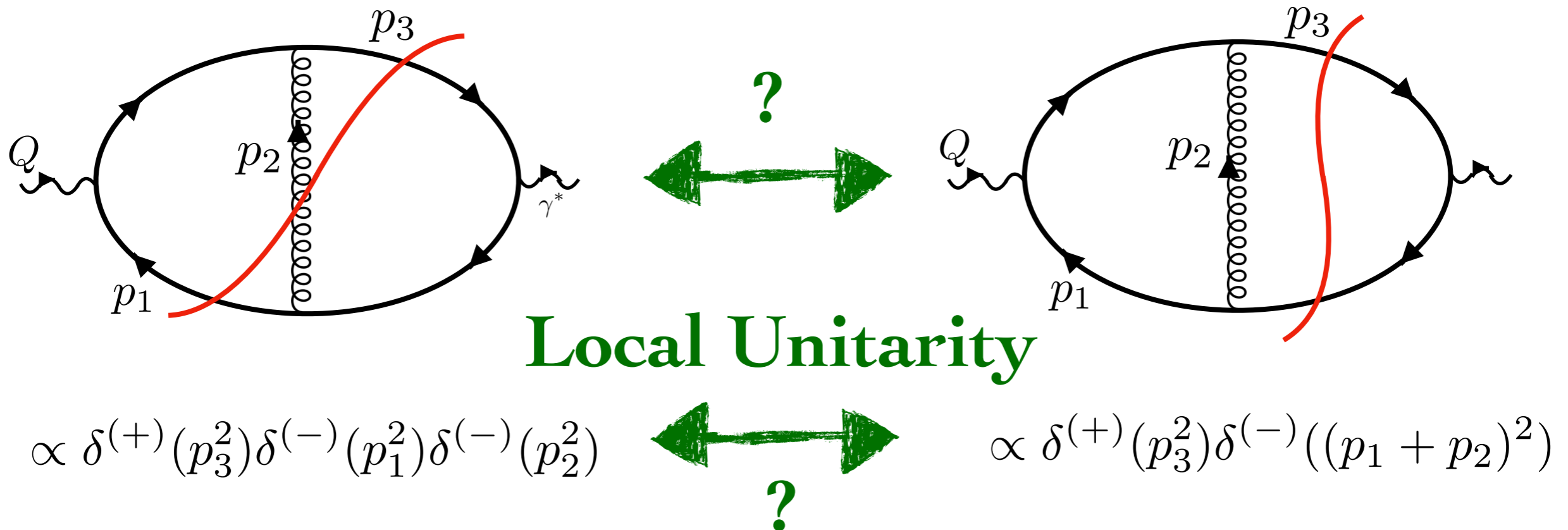
$$\propto \delta^{(+)}(p_3^2) \delta^{(-)}(p_1^2) \delta^{(-)}(p_2^2) \quad \longleftrightarrow \quad \propto \delta^{(+)}(p_3^2) \delta^{(-)}((p_1 + p_2)^2)$$

?
 ?

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

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MAIN INGREDIENT: LOOP TREE DUALITY

$$\int d^4 k \text{ (triangle diagram) } = \int d^3 \vec{k} \left[\text{(triangle diagram with green lines and red +)} + \text{(triangle diagram with green lines and red -)} + \text{(triangle diagram with green lines and red +)} \right]$$

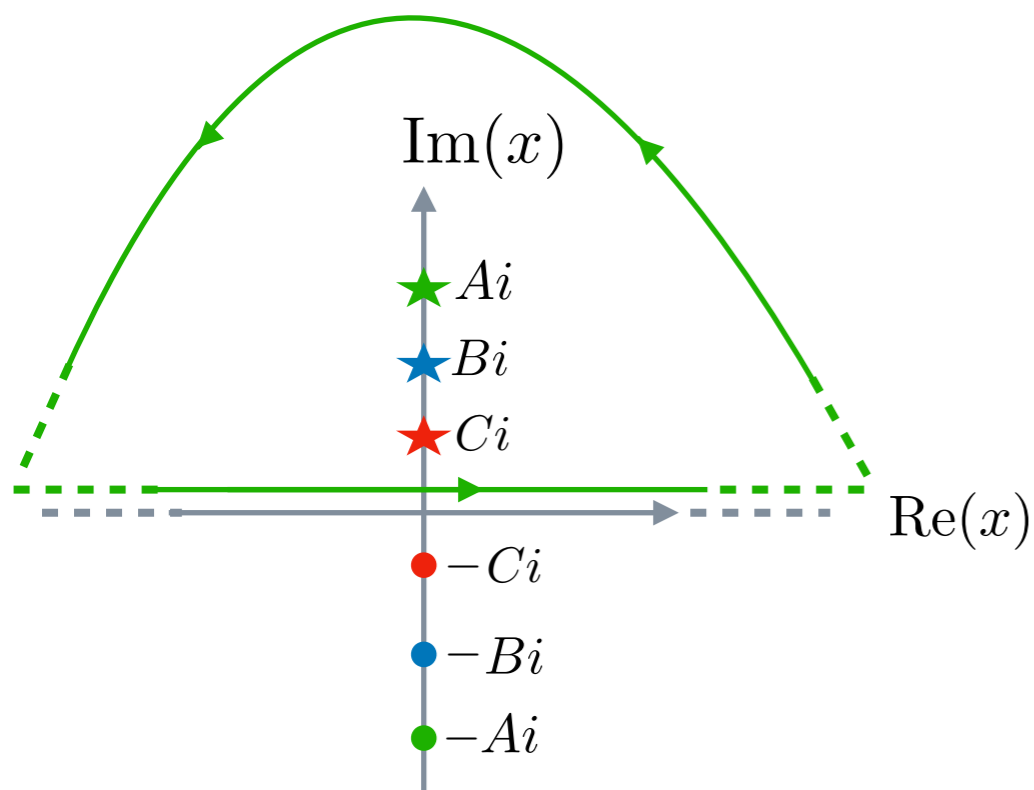
(CLOSELY RELATED TO THE FEYNMAN-TREE THEOREM
AND TIME-ORDERED PERTURBATION THEORY)

(ONE)-LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx F(x) \quad F(x) = \frac{1}{x^2 + A^2} \frac{1}{x^2 + B^2} \frac{1}{x^2 + C^2}$$

$$F(x) = \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(x - Bi)(x + Bi)} \frac{1}{(x - Ci)(x + Ci)}$$

(Assumptions $\rightarrow \{A > 0, B > 0, C > 0\}$)



Cauchy: $(R(x^*) \equiv \text{Res}(F, x = x^*))$

$$I = (-2\pi i) [R(Ai) + R(Bi) + R(Ci)]$$

What does it correspond to for a one-loop integral?

(ONE-)LOOP TREE DUALITY

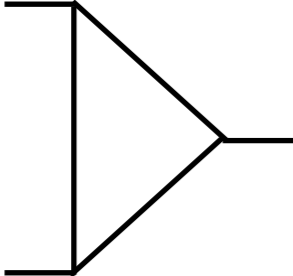
$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

Pole selected for each propagator

Then integrate the energy component using residue theorem

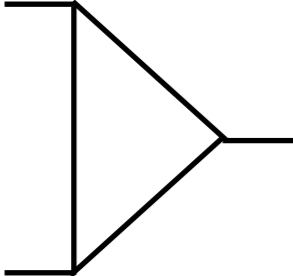

$$= \int d^3\vec{k} \left[\text{Res}_1 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_2 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_3 \left[\frac{N}{D_1 D_2 D_3} \right] \right]$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

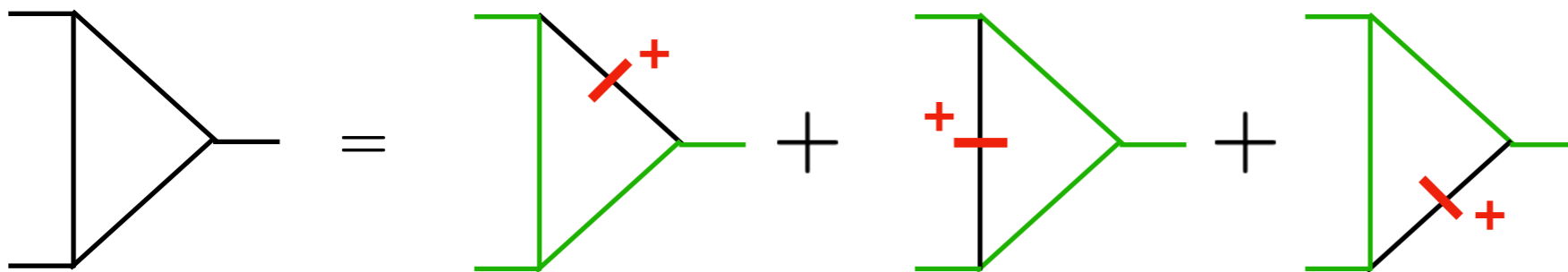
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$$= \int d^3 \vec{k} \left[\text{Res}_1 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_2 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_3 \left[\frac{N}{D_1 D_2 D_3} \right] \right]$$

Residues can be represented as cuts:



$$= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

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$$\text{Diagram} = \int d^3 \vec{k} \left[\text{Res}_1 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_2 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_3 \left[\frac{N}{D_1 D_2 D_3} \right] \right]$$

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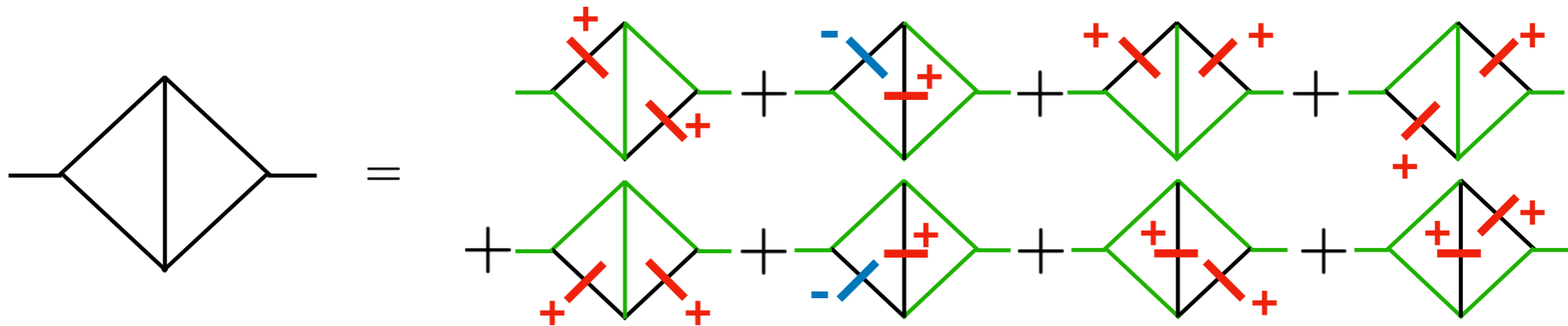
$$\text{Diagram} = \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3$$

Energy flow

$$= \int d^4 k \frac{N}{D_1 D_2 D_3} (D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3))$$

(MULTI-)LOOP TREE DUALITY

Applying LTD to a two-loop double-triangle: one residue per spanning tree



Interplay of momentum conservation and causal prescription is key to obtain the energy flow

- **Distributional identities:** [Bierenbaum, Catani, Draggiotis, Rodrigo, arxiv: 1007.0194]
- **Averaging procedure:** [Runkel, Scór, Vesga, Weinzierl, arxiv: 1902.02135]
- **Iterative procedure:** [Capatti, VH, Kermanschah, Ruijl, arxiv: 1906.06138]
- **Manifestly causal:** [Capatti, VH, Kermanschah, Pelloni, Ruijl, arxiv: 2009.05509]
- **Cross-Free Family** [Capatti, arxiv: 2211.09653]
(the best 3D repr.!

Codes : [<https://github.com/apelloni/cLTD>]
[<https://bitbucket.org/wjtorresb/lotty>]

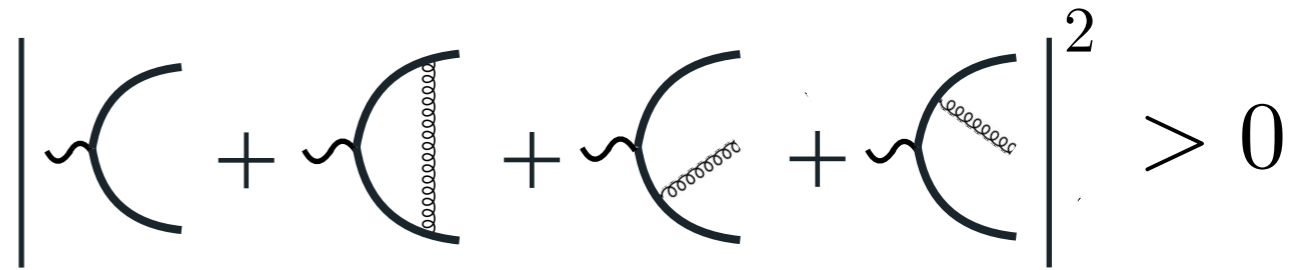
{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

POSITIVITY

$$\left| \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \right|^2 > 0$$

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~~POSITIVITY~~

$$\left| \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right|^2 > 0$$


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LORENTZ INVARIANCE

$$\left| \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right|^2 > 0$$

$$\mathcal{M}^\mu (\{p_i^\mu\}) = \Lambda_\nu^\mu \mathcal{M}^\nu (\{\Lambda_\nu^\mu p_i^\nu\})$$

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$$k^\mu \mathcal{M}_\mu = 0$$

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$$i(T^\dagger - T) = T^\dagger T$$

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~~POSITIVITY~~

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UNITARITY

$$i(T^\dagger - T) = T^\dagger T$$

LOCALLY FINITE

$$\lim_{k \rightarrow \text{soft, colli, UV}} I(k) = \mathcal{O}(1)$$

FOUR TYPES OF SINGULARITIES :

THRESHOLDS

**CONTOUR-DEFORMATION
OR
THRES. SUBTRACTION**

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(TROPICAL) SAMPLING

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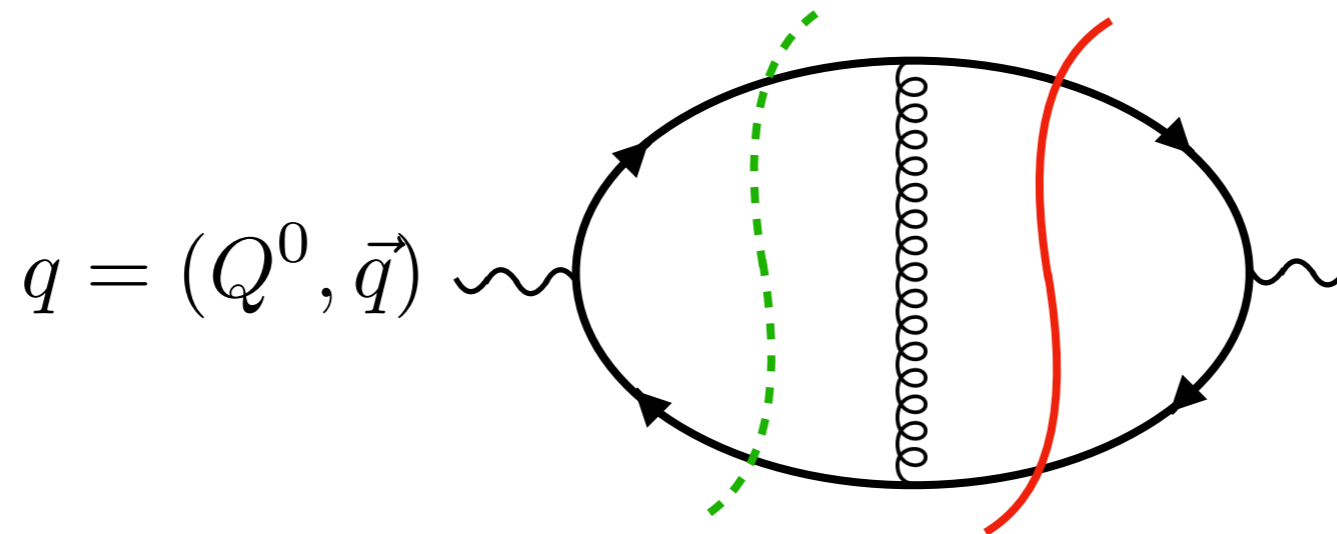
(TROPICAL) SAMPLING

ULTRAVIOLET

LOCAL BPHZ

THRESHOLDS

$$E_1 = \sqrt{|\vec{k}|^2 + m^2 - i\epsilon}$$



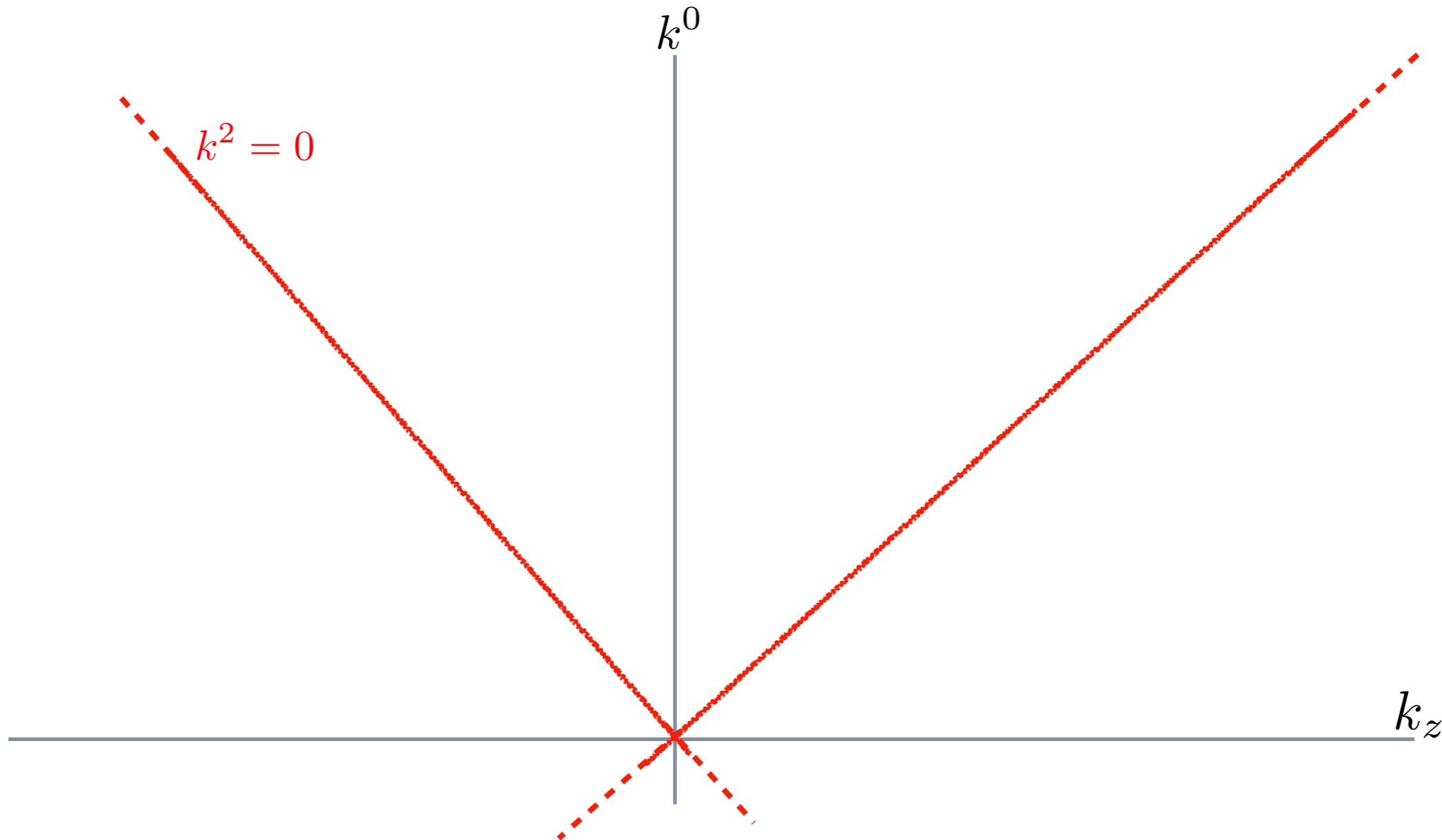
$$E_2 = \sqrt{|\vec{k} - \vec{q}|^2 + m^2 - i\epsilon}$$

$$\int d^3\vec{k} I^{(\text{Local Unitarity})} \supset \int d^3\vec{k} \frac{1}{E_1 E_2 E_3} \left(\frac{1}{(E_1 + E_2 - Q^0)(E_1 + E_2 + Q^0)} \right)$$

$$\eta(\vec{k}) = E_1 + E_2 - Q^0 \stackrel{\vec{Q}=0 \quad m=0}{=} 2|\vec{k}| - Q^0$$

SINGULAR SURFACES IN MINKOWSKI SPACE

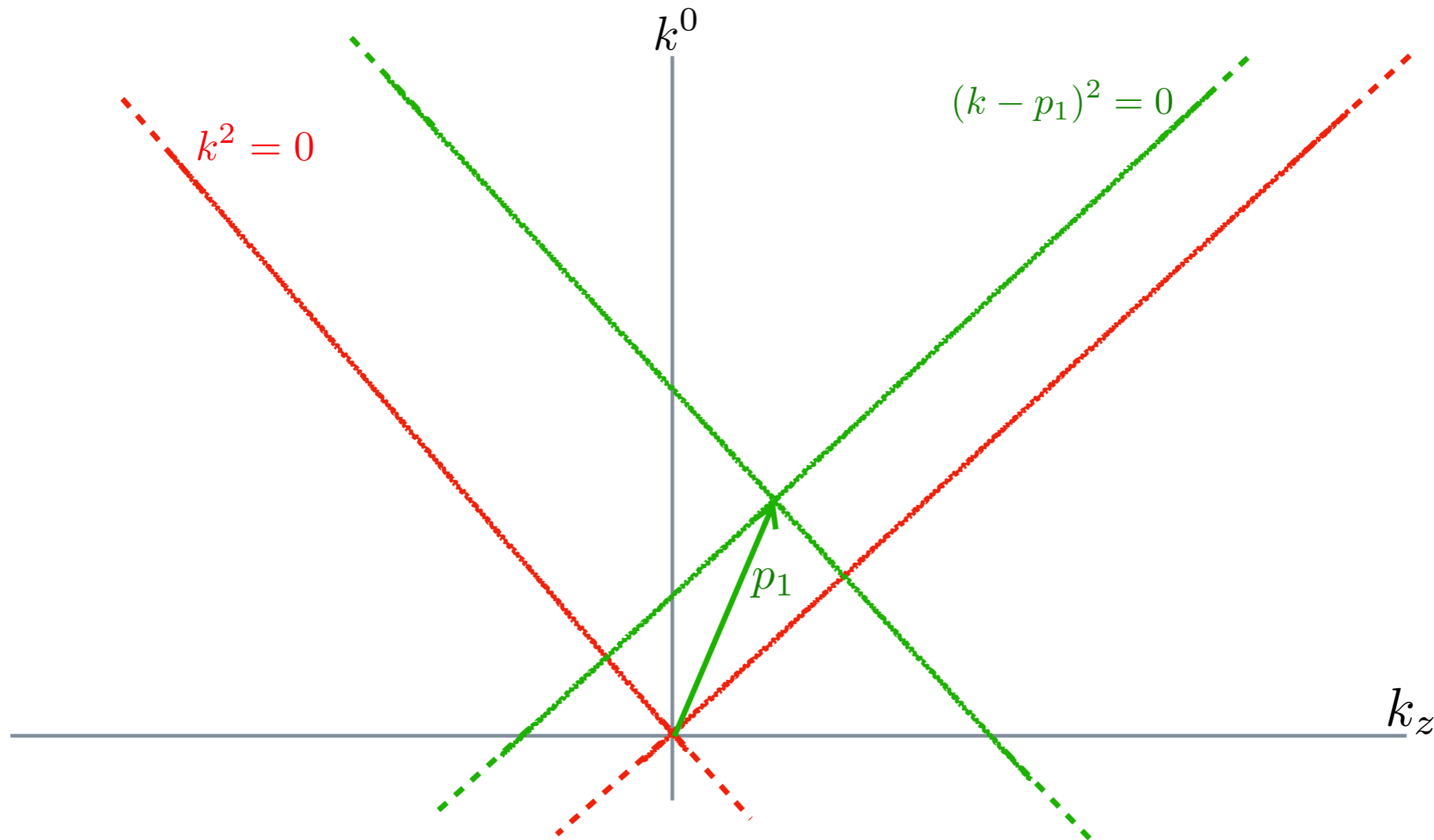
$$\int d^4 k \frac{1}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2}$$



The integrand is singular along each of the coloured surface

SINGULAR SURFACES IN MINKOWSKI SPACE

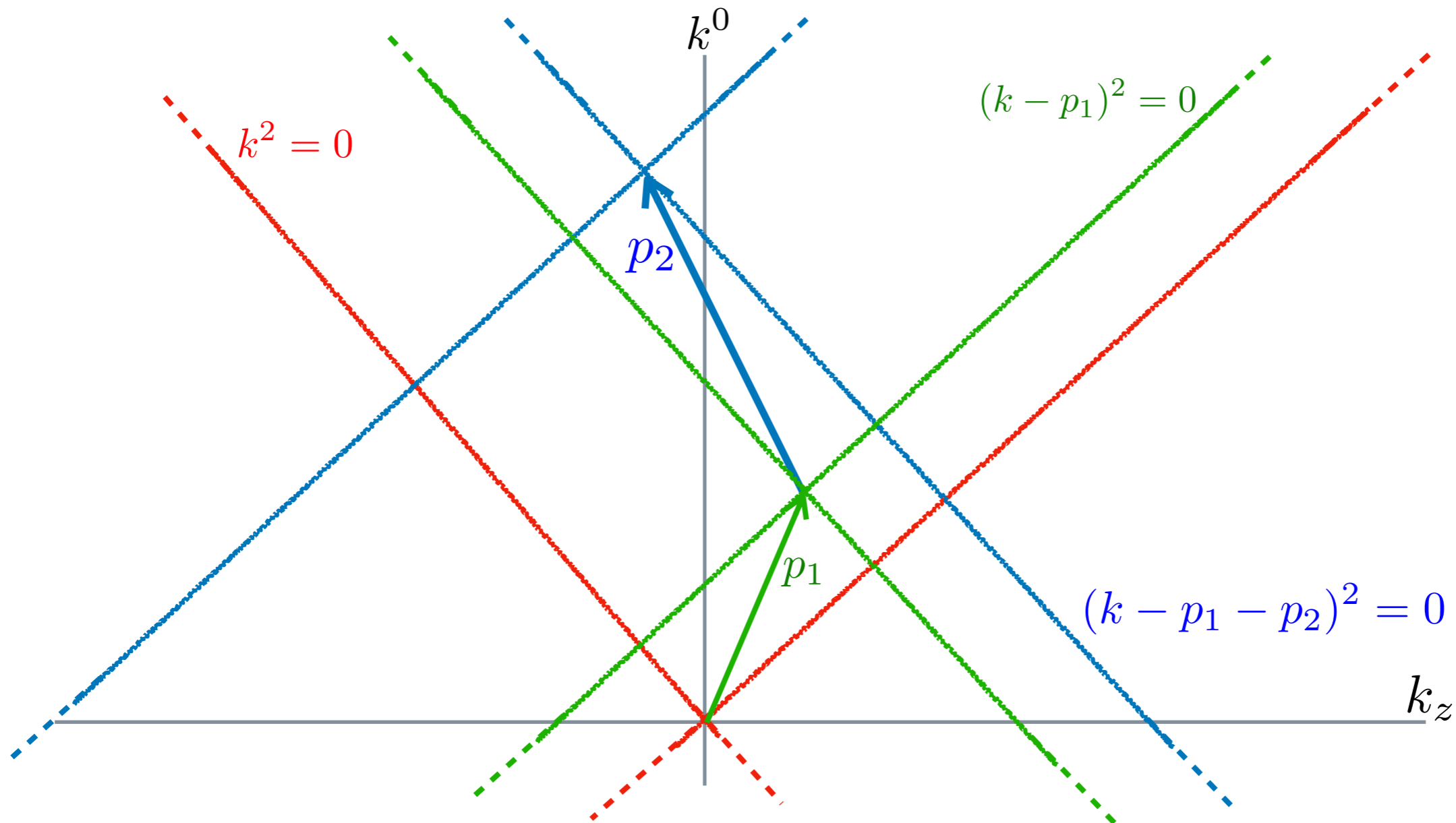
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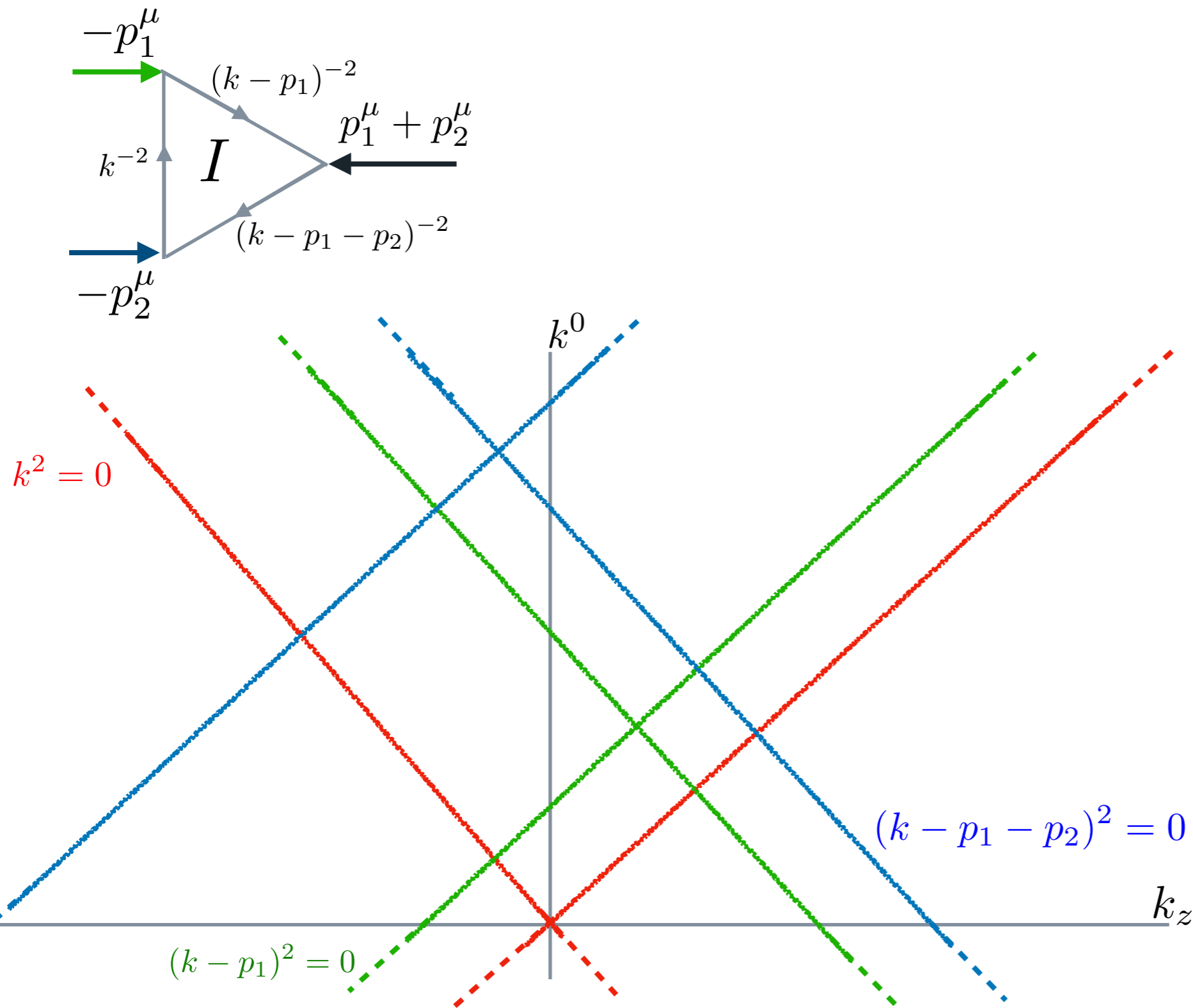
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SINGULAR SURFACES OF THE LTD REPRESENTATION

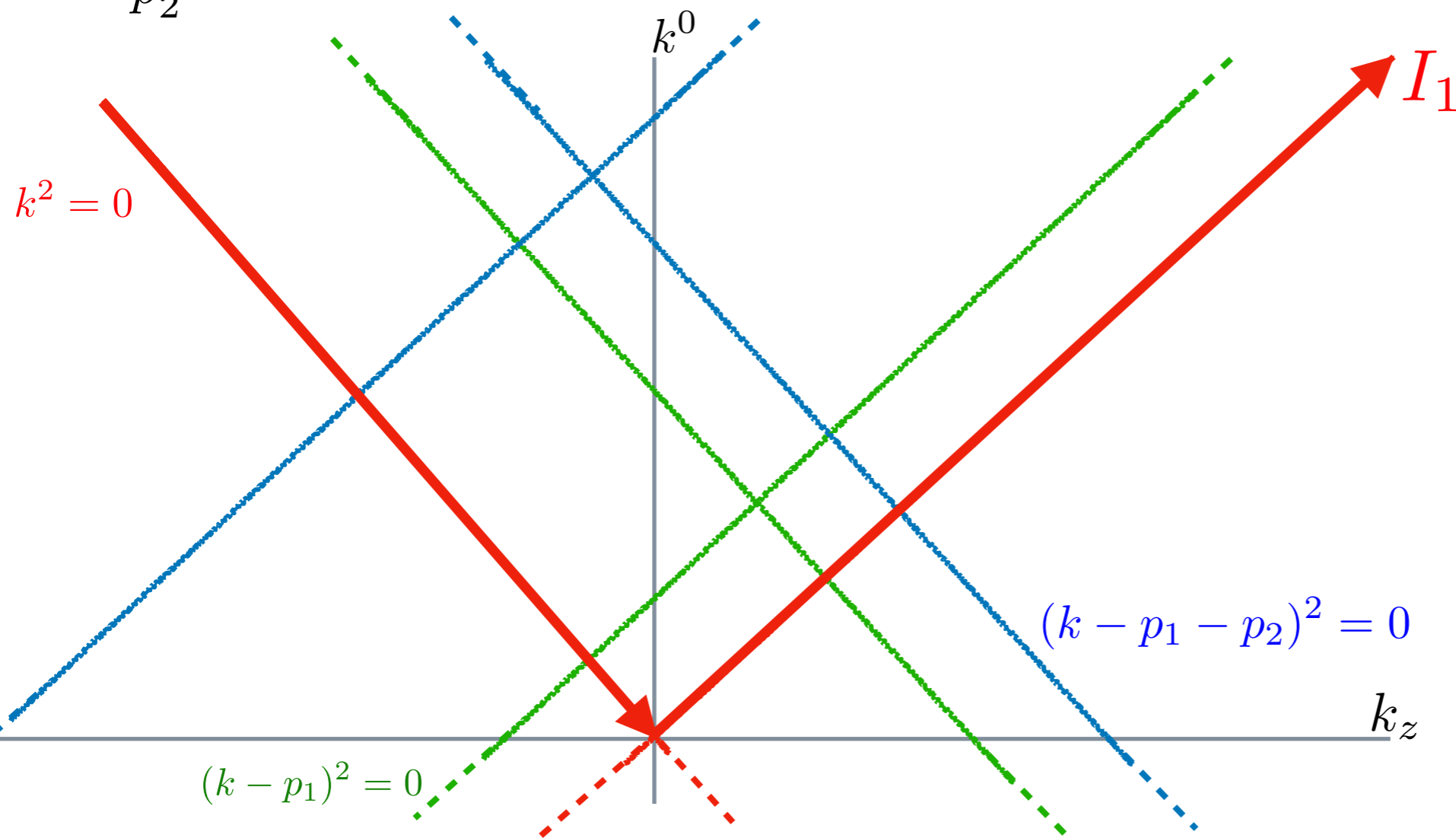
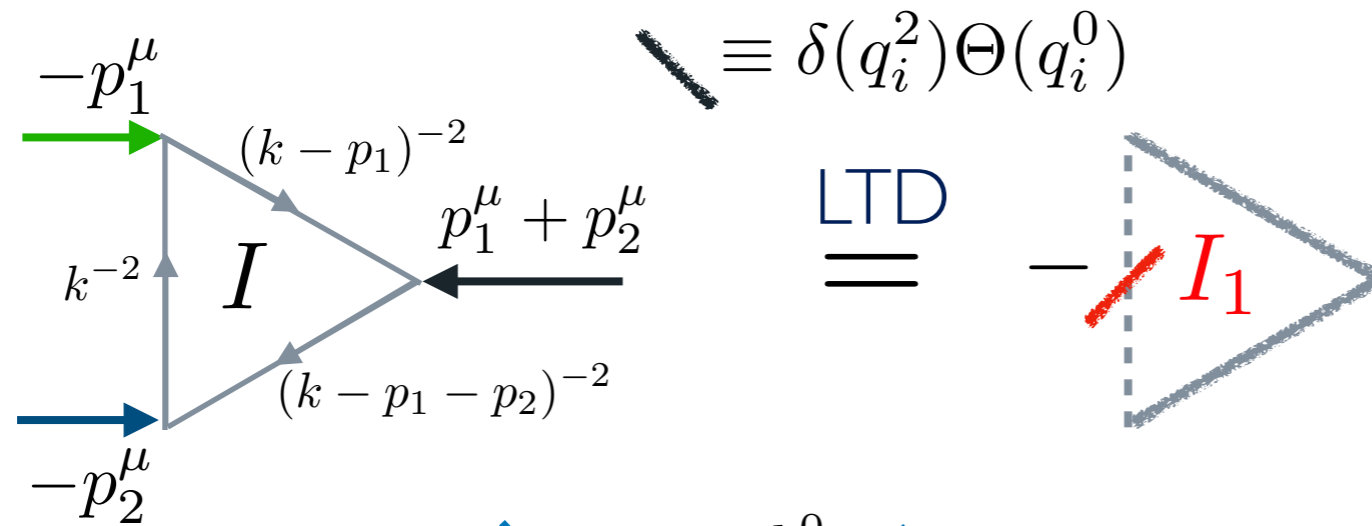
Analytically integrate over the loop energies using Cauchy's theorem (LTD):



$$p_i^2 > 0 \quad \forall i$$

SINGULAR SURFACES OF THE LTD REPRESENTATION

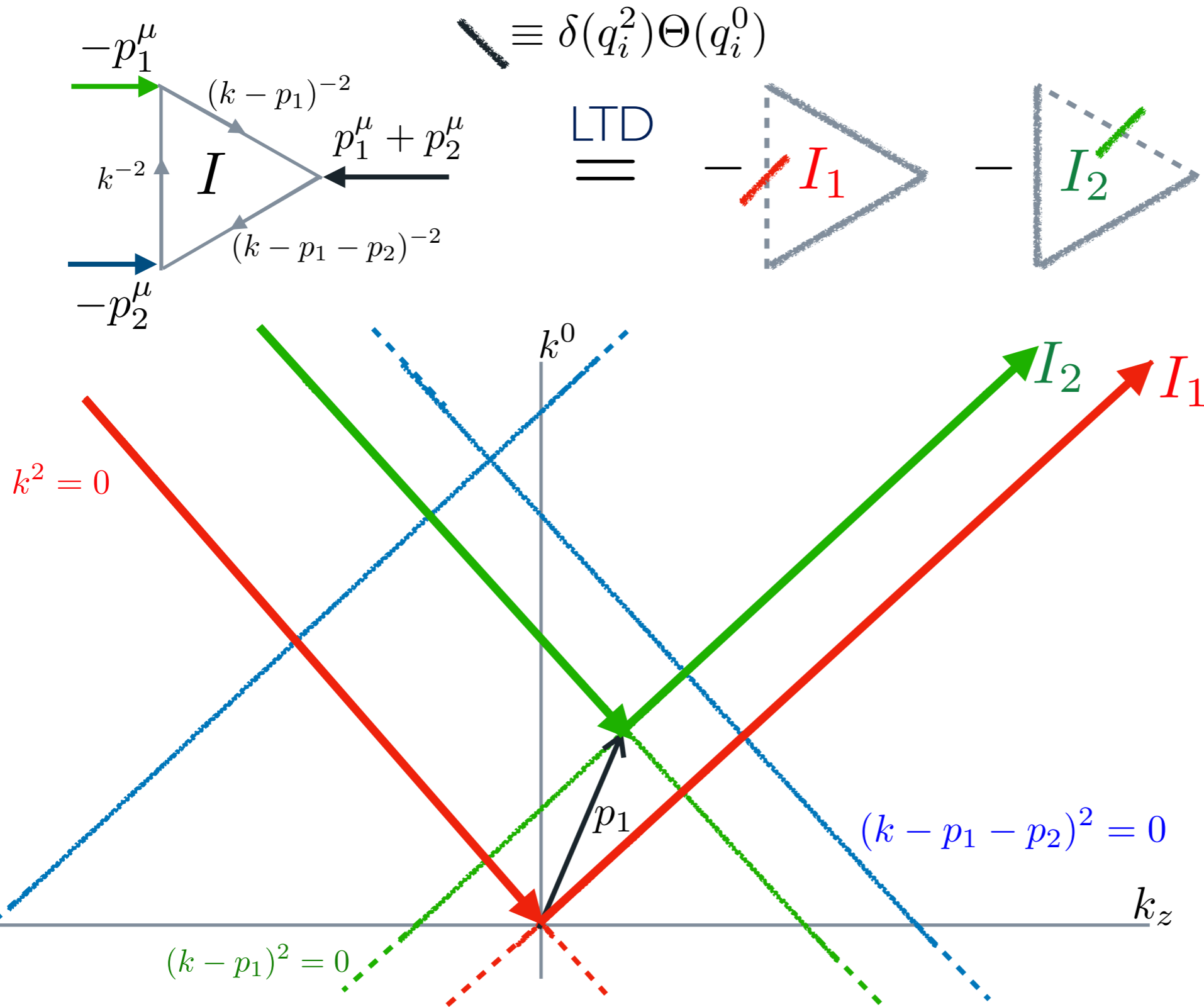
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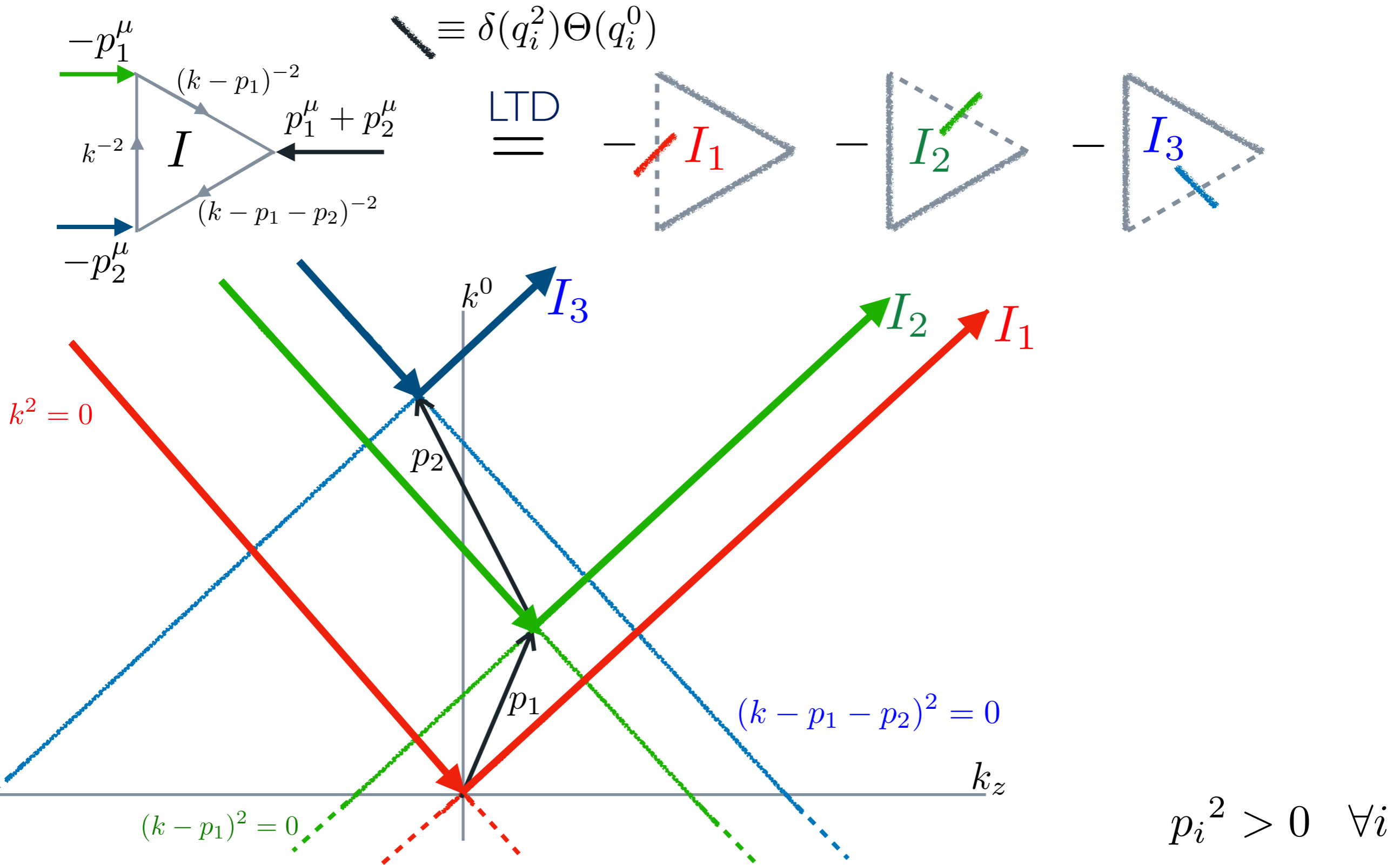
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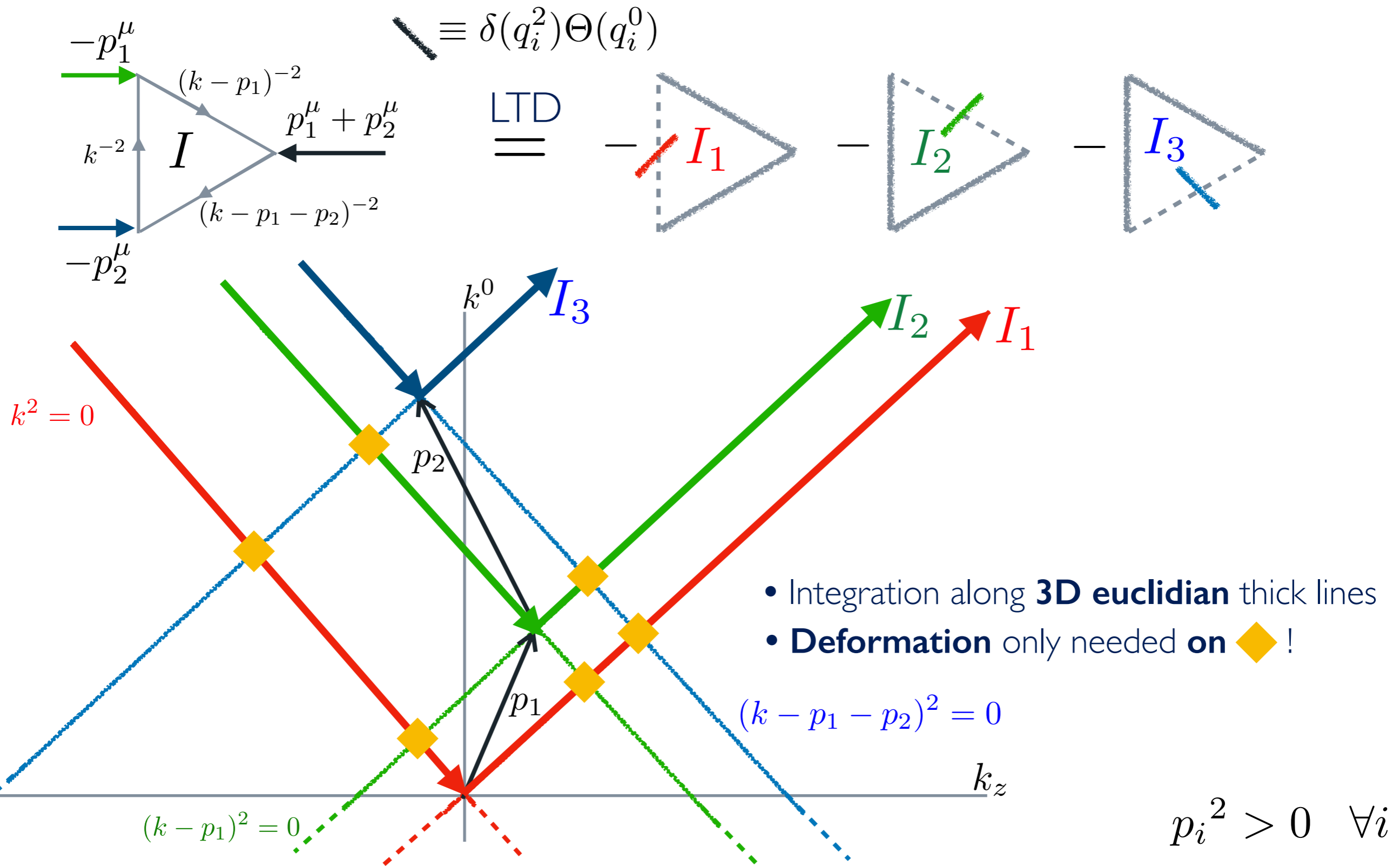
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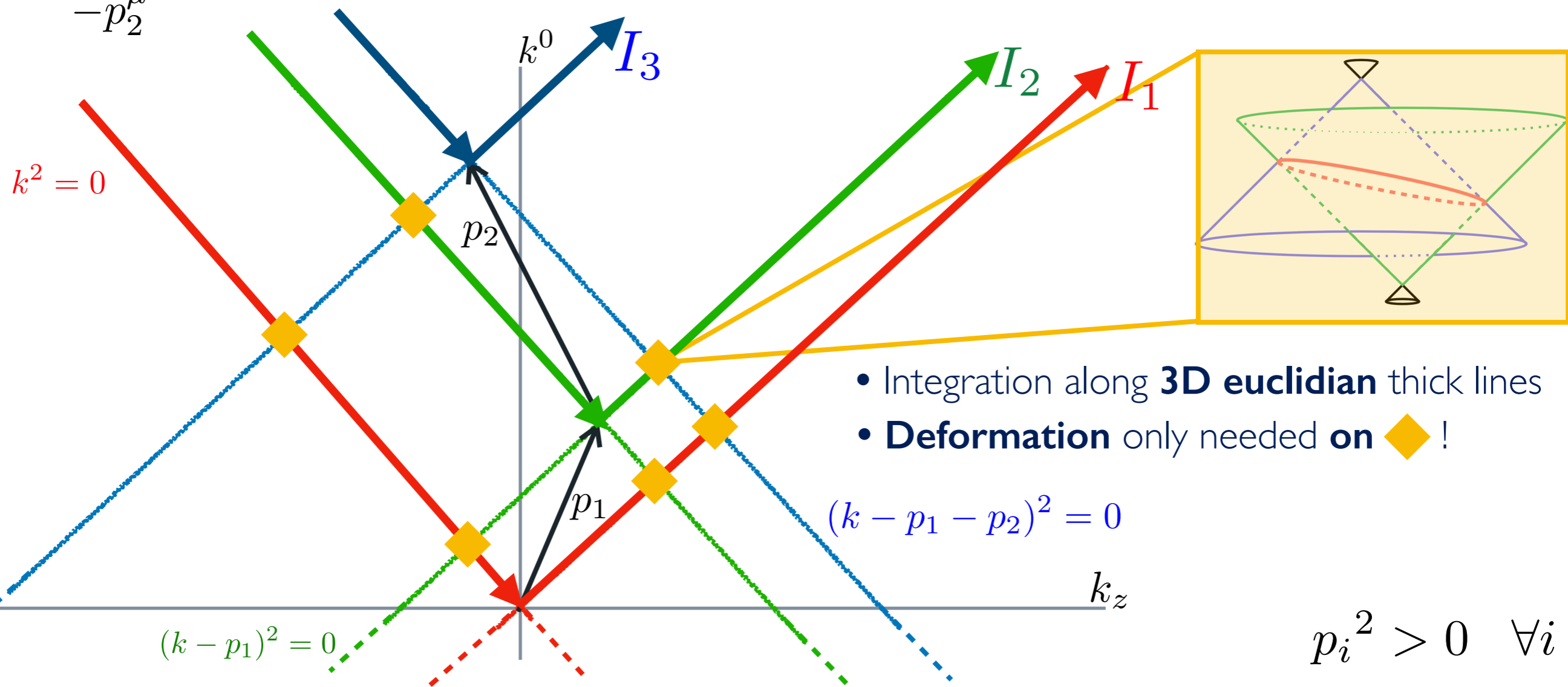
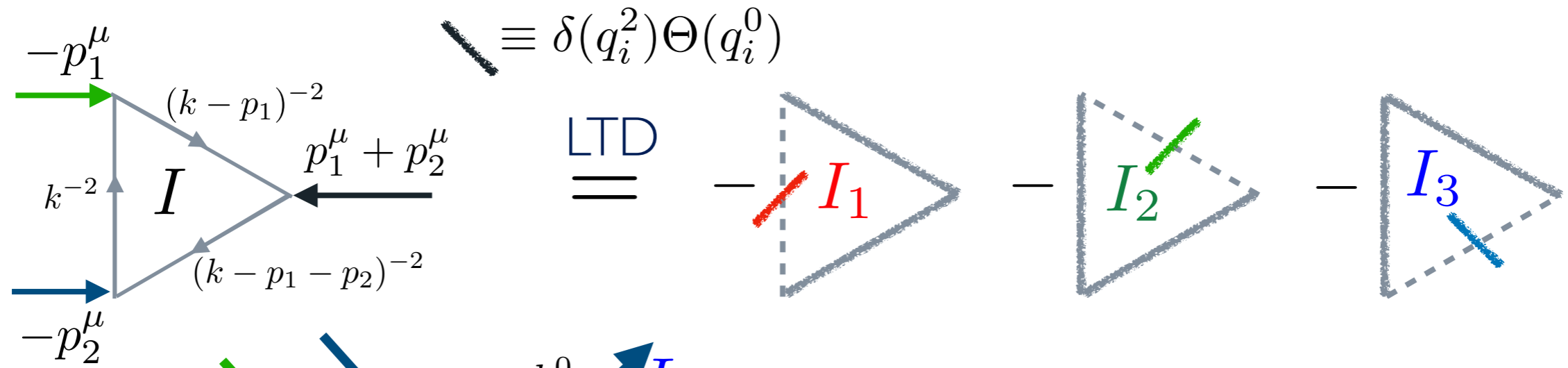
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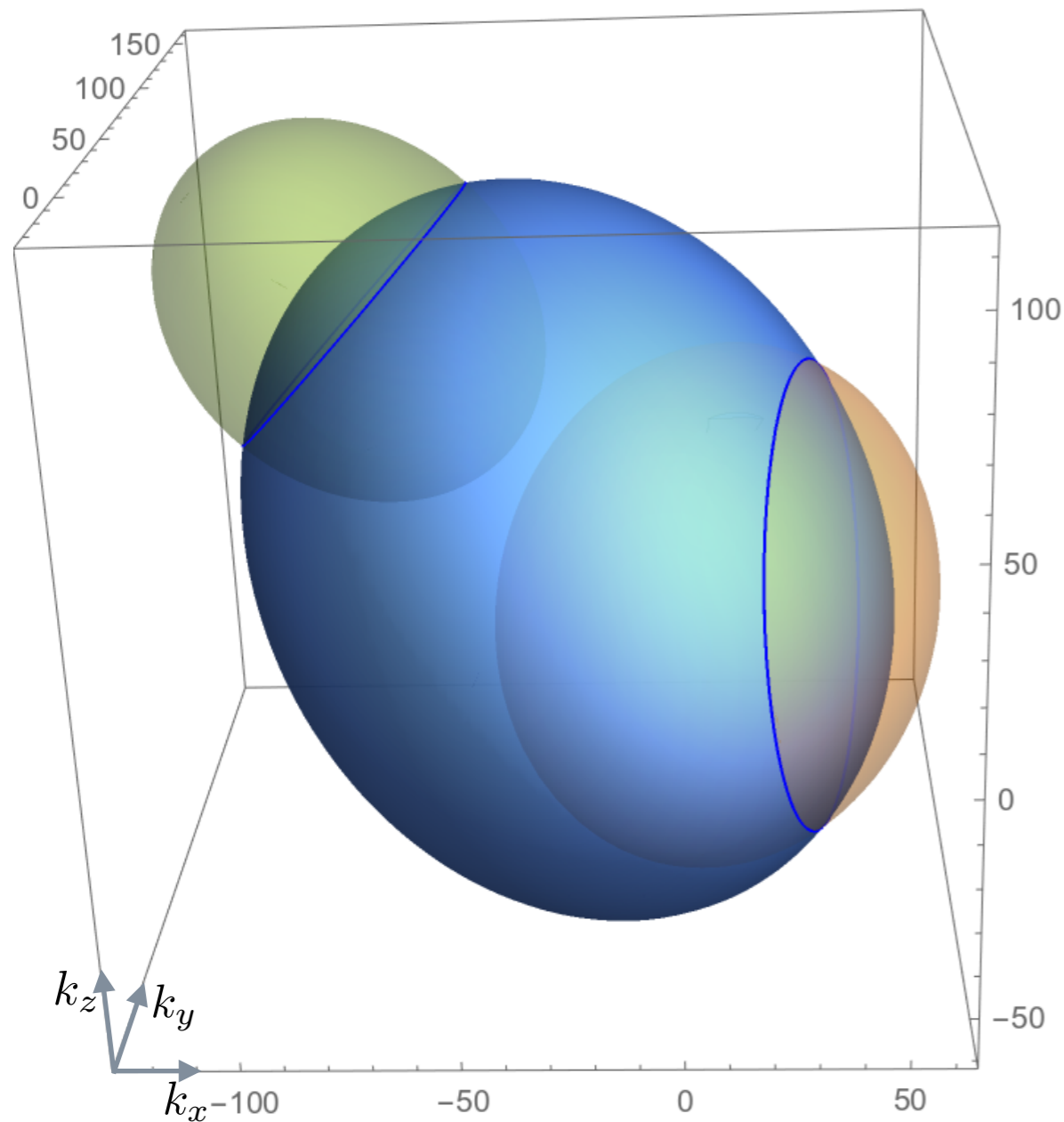


SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):



SINGULAR SURFACES - 2D ELLIPSOIDS



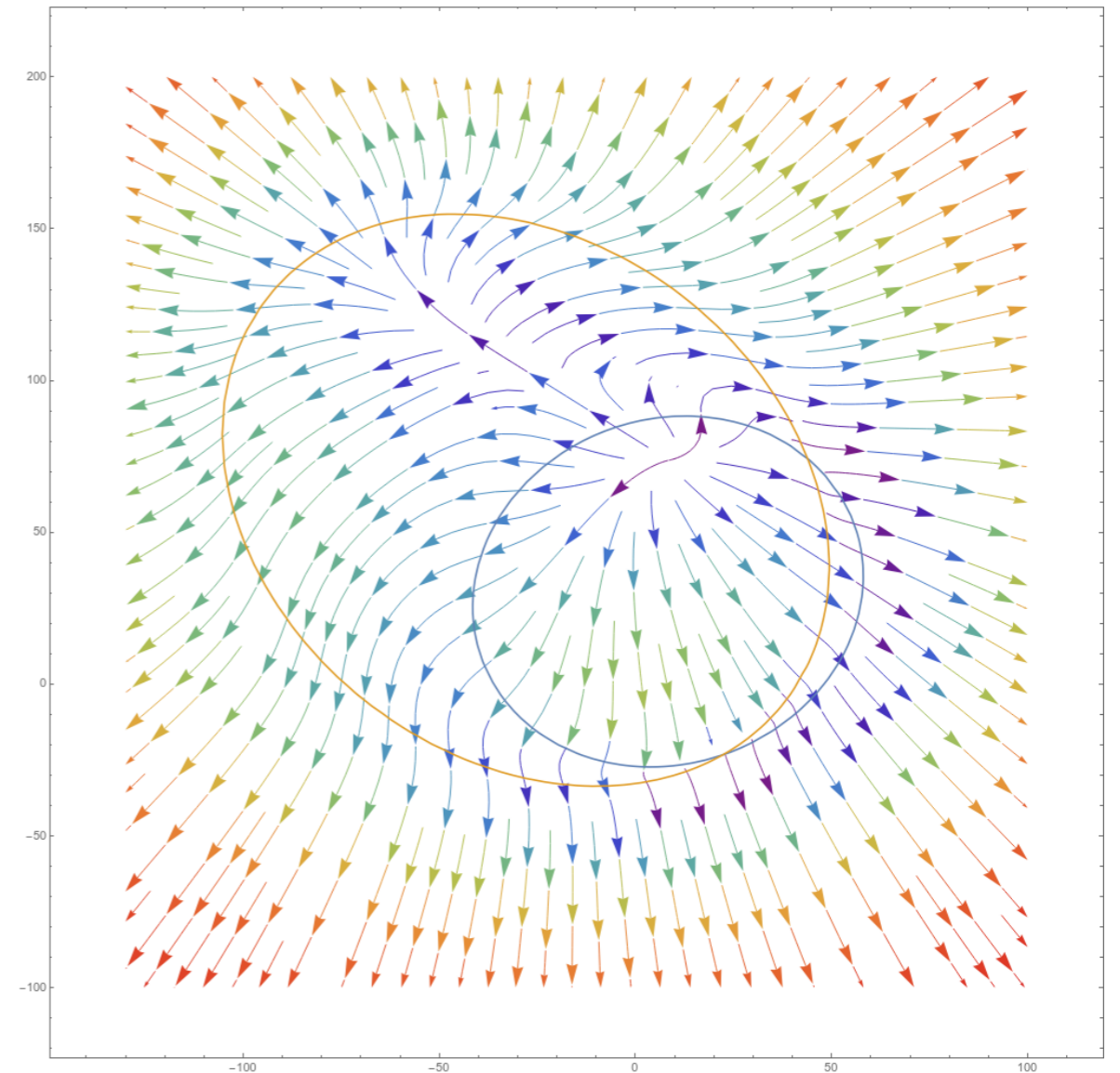
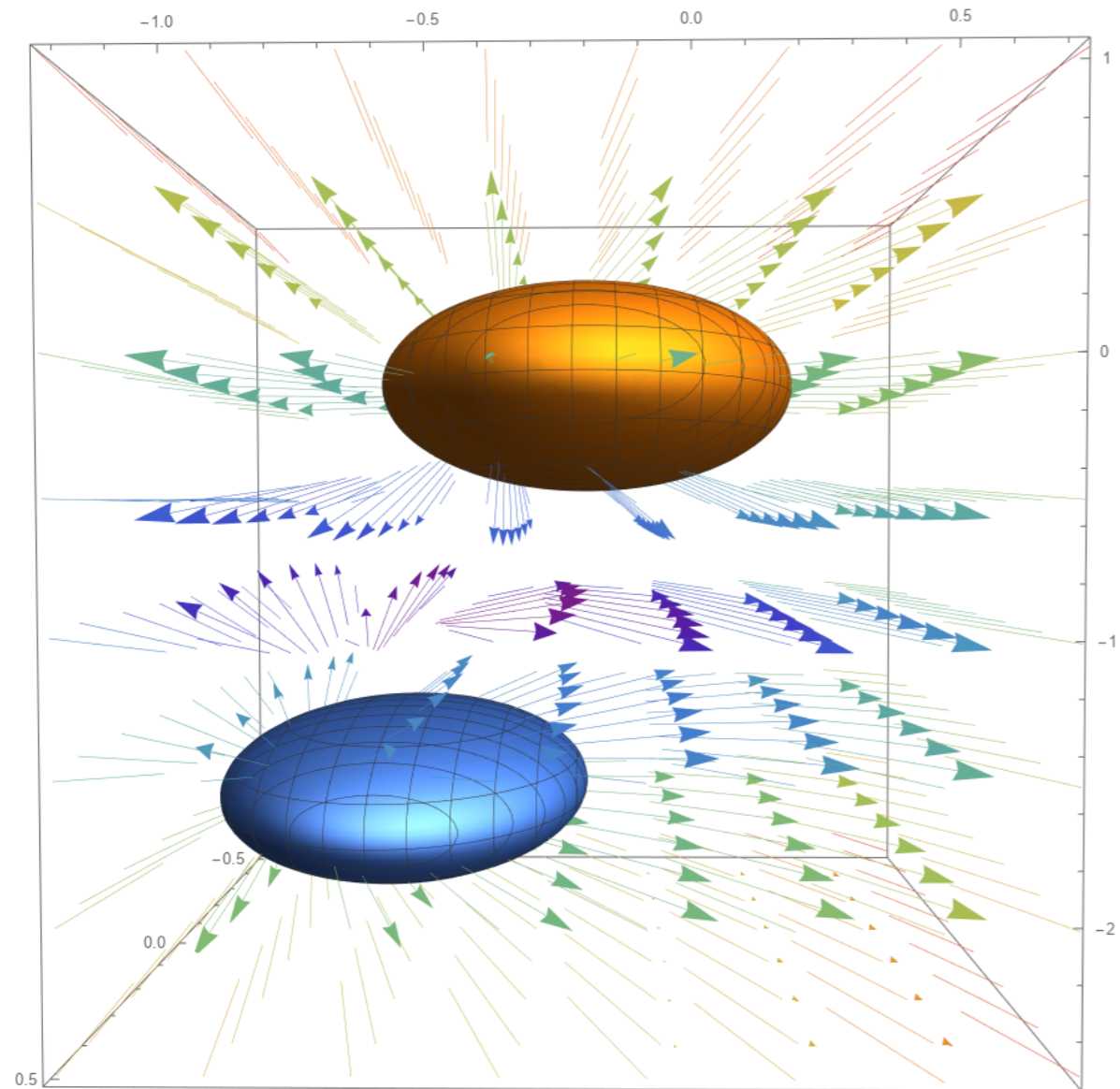
- General **one-loop** ellipsoid **expression**:

$$E_{ij}(\vec{k}) = \sqrt{(\vec{k} + \vec{p}_i)^2 + m_i^2 - i\delta} + \sqrt{(\vec{k} + \vec{p}_j)^2 + m_j^2 - i\delta} - p_i^0 + p_j^0$$

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

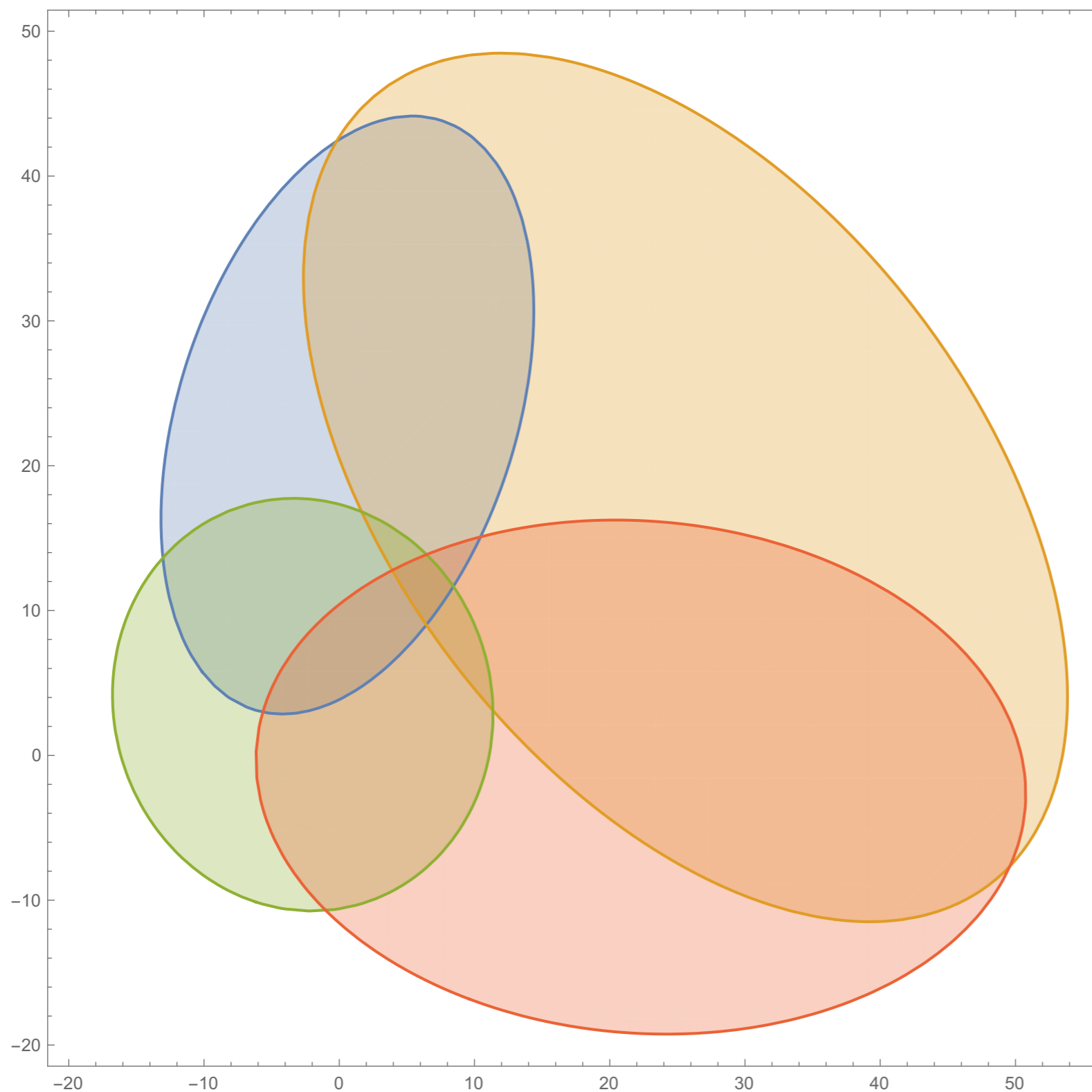
Deformation: $\vec{k} \rightarrow \vec{k} - i\vec{\kappa}$

Causal prescription imposes: $\vec{\kappa} \cdot \vec{n}_{E_{ij}} > 0$



DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

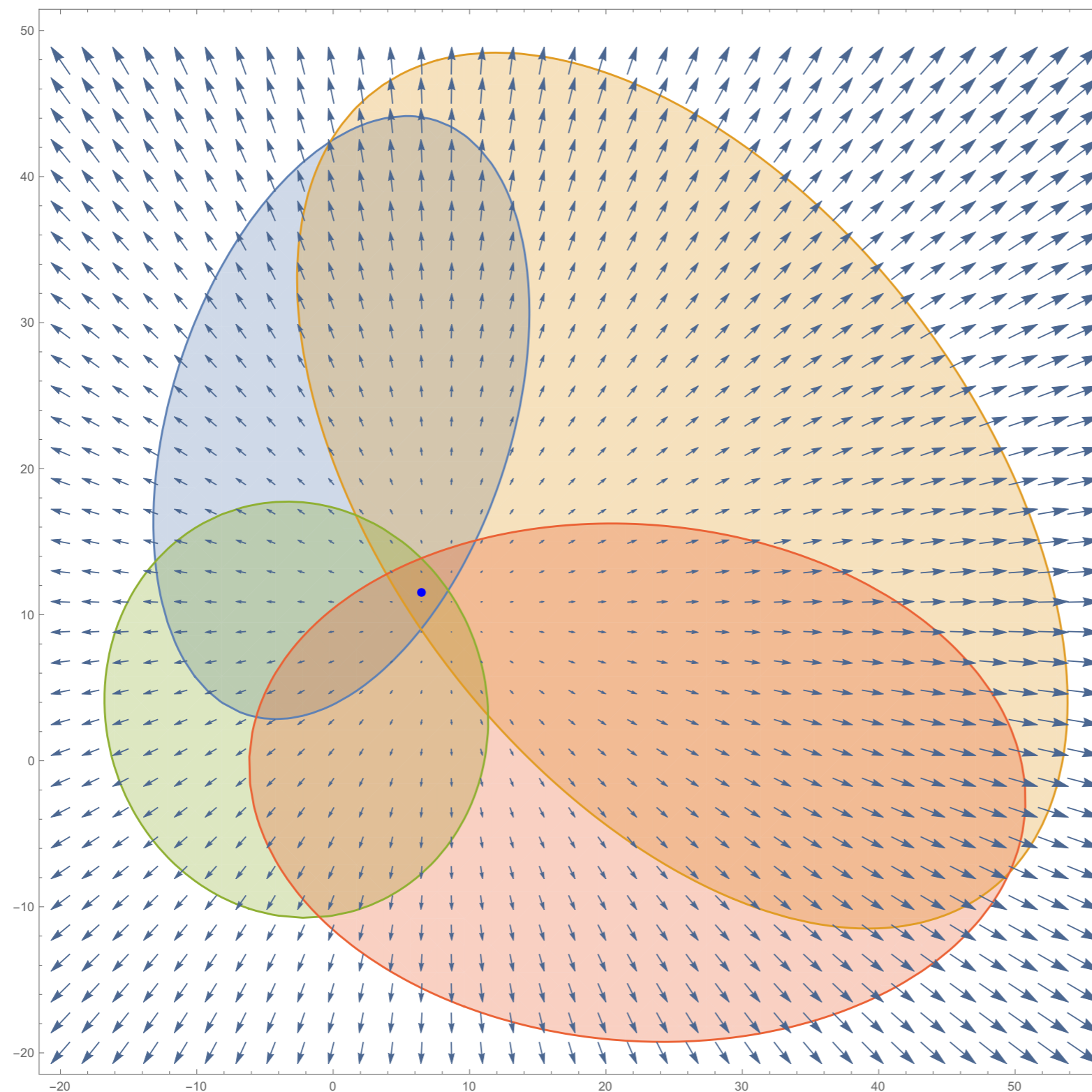
How to construct such a field? For example for this case:



[Capatti, VH, Kermanschah, Pelloni, Ruijl]
[arxiv:1906.06138]

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

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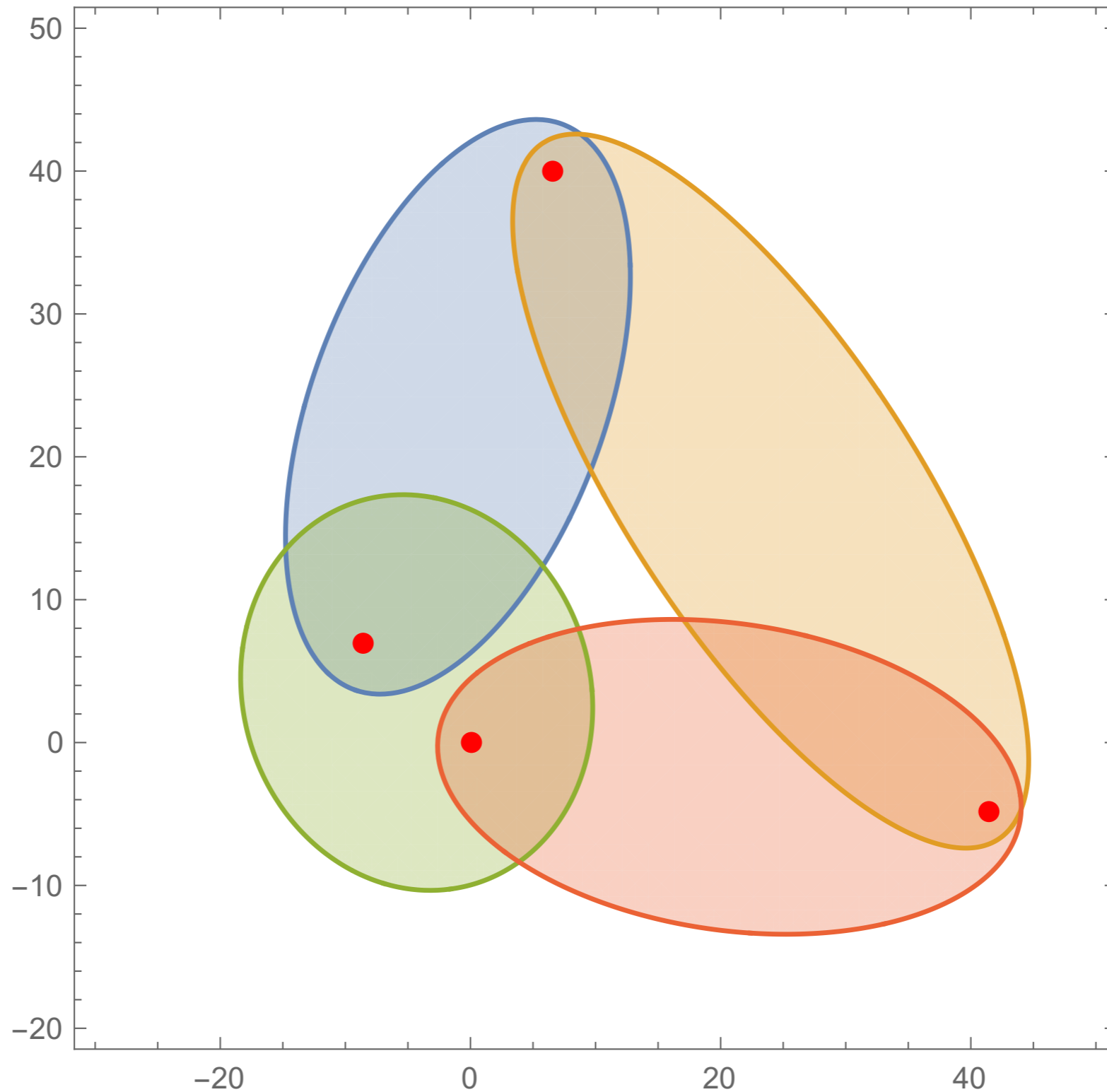


A **radial field** centered in the inside of all ellipsoids!

[Capatti, VH, Kermanschah, Pelloni, Ruijl]
[arxiv:1906.06138]

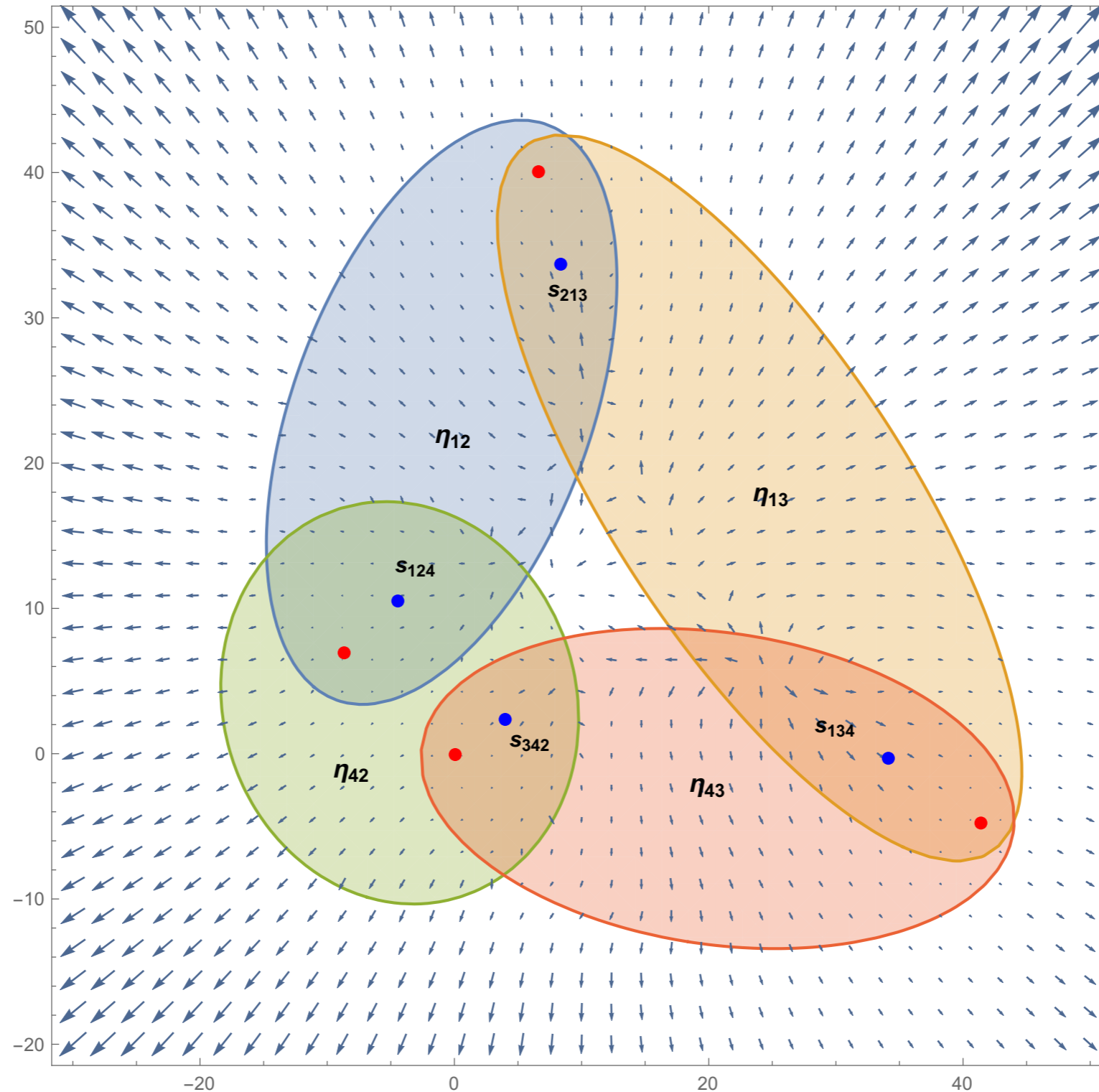
DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

But then what if there is no point in the inside of **all ellipsoids** (Box4E example) ?



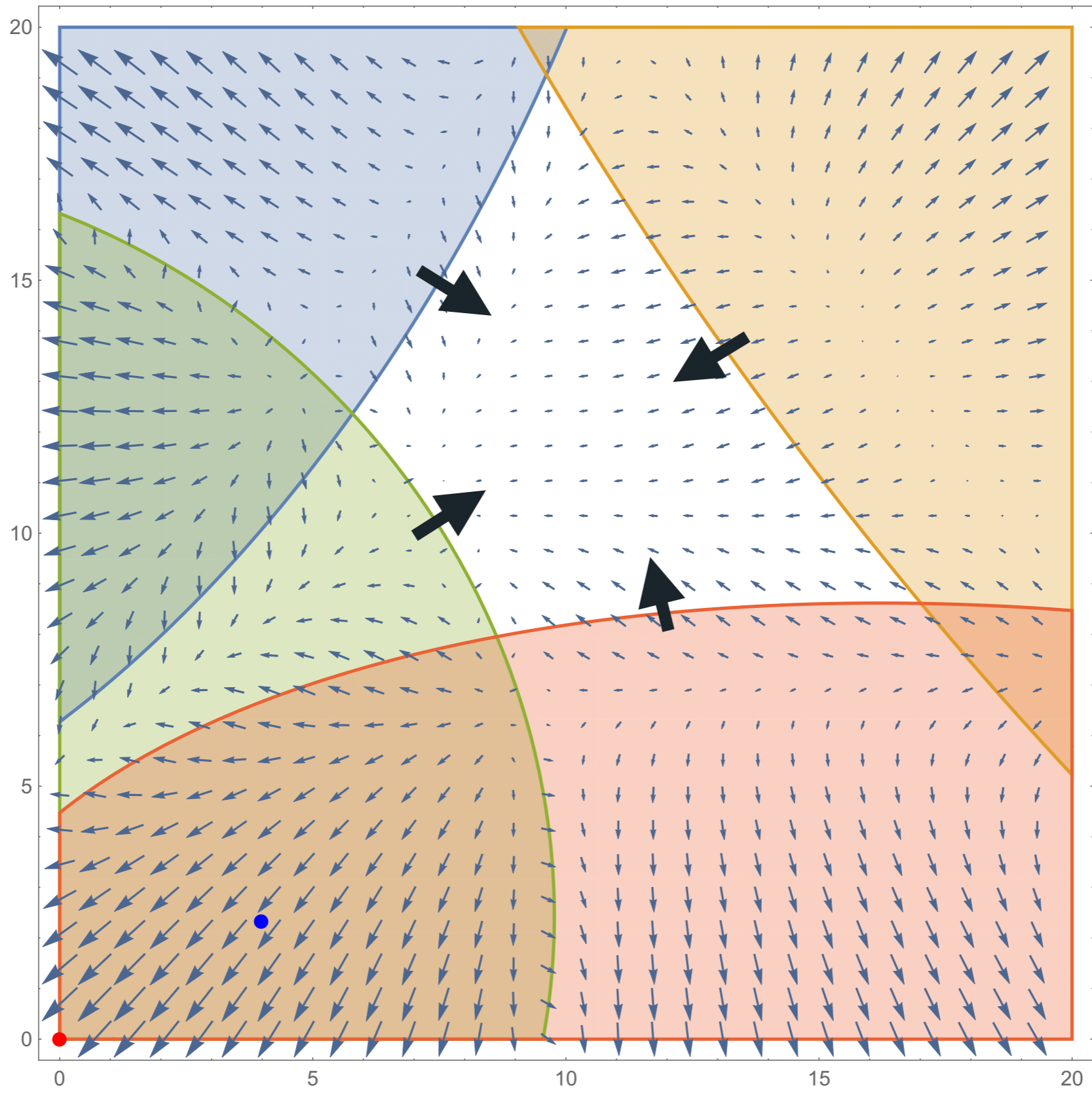
DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

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DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

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THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : [2110.06869](https://arxiv.org/abs/2110.06869)]

$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0}$$
$$\underset{p_1^\mu = (2, \vec{0})}{=} \frac{1}{2|\vec{k}| - 2} \propto \frac{1}{\sqrt{k_x^2 + k_y^2} - 1}$$

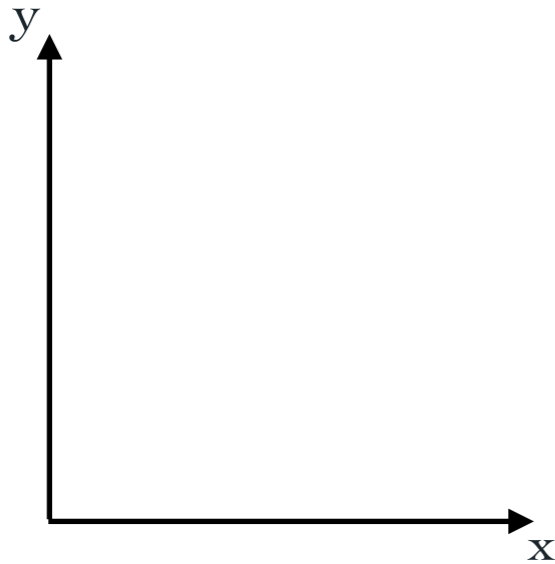
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$$\lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} dx dy \frac{2}{\pi^2} \frac{1}{x^2 + y^2 + 1} \frac{1}{\sqrt{x^2 + y^2} - 1 \pm i\delta} = 1 \mp 2i$$



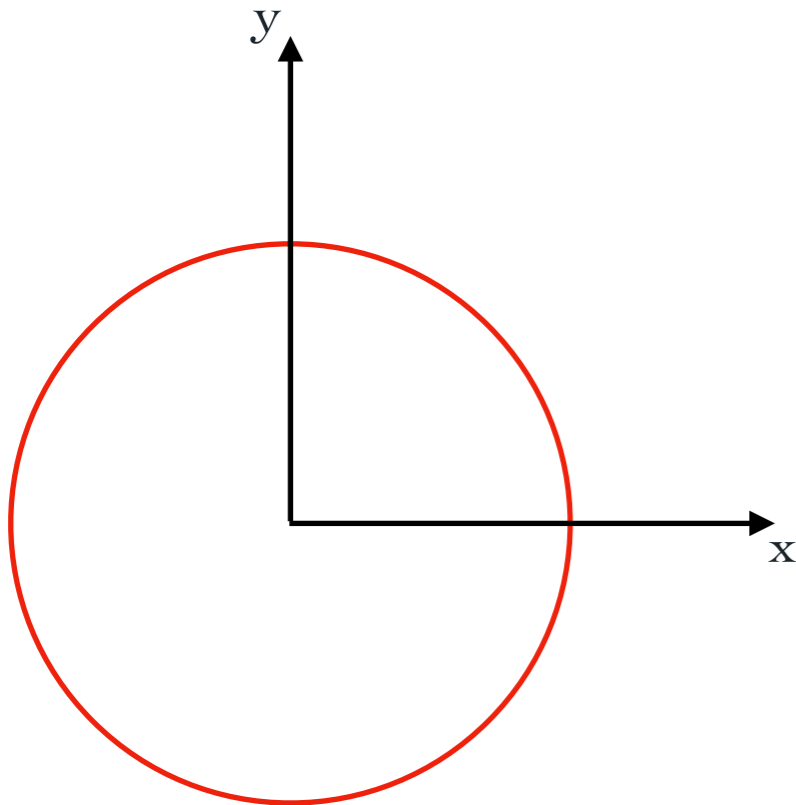
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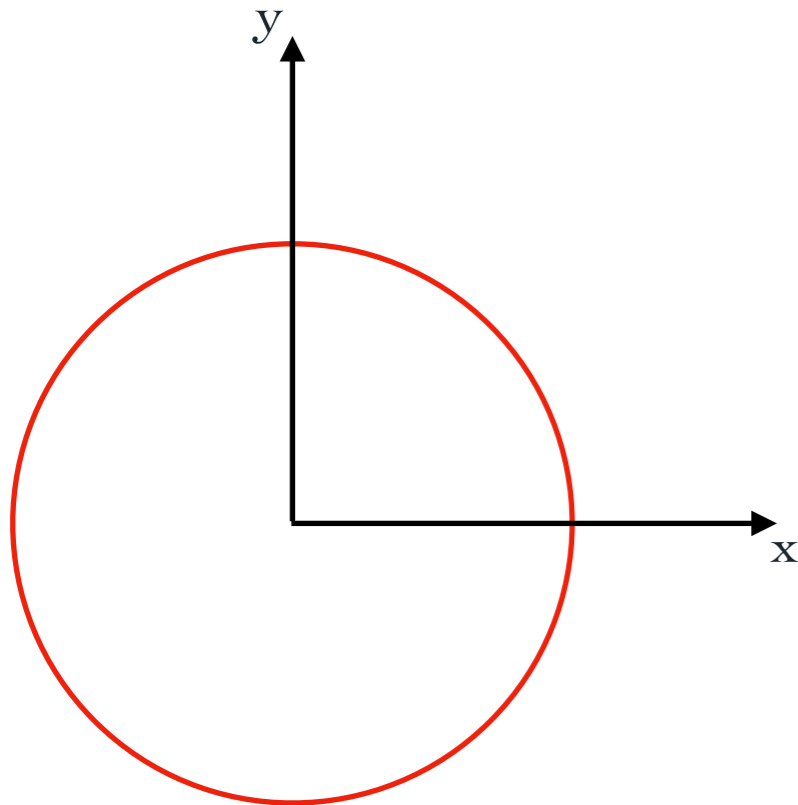
[D. Kermanschah, arXiv : [2110.06869](https://arxiv.org/abs/2110.06869)]

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THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

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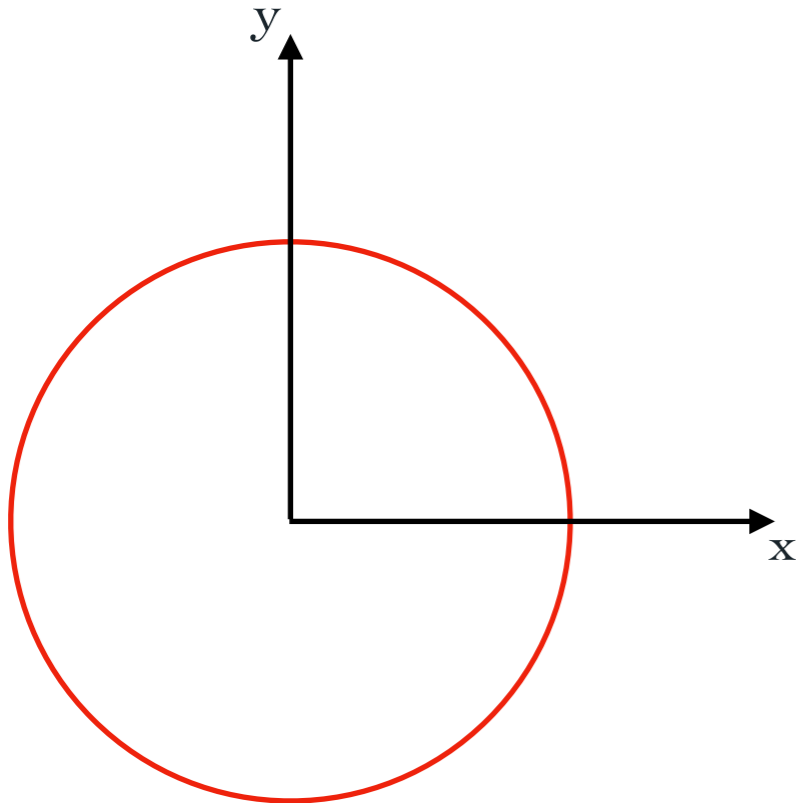
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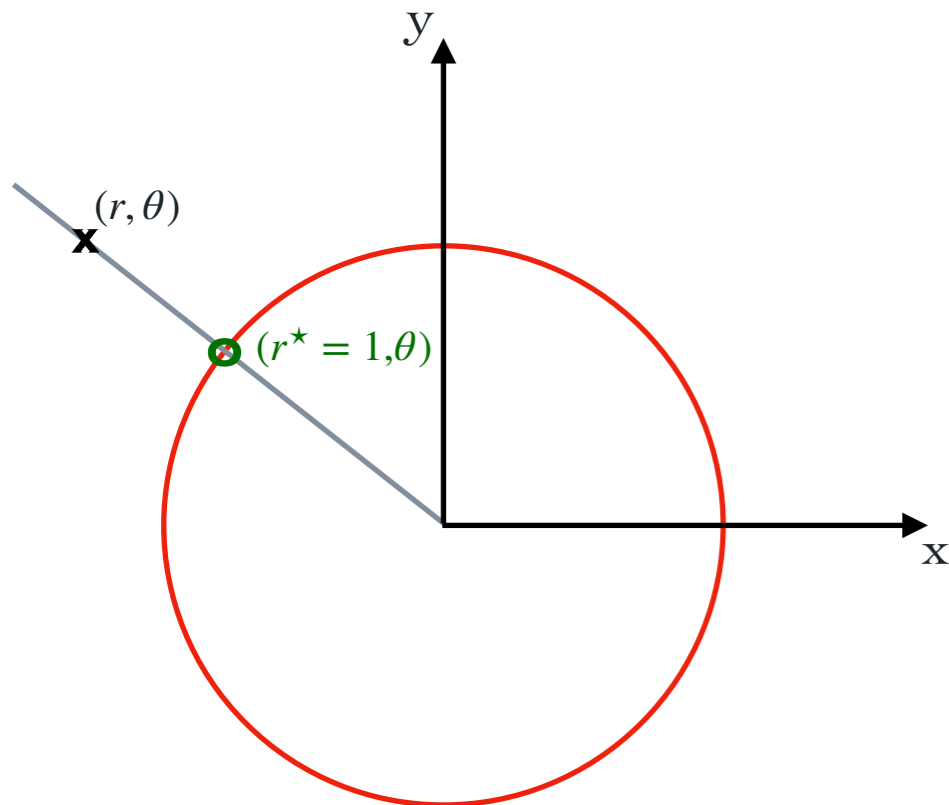
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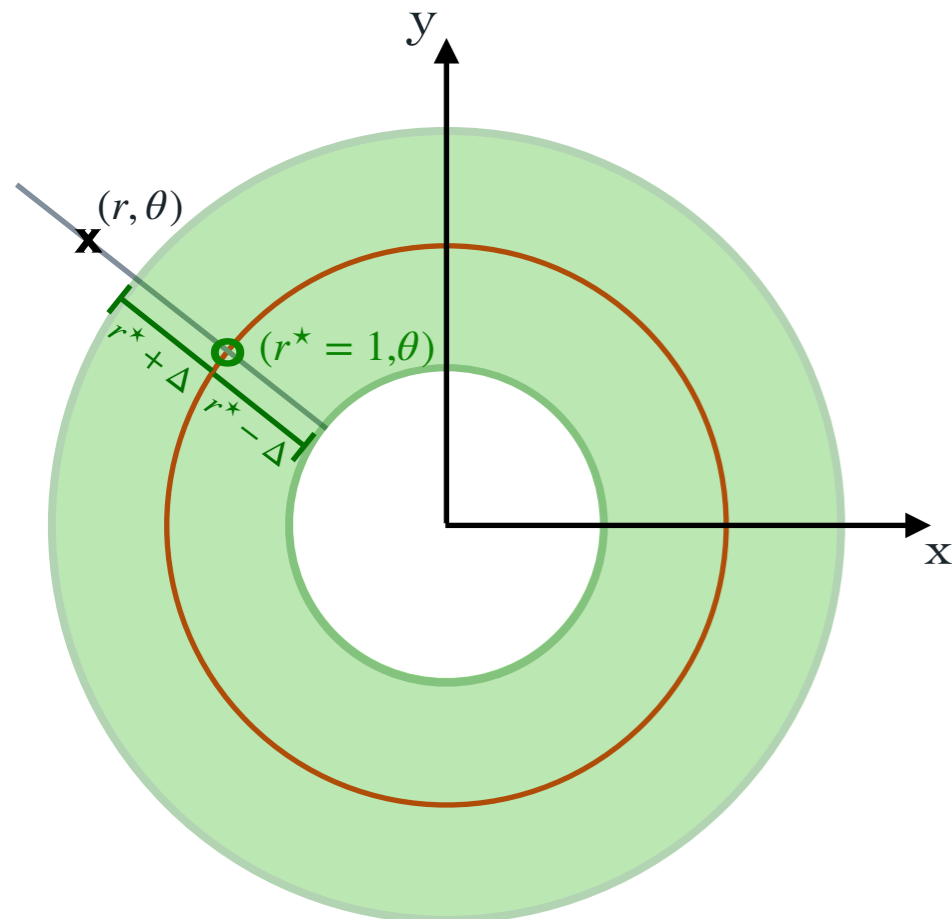
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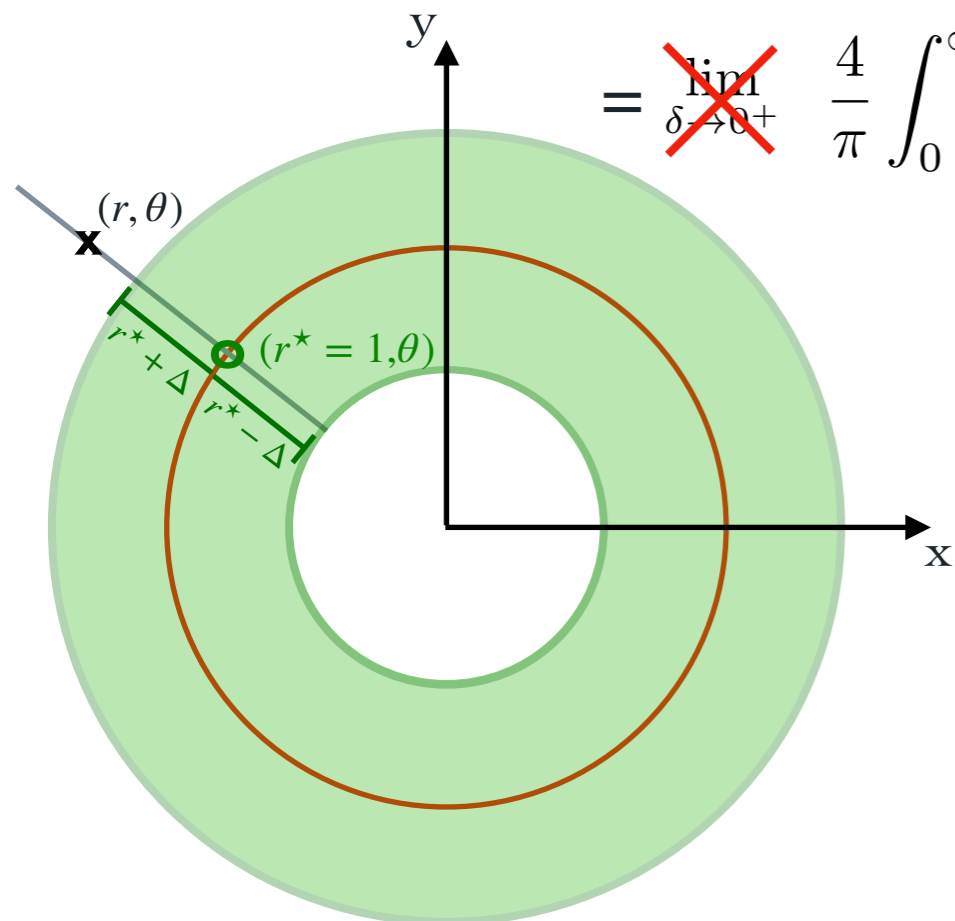
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$\delta = 0$ can be taken safely !



THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

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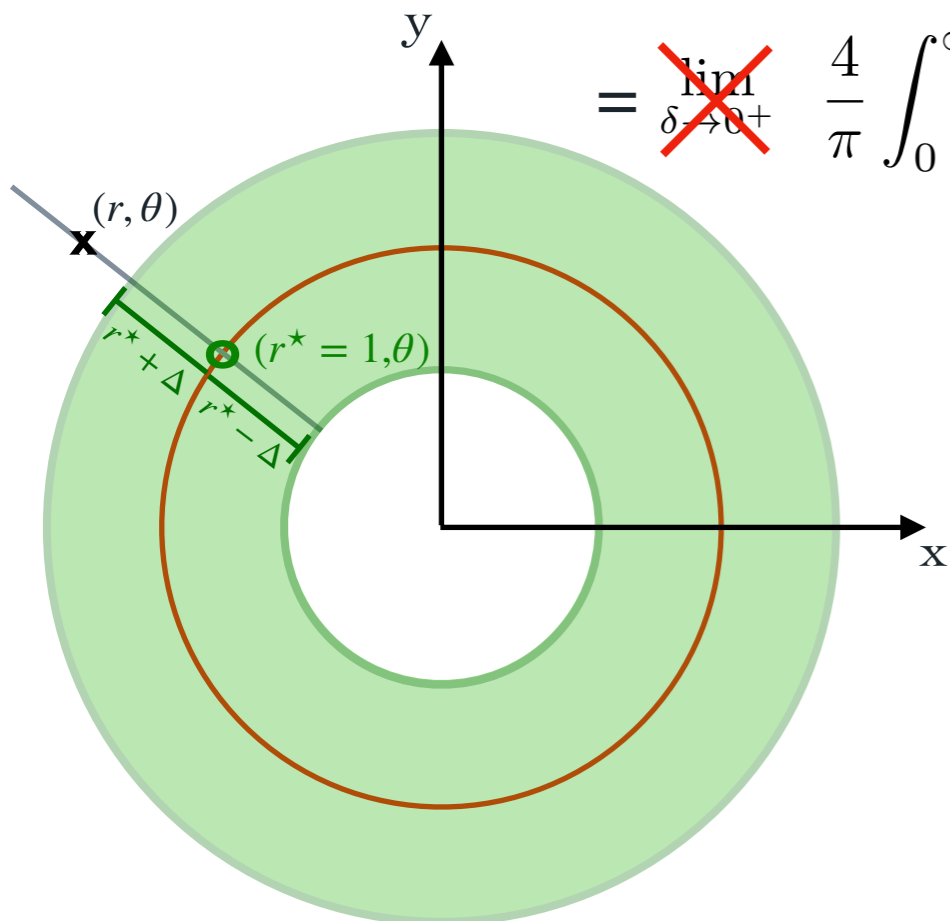
Local threshold counterterm

Integrated threshold counterterm

$$= \lim_{\delta \rightarrow 0^+} \frac{4}{\pi} \left[\int_0^\infty dr \left(\frac{1}{r^2 + 1} \right) \frac{1}{1 - r \pm i\delta} - \int_{1-\Delta}^{1+\Delta} dr \left(\frac{1}{(1)^2 + 1} \right) \frac{1}{1 - r \pm i\delta} \right] + \frac{4}{\pi} \int_{1-\Delta}^{1+\Delta} dr \left(\frac{1}{(1)^2 + 1} \right) \frac{1}{1 - r \pm i\delta}$$

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[D. Kermanschah, arXiv : 2110.06869]

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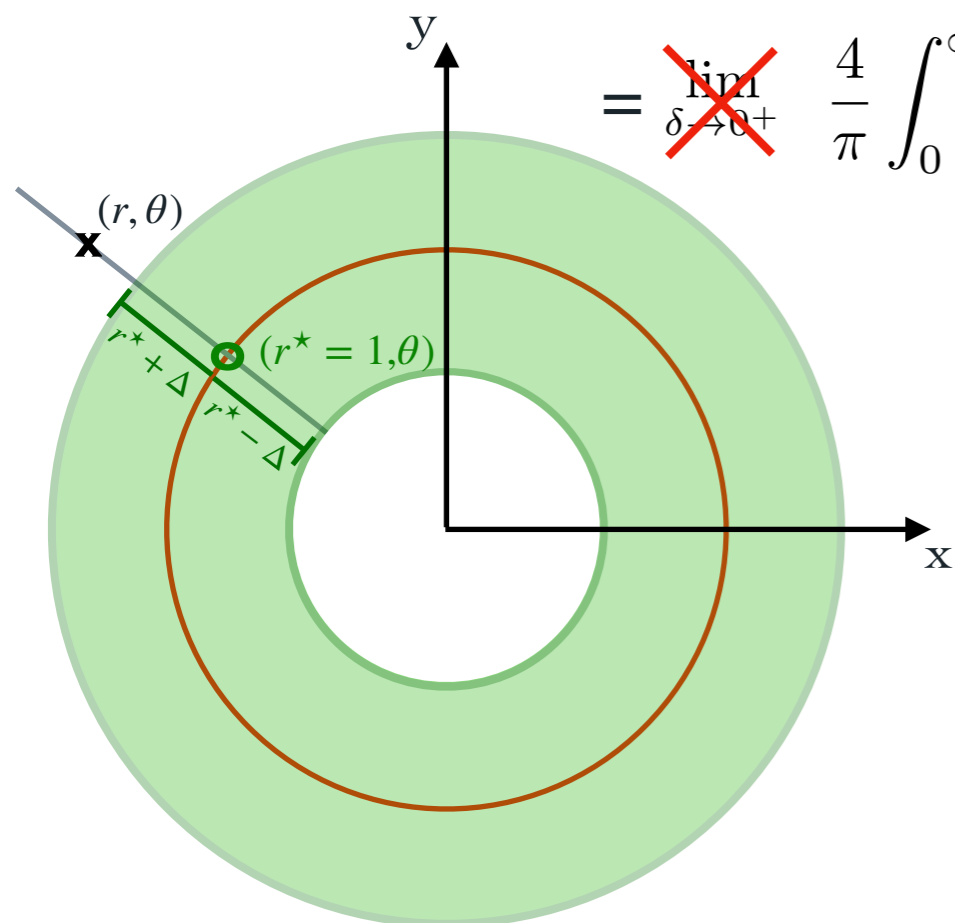
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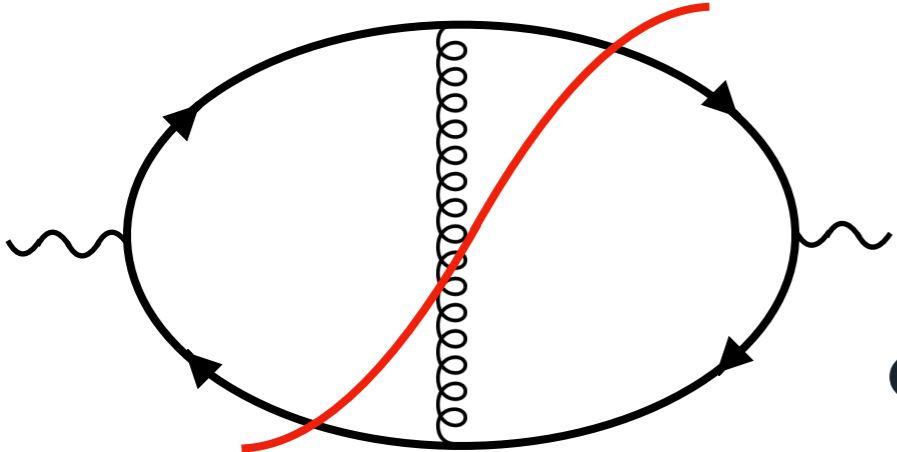
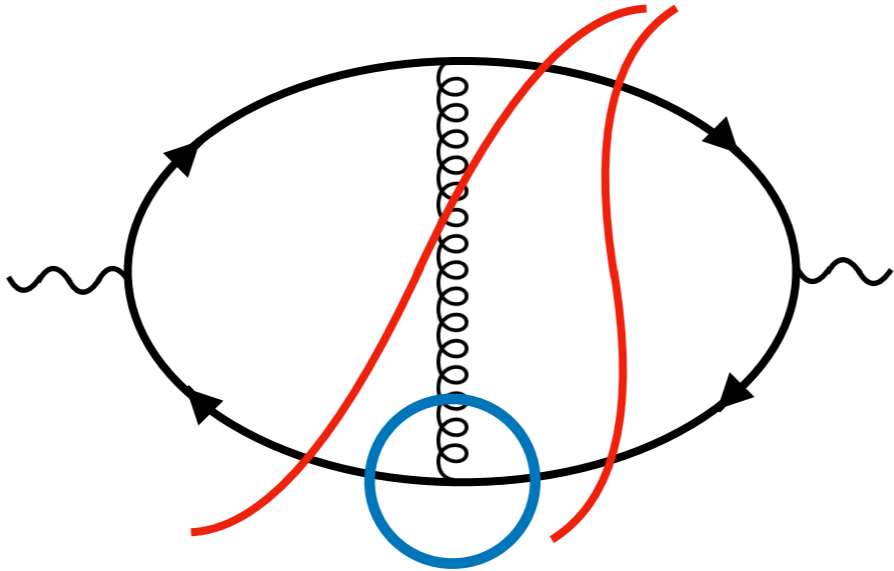
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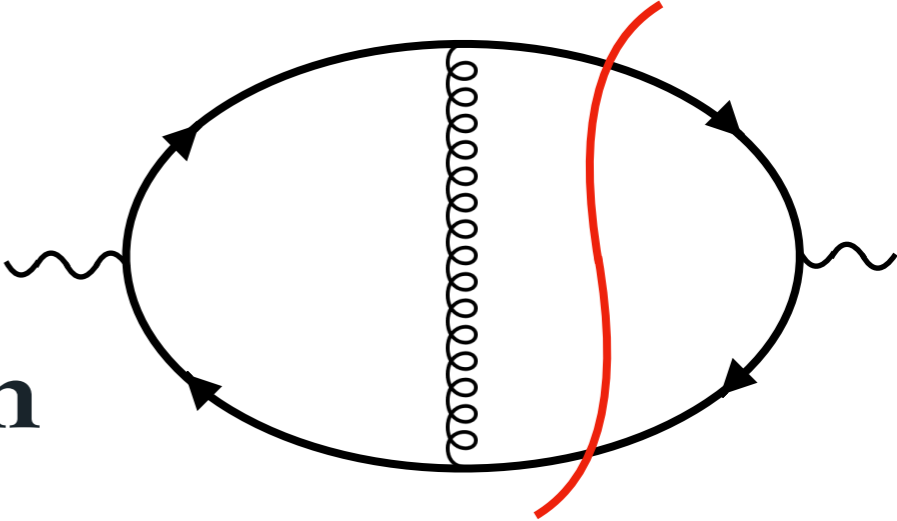
$$+ \lim_{\delta \rightarrow 0^+} \int_{1-\Delta}^{1+\Delta} dr \frac{4}{\pi} \frac{1}{2} \frac{1}{r - 1 \pm i\delta} \stackrel{\Delta \leq 1}{=} \underbrace{\frac{2}{\pi} \text{PV} \left[\frac{1}{r - 1 \pm i\delta} \right]}_0 \mp 2i$$

Easy to compute since Principal Value is zero by construction !

LOCAL UNITARITY



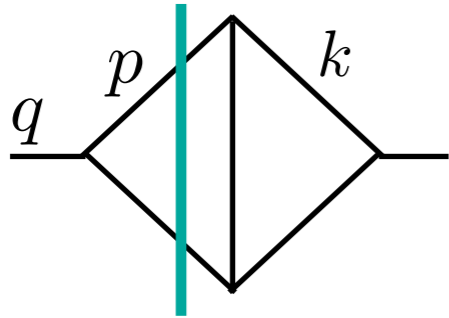
local
↔
cancellation



LOCALITY UNITARITY

We convert the **four-dimensional Minkowski loop integration measure** into a **three-dimensional Euclidean phase-space measure**:

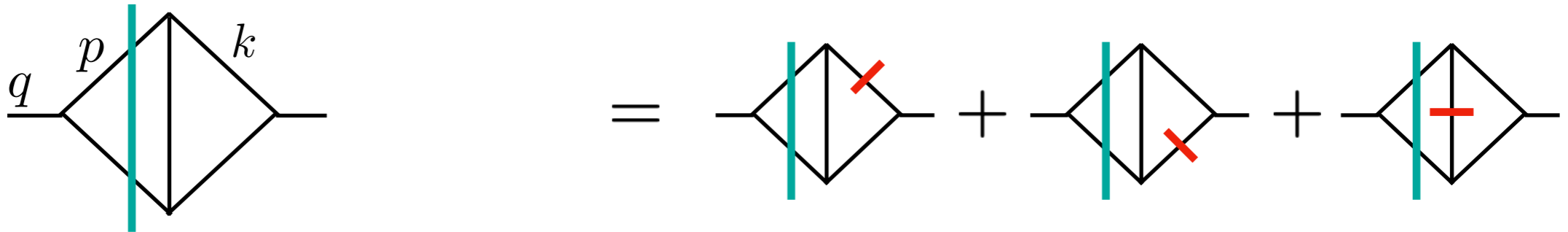
$$\frac{d^3\vec{p}}{2|\vec{p}|} d^4k \delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0)$$



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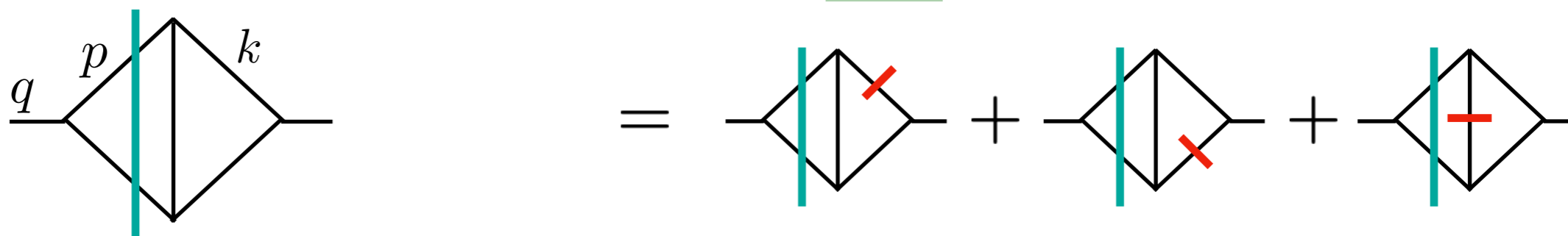
$$\frac{d^3\vec{p}}{2|\vec{p}|} d^4k \delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0) \rightarrow \frac{d^3\vec{p}}{2|\vec{p}|} \frac{d^3\vec{k}}{2|\vec{k}|} \delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0)$$



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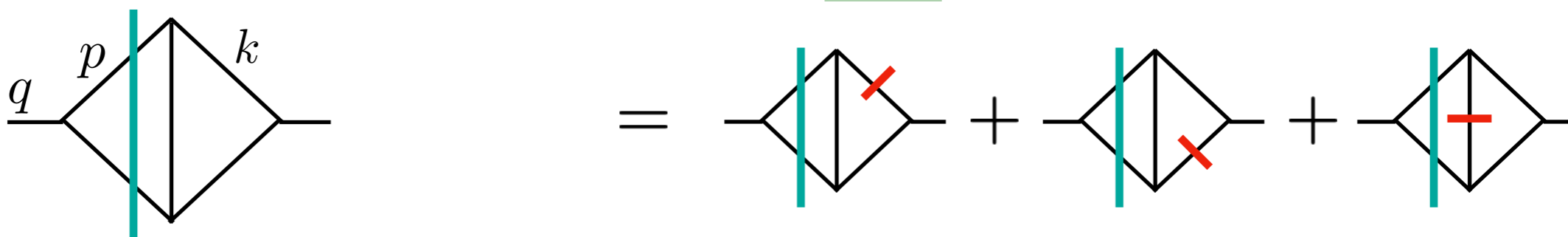
But the measure is **not yet fully aligned**:

$$\left. \begin{array}{c} E_2 \\ E_5 \\ E_3 \\ E_1 \\ E_4 \end{array} \right\} \left| \begin{array}{c} \text{loop with teal line} \\ \text{loop with teal slash} \end{array} \right. = \int d^3\vec{k} d^3\vec{p} (\delta(E_1 + E_2 - Q_0) f_{\text{virt}} + \delta(E_1 + E_3 + E_5 - Q_0) f_{\text{real}})$$

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$\eta_v(\vec{k}, \vec{p})$

$\eta_r(\vec{k}, \vec{p})$

(on-shell energies: $E_i(\vec{k}_i) = \sqrt{\vec{k}_i^2 + m_i^2 - i\delta}$)

CAUSAL FLOW

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The measure now differs only in the **delta enforcing on shell energy conservation**

$$\text{Diagram 1} \sim \delta(E_1 + E_2 - Q_0)$$

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Objective: find a common variable to solve both deltas.

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A different perspective on the usual phase space mapping problem

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Soper,
arXiv: [9804454](https://arxiv.org/abs/9804454) (1998)

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**General FSR cancellations
For N to M N^kLO processes**

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**General FSR cancellations
For N to M N^kLO processes**

A toy example:

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k})$$

CAUSAL FLOW : TOY INTEGRAL

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$$= \int d^3 \vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k}) \quad \text{using} \quad 1 = \int dt h(t)$$

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$$= \int d^3 \vec{k} \frac{Q_0^3}{|\vec{k}|^4} h(Q_0/|\vec{k}|) f(Q_0\vec{k}/|\vec{k}|) \quad \text{with} \quad t^* = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

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When applying this construction to LU we get:

$$\text{Diagram} = \int d^3 \vec{k} d^3 \vec{p} \delta(E_1 + E_2 - Q_0) f_{\text{virt}} = \int d^3 \vec{k} d^3 \vec{p} g_v(t_v^*)$$

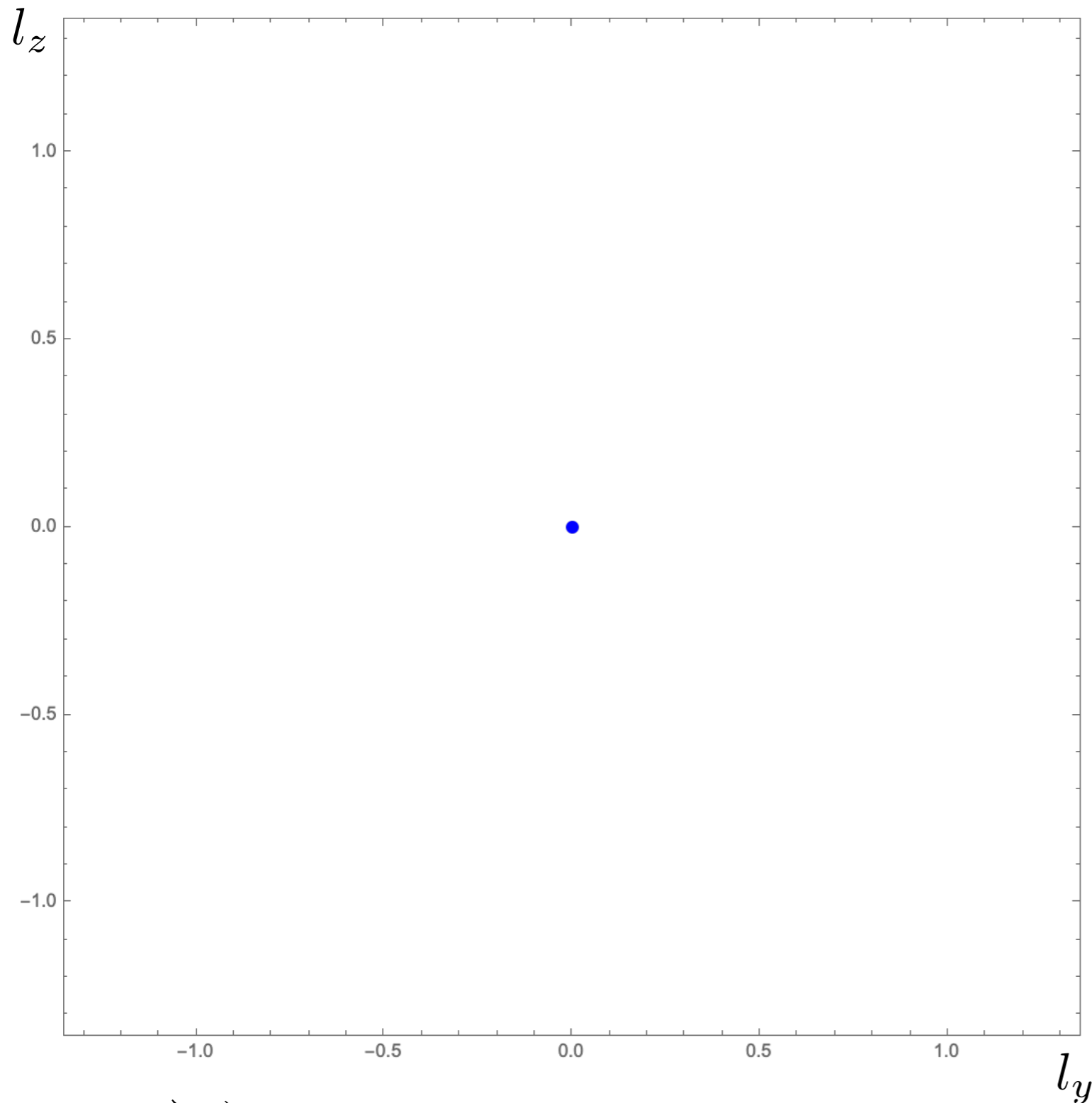
where $t_v^* = t_v^*(\vec{k}, \vec{p}) = \frac{Q_0}{E_1 + E_2}$

$(\vec{p}, \vec{k}) \rightarrow \vec{\phi}(t, (\vec{p}, \vec{k}))$

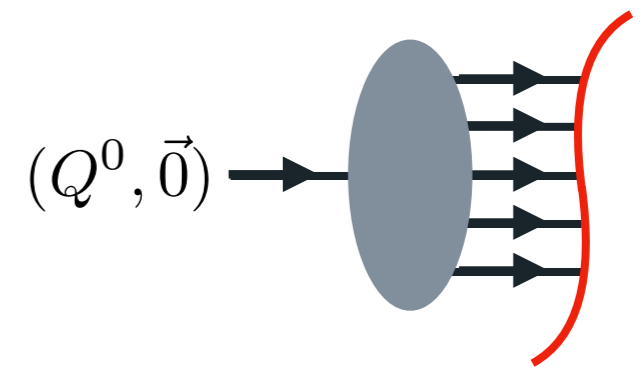
“Causal flow” is called like this because it is the generalisation of the Soper, derive from a contour deformation field satisfying the causal constraints.

$$\begin{cases} \partial_t \vec{\phi} = \vec{k} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$

LOCALITY UNITARITY: VISUALISATION

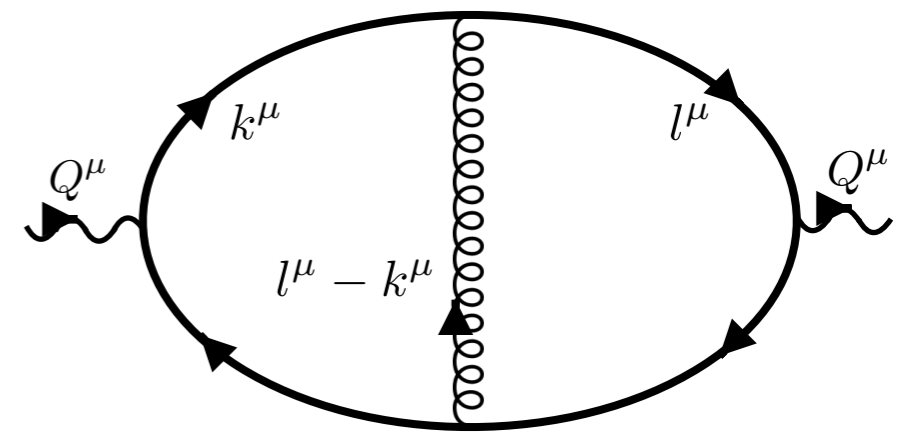


$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$



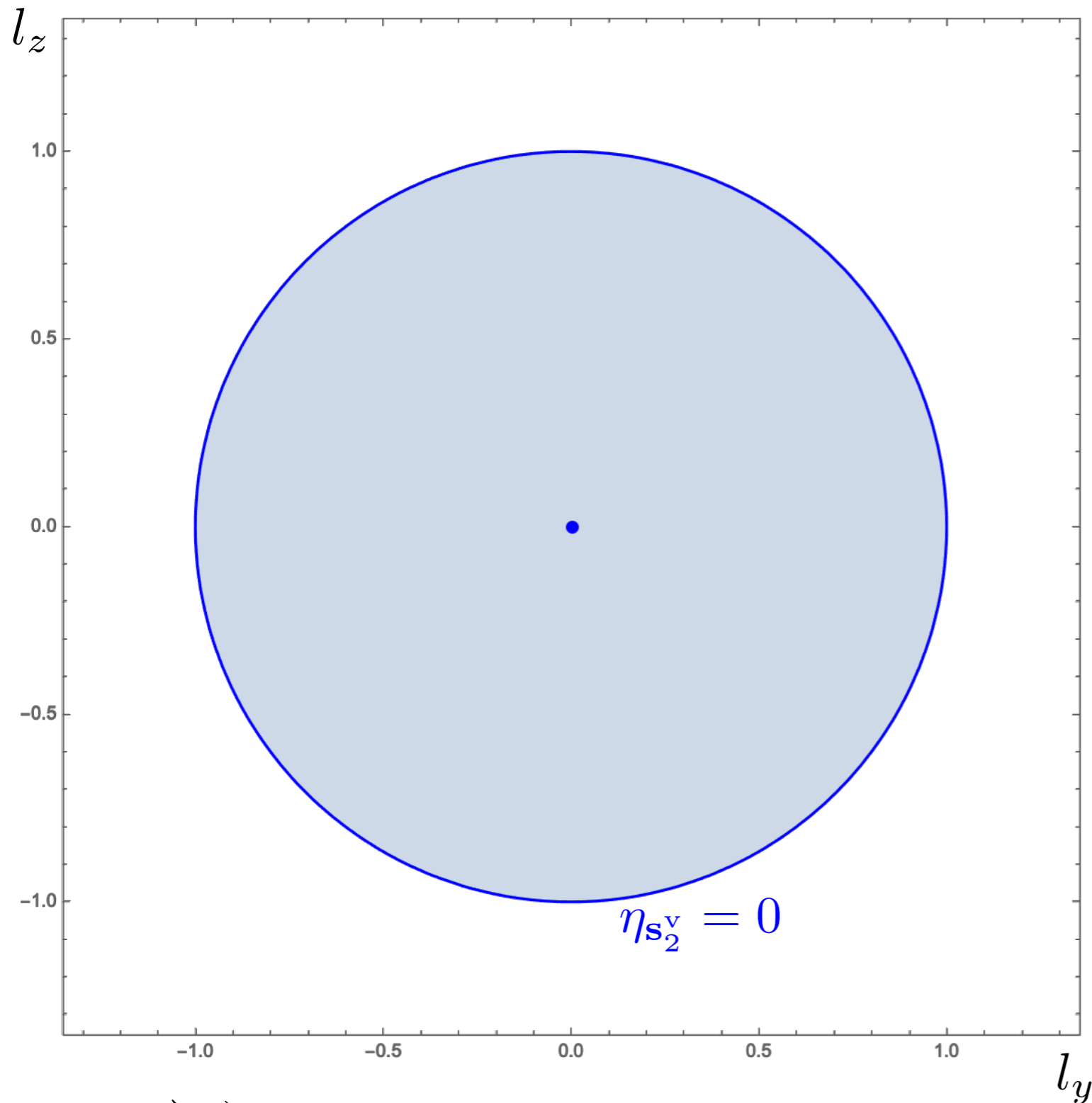
$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = ((0, 0.5, 0.5), (0, l_y, l_z))$$

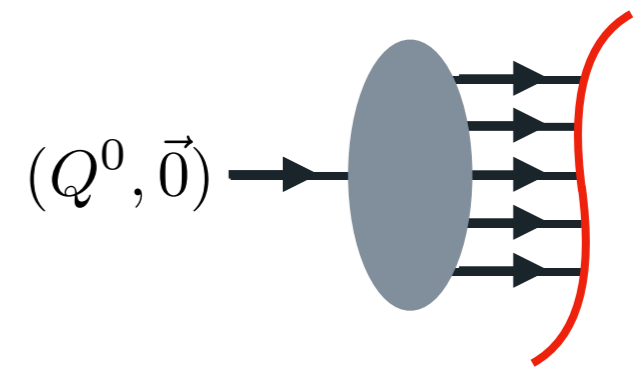


— = Cutkosky cut \equiv threshold

LOCALITY UNITARITY: VISUALISATION



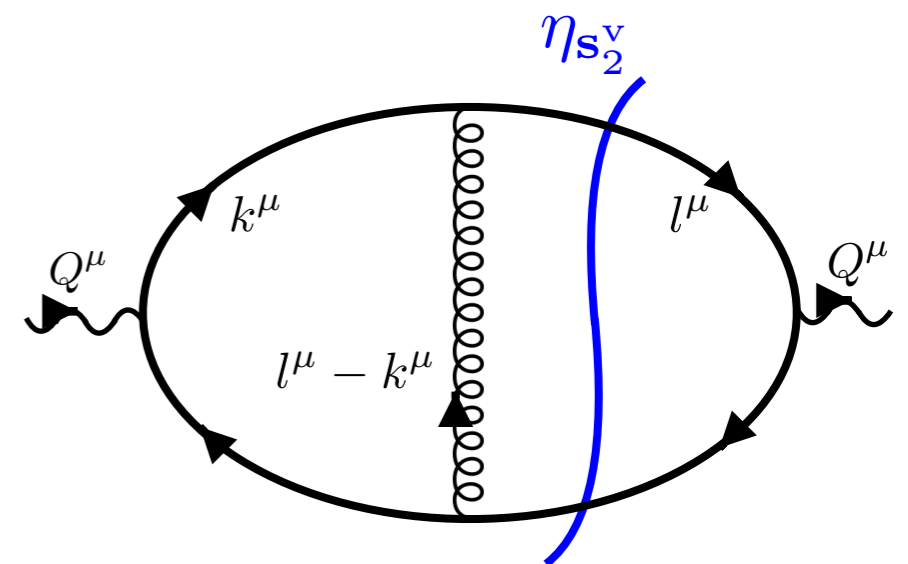
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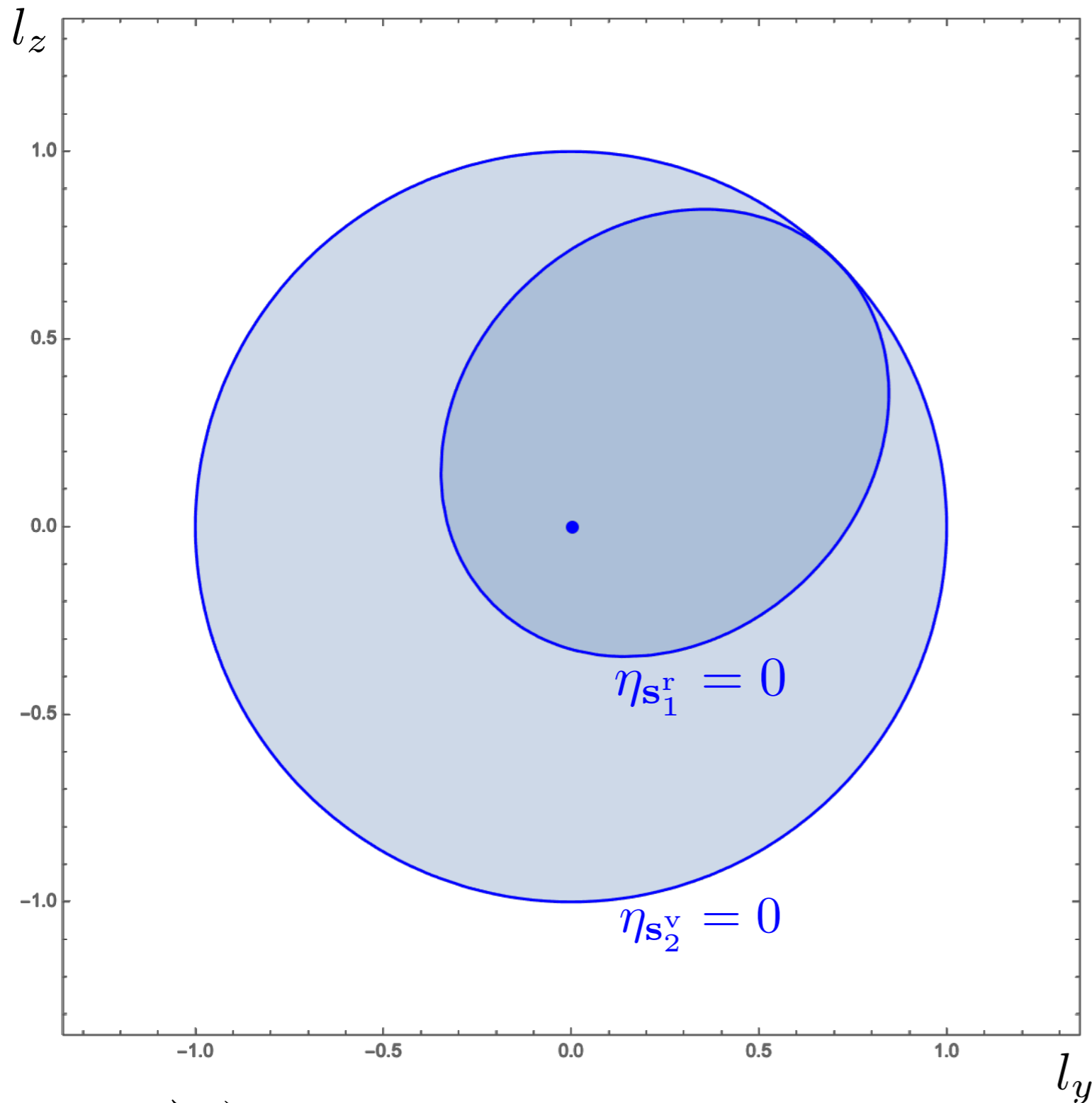
$$(\vec{k}, \vec{l}) = ((0, 0.5, 0.5), (0, l_y, l_z))$$

$$\eta_{s_2^v} \rightarrow 2|\vec{l}'| = Q^0$$

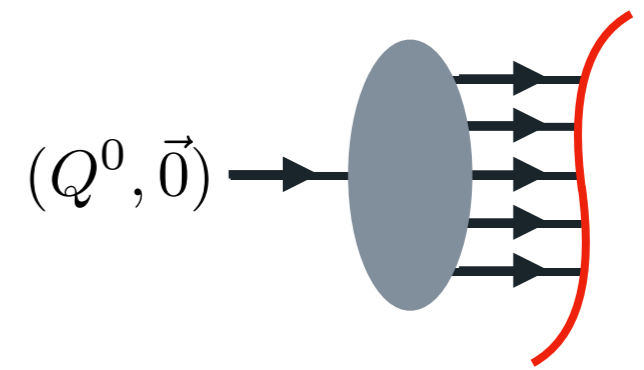


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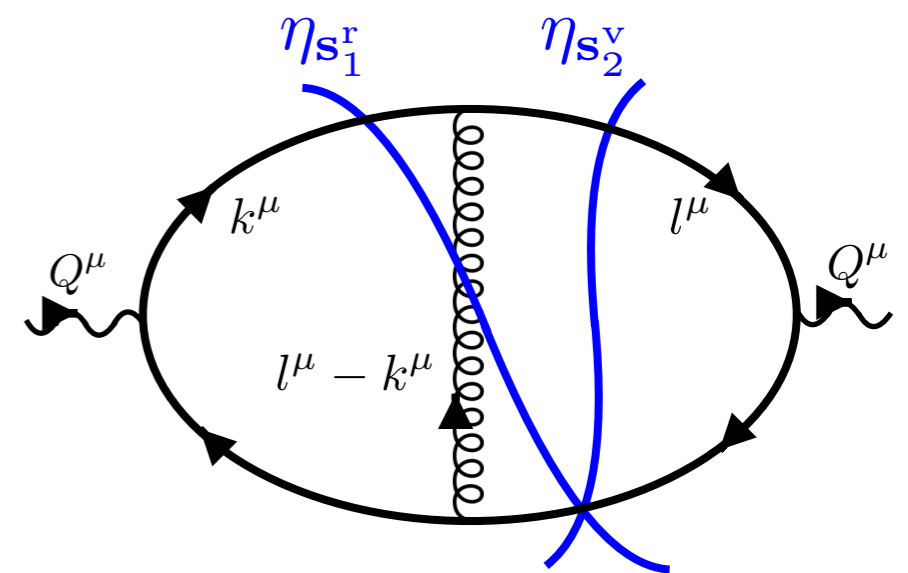


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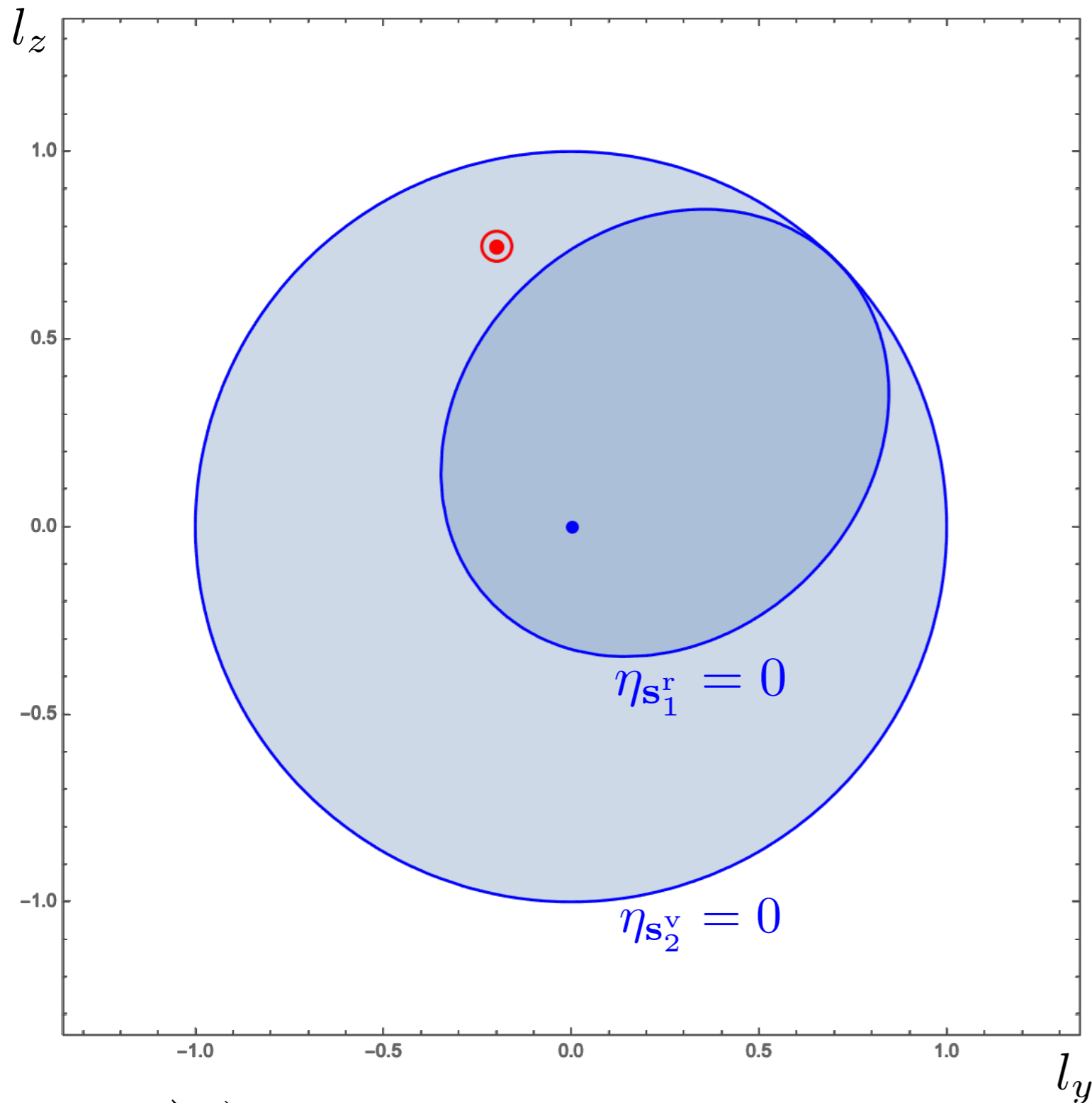
$$\eta_{s_2^v} \rightarrow 2|\vec{l}'| = Q^0$$

$$\eta_{s_1^r} \rightarrow |\vec{l}'| + |\vec{l}' - \vec{k}'| = Q^0 - |\vec{k}'|$$

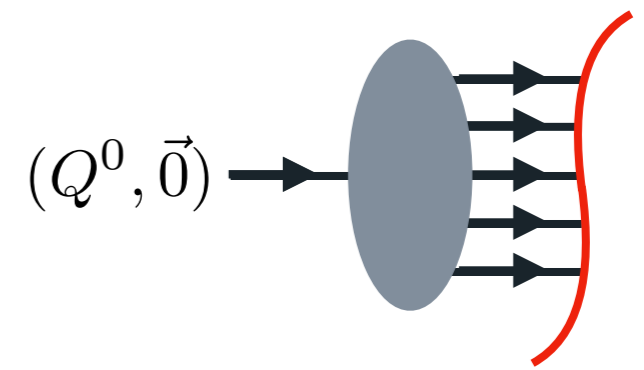


— = Cutkosky cut \equiv threshold

LOCALITY UNITARITY: VISUALISATION



$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$

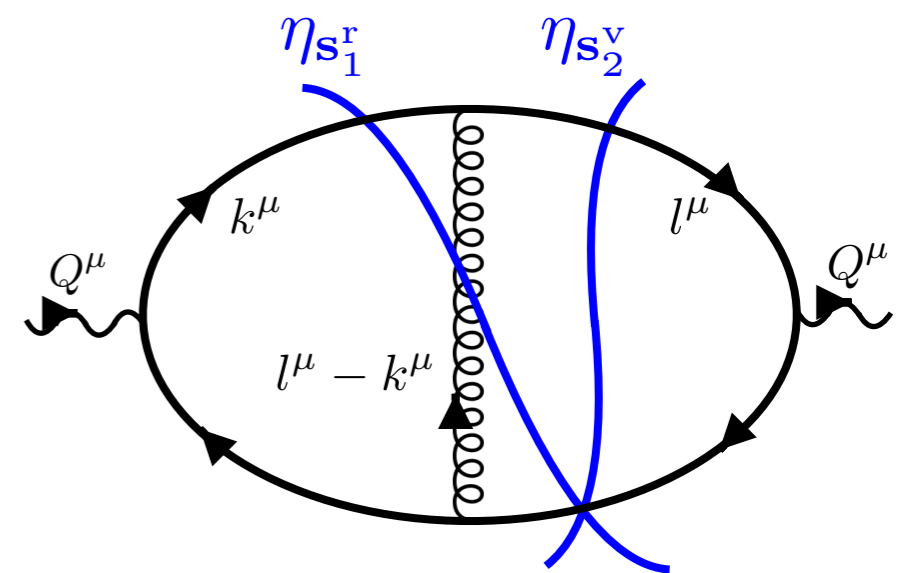


$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = ((0, 0.5, 0.5), (0, l_y, l_z))$$

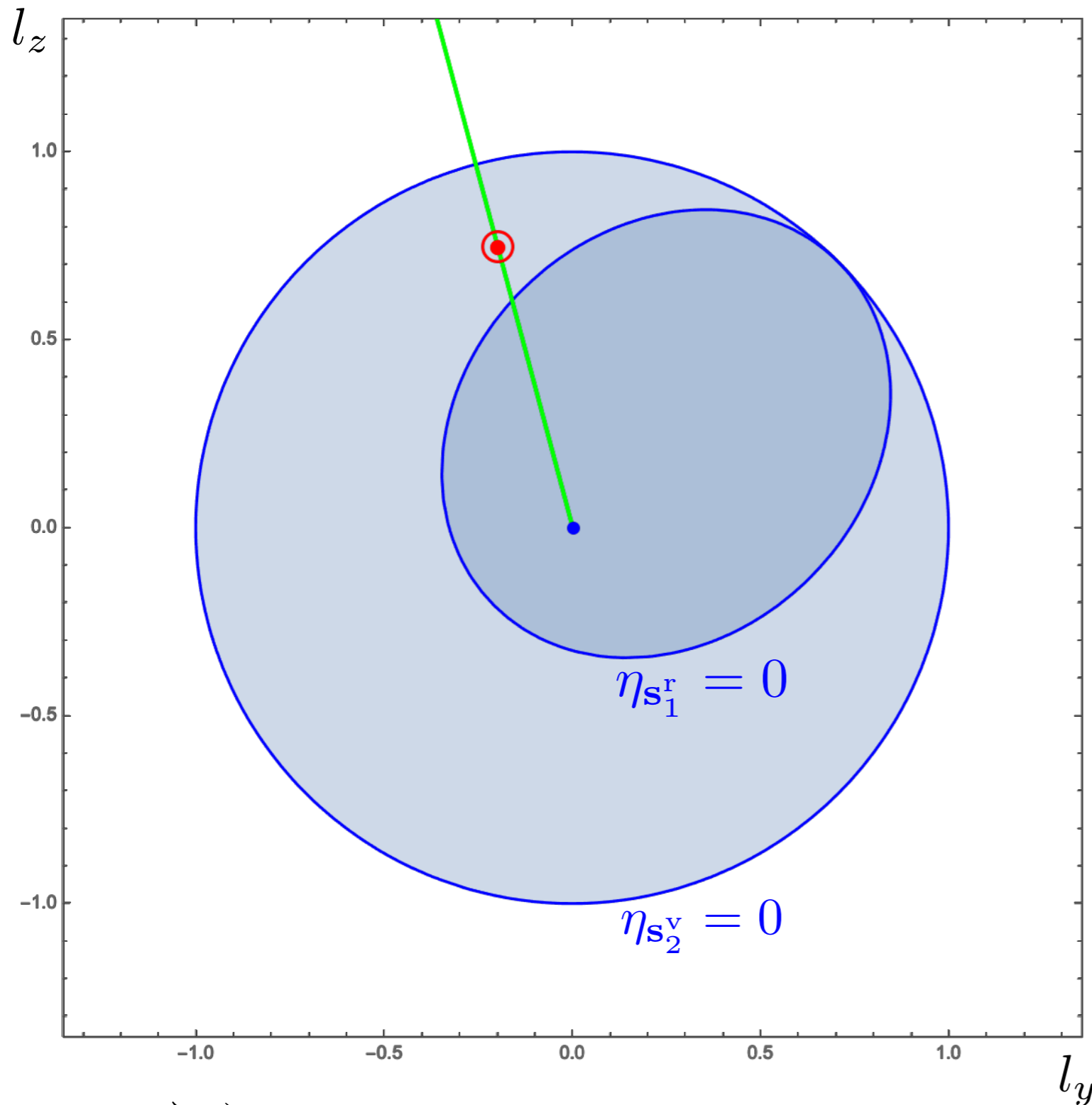
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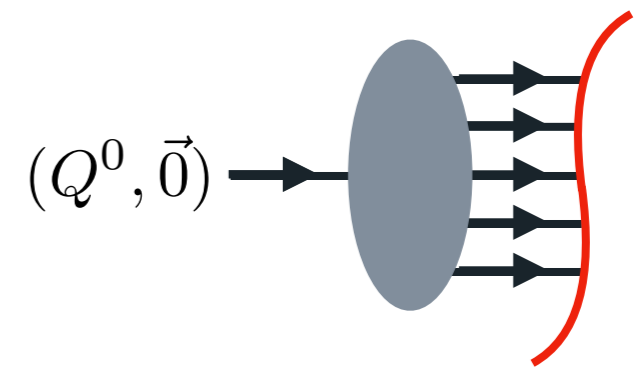


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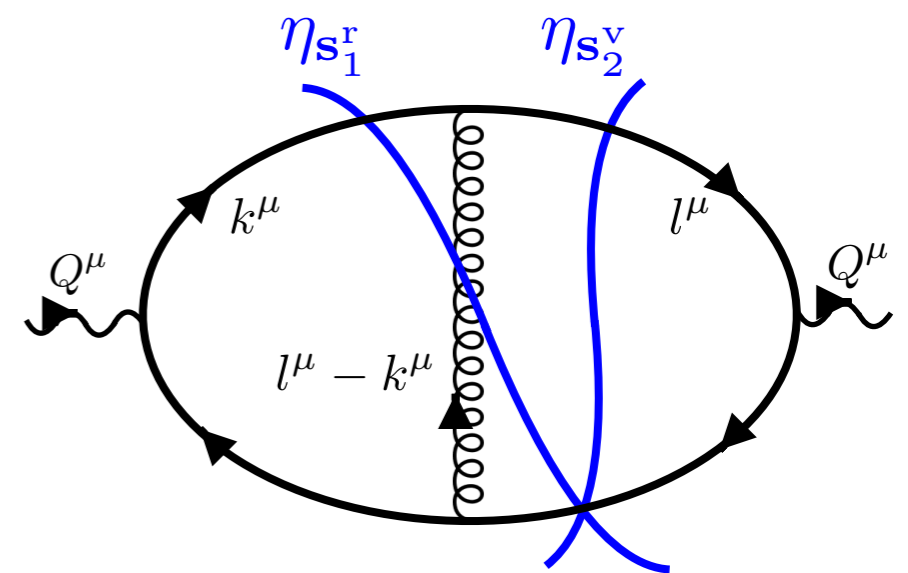


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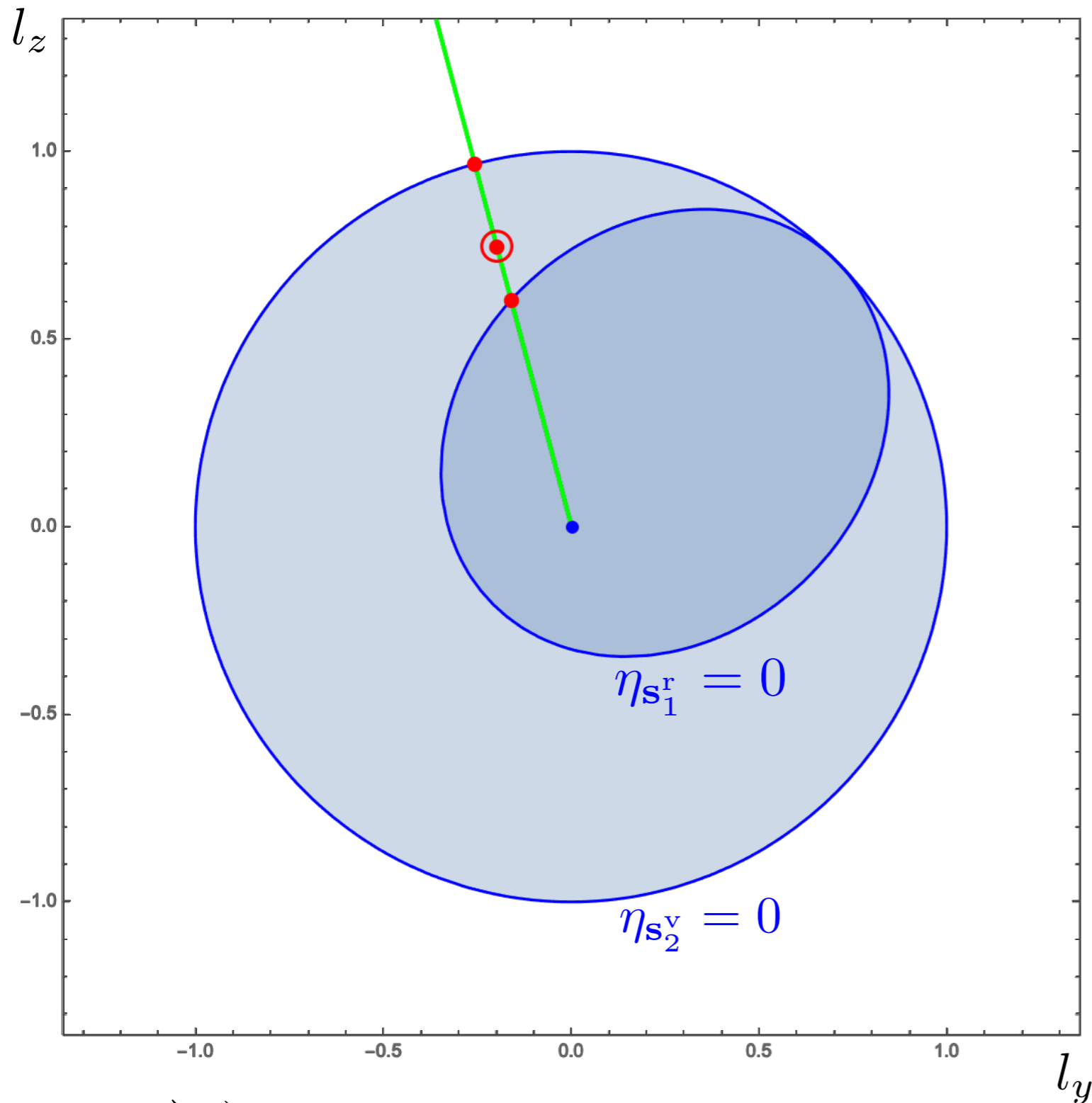
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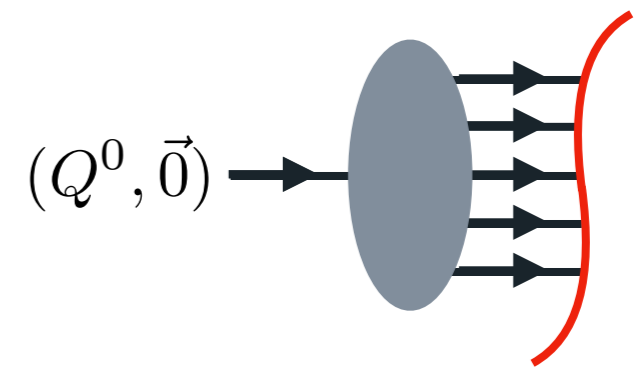


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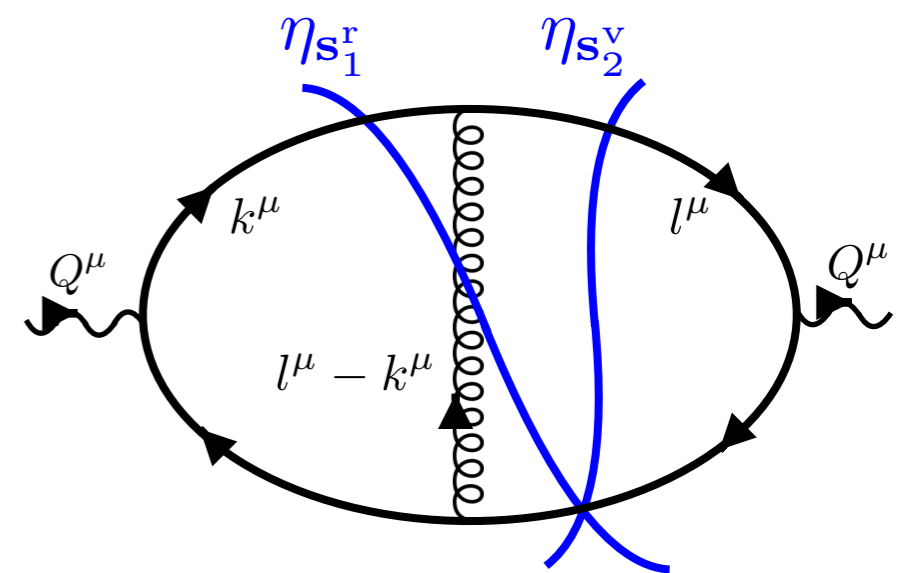


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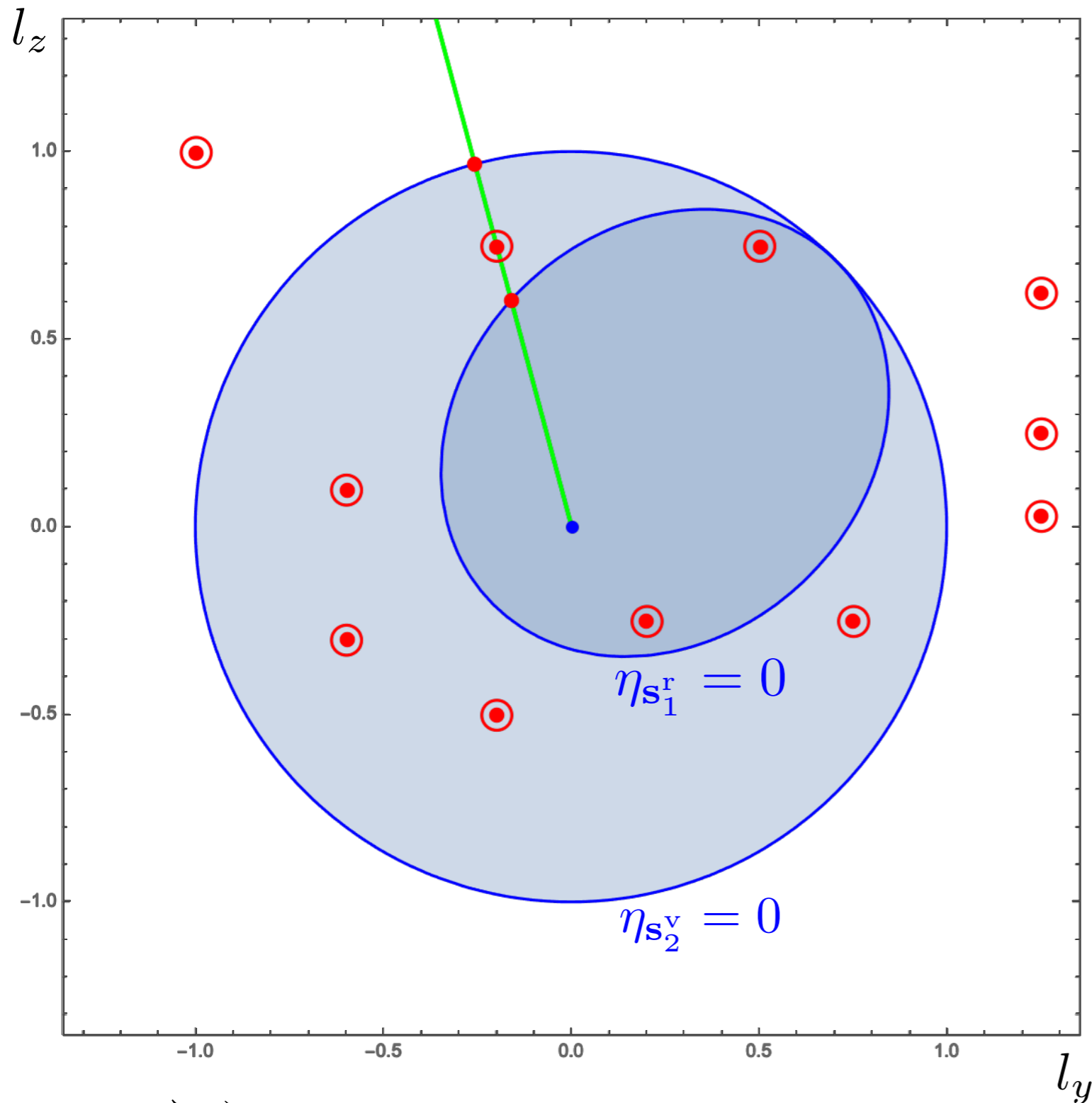
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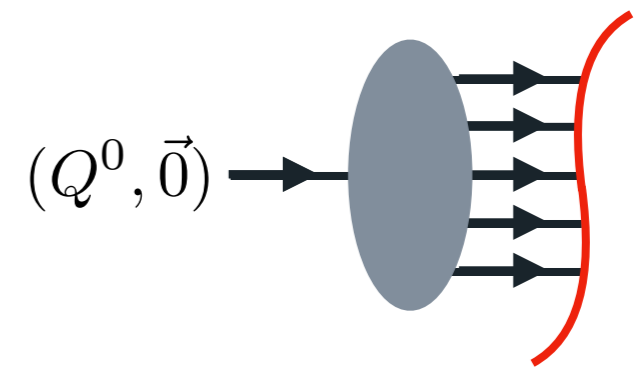


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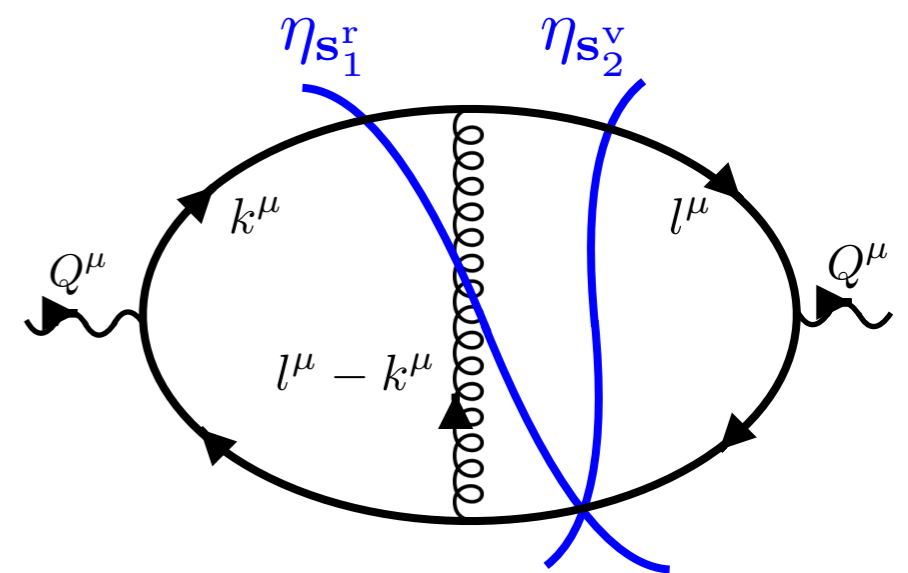


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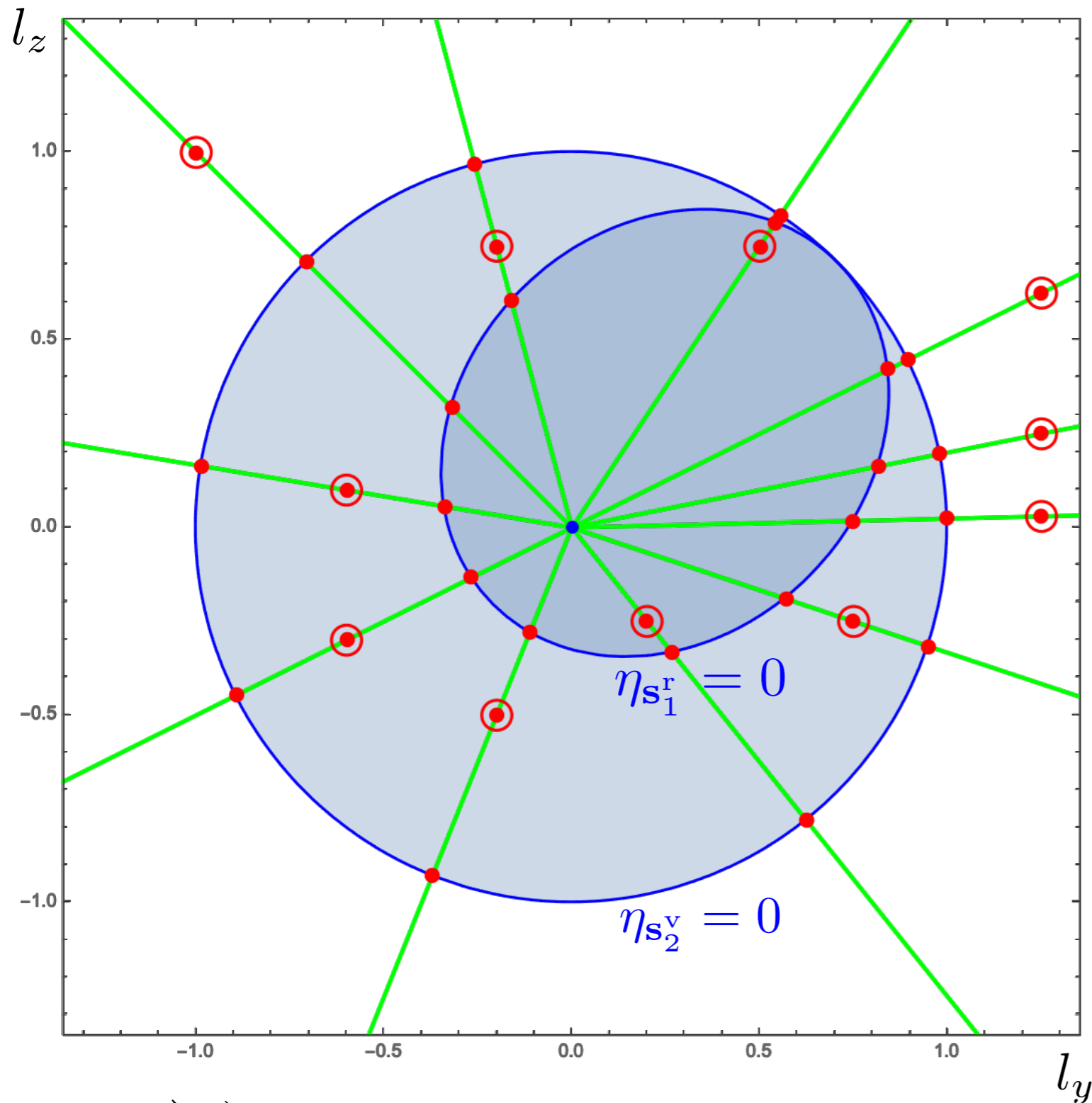
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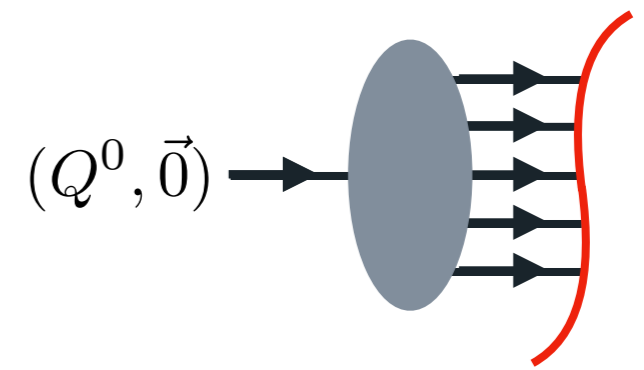


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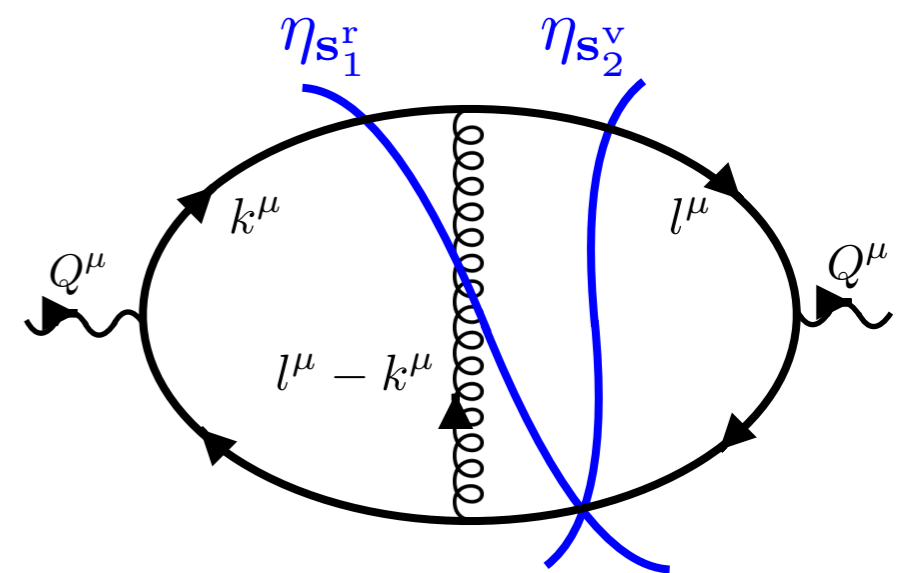


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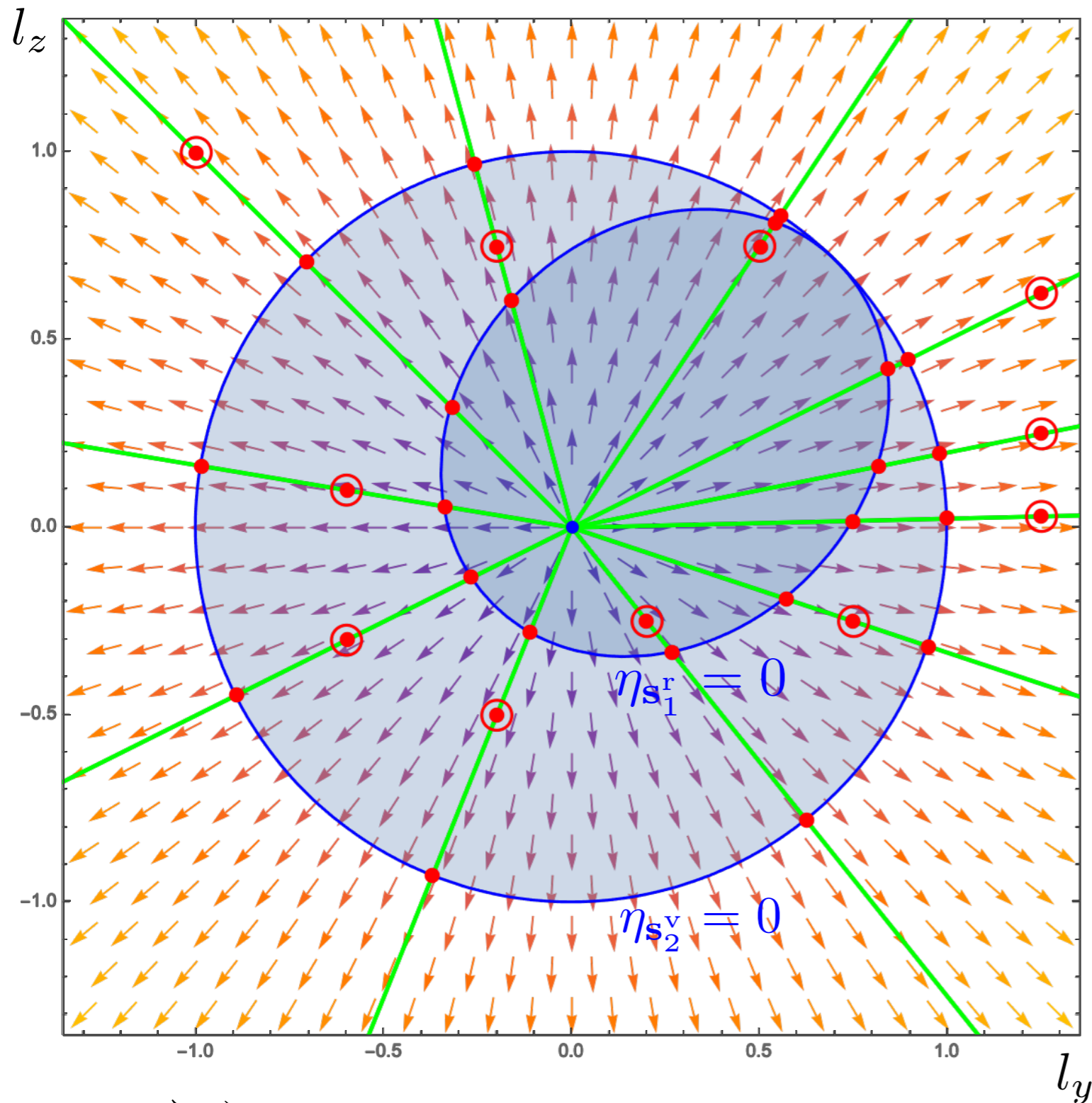
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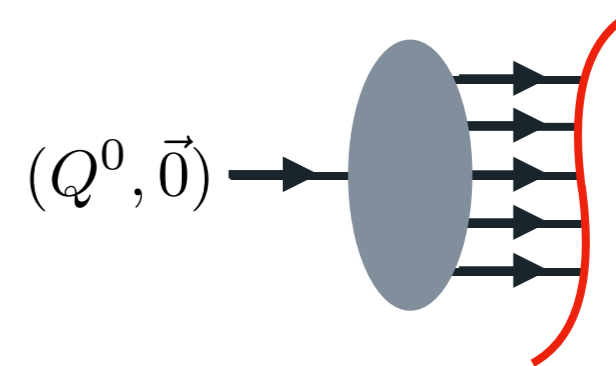


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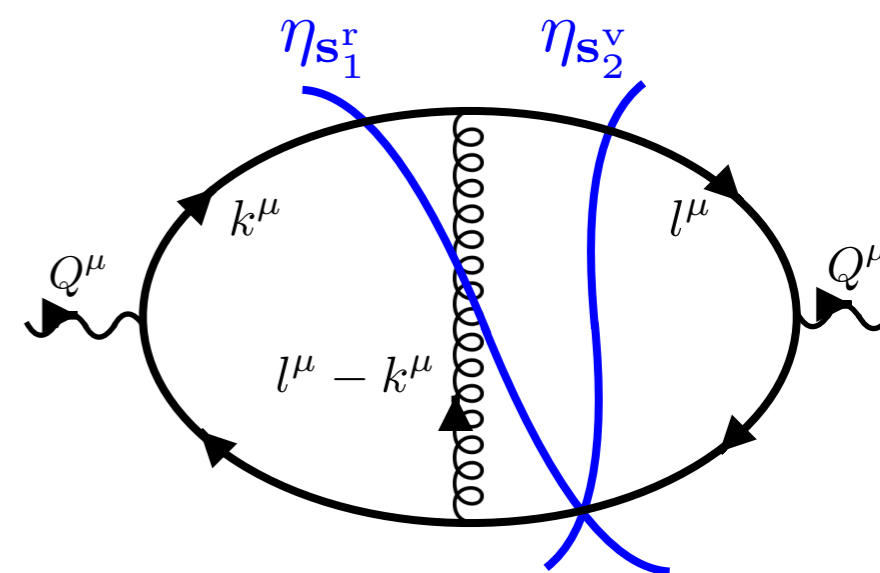


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LOCALITY UNITARITY: ALL-ORDERS PROOF

[Z. Capatti, VH, A. Pelloni, B. Ruijl, arXiv : [2010.01068](https://arxiv.org/abs/2010.01068)] [Summary in proceedings, arXiv : [2110.15662](https://arxiv.org/abs/2110.15662)]

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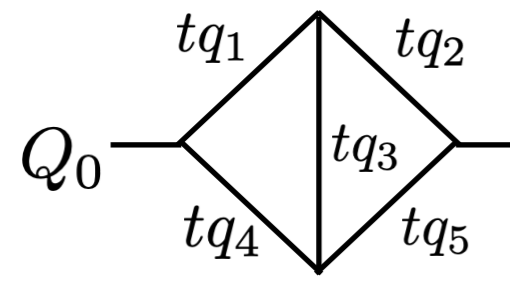
The **LTD representation** of the double triangle with rescaled momenta is

$$\begin{array}{c}
 tq_1 \quad tq_2 \\
 \diagdown \quad / \\
 Q_0 \quad \text{---} \quad \text{---} \quad tq_3 \\
 / \quad \diagdown \\
 tq_4 \quad tq_5
 \end{array}
 \quad
 f_{\text{ltd}} \left(\text{triangle} \right) \Big|_{tq_i} = \left[\begin{array}{cccc}
 \text{triangle with red cross} & + & \text{triangle with red cross} & + & \text{triangle with red cross} & + & \text{triangle with red cross} \\
 + & \text{triangle with red cross} & + & \text{triangle with red cross} & + & \text{triangle with red cross} & + & \text{triangle with red cross}
 \end{array} \right] q_i \rightarrow tq_i$$

LOCALITY UNITARITY: ALL-ORDERS PROOF

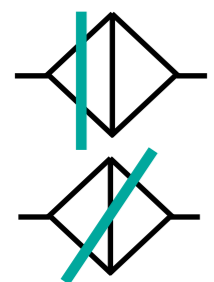
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Then one can capture the **thresholds** of this forward-scattering graphs with



$$= \int d^3 \vec{p} d^3 \vec{k} \left[\lim_{t \rightarrow t_v^*} (t - t_v^*) f_{\text{ltd}} \left(\text{triangle} \right) \Big|_{tq_i} + \lim_{t \rightarrow t_r^*} (t - t_r^*) f_{\text{ltd}} \left(\text{triangle} \right) \Big|_{tq_i} \right]$$

g_v , g_r can be written as different limits of the same function!

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The **LTD representation** of the double triangle with rescaled momenta is

$$Q_0 \left(\begin{array}{c} tq_1 \quad tq_2 \\ \diagdown \quad / \\ tq_3 \\ / \quad \diagdown \\ tq_4 \quad tq_5 \end{array} \right) \Big|_{tq_i} = \left[\begin{array}{c} \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \\ + \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} \end{array} \right] q_i \rightarrow tq_i$$

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g_v , g_r can be written as different limits of the same function!

Solving delta in the scaling variable \Rightarrow **1d residue theorem along the line** $\gamma(t) = (t\vec{k}, t\vec{p})$

$$= \sigma_d = \int d^3 \vec{p} d^3 \vec{k} \left[\sum_{i=1}^4 \lim_{t \rightarrow t_i^*} (t - t_i^*) f_{\text{ltd}} \left(\begin{array}{c} \text{Diagram} \end{array} \right) \Big|_{tq_i} \right] \text{LU representation}$$

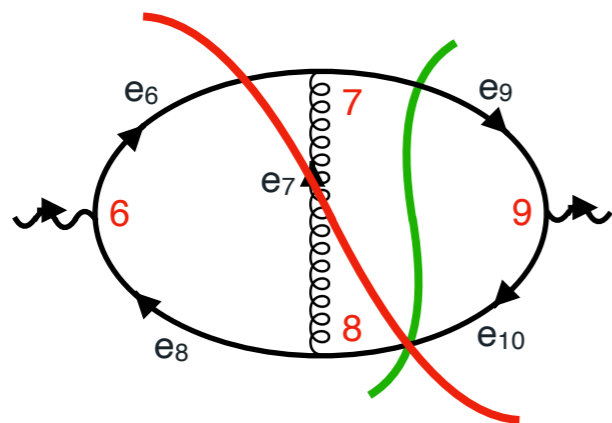
Cutkosky, but at the local level! We prove cancellations by studying the limit $t_r^* \rightarrow t_v^*$

LOCALITY UNITARITY

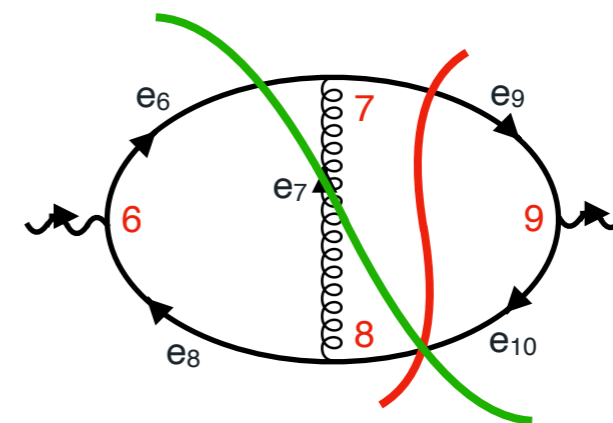
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

— = Cutkosky cut — = threshold singularity



cancel

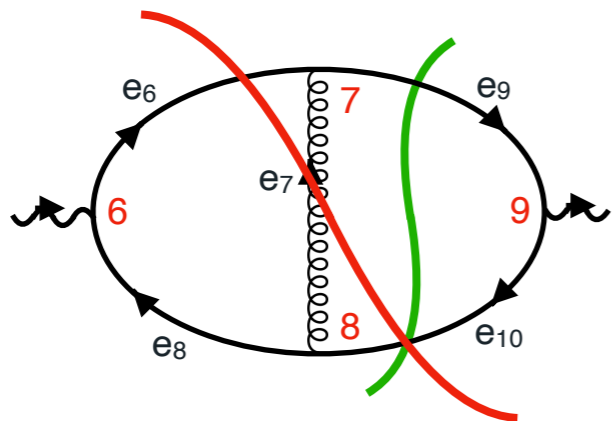


LOCALITY UNITARITY

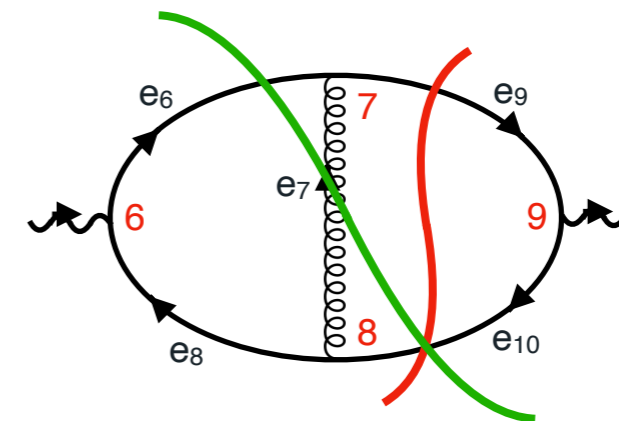
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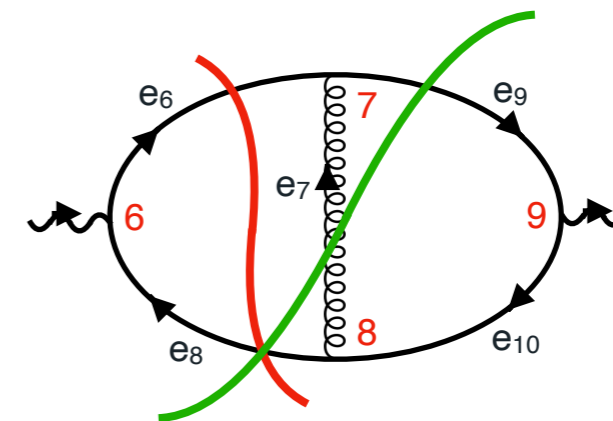
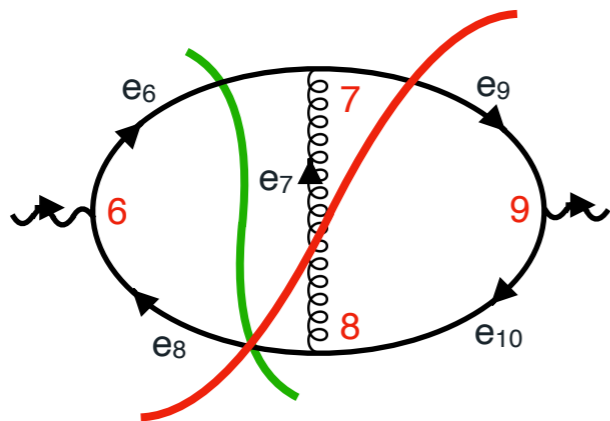
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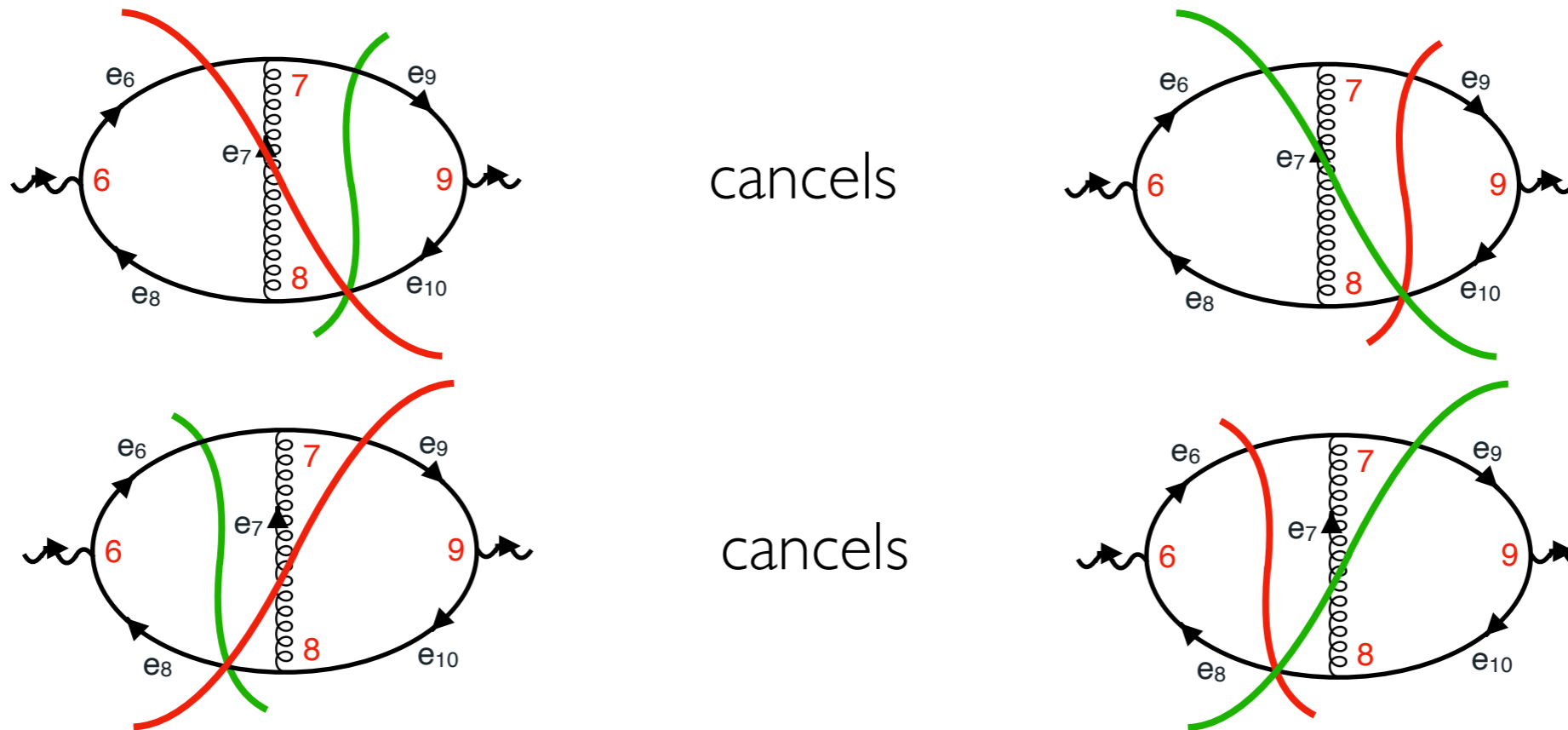


LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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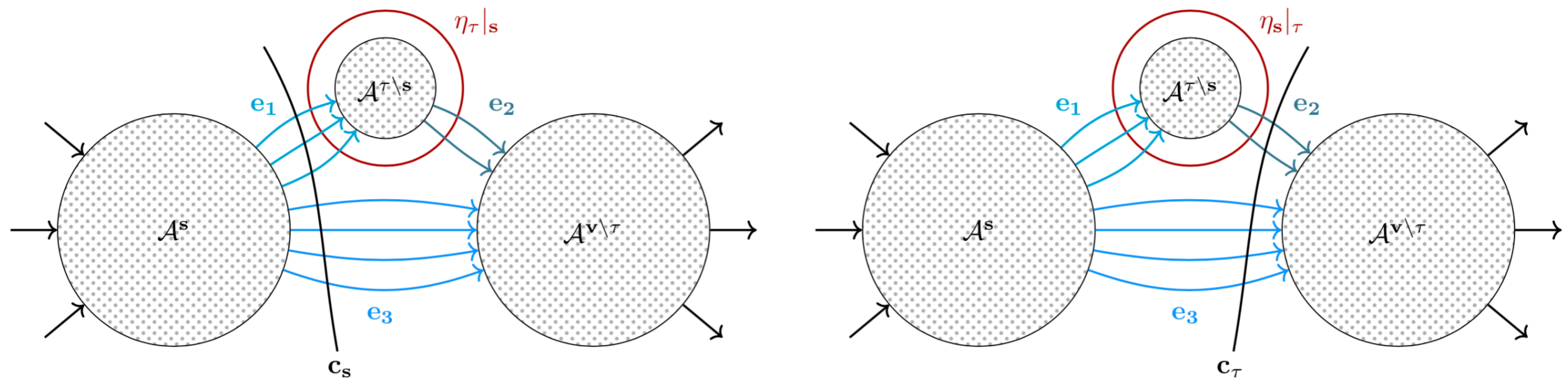
Even for **non-pinched singular threshold** ! (when $\mathcal{O}_s \equiv 1$) :



LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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TROPICAL SAMPLING IN MOMENTUM SPACE

FOR TAMING INTEGRABLE SINGULARITIES

$$I^{(\text{eucl.})}[f] = \int d^{DL} \mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}}$$

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$$= \int_{[0,1]^{(DL+2|E|+1)}} dz \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

TROPICAL SAMPLING OF EUCLIDEAN FEYNMAN INTEGRALS

$$I^{\text{eucl.}} [f = 1]$$

E	$\ell(G)$	σ_I/I	samples per second	preprocessing time	RAM
6	3	0.9	$1.1 \cdot 10^6 / s$	$3.0 \cdot 10^{-5} s$	1 KB
8	4	1.1	$7.5 \cdot 10^5 / s$	$1.3 \cdot 10^{-4} s$	4 KB
10	5	1.3	$5.1 \cdot 10^5 / s$	$6.0 \cdot 10^{-4} s$	16 KB
12	6	1.6	$4.1 \cdot 10^5 / s$	$2.7 \cdot 10^{-3} s$	64 KB
14	7	1.8	$3.2 \cdot 10^5 / s$	$1.2 \cdot 10^{-2} s$	256 KB
16	8	2.1	$2.6 \cdot 10^5 / s$	$5.3 \cdot 10^{-2} s$	1 MB
18	9	2.5	$2.1 \cdot 10^5 / s$	$2.3 \cdot 10^{-1} s$	4 MB
20	10	2.8	$1.4 \cdot 10^5 / s$	$1.1 \cdot 10^0 s$	16 MB
22	11	3.2	$1.0 \cdot 10^5 / s$	$4.7 \cdot 10^0 s$	64 MB
24	12	3.7	$8.6 \cdot 10^4 / s$	$2.1 \cdot 10^1 s$	256 MB
26	13	4.2	$6.9 \cdot 10^4 / s$	$9.5 \cdot 10^1 s$	1 GB
28	14	4.8	$5.9 \cdot 10^4 / s$	$4.4 \cdot 10^2 s$	4 GB
30	15	5.3	$5.1 \cdot 10^4 / s$	$1.9 \cdot 10^3 s$	16 GB
32	16	6.3	$4.3 \cdot 10^4 / s$	$8.7 \cdot 10^3 s$	64 GB
34	17	7.2	$3.6 \cdot 10^4 / s$	$3.9 \cdot 10^4 s$	256 GB

Table 1: Benchmark of Feynman integral evaluations with different numbers of edges.

[M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

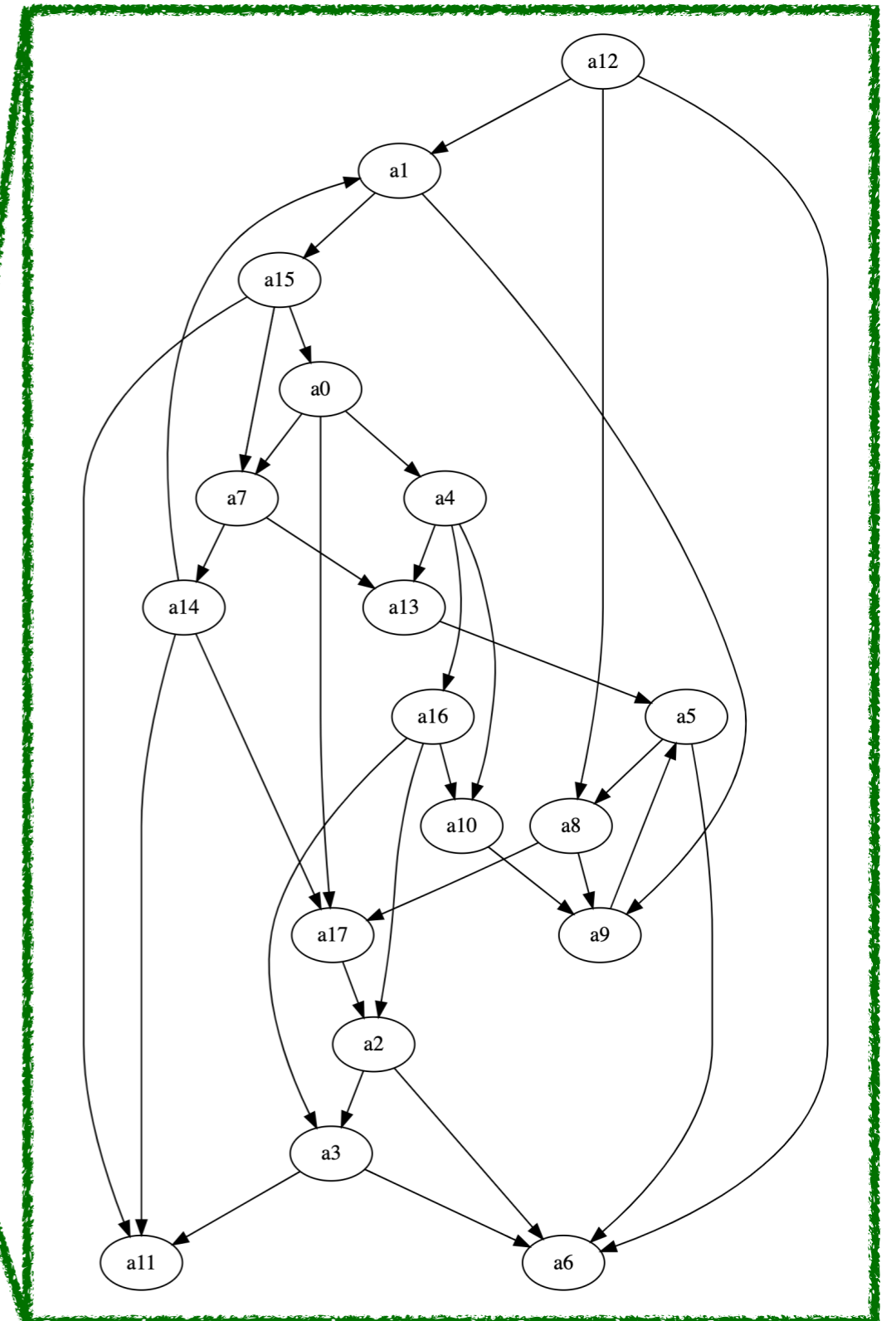
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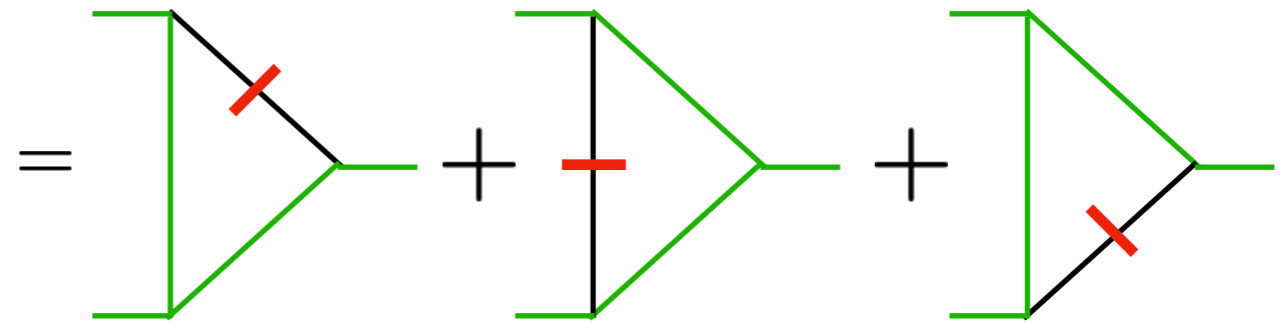
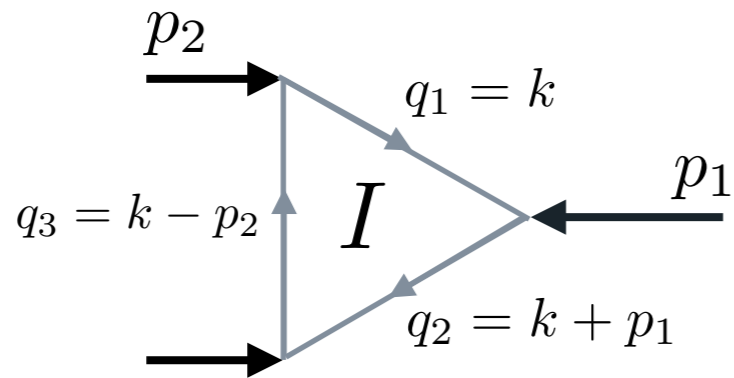


TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

[Inspired from and in collaboration with M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

[Investigation from Mathijs Fraije]

Recall:



TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

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Recall:

$$= \int d^4k \frac{1}{D_1 D_2 D_3} (D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)) \supset \int d^3\vec{k} \frac{1}{E_1 E_2 E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \stackrel{m=0}{=} |\vec{q}_i|$.

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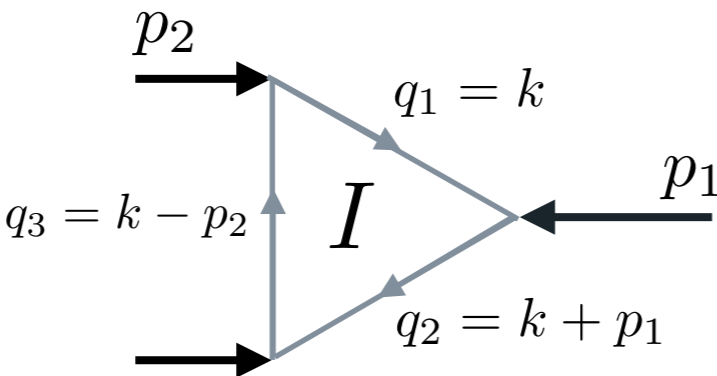
With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \stackrel{m=0}{=} |\vec{q}_i|$. The prefactor $\frac{1}{E_1 E_2 E_3}$ contains point-like integrable singularities :

TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

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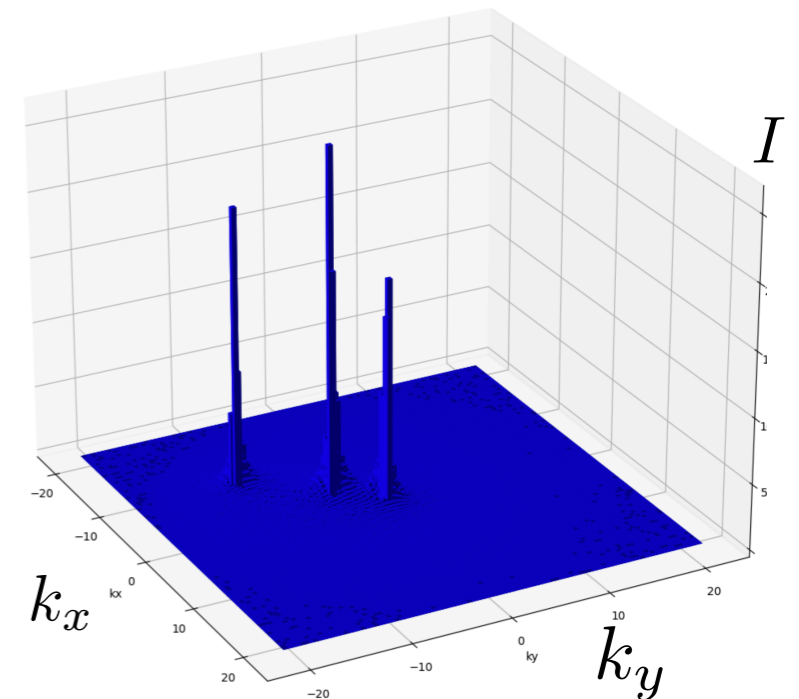
$$= \text{[Three diagrams with red crosses on different internal lines, representing tropical sampling terms.]}$$

$$= \int d^4k \frac{1}{D_1 D_2 D_3} (D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)) \supset \int d^3\vec{k} \frac{1}{E_1 E_2 E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \stackrel{m=0}{=} |\vec{q}_i|$. The prefactor $\frac{1}{E_1 E_2 E_3}$ contains point-like integrable singularities :

$$\frac{1}{E_1 E_2 E_3} = \frac{1}{|\vec{k}| |\vec{k} + \vec{p}_1| |\vec{k} - \vec{p}_2|}$$

2D example: 

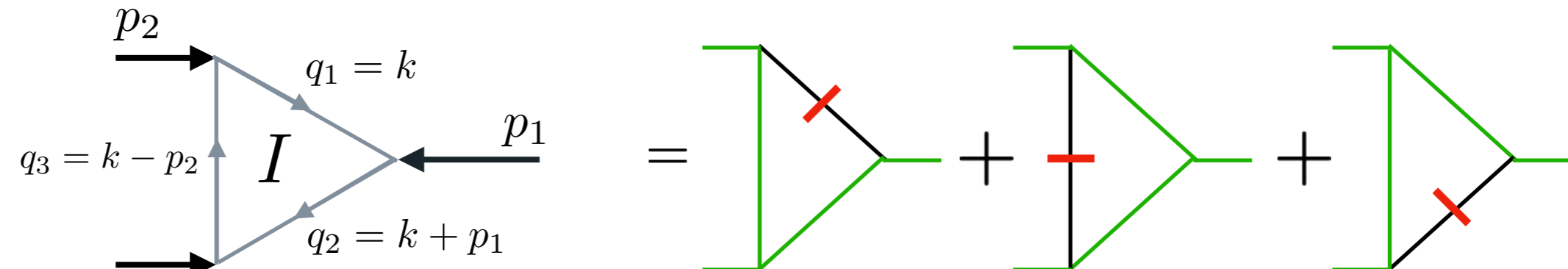


TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

[Inspired from and in collaboration with M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

[Investigation from Mathijs Fraije]

Recall:

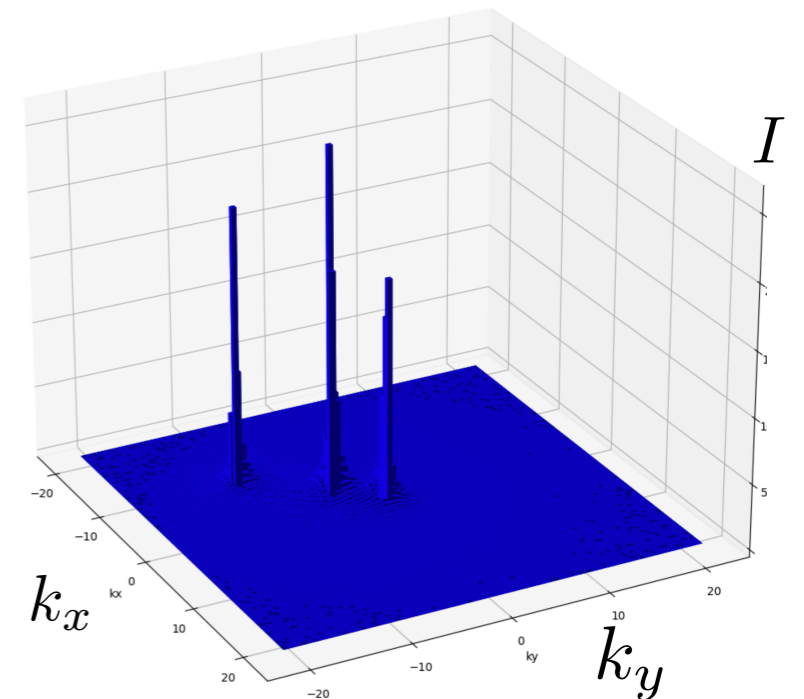


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2D example: 



[Originates from $\frac{1}{(k^0 - E(\vec{k})) (k^0 + E(\vec{k}))} = \frac{1}{2E(\vec{k})} \left(\frac{1}{E(\vec{k}) - k^0} + \frac{1}{E(\vec{k}) + k^0} \right)]$

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$$I^{(\text{LTD})} \supset \int d^3 \vec{k} \frac{1}{|\vec{k}| |\vec{k} + \vec{p}_1| |\vec{k} - \vec{p}_2|} \frac{\mathcal{N}(\vec{k})}{((E_1 + E_2 - p_1^0)(E_1 + E_2 + p_1^0))}$$

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Multi-channeling is the canonical approach to flatten these three integrable singularities.

But it would be far better to build a single parameterisation whose Jacobian vanishes

simultaneously at *all three points* $\vec{k} = \{ \vec{0}, -\vec{p}_1, \vec{p}_2 \}$:

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simultaneously at all three points $\vec{k} = \{ \vec{0}, -\vec{p}_1, \vec{p}_2 \}$:

$$= \int d^3 \vec{k} \frac{1}{D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} D_3^{\frac{1}{2}}} f(\vec{k})$$

with **euclidean** propagators :

$$D_1 = \vec{k} \cdot \vec{k}$$

$$D_2 = (\vec{k} + \vec{p}_1) \cdot (\vec{k} + \vec{p}_1)$$

$$D_3 = (\vec{k} - \vec{p}_2) \cdot (\vec{k} - \vec{p}_2)$$

Ready for being tropical-sampled !

$$= \int_{[0,1]^{(DL+2|E|+1)}} dz \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

TROPICAL SAMPLING IN MOMENTUM SPACE

$$I^{\text{eucl.}}[f] = \int d^{DL} \mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}}$$

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Easy to sample momenta from this PDF **exactly**

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Tropical sampling [M. Borinsky 2008.12310]

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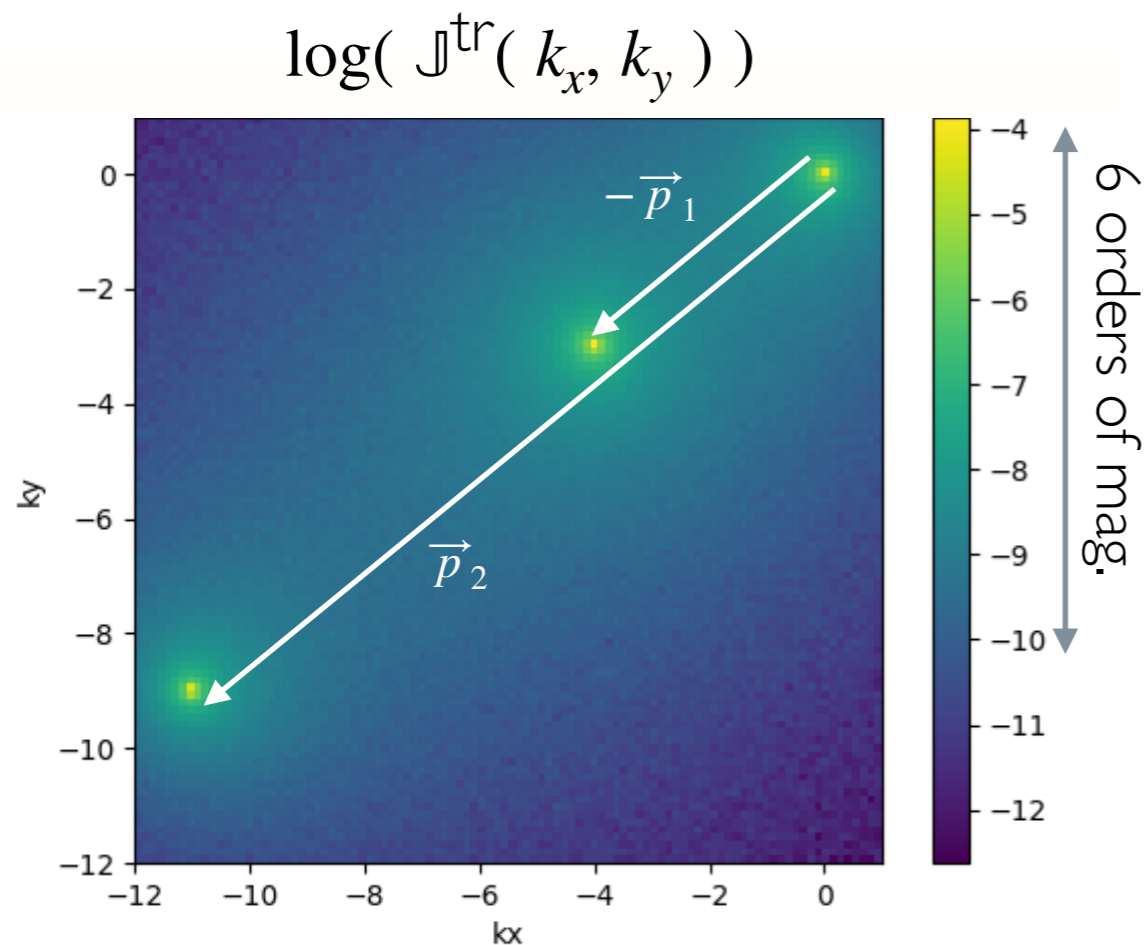
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- M. Fraije and M. Borinsky will soon publish a **Rust** implementation of the map: $\phi^{\text{tr}}[\Gamma](\mathbf{z}) \rightarrow (K(\mathbf{z}), \mathbf{k}(\mathbf{z}))$

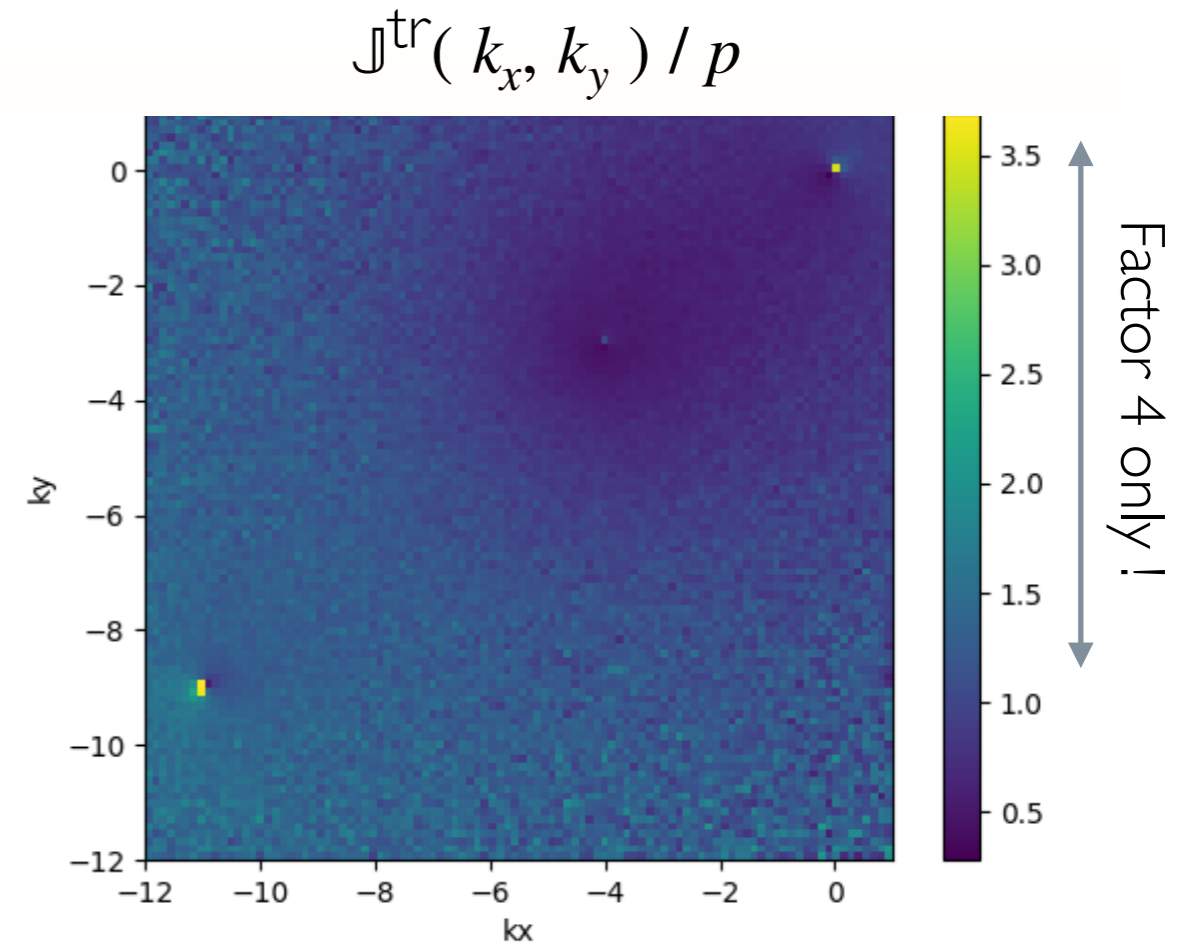
TROPICAL SAMPLING OF THE LTD TRIANGLE INTEGRAL

Let's apply this approach to our example.

In 2D with tropical sampling for removing blow ups from $p := \frac{1}{|\vec{k}||\vec{k} + \vec{p}_1||\vec{k} - \vec{p}_2|}$:



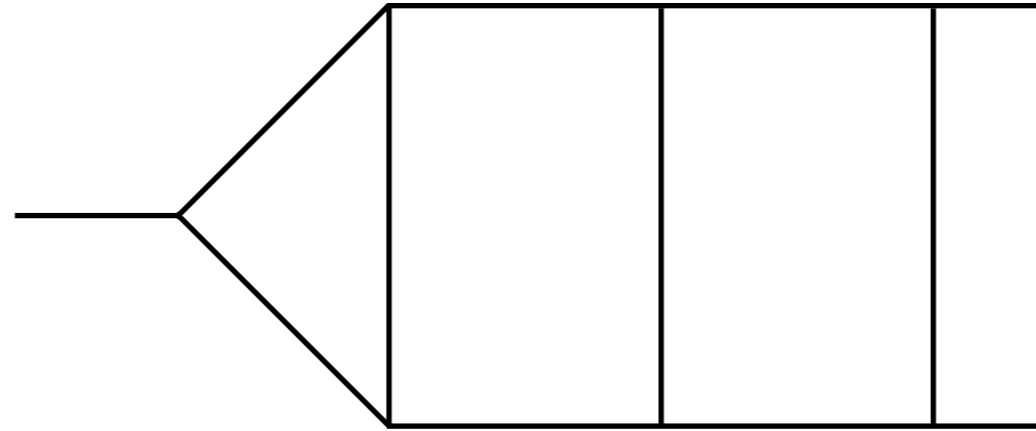
Log of tropical ~sampling density



Integrable singularities conquered !

- Many details of this approach omitted here. In our implementation: **arbitrary numerators** supported!

TROPICAL SAMPLING RESULTS

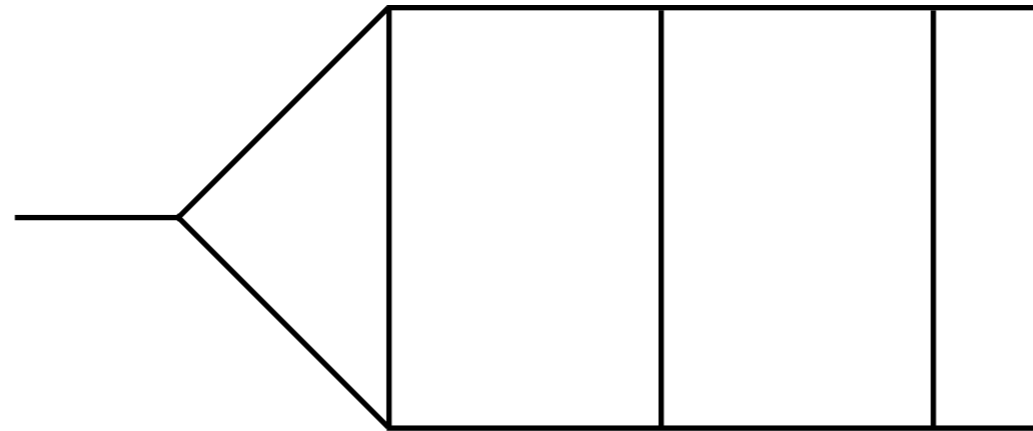


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/18$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	3.51(63)e-8 18%	
0.1M 0.3 s	3.78(24)e-8 6%	
1M 3 s	3.99(11)e-8 2.7%	
10M 30 s	4.045(36)e-8 0.9%	

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

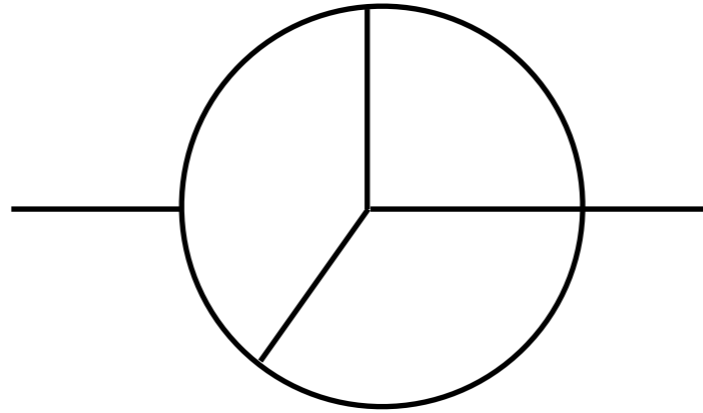


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/18$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	3.51(63)e-8 18%	4.050(35)e-8 0.9%
0.1M 0.3 s	3.78(24)e-8 6%	4.030(11)e-8 0.3%
1M 3 s	3.99(11)e-8 2.7%	4.0379(35)e-8 0.09%
10M 30 s	4.045(36)e-8 0.9%	4.0358(11)e-8 0.03%

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

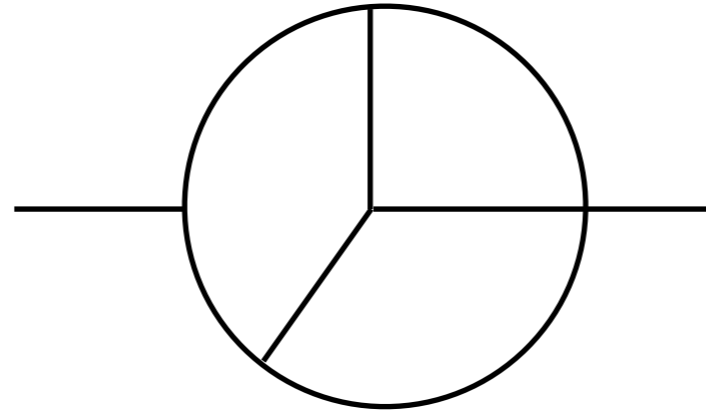


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/14$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	$8.9(2.9)e^{-7}$ 33%	
0.1M 0.3 s	$3.5(1.0)e^{-6}$ 29%	
1M 3 s	$5.6(1.2)e^{-6}$ 22%	
10M 30 s	$1.23(41)e^{-5}$ 34%	

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

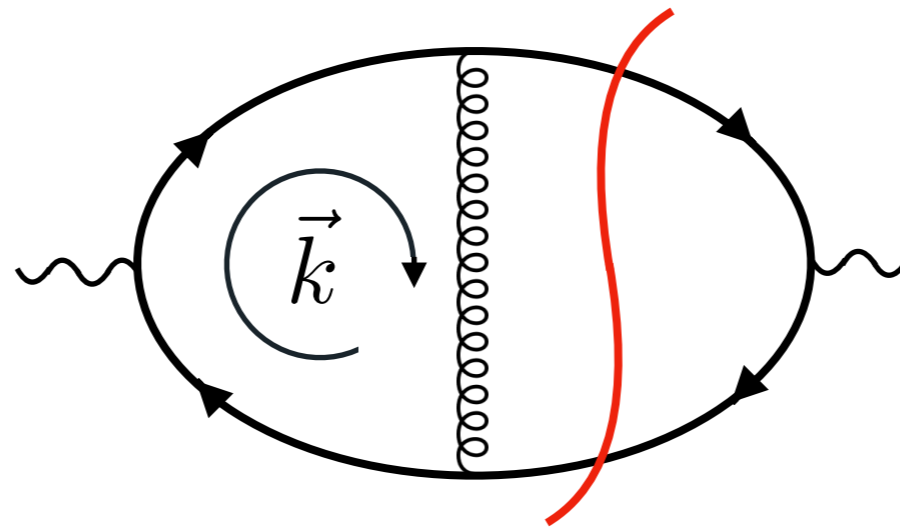


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/14$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	8.9(2.9)e-7 33%	9.403(60)e-6 0.6%
0.1M 0.3 s	3.5(1.0)e-6 29%	9.518(19)e-6 0.2%
1M 3 s	5.6(1.2)e-6 22%	9.4984(60)e-6 0.06%
10M 30 s	1.23(41)e-5 34%	9.4986(19)e-6 0.02%

Credits: Mathijs Fraaije

LOCALISED RENORMALISATION: BPHZ



$$\lim_{|\vec{k}| \rightarrow \infty} I(\text{Local Unitarity}) \rightarrow \infty$$

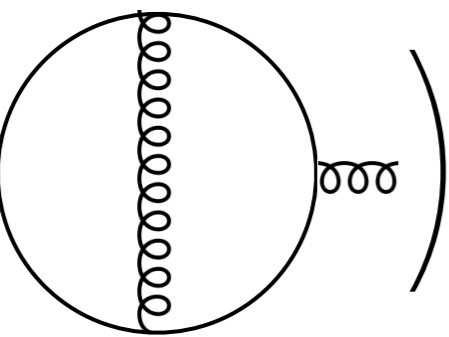
LOCALISED RENORMALISATION

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ [refs.](#)]

$$R(\Gamma) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

LOCALISED RENORMALISATION

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$$R \left(\Gamma = \text{---} \left(\text{---} \bigcirc \text{---} \right) \text{---} \right) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$


The diagram on the left shows a circle with a vertical chain of vertices (represented by small circles) inside it. The circle has external lines on the left and right sides, and the chain of vertices is connected to the circle's boundary. The diagram is enclosed in large parentheses.

LOCALISED RENORMALISATION

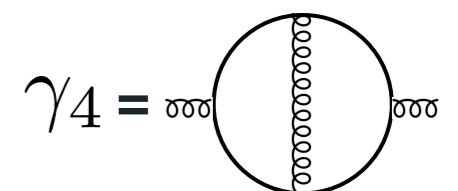
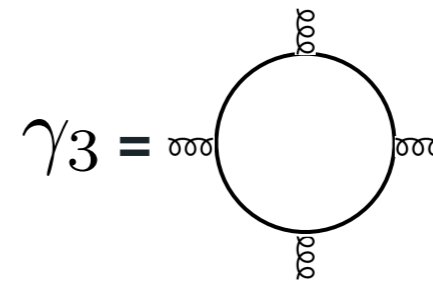
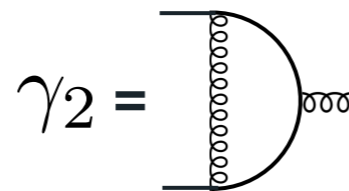
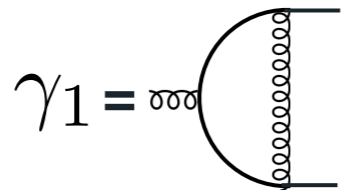
[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ [refs.](#)]

$$R \left(\Gamma = \text{circle with vertical wavy line} \right) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

UV subgraphs :

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LOCALISED RENORMALISATION

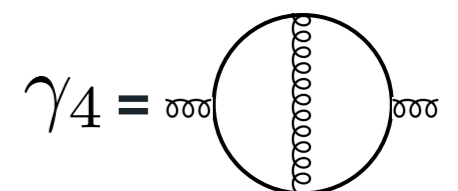
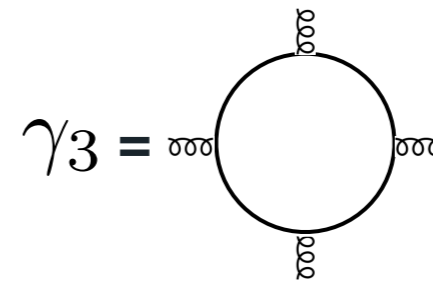
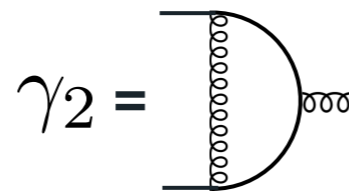
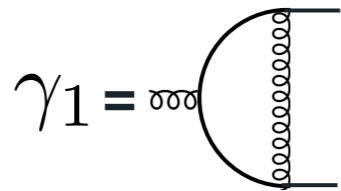
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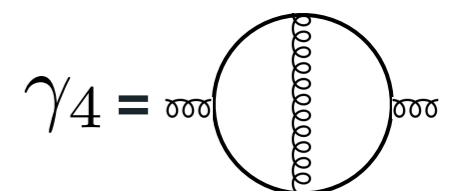
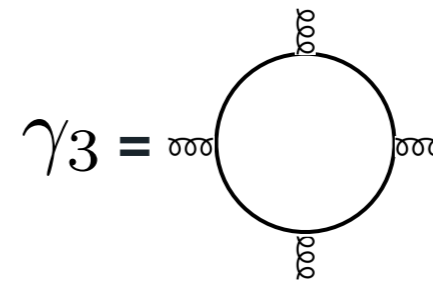
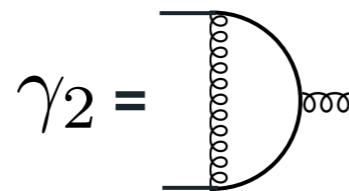
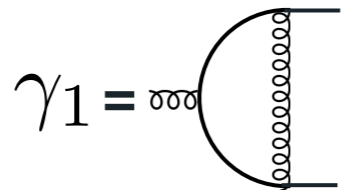
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What is the operator $K(\gamma)$? Anything we want ! so long as it:

- Locally cancels UV divergences of γ , even in the presence of nestings
- Yields results immediately renormalised in the chosen scheme ($\overline{\text{MS}} + \text{OS}$)
- Minimal analytics: at most single-scale all-massive vacuum integrals

LOCAL RENORMALISATION OPERATOR K

Our solution: $K(\gamma) := T(\gamma)$

$T(\gamma) :=$ **Local CT** : Taylor expansion around the “UV point” up to $\text{dod}(\gamma)$

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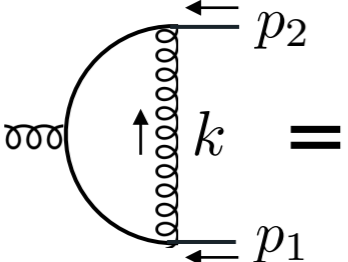
$$\gamma_1 = \text{diagram} = \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{((k - p_1)^2 - m^2)(k^2)((k + p_2)^2 - m^2)}$$

The diagram shows a loop with a vertical internal line. The left side of the loop is a semi-circle with three small circles on its outer edge. The top horizontal line has an arrow pointing left and is labeled p_2 . The bottom horizontal line has an arrow pointing left and is labeled p_1 . The vertical internal line has an arrow pointing up and is labeled k .

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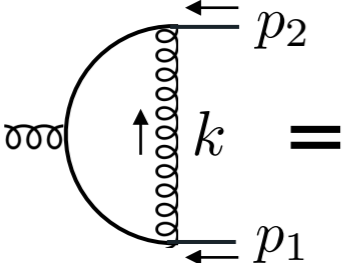
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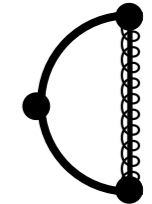
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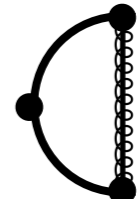
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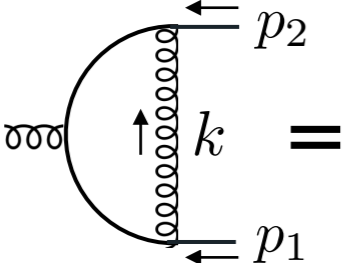
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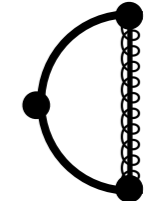
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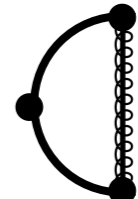
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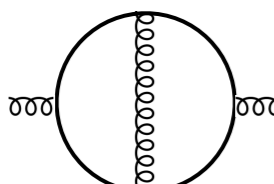
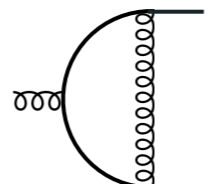
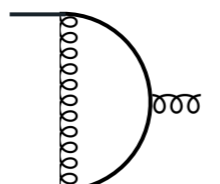
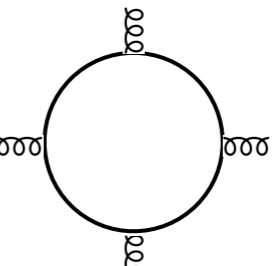
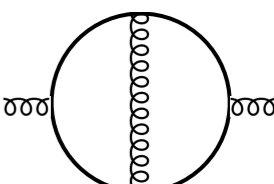
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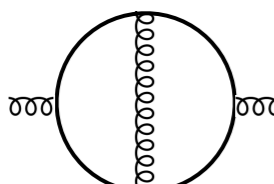
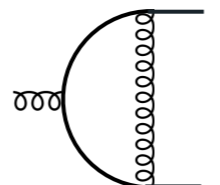
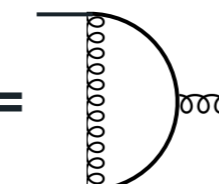
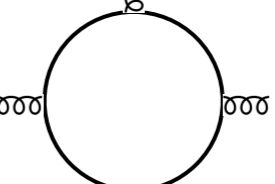
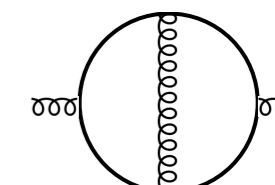
$$\delta^X(\gamma) := \text{Renormalisation CT in scheme X}, \quad (-[T] + \delta^{\overline{\text{MS}}}) := \bar{K}, \quad \bar{K}(\gamma_1) = \sum_{k=0}^{+\infty} \alpha_k \epsilon^k$$

R-OPERATOR UNFOLDING


$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$\begin{aligned}
 R(\Gamma) = & \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
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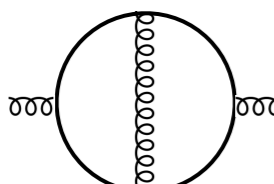
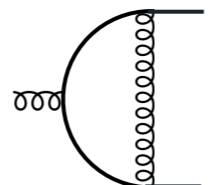
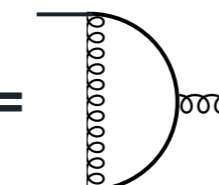
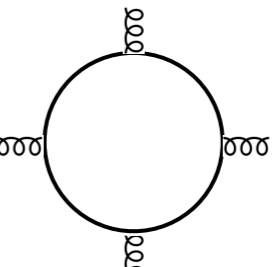
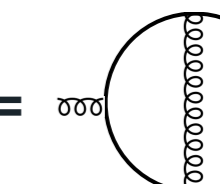
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Valentin Hirschi, U. Bern$$

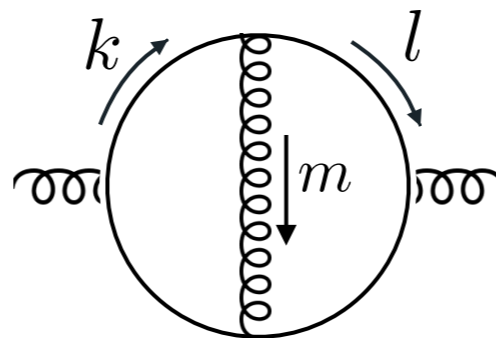
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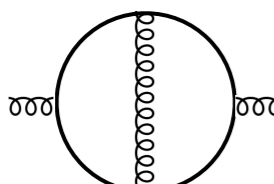
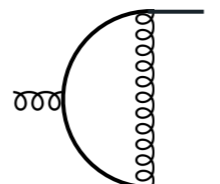
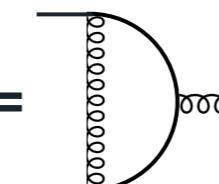
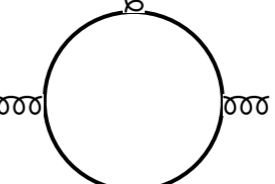
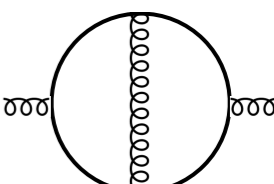
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The four different types of UV limits are now **finite** !$$



- $k, m \rightarrow \infty, l \text{ finite}$
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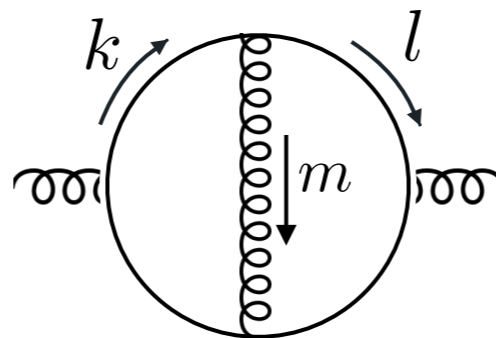
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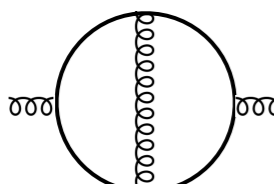
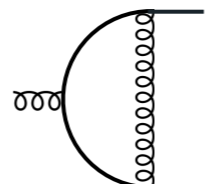
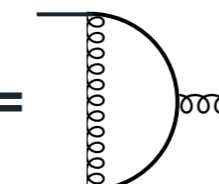
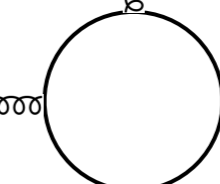
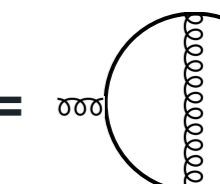
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- $k, l, m \rightarrow \infty$

R-OPERATOR UNFOLDING

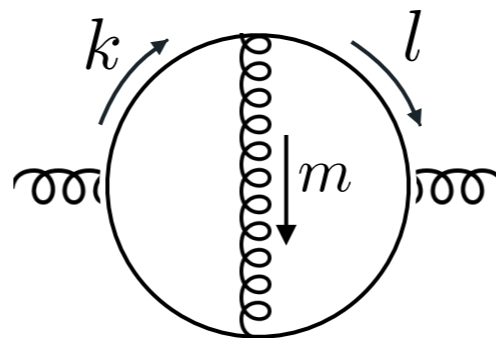
$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$\begin{aligned}
 R(\Gamma) = & \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
 & + T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}
 \end{aligned}$$

$$\begin{aligned}
 = & \text{~~Diagram 1~~} - \text{Diagram 2} - \text{Diagram 3} - \text{Diagram 4} - \text{Diagram 5} \\
 & + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \bar{K} \text{ terms}
 \end{aligned}$$

(Note: Diagram 1 is crossed out with a red line, and Diagram 2 is crossed out with a blue line.)

The four different types of UV limits are now **finite**!



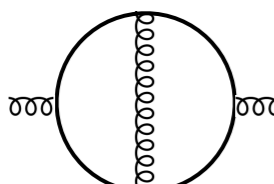
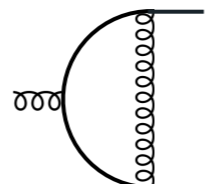
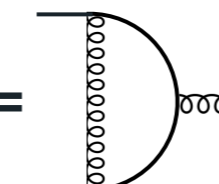
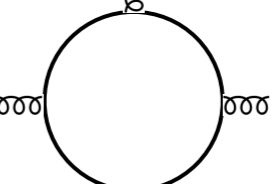
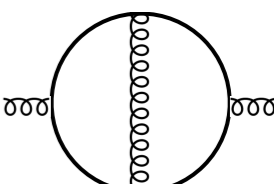
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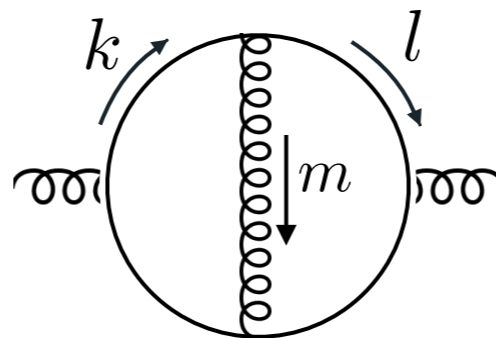
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 \end{aligned}$$

$$\begin{aligned}
 = & \text{~~Diagram of } \Gamma \text{ with a red diagonal line through it}~~ - \text{~~Diagram of } \Gamma \setminus \gamma_1 \text{ with a red diagonal line through it}~~ - \text{~~Diagram of } \Gamma \setminus \gamma_2 \text{ with a blue diagonal line through it}~~ - \text{~~Diagram of } \Gamma \setminus \gamma_3 \text{ with a green diagonal line through it}~~ - \text{~~Diagram of } \Gamma \setminus \gamma_4 \text{ with a red diagonal line through it}~~ \\
 & + \text{Diagram of } T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + \text{Diagram of } T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + \text{Diagram of } T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}
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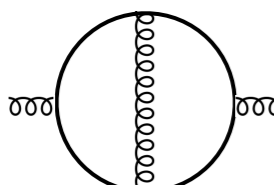
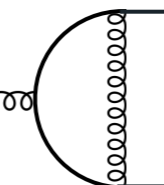
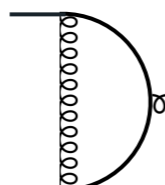
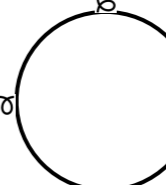
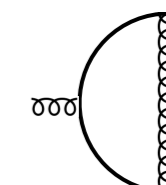
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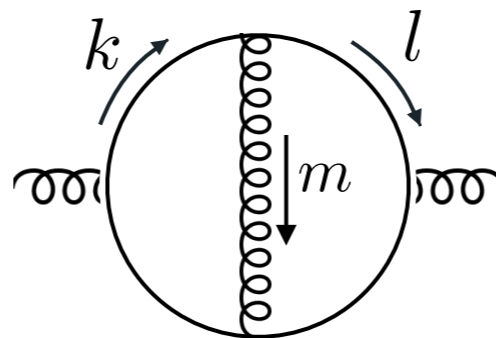
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 = & \text{

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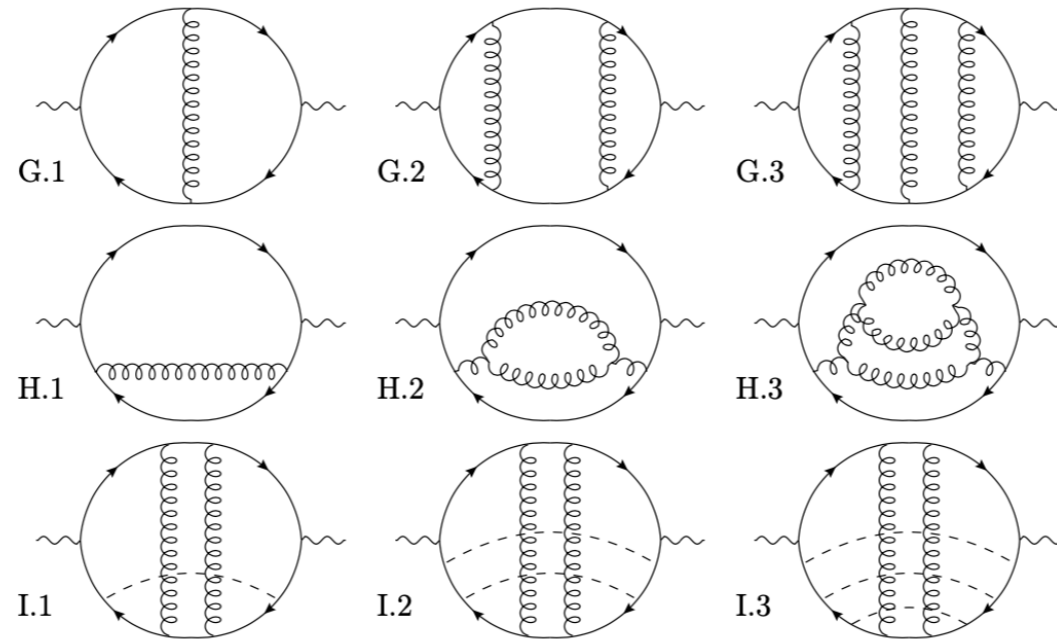
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$$k, l, m \rightarrow \infty$$

EFFICIENCY IN LU IMPLEMENTATION

α Loop \longrightarrow γ Loop



SG	proc.	order	t_{gen} [s]	M_{disk} [MB]	N_{sg} [-]	N_{cuts} [-]	t_{eval} [ms]	$t_{\text{eval}}^{(\text{f128})}$ [ms]
G.1	1 \rightarrow 2	NLO	0.1	0.13	2	4	0.004	0.13
G.2	1 \rightarrow 2	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	1 \rightarrow 2	N3LO	36K	509	220	16	17.6	281
H.1	1 \rightarrow 2	NLO	0.07	0.12	2	2	0.006	0.14
H.2	1 \rightarrow 2	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	1 \rightarrow 2	N3LO	255	43	220	4	2.35	56
I.1	1 \rightarrow 3	NNLO	126	22	266	9	0.32	12.4
I.2	1 \rightarrow 4	NNLO	1.9K	120	4492	9	4.4	67
I.3	1 \rightarrow 5	NNLO	36K	20K	$\mathcal{O}(100\text{K})$	9	3.6K	17.3K

NB: most recent version of α Loop does better, but scaling is similar

NUMERICAL GAMMA CHAINS WITH SPENSO.RS

spenso.rs

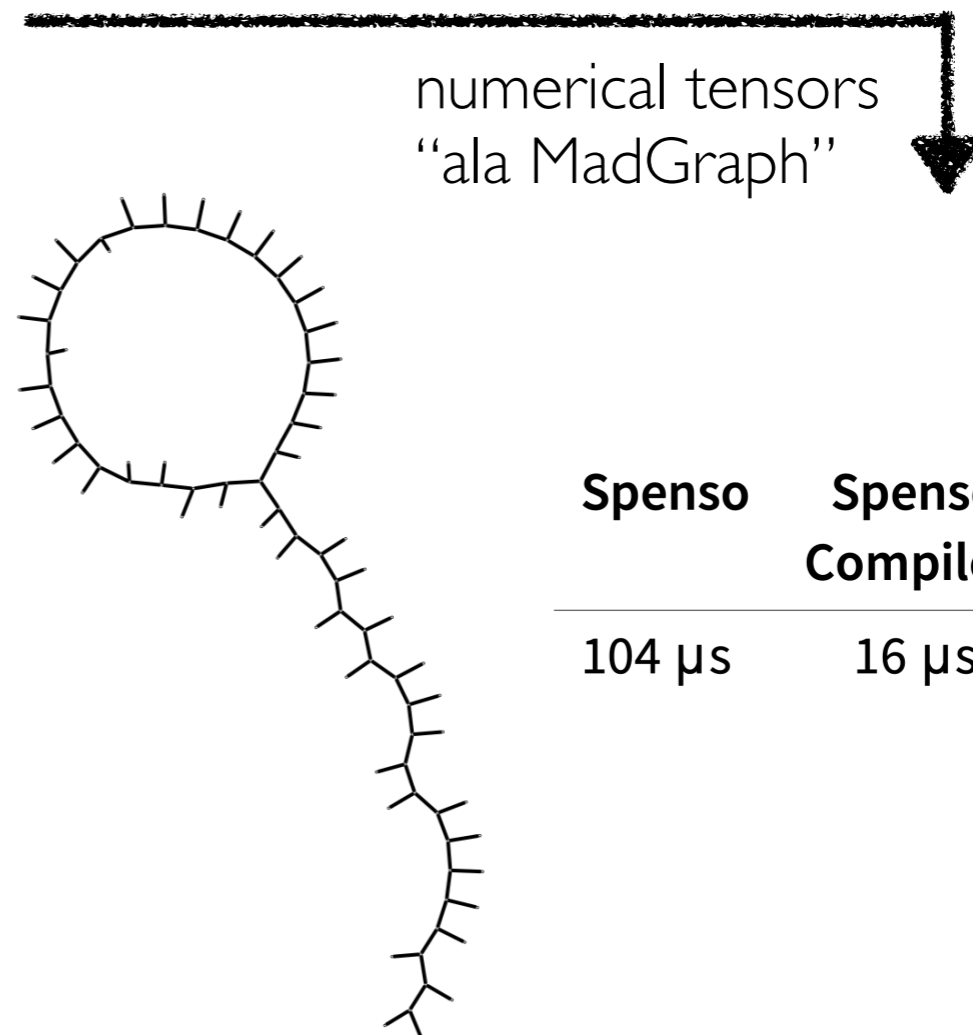
[Lucien Huber : crates.io/crates/spenso]

$$p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} \text{Tr}(\gamma_\mu \gamma_{\mu_1} \gamma_\nu \gamma_{\mu_2} \gamma_\rho \gamma_{\mu_3})$$

↓ γ -algebra

$$\begin{aligned} & -4(p_1 \cdot p_2) p_{3,\mu} \eta_{\nu\rho} \\ & + 4(p_1 \cdot p_2) p_{3,\nu} \eta_{\mu\rho} \\ & - 4(p_1 \cdot p_2) p_{3,\rho} \eta_{\mu\nu} \\ & + 4(p_1 \cdot p_3) p_{2,\mu} \eta_{\nu\rho} \\ & - 4(p_1 \cdot p_3) p_{2,\nu} \eta_{\mu\rho} \\ & - 4(p_1 \cdot p_3) p_{2,\rho} \eta_{\mu\nu} \\ & - 4p_{1,\mu} (p_2 \cdot p_3) \eta_{\nu\rho} \\ & + 4p_{1,\mu} p_{2,\nu} p_{3,\rho} \\ & + 4p_{1,\mu} p_{2,\rho} p_{3,\nu} \\ & - 4p_{1,\nu} (p_2 \cdot p_3) \eta_{\mu\rho} \end{aligned}$$

necessary for d-dimensions
but scales badly for $d = 4$



Spenso	Spenso Compiled	Hardcode Fortran
104 μs	16 μs	31 μs

SYMBOLICA

BEN RUIJL'S SUCCESSOR TO FORM

The screenshot shows the Symbolica website homepage. At the top, there is a navigation bar with links for 'Symbolica', 'Documentation', 'Pricing', 'Blog', and 'About'. The main header features the word 'Symbolica' in a large, teal font, with the tagline 'A blazing fast computer algebra system' below it. A list of features includes: 'Born of a need for advanced computations for CERN', 'State-of-the art algorithms and novel features', and 'Easily integrates into your existing and new projects'. A prominent teal button says 'Try our live demo!'. The background is dark with various mathematical symbols like \int , i , 9 , ∞ , Σ , σ , π , and ε scattered across it.

```
1 x, x_ = Expression.symbols('x', 'x_')
2 f = Expression.symbol('f')
3 e = f(x**2)
4 r = e.replace_all(f(x_), f(x_ + 1, 3))
5 display(r)
```

$$f(x^2 + 1, 3)$$

Try it for yourself in this [colab notebook](#) !

SUMMARY - POSSIBLE OVERLAP WITH LSS

Formalism :

- **Local** cancellation of all **final-state IR singularities**
- Completely **generic** (masses, kinematics, topologies, observables...)
- New **theoretical perspectives** on perturbative expansions
- **Automated renormalisation** with minimal analytical computations
- **Falsehood** : “Analytical = solved && Numerical = black box”

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Outlook :

- Rewrite **LU** impl. **cleaner and more efficient**: $\alpha\text{Loop} \rightarrow \gamma\text{Loop}$
- Apply **LU** to **new unknown corrections** : e.g. $b\bar{b}$ N³LO AFB or decays
- Applications in finite **T** and finite μ pQFT : see [Navarrete & al.: [2403.02180](#)]
- Generalise **LU** so as to apply of **initial-state singularities** as well
- Match **LU** to some form of **numerical resummation** (PSMC).
- Method likely well-suited for deployment on GPU.



BACK-UP SLIDES

LOCAL UNITARITY: X-SEC RESULTS

NUMERICAL RESULTS FOR $e^+e^- \rightarrow \gamma^* \rightarrow d\bar{d}$ @ NLO

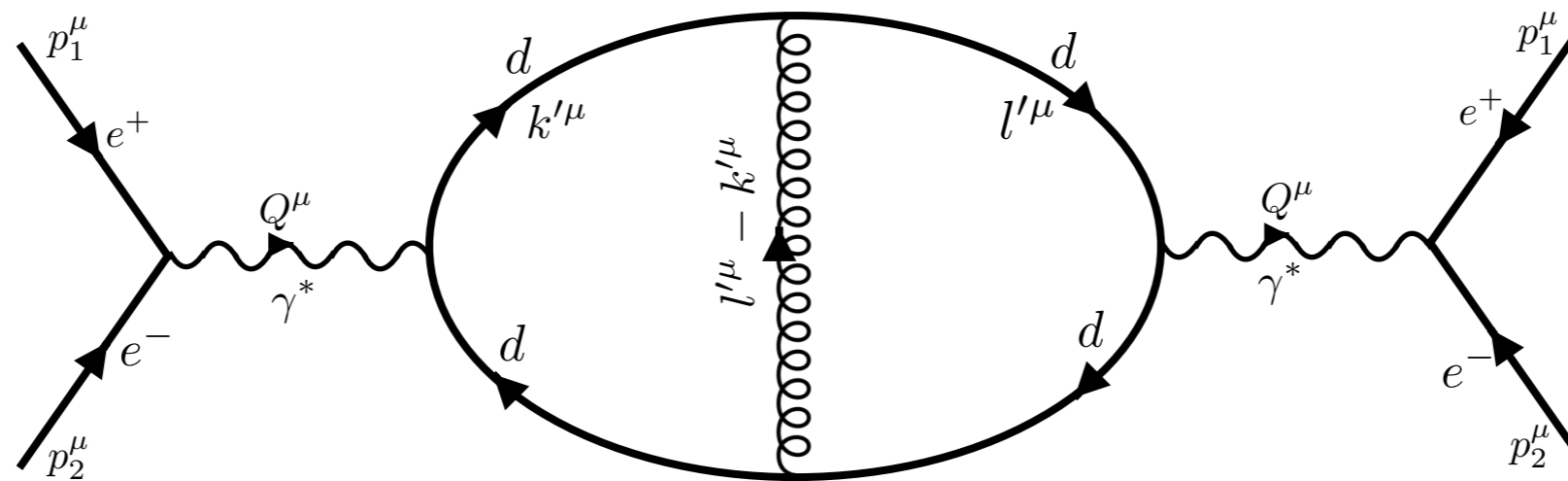
$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{tree} + \text{loop} + 2 \times \text{box} \right]$$

The equation shows the numerical results for the cross-section $\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})}$ at NLO. It is expressed as a sum of three terms enclosed in large red square brackets, all multiplied by the text "LU" in red. The first term is a tree-level diagram: a circle with two wavy external lines and two arrows on the circle. The second term is a loop-level diagram: a circle with two wavy external lines, two arrows on the circle, and a vertical dashed line representing a gluon loop. The third term is a box-level diagram: a circle with two wavy external lines, two arrows on the circle, and a small loop on top of the circle.

NUMERICAL RESULTS FOR $e^+e^- \rightarrow \gamma^* \rightarrow d\bar{d}$ @ NLO

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Loop 1} + \text{Loop 2} + 2 \times \text{Loop 3} \right]$$

Visualisation of the LU integrand for the Double-Triangle supergraph and :



$$p_1^\mu = (1, 0, 0, 1)$$

$$0.4 < p_{t,j_1} < 0.8$$

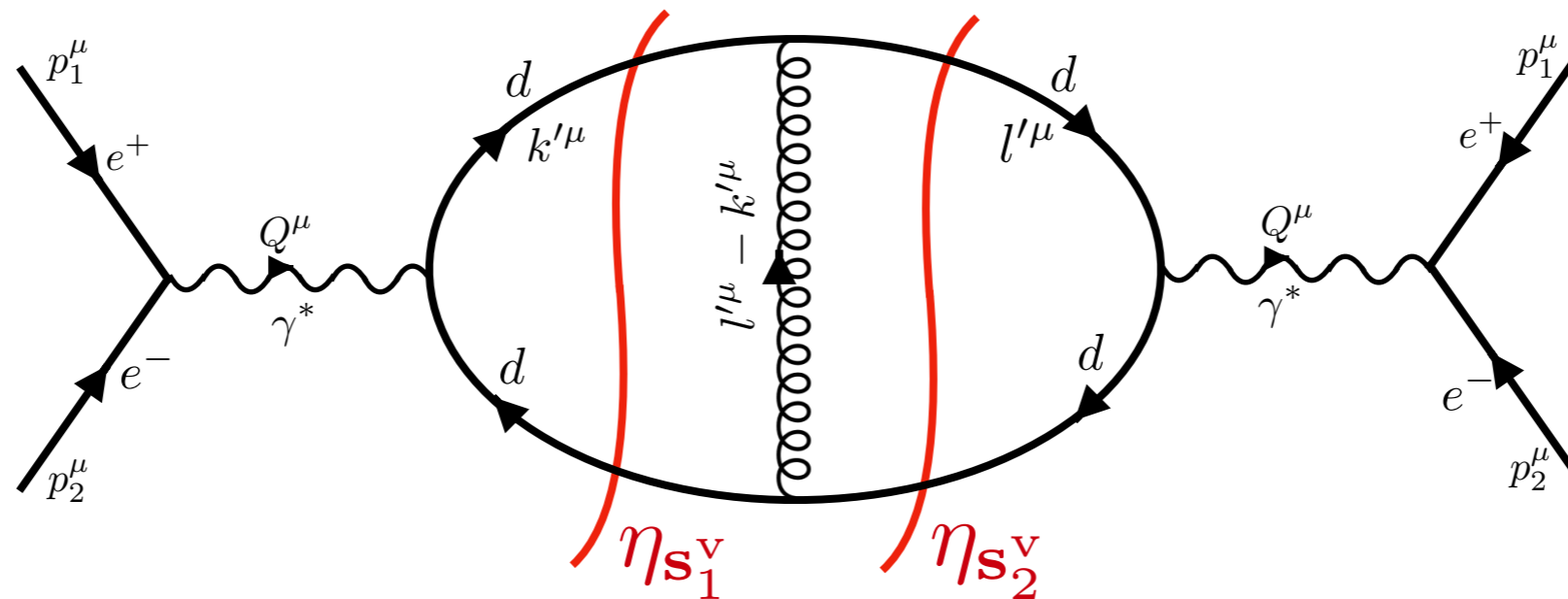
$$p_2^\mu = (1, 0, 0, -1)$$

$$(\vec{k}, \vec{l}) = \left(\left(0, k_y, \frac{1}{\sqrt{2}} \right), \left(0, \frac{1}{\sqrt{2}}, l_z \right) \right)$$

NUMERICAL RESULTS FOR $e^+e^- \rightarrow \gamma^* \rightarrow d\bar{d}$ @ NLO

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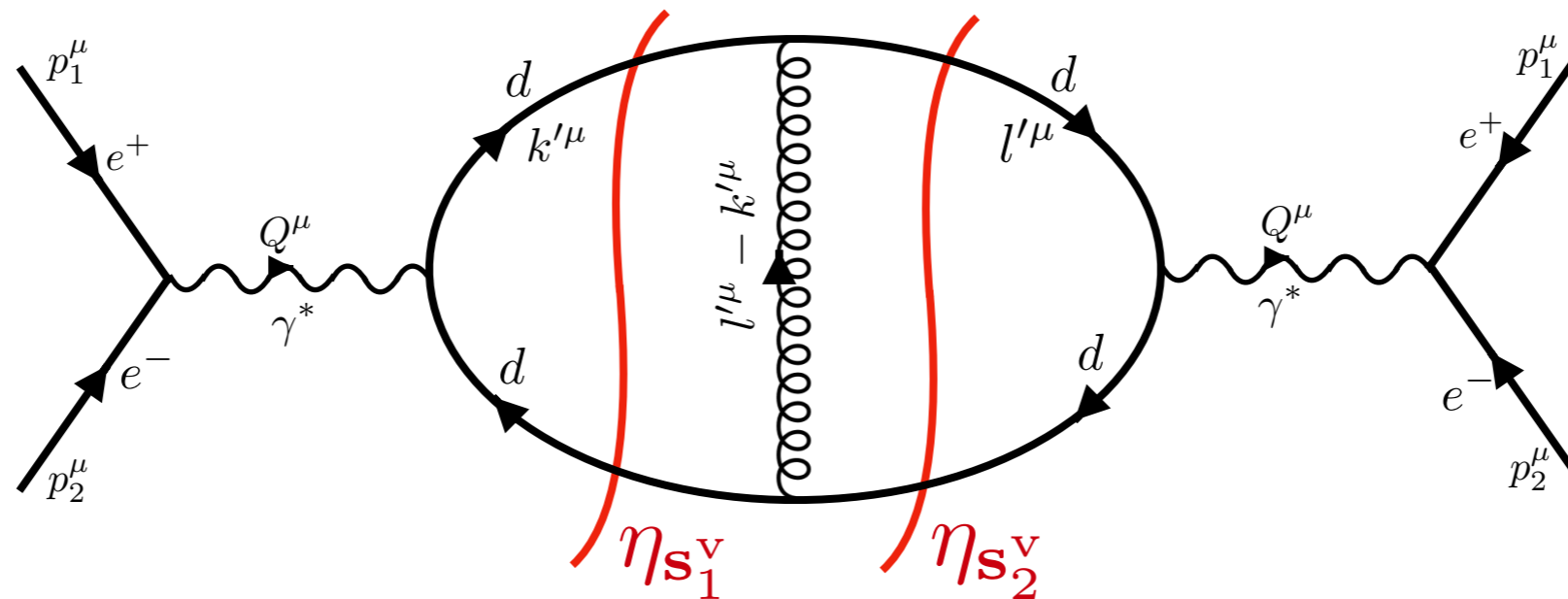
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Cancellation of non-pinched E-surfaces for : $\eta_{S_1^v} = \eta_{S_2^v} \rightarrow k'_y = l'_z$

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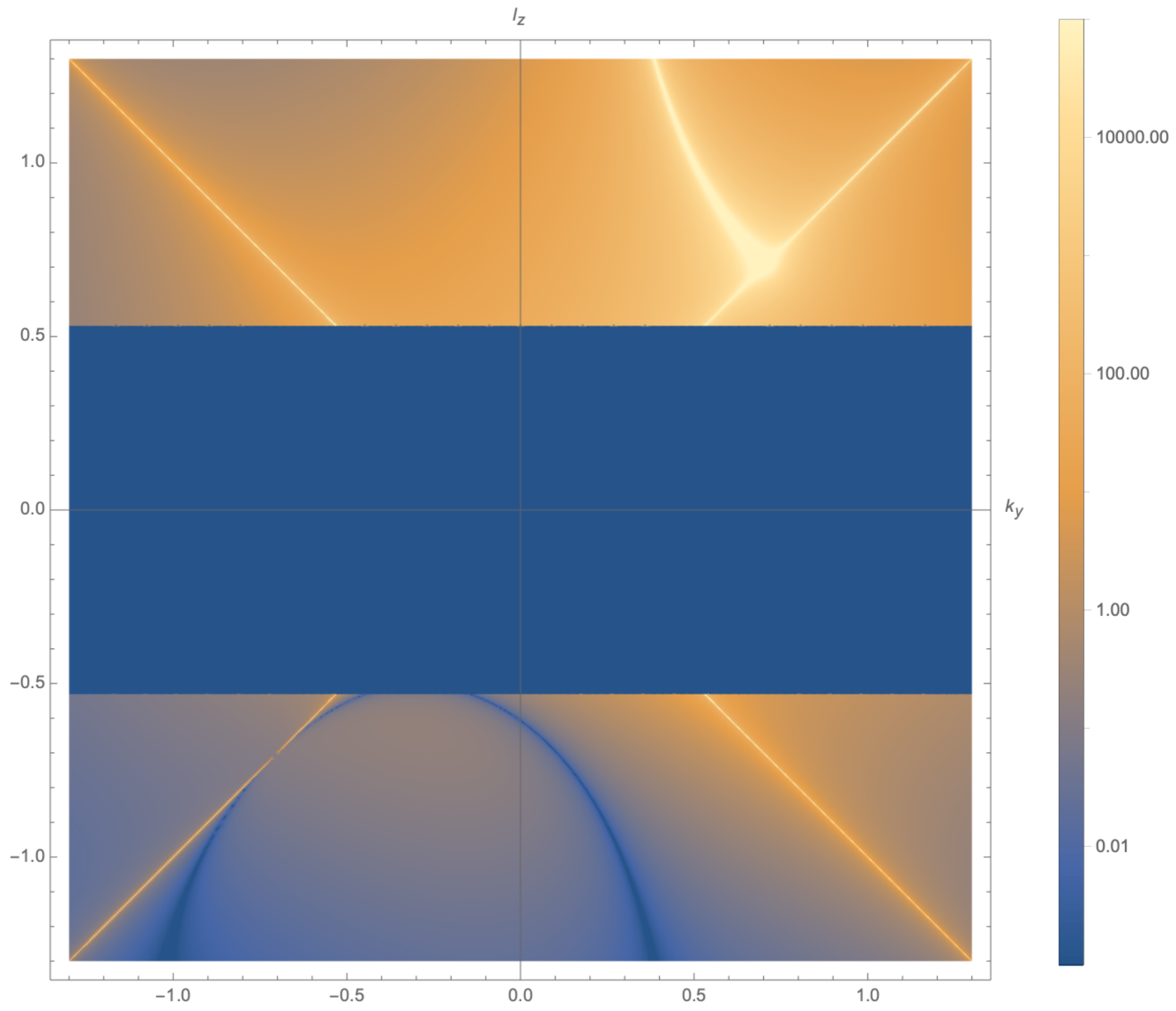
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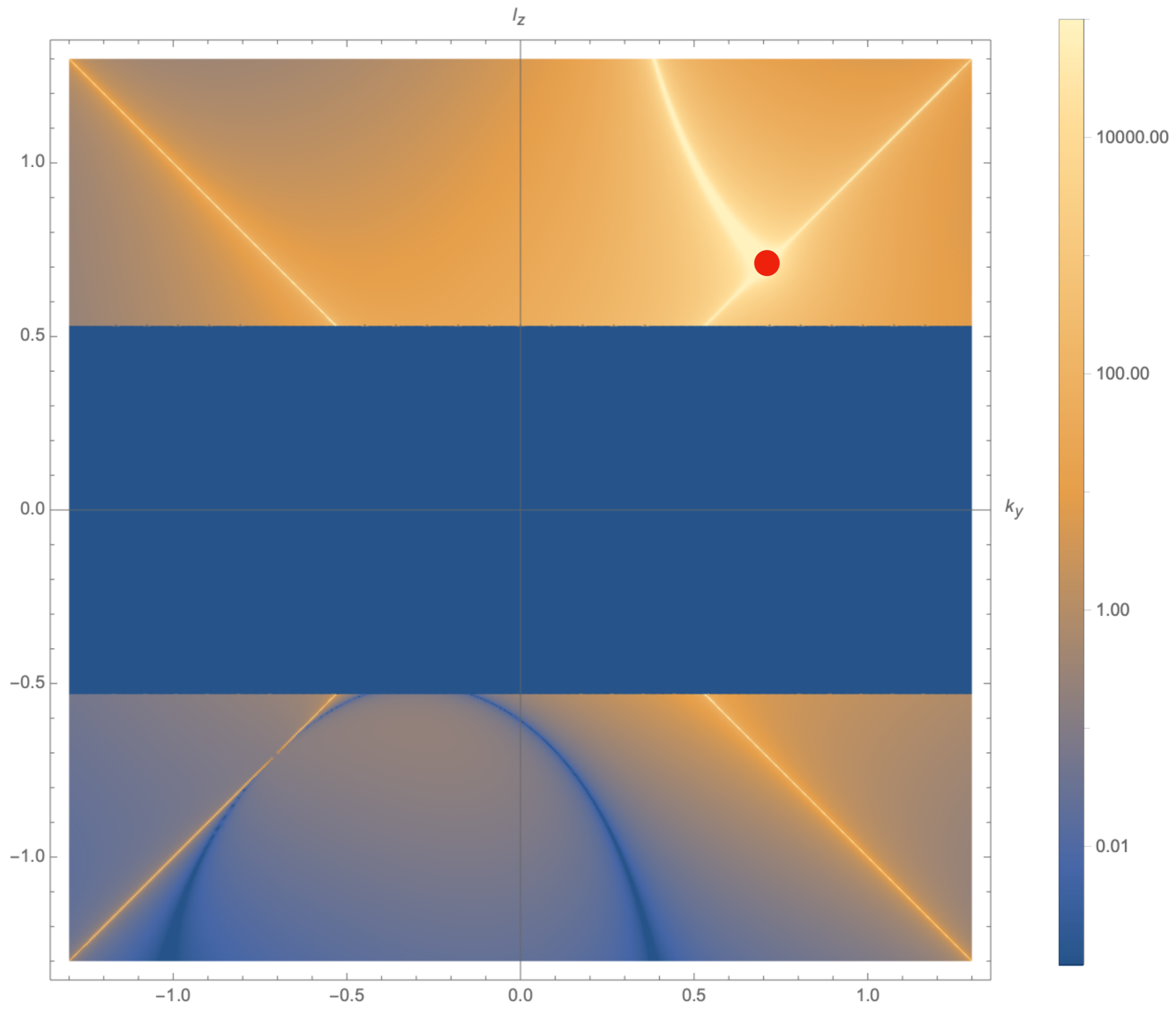
Soft configuration for : $|\vec{l}' - \vec{k}| = 0 \rightarrow k'_y = l'_z = \frac{1}{\sqrt{2}}$

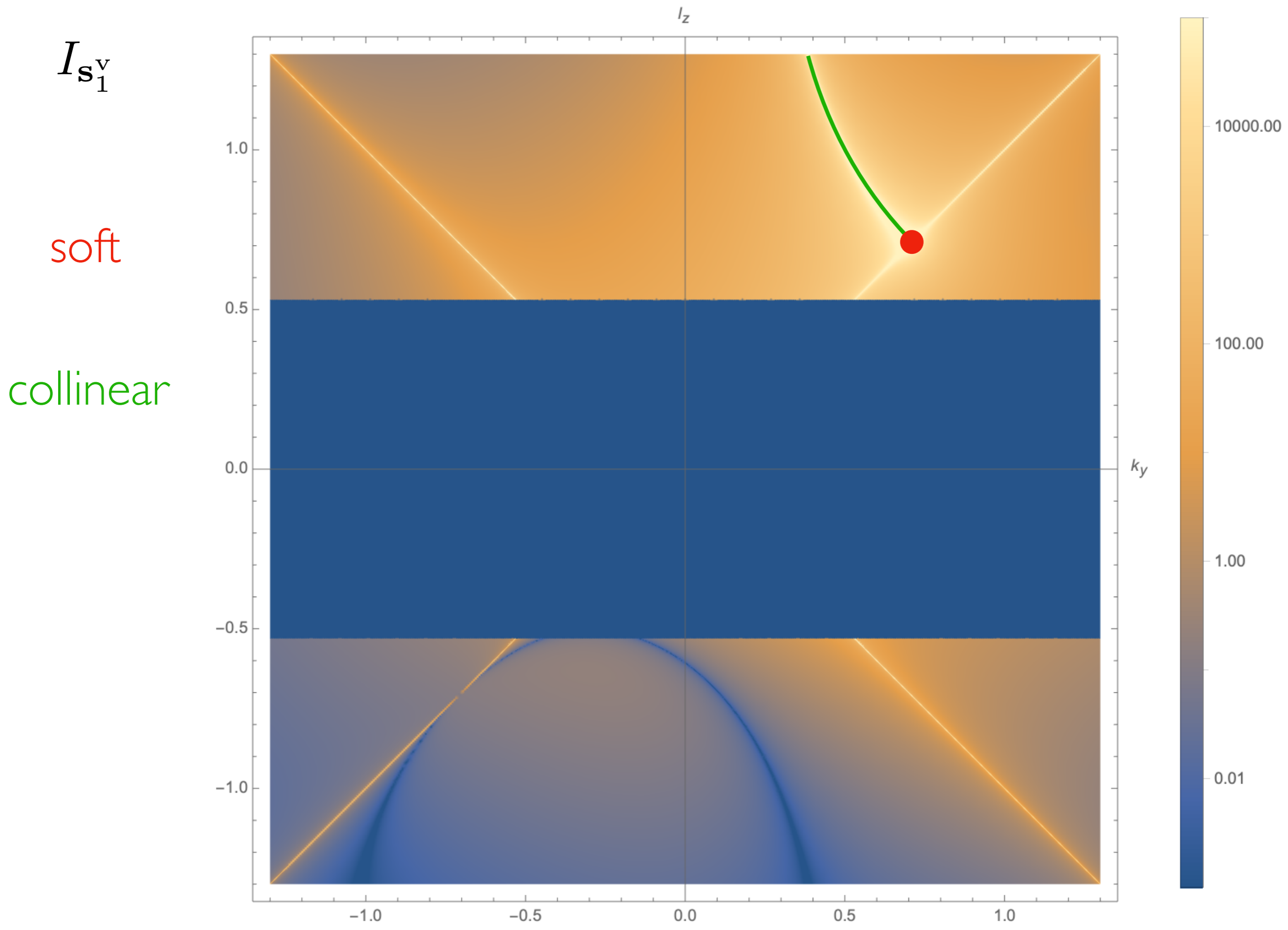
$I_{s_1^v}$

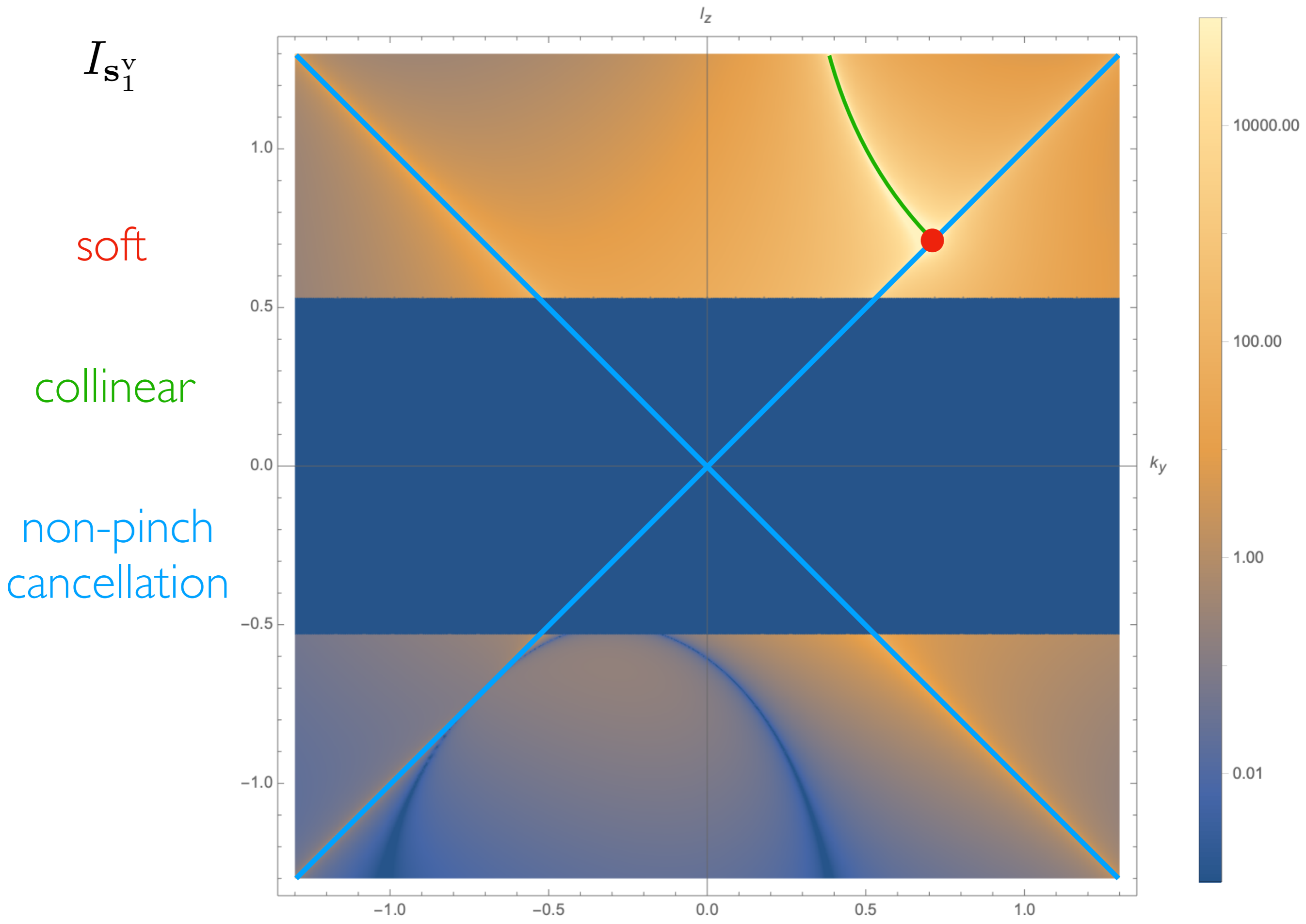


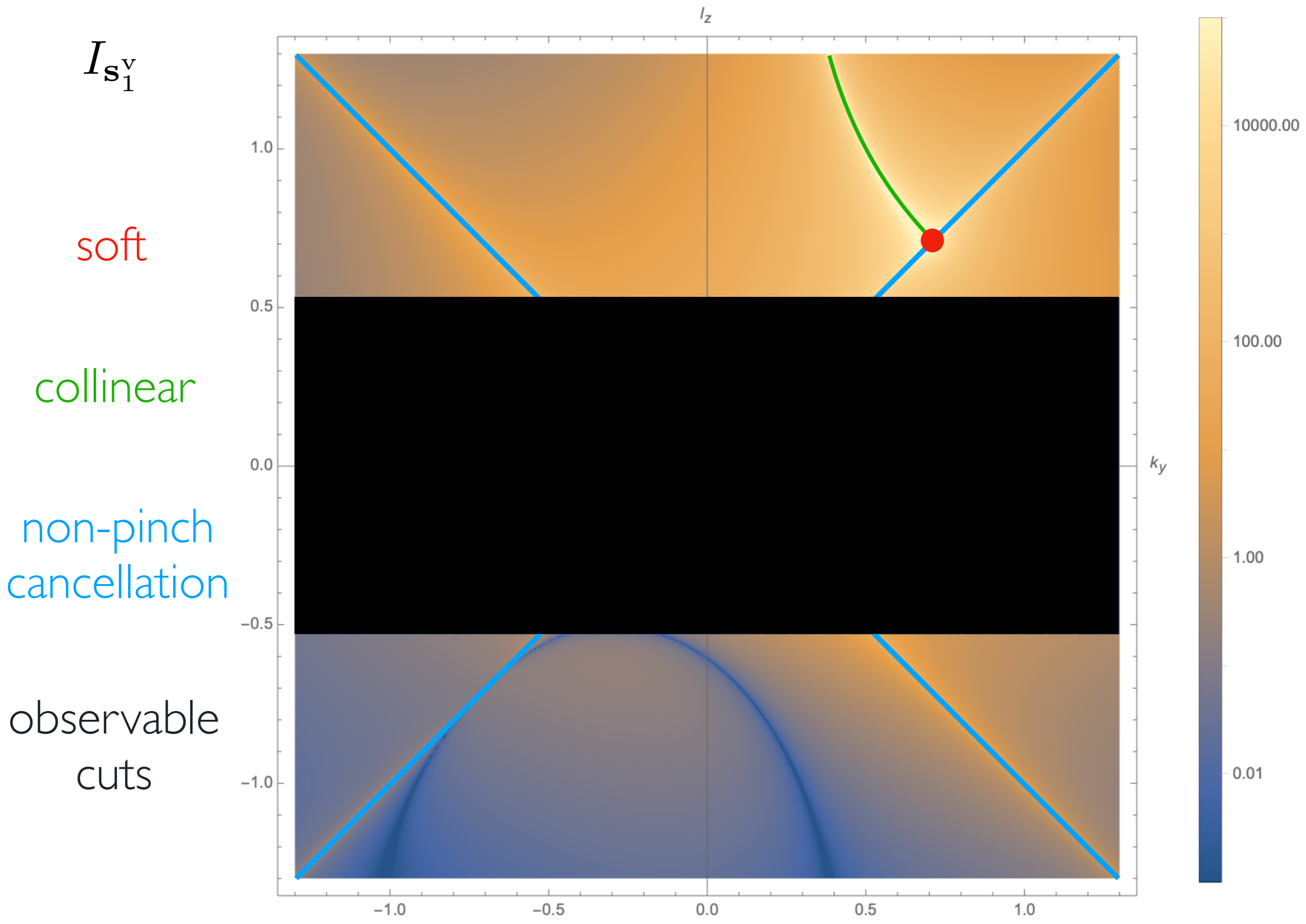
$I_{s_1^v}$

soft

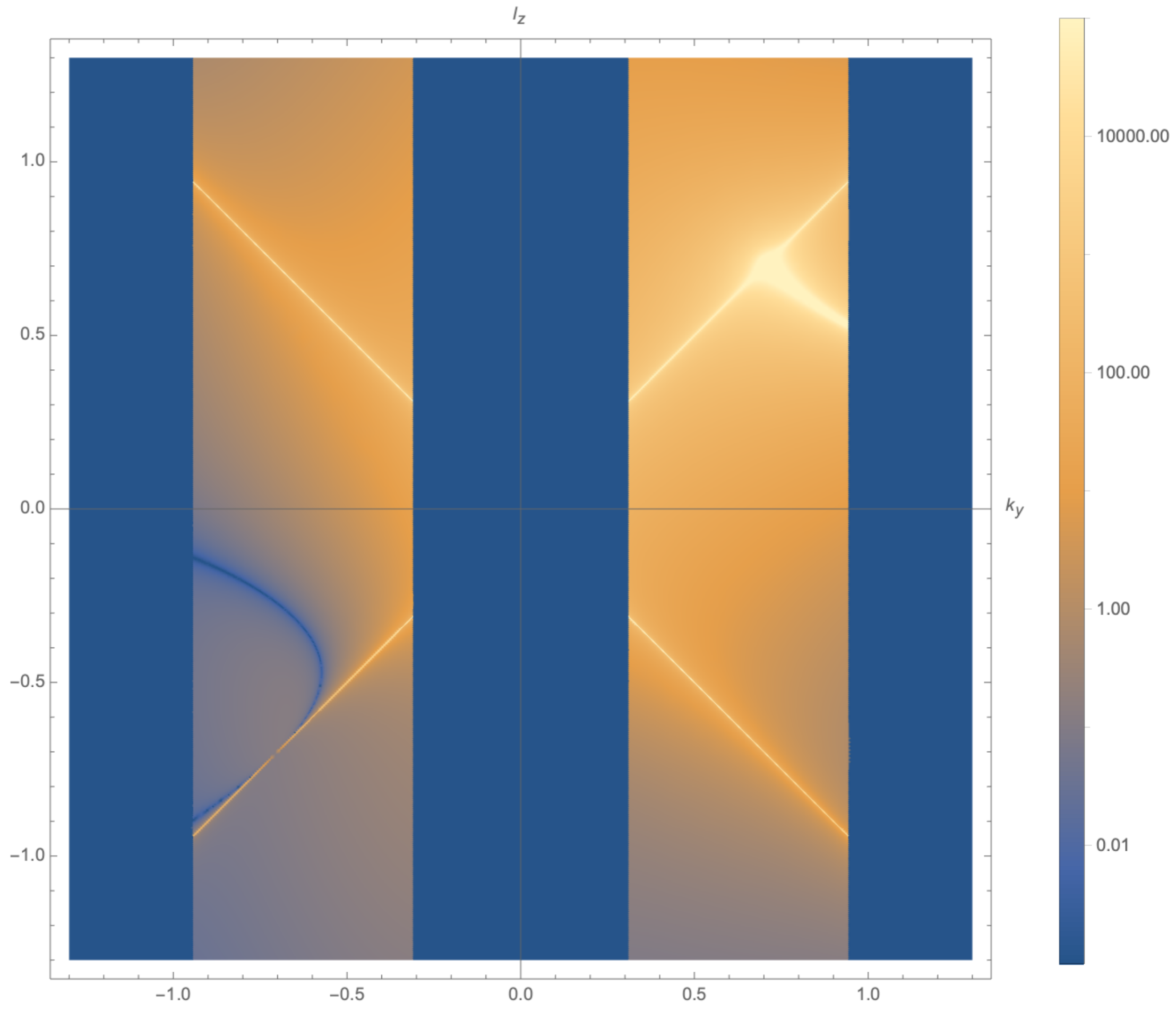




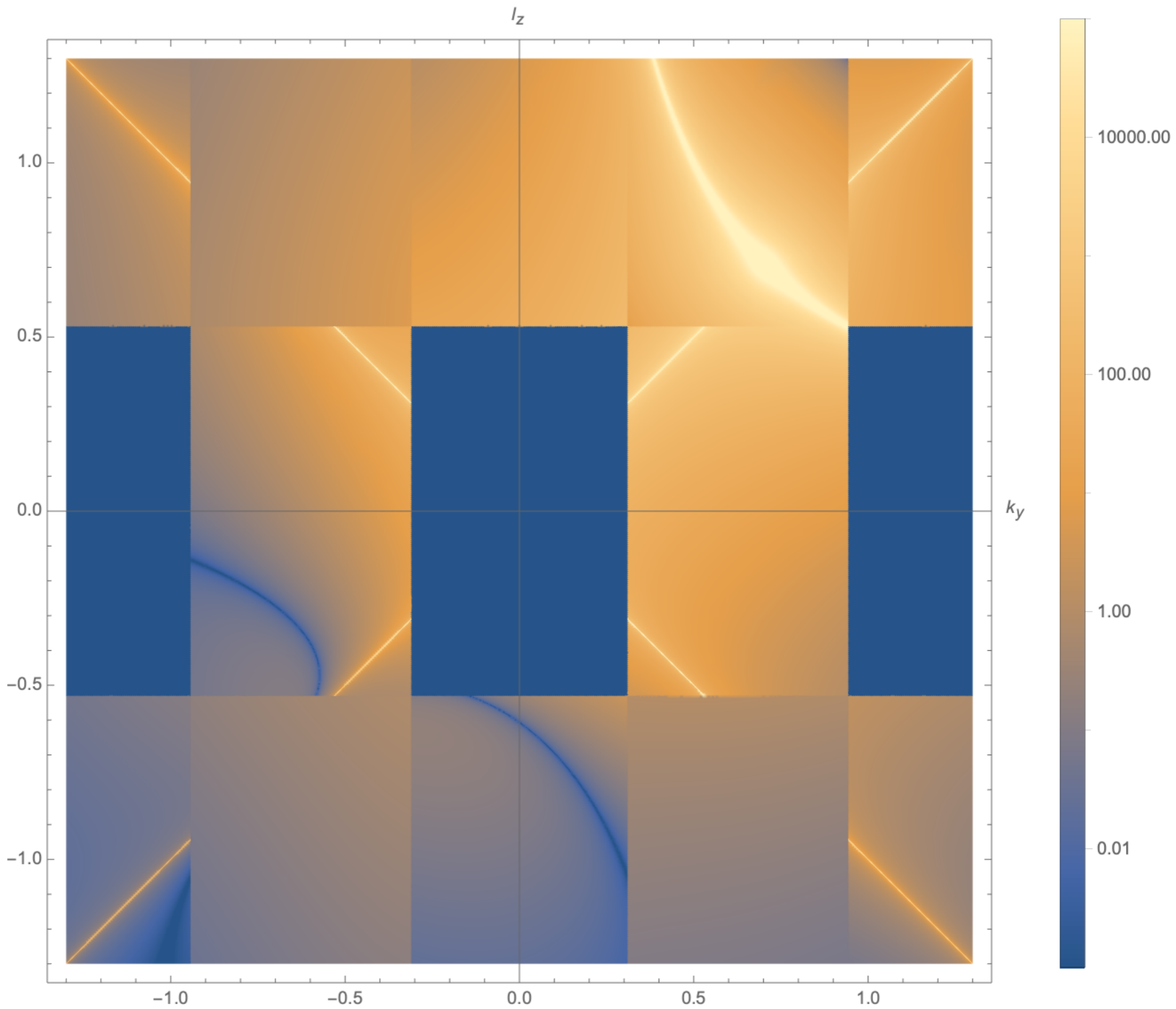




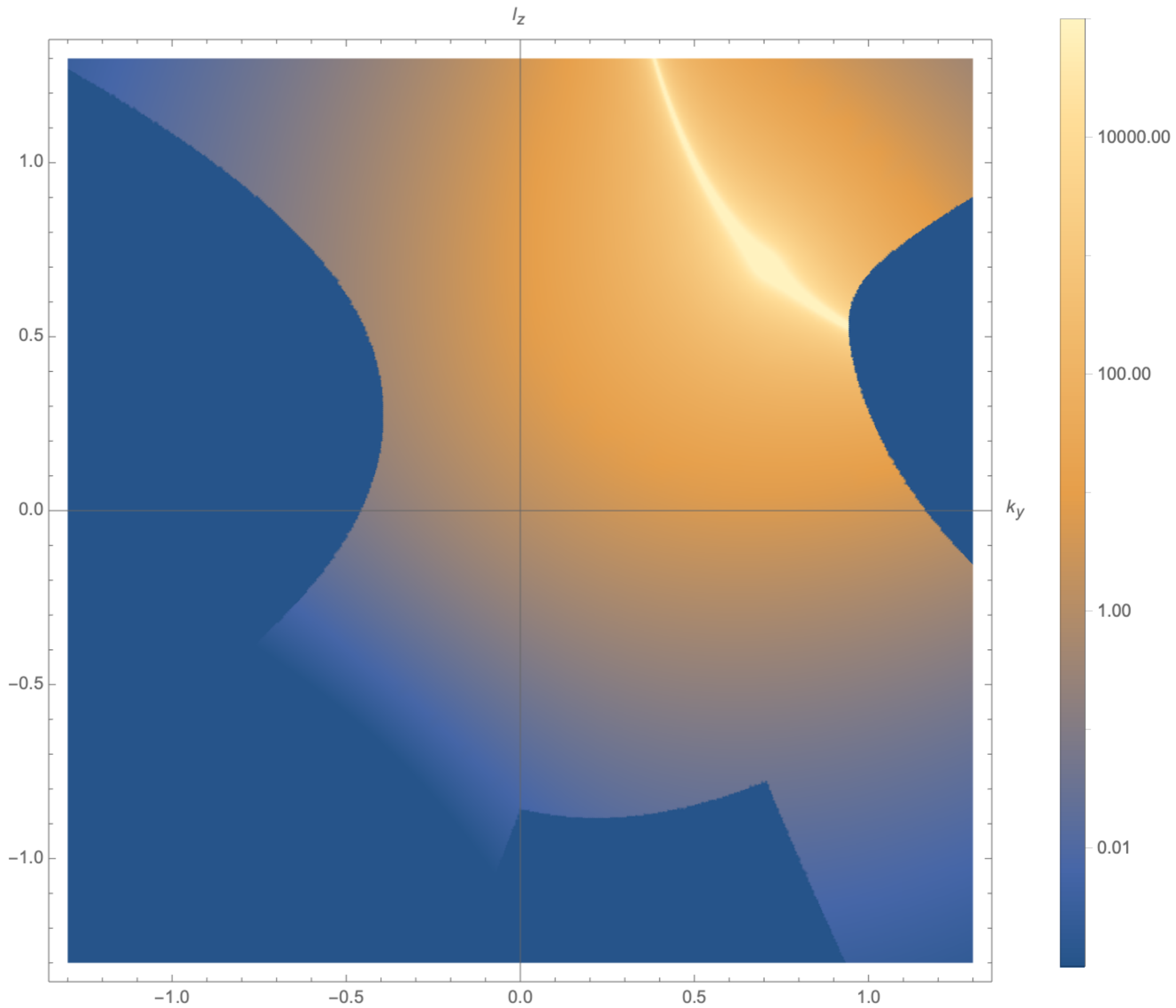
$I_{S_2^v}$

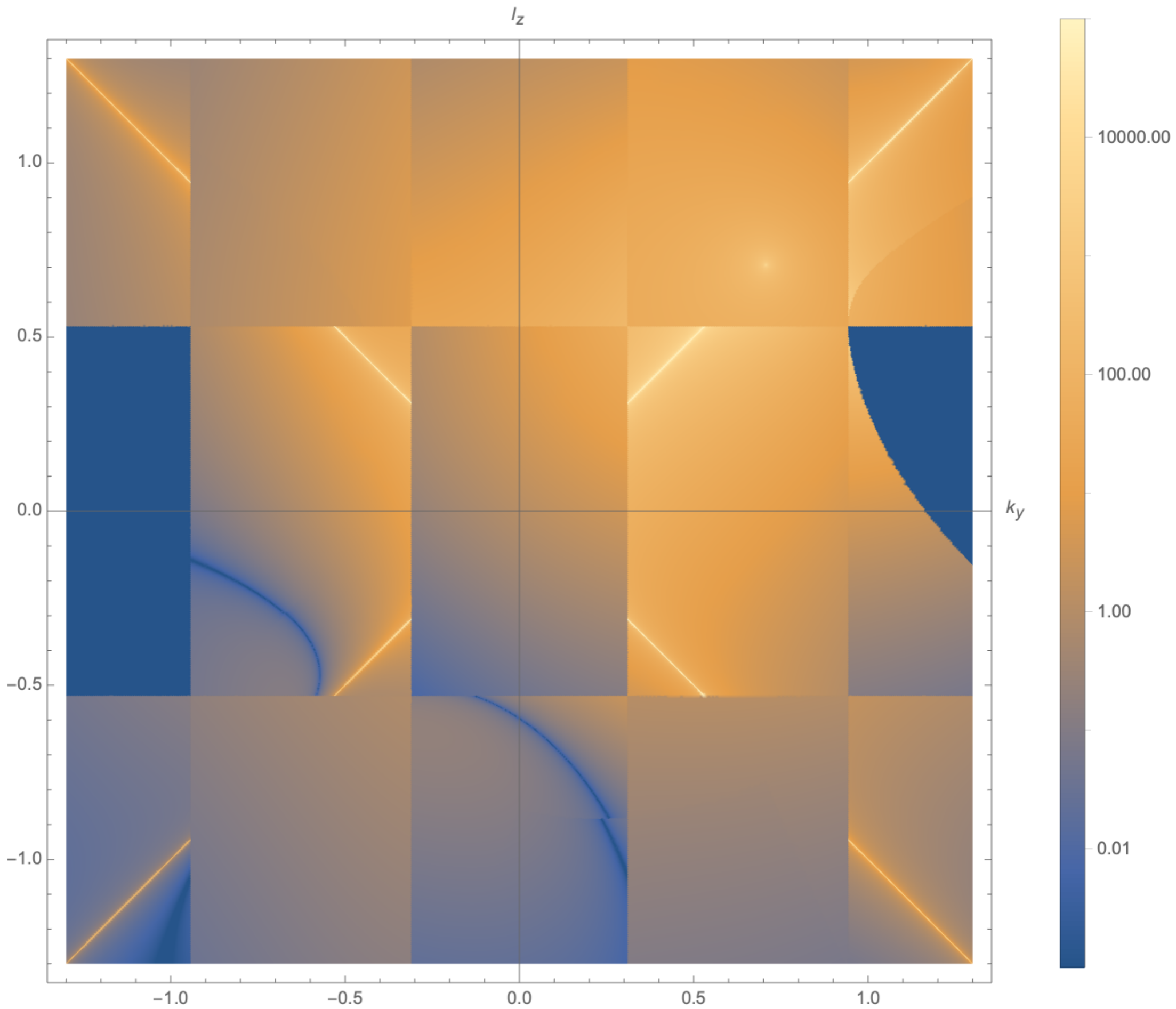


$$I_{\mathbf{s}_1^v} + I_{\mathbf{s}_2^v}$$

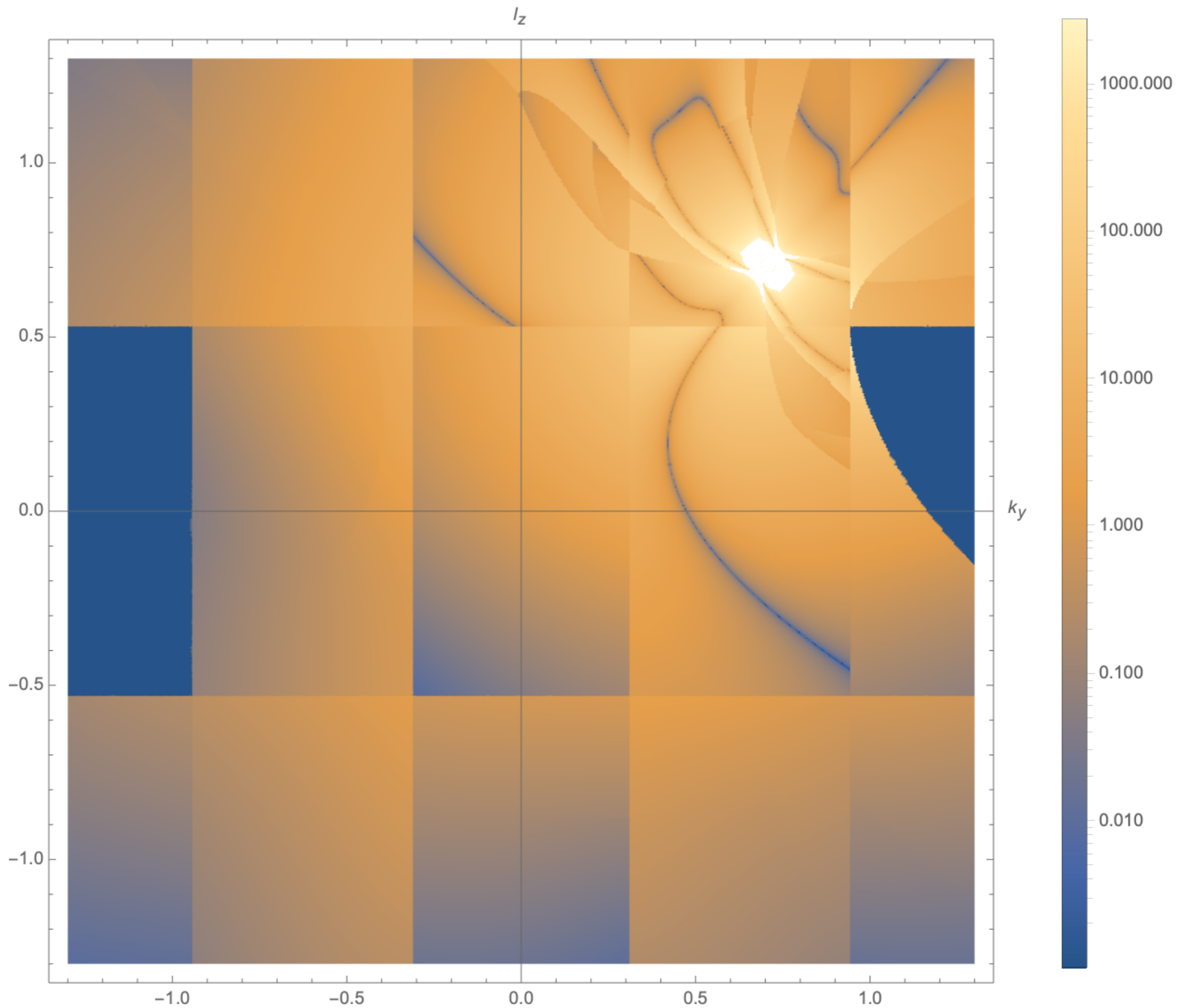


$$I_{\mathbf{s}_1^r} + I_{\mathbf{s}_2^r}$$

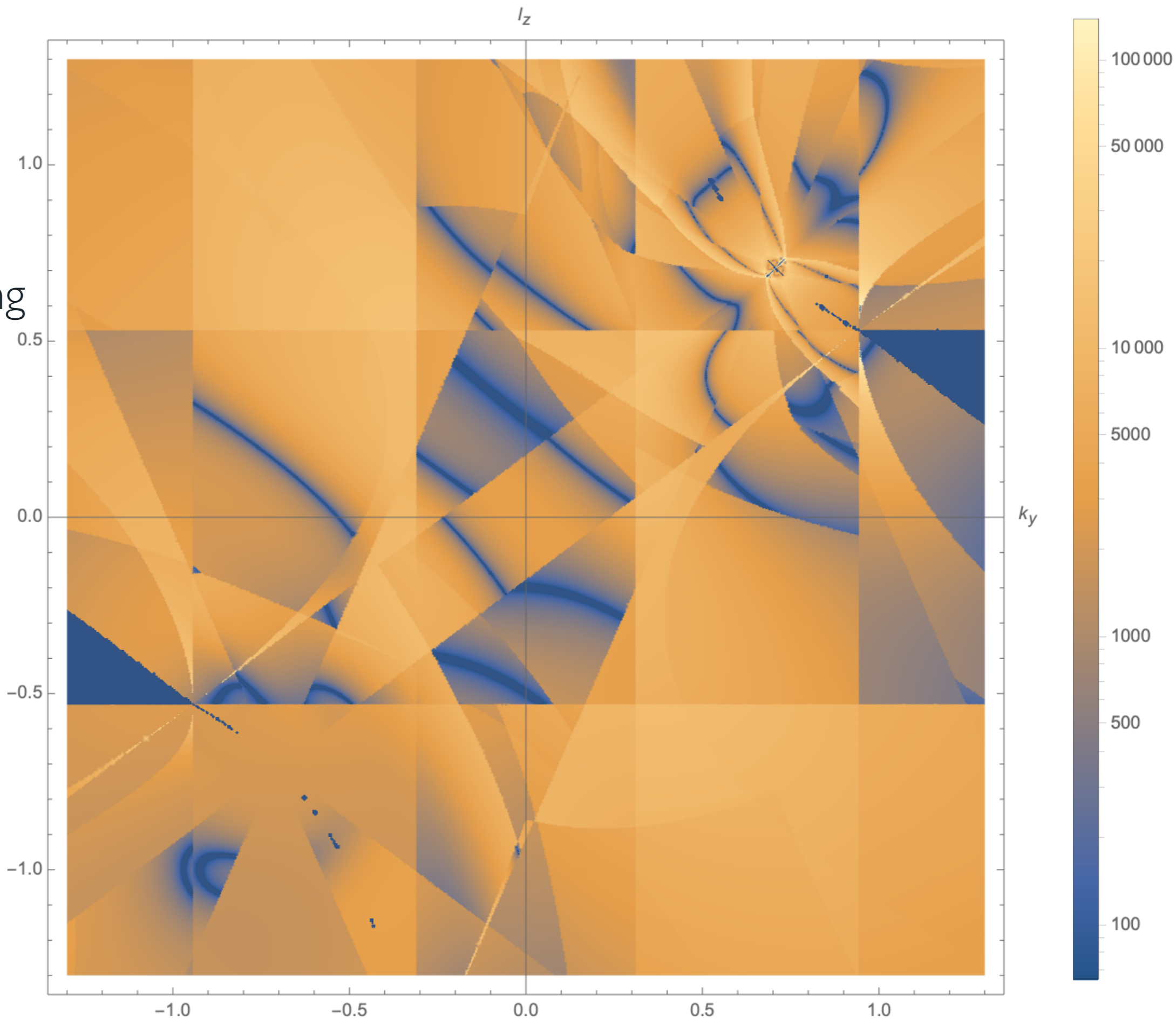


I_Σ 

$\text{Re} [I_\Sigma]$
with
deformation



$\text{Re} [I_\Sigma]$
with
deformation
and
multichanneling

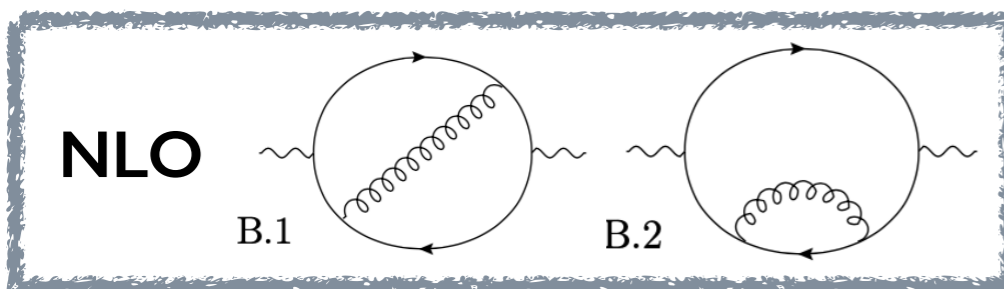
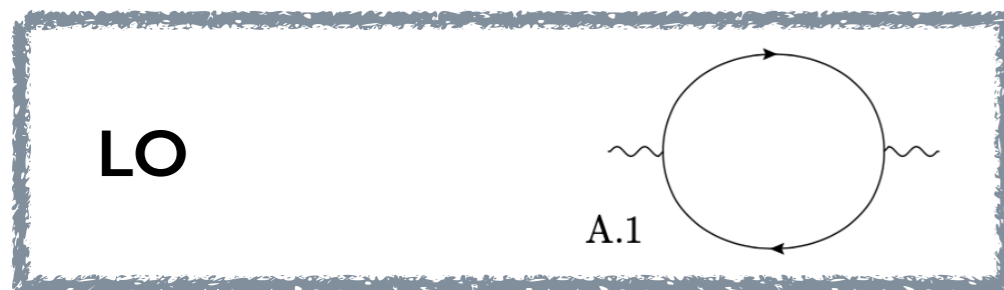


NUMERICAL RESULTS

We have computed the following fixed-order processes with **Local Unitarity**:

NLO	$e^+e^- \rightarrow \gamma \rightarrow jj$	$p_t(j_1)$ distribution	NNLO	$\gamma^* \rightarrow jj$	inclusive
	$e^+e^- \rightarrow \gamma \rightarrow jjj$	semi-inclusive		$\gamma^* \rightarrow t\bar{t}$	inclusive
	$e^+e^- \rightarrow \gamma \rightarrow t\bar{t}h$	(semi-)inclusive			

First **NNLO** cross-sections computed **fully numerically** in momentum space.

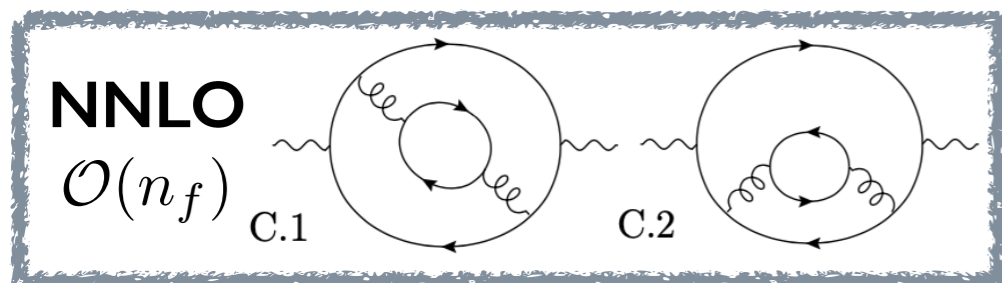
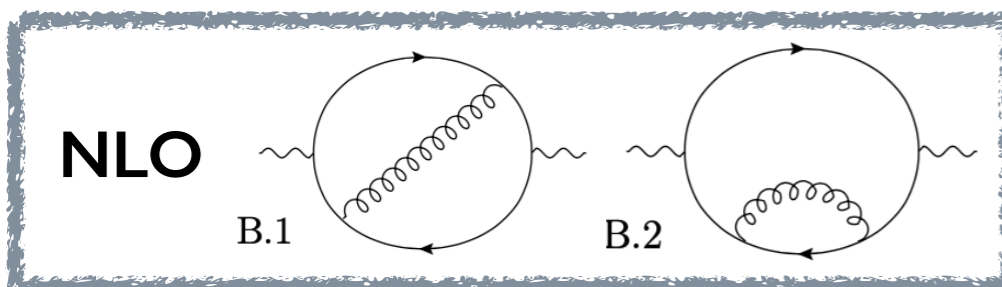
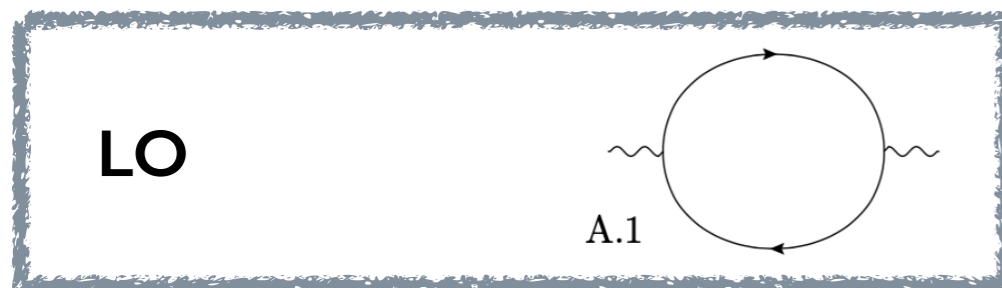


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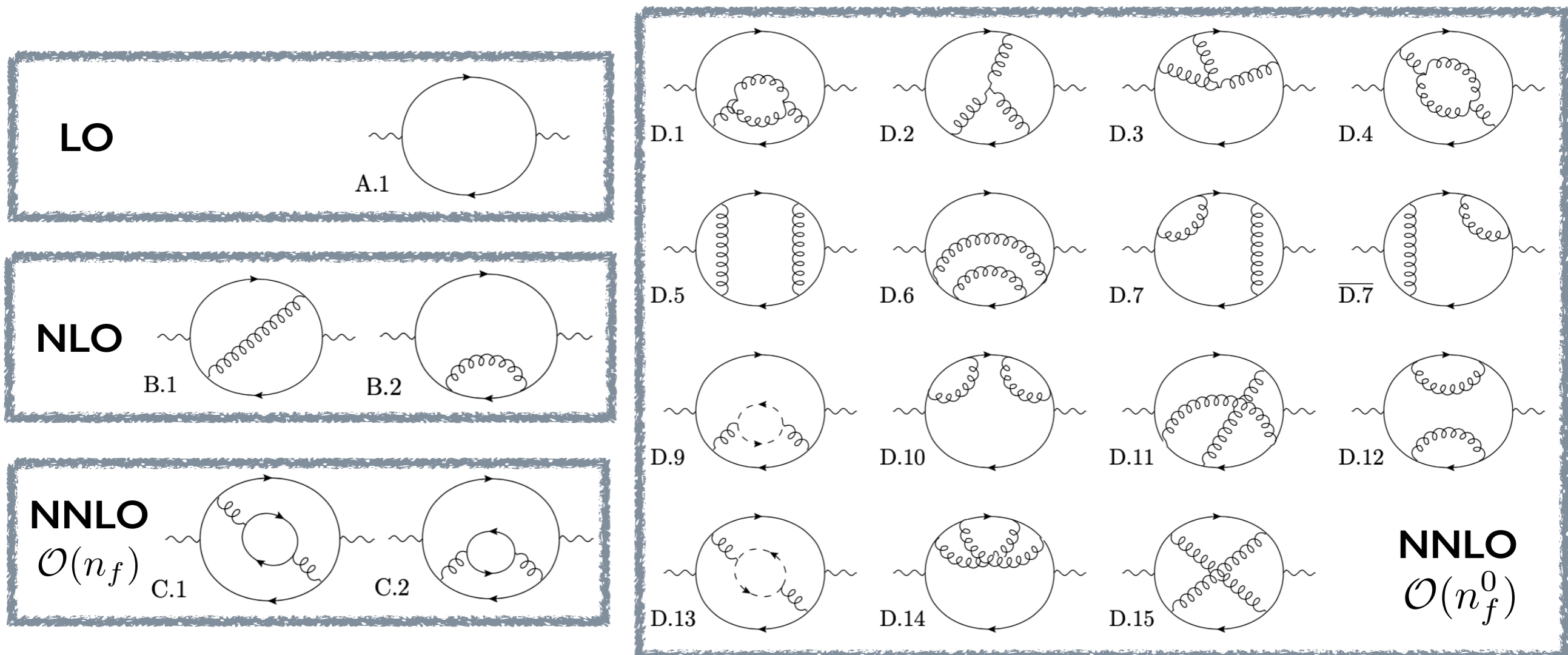


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NUMERICAL RESULTS

SG id	Ξ	$\sigma_{\gamma^* \rightarrow jj}^{(\overline{\text{MS}})}$ [GeV ⁻²] $p_{\gamma^*}^2 = \mu_r^2 = (400 \text{ GeV})^2$	Δ [%]	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]}$ [GeV ⁻²] $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (400 \text{ GeV})^2$	Δ [%]
LO $\mathcal{O}(\alpha_s^0)$					
A.1	1	$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
Total		$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
NLO $\mathcal{O}(\alpha_s^1)$					
B.1	1	$5.03926 \cdot 10^{-02}$	0.0075	$2.52705 \cdot 10^{-01}$	0.034
B.2	2	$-3.14956 \cdot 10^{-02}$	0.018	$1.80050 \cdot 10^{-01}$	0.049
Total		$1.88970 \cdot 10^{-02}$	0.036	$4.3276 \cdot 10^{-01}$	0.028
Benchmark		$1.889690 \cdot 10^{-02}$	0.00053	$4.32831 \cdot 10^{-01}$	-0.018
NNLO $\mathcal{O}(\alpha_s^2 n_f)$					
C.1	1	$-4.66342 \cdot 10^{-04}$	0.019	$-1.0022 \cdot 10^{-03}$	0.17
C.2	2	$3.8448 \cdot 10^{-04}$	0.036	$-4.6982 \cdot 10^{-03}$	0.081
Total		$-8.186 \cdot 10^{-05}$	0.20	$-5.7004 \cdot 10^{-03}$	0.073
Benchmark		$-8.1834 \cdot 10^{-05}$	0.036	$-5.6982 \cdot 10^{-03}$	0.038
NNLO $\mathcal{O}(\alpha_s^2)$					
D.1	2	$-2.30886 \cdot 10^{-03}$	0.017	$3.8886 \cdot 10^{-02}$	0.031
D.2	2	$6.42018 \cdot 10^{-03}$	0.0055	$5.6351 \cdot 10^{-03}$	0.14
D.3	2	$-6.91254 \cdot 10^{-03}$	0.0046	$1.76075 \cdot 10^{-02}$	0.055
D.4	1	$3.20278 \cdot 10^{-03}$	0.0084	$8.8163 \cdot 10^{-03}$	0.078
D.5	1	$1.68148 \cdot 10^{-03}$	0.013	$9.200 \cdot 10^{-04}$	0.79
D.6	2	$6.6698 \cdot 10^{-04}$	0.027	$5.1058 \cdot 10^{-03}$	0.15
D.7	2	$-1.30381 \cdot 10^{-03}$	0.013	$6.7284 \cdot 10^{-03}$	0.10
$\overline{\text{D.7}}$	2	$-1.30395 \cdot 10^{-03}$	0.013	$6.7300 \cdot 10^{-03}$	0.10
D.9	2	$-1.6661 \cdot 10^{-04}$	0.064	$2.3361 \cdot 10^{-03}$	0.12
D.10	2	$6.64155 \cdot 10^{-04}$	0.012	$3.7418 \cdot 10^{-03}$	0.14
D.11	2	$2.34300 \cdot 10^{-04}$	0.031	$2.0845 \cdot 10^{-03}$	0.083
D.12	1	$4.11063 \cdot 10^{-04}$	0.017	$3.5114 \cdot 10^{-03}$	0.12
D.13	1	$2.41514 \cdot 10^{-04}$	0.026	$8.222 \cdot 10^{-04}$	0.19
D.14	2	$5.8386 \cdot 10^{-05}$	0.088	$1.76075 \cdot 10^{-02}$	0.055
D.15	1	$-1.75957 \cdot 10^{-04}$	0.022	$-7.242 \cdot 10^{-04}$	0.14
Total		$1.40910 \cdot 10^{-03}$	0.056	$1.04214 \cdot 10^{-01}$	0.024
Benchmark		$1.40941 \cdot 10^{-03}$	-0.022	$1.0386 \cdot 10^{-01}$	0.34

Analytic benchmarks :

$$\sigma_{\gamma^* \rightarrow X}^{\overline{\text{MS}}}(p^2, \mu_r^2 = p^2) \supset K_X n_f \left(\frac{\alpha_s}{4\pi}\right)^2 \sigma_{\gamma^* \rightarrow X}^{(\text{LO})}(p^2)$$

NUMERICAL RESULTS

SG id	Ξ	$\sigma_{\gamma^* \rightarrow jj}^{(\overline{\text{MS}})}$ [GeV ⁻²] $p_{\gamma^*}^2 = \mu_r^2 = (400 \text{ GeV})^2$	Δ [%]	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]}$ [GeV ⁻²] $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (400 \text{ GeV})^2$	Δ [%]
LO $\mathcal{O}(\alpha_s^0)$					
A.1	1	$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
Total		$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
NLO $\mathcal{O}(\alpha_s^1)$					
B.1	1	$5.03926 \cdot 10^{-02}$	0.0075	$2.52705 \cdot 10^{-01}$	0.034
B.2	2	$-3.14956 \cdot 10^{-02}$	0.018	$1.80050 \cdot 10^{-01}$	0.049
Total		$1.88970 \cdot 10^{-02}$	0.036	$4.3276 \cdot 10^{-01}$	0.028
Benchmark		$1.889690 \cdot 10^{-02}$	0.00053	$4.32831 \cdot 10^{-01}$	-0.018
NNLO $\mathcal{O}(\alpha_s^2 n_f)$					
C.1	1	$-4.66342 \cdot 10^{-04}$	0.019	$-1.0022 \cdot 10^{-03}$	0.17
C.2	2	$3.8448 \cdot 10^{-04}$	0.036	$-4.6982 \cdot 10^{-03}$	0.081
Total		$-8.186 \cdot 10^{-05}$	0.20	$-5.7004 \cdot 10^{-03}$	0.073
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[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

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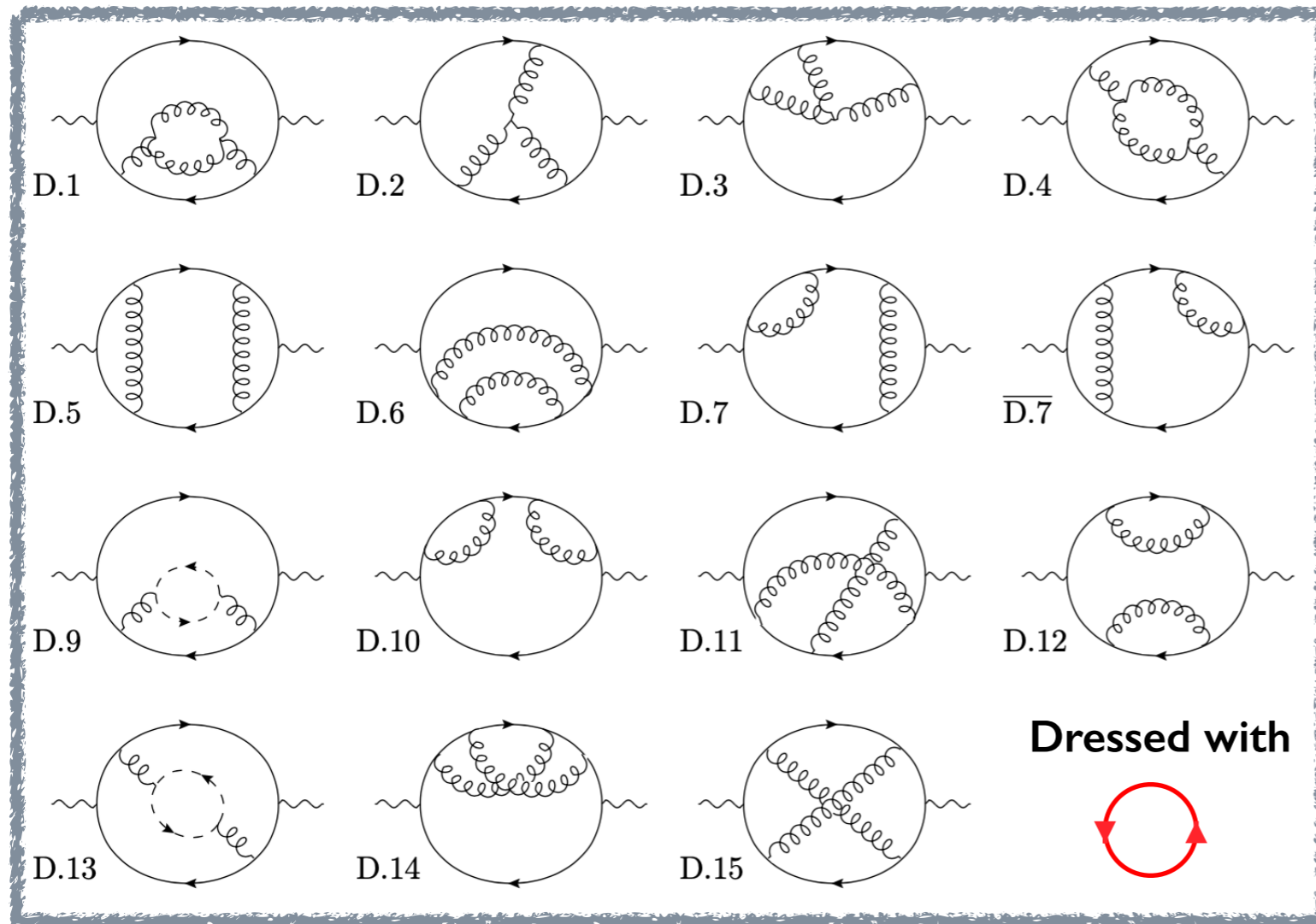
$$\gamma^* \rightarrow t\bar{t}$$

$$K_{t\bar{t}} = \delta^{(2)} = -\frac{(3-v^2)(1+v^2)}{6} \times \left\{ \text{Li}_3(p) - 2\text{Li}_3(1-p) - 3\text{Li}_3(p^2) - 4\text{Li}_3\left(\frac{p}{1+p}\right) - 5\text{Li}_3(1-p^2) + \frac{11}{2}\zeta(3) \right. \\ \left. + \text{Li}_2(p) \ln\left(\frac{4(1-v^2)}{v^4}\right) + 2\text{Li}_2(p^2) \ln\left(\frac{1-v^2}{2v^2}\right) + 2\zeta(2) \left[\ln p - \ln\left(\frac{1-v^2}{4v}\right) \right] \right. \\ \left. - \frac{1}{6} \ln\left(\frac{1+v}{2}\right) \left[36 \ln 2 \ln p - 44 \ln^2 p + 49 \ln p \ln\left(\frac{1-v^2}{4}\right) + \ln^2\left(\frac{1-v^2}{4}\right) \right] \right. \\ \left. - \frac{1}{2} \ln p \ln v \left[36 \ln 2 + 21 \ln p + 16 \ln v - 22 \ln(1-v^2) \right] \right\} \\ + \frac{1}{24} \left\{ (15 - 6v^2 - v^4) (\text{Li}_2(p) + \text{Li}_2(p^2)) + 3(7 - 22v^2 + 7v^4) \text{Li}_2(p) \right. \\ \left. - (1-v)(51 - 45v - 27v^2 + 5v^3) \zeta(2) \right. \\ \left. + \frac{(1+v)(-9 + 33v - 9v^2 - 15v^3 + 4v^4)}{v} \ln^2 p \right. \\ \left. + \left[(33 + 22v^2 - 7v^4) \ln 2 - 10(3-v^2)(1+v^2) \ln v \right. \right. \\ \left. \left. - (15 - 22v^2 + 3v^4) \ln\left(\frac{1-v^2}{4v^2}\right) \right] \ln p \right. \\ \left. + 2v(3-v^2) \ln\left(\frac{4(1-v^2)}{v^4}\right) \left[\ln v - 3 \ln\left(\frac{1-v^2}{4v}\right) \right] \right. \\ \left. + \frac{237 - 96v + 62v^2 + 32v^3 - 59v^4}{4} \ln p - 16v(3-v^2) \ln\left(\frac{1+v}{4}\right) \right. \\ \left. - 2v(39 - 17v^2) \ln\left(\frac{1-v^2}{2v^2}\right) - \frac{v(75 - 29v^2)}{2} \right\} \dots \quad (\text{B.3})$$

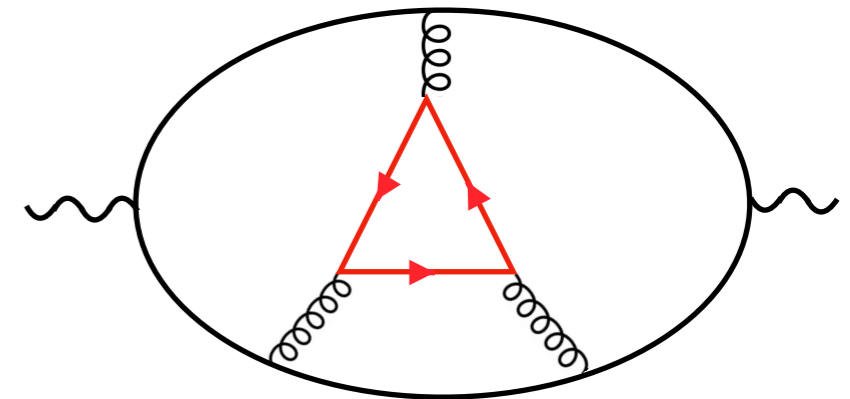
[Chetyrin, Kuehn, Steinhauser, arxiv : 9606230]

PRELIMINARY N3LO RESULTS

n_f contributions :



+ new topologies, such as:



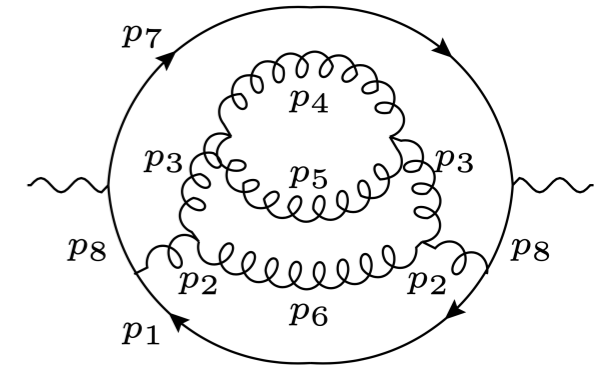
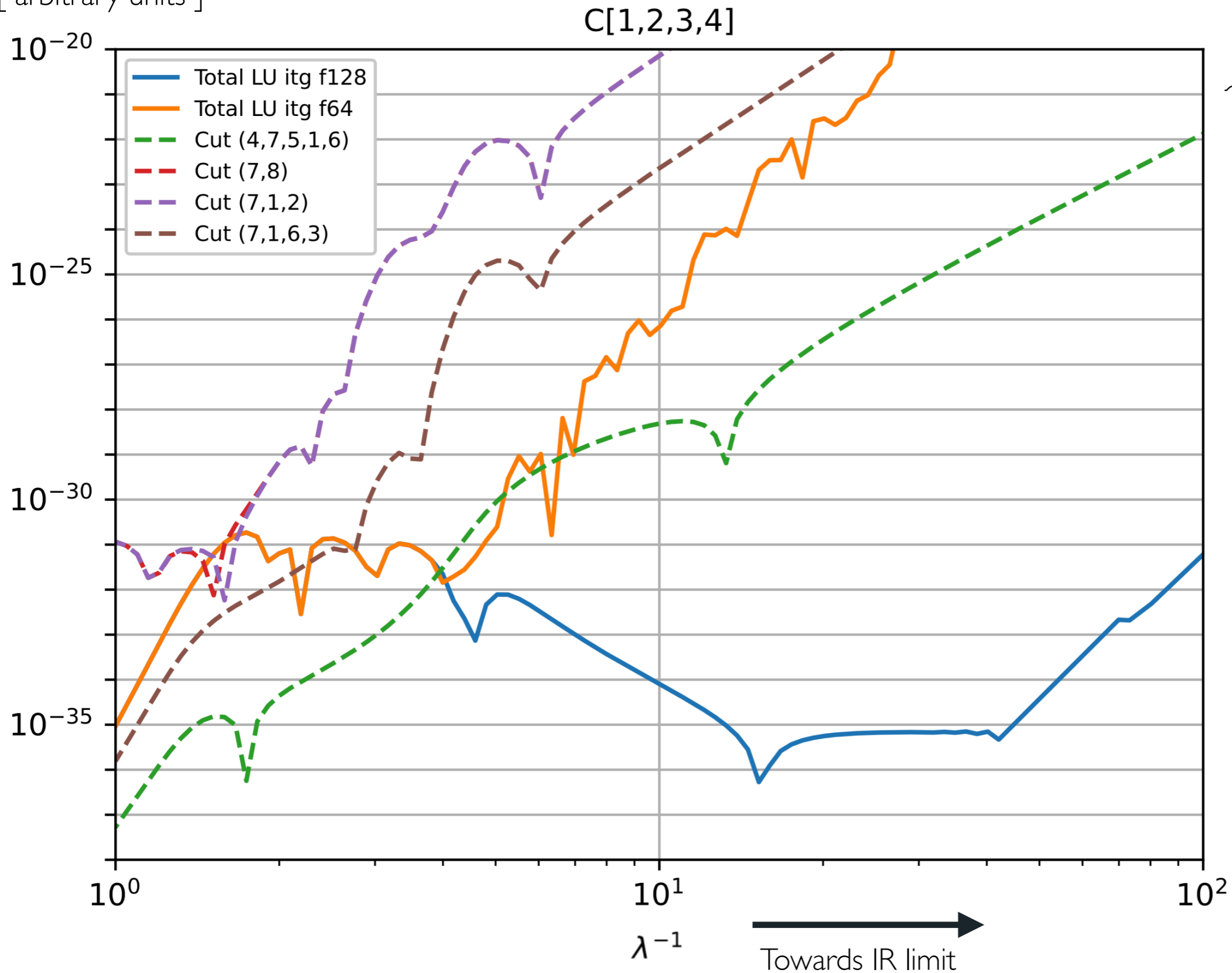
$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f), (\text{MC LU})} = -77.1(1.7)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f)} = -C_F^2 \left(\frac{29}{2} - 152\zeta_3 + 160\zeta_5 \right) - C_F C_A \left(\frac{15520}{27} - \frac{88}{3}\zeta_2 - \frac{3584}{9}\zeta_3 - \frac{80}{3}\zeta_5 \right) = -76.8086$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

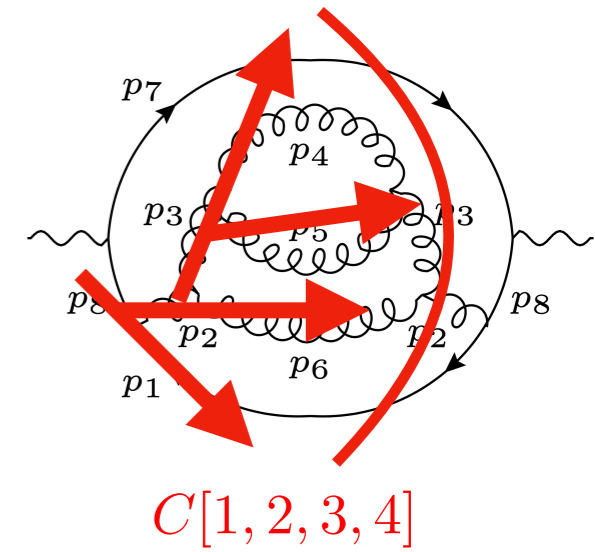
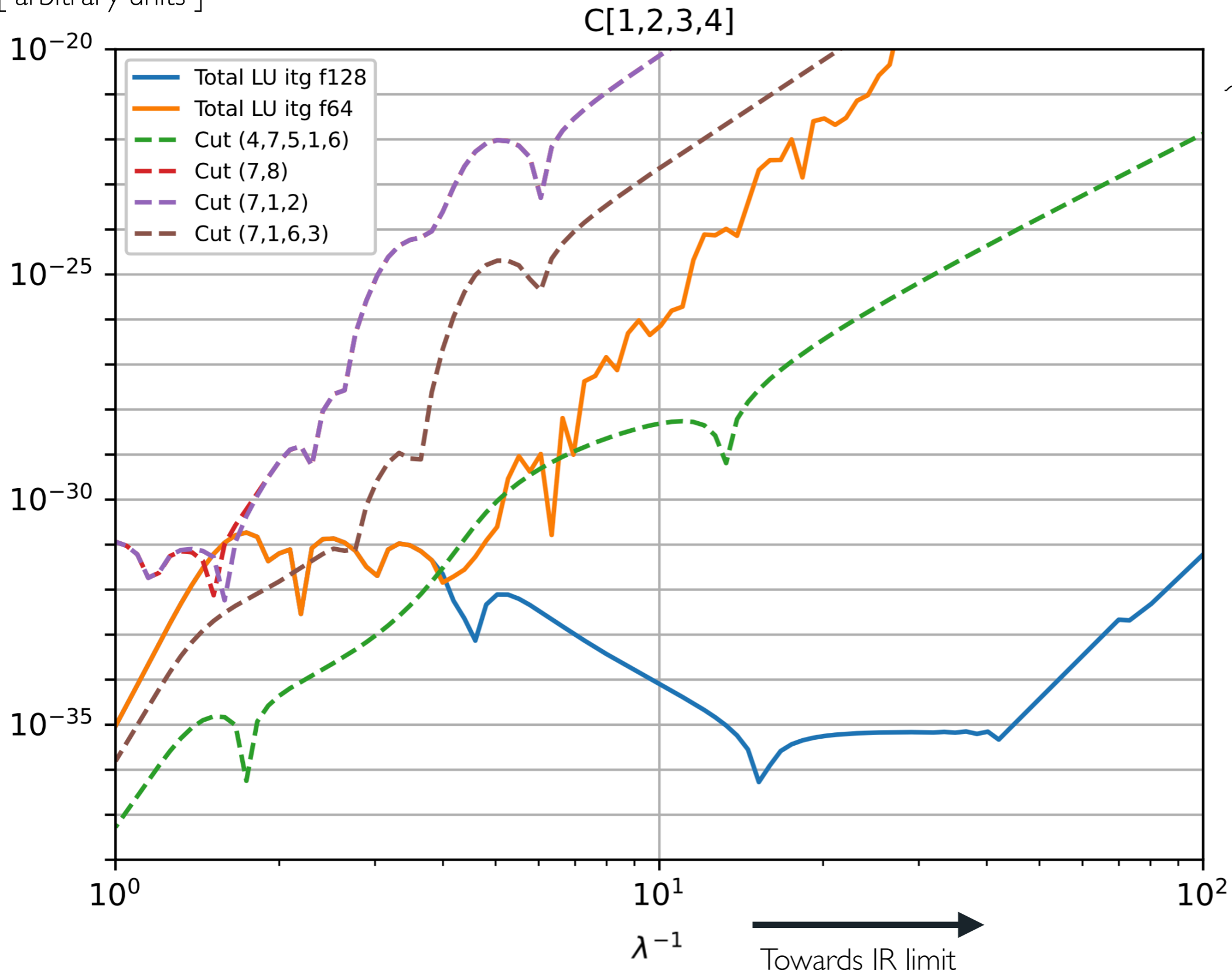
TESTING IR QUADRUPLE COLLINEAR LIMITS

[arbitrary units]



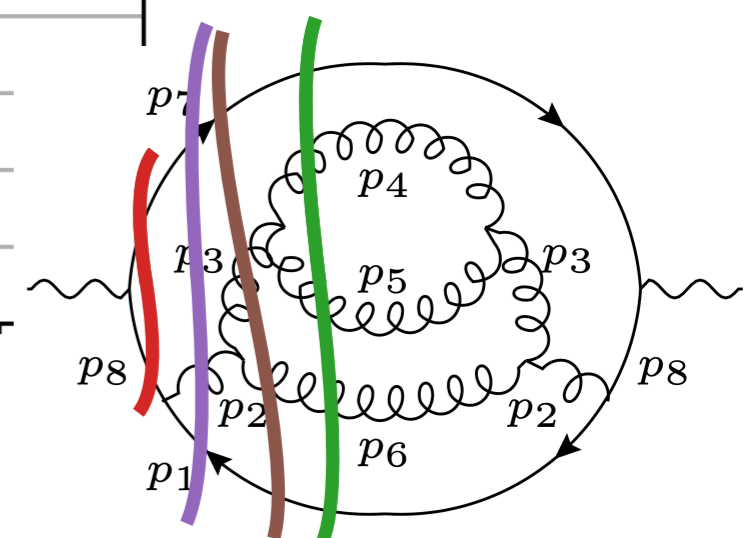
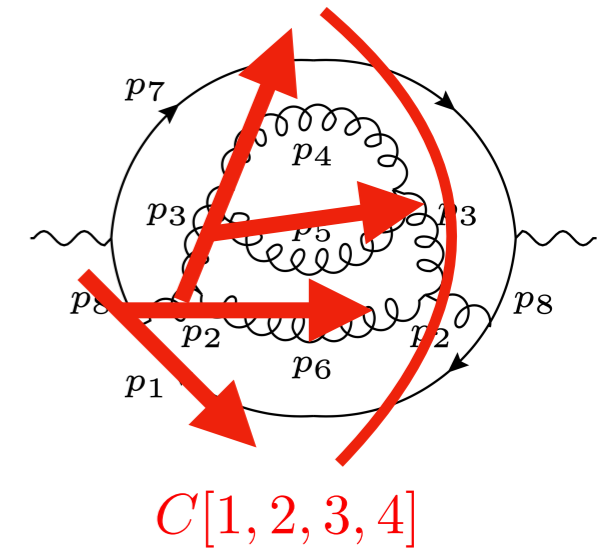
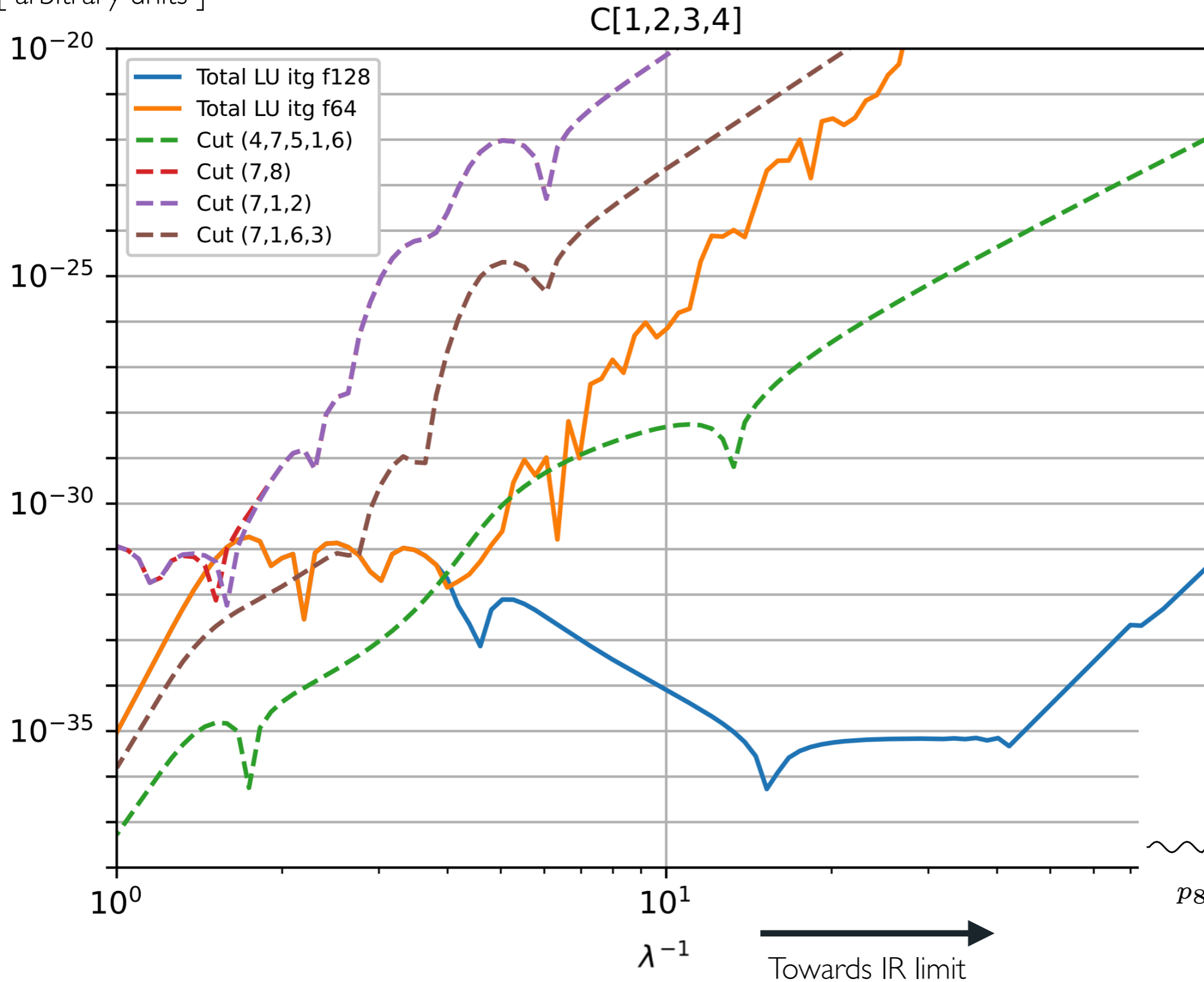
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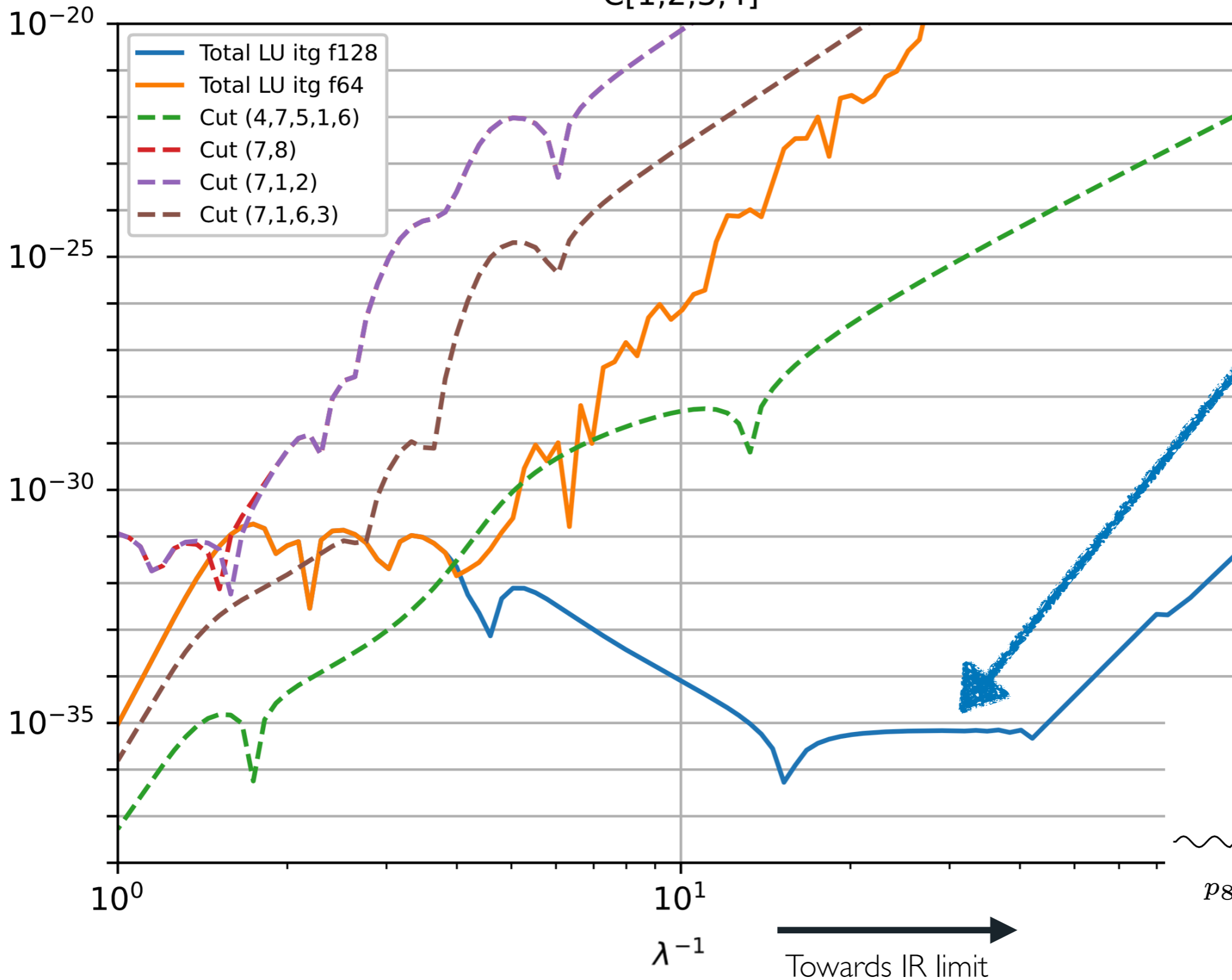
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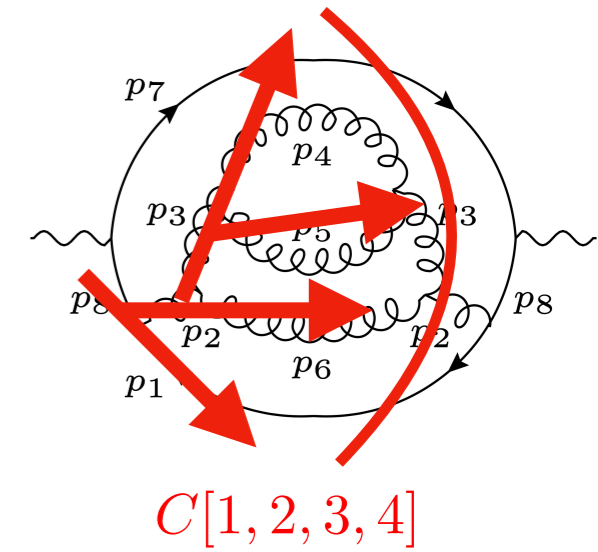


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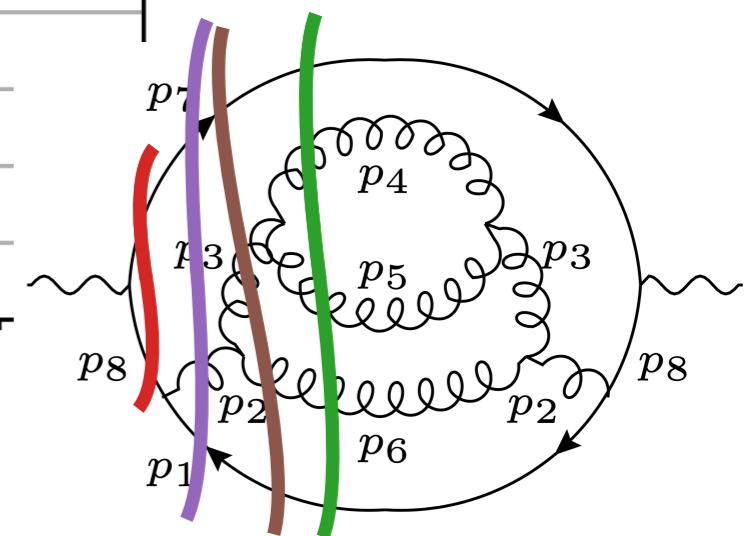
[arbitrary units]



C[1,2,3,4]



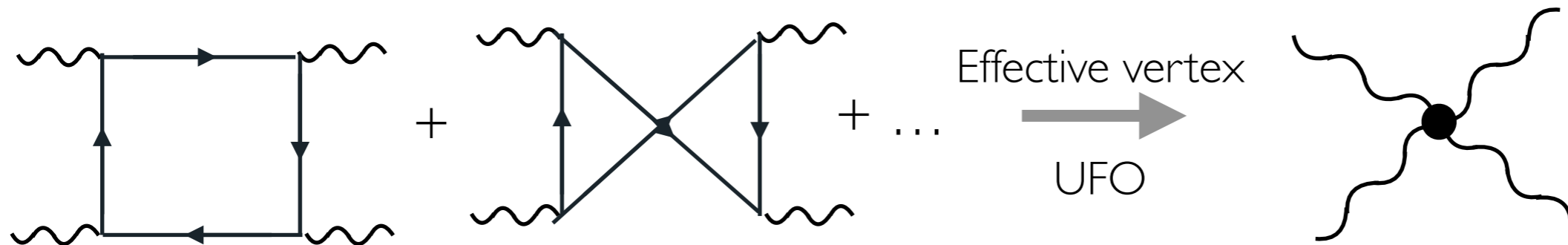
LU integrand always goes to a constant on **collinear limits** without incl. *any* scaling of the measure ! No residual integrable singularity.



PHOTON-PHOTON SCATTERING IN HI UPC

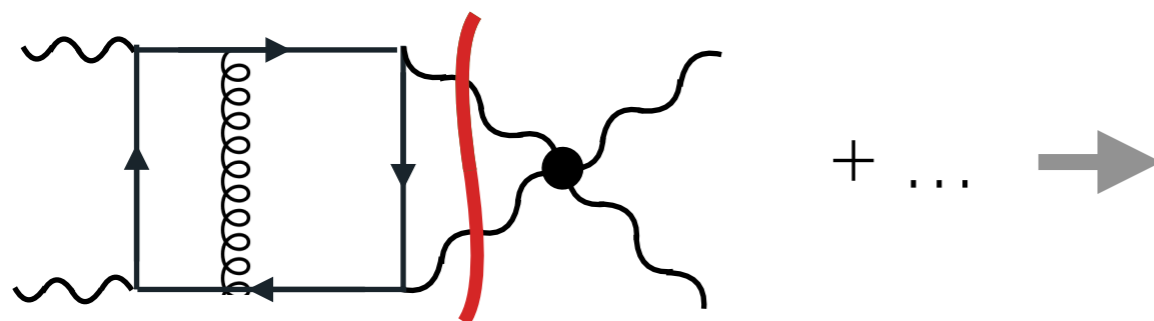
[In collaboration with H.S. -Shao, M. Fraaije, E. Chaubet]

- Too early to present / show much, but alphaLoop delivered a first NLO unknown x-sec!



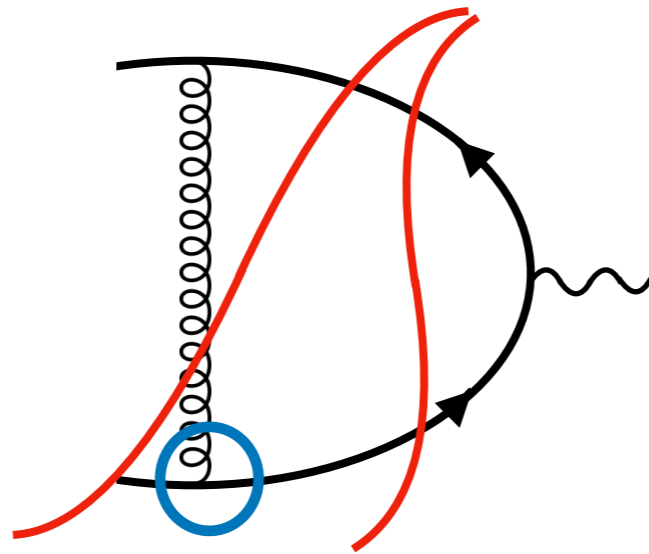
$$\sim \sum_{i=1}^3 C_i(s, t, u, m_f^2) T_i^{\mu_1 \mu_2 \mu_3 \mu_4}(\{p_i\})$$

- In alphaloop we then do



Yielding our first “prediction” for a piece of a yet unknown cross-section: NLO photon scattering.
Successful validation vs analytics!

INITIAL-STATE SINGULARITIES

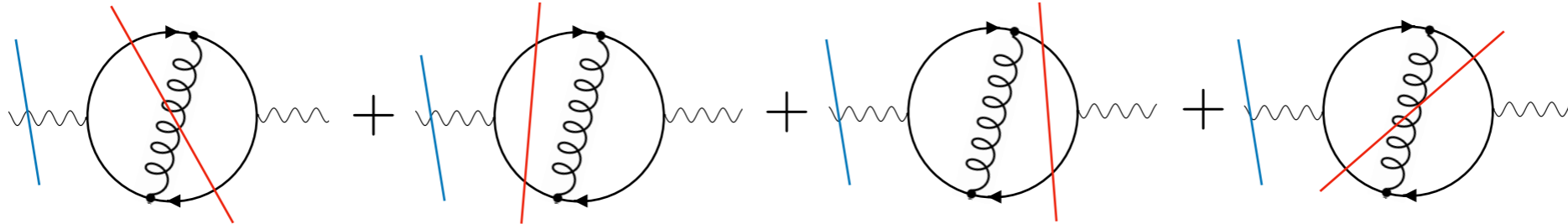


KLN CAN WORK FOR INITIAL-STATE !

INITIAL-STATE SINGULARITIES: IDEA

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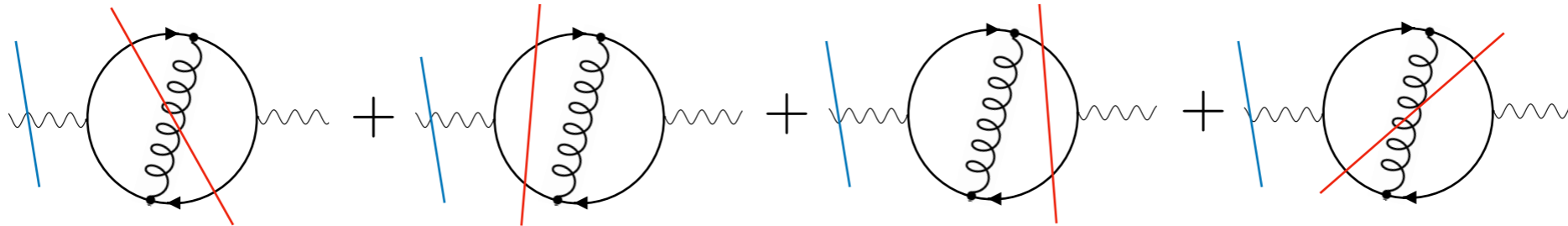
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(Including all degenerate configurations, higher final-state multiplicities)

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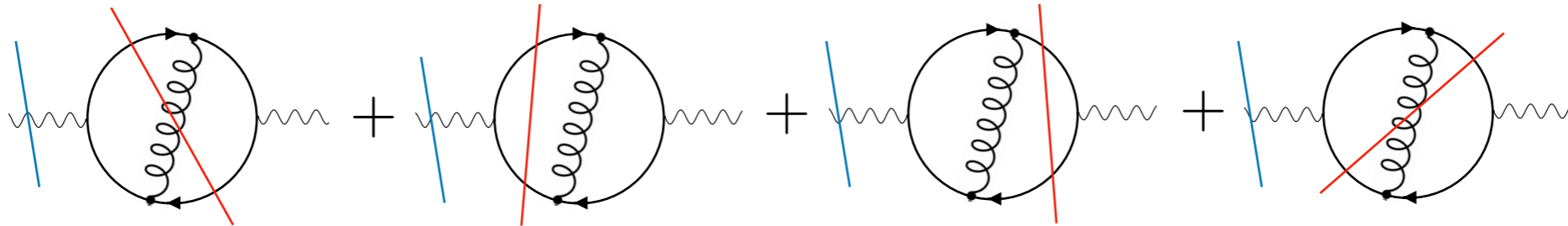


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Flip it, and obtain the answer for Drell-Yan, $2j \rightarrow e^+e^-$

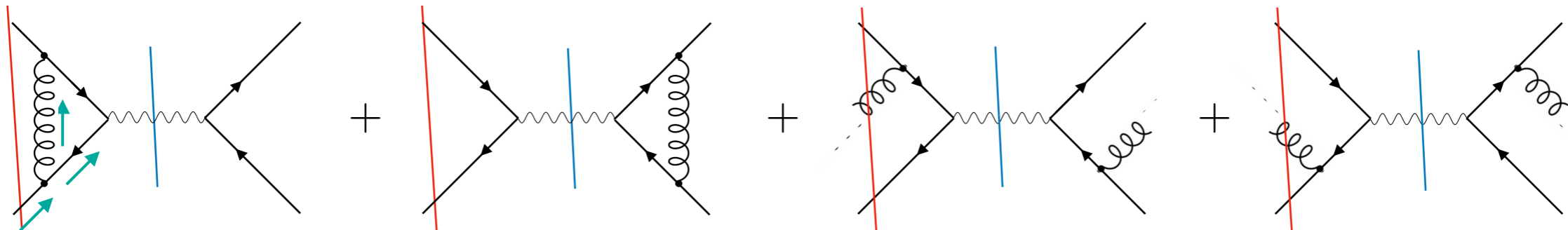
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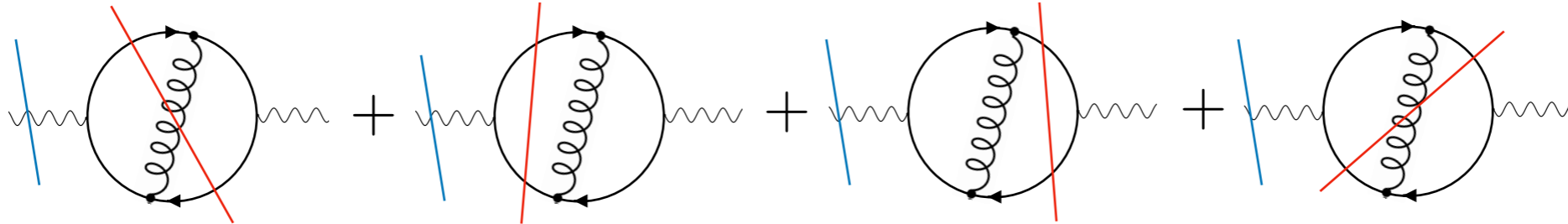
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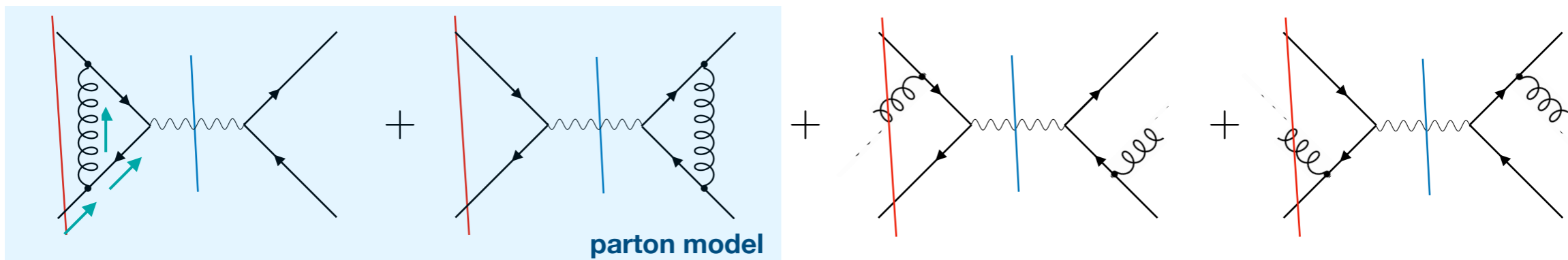
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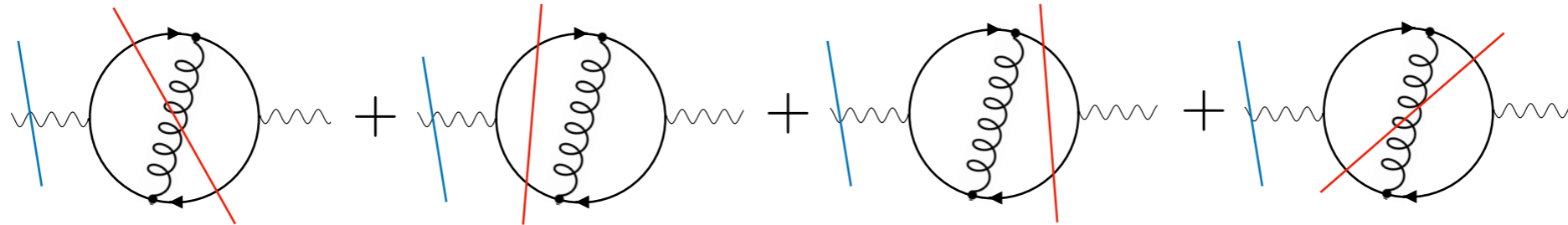
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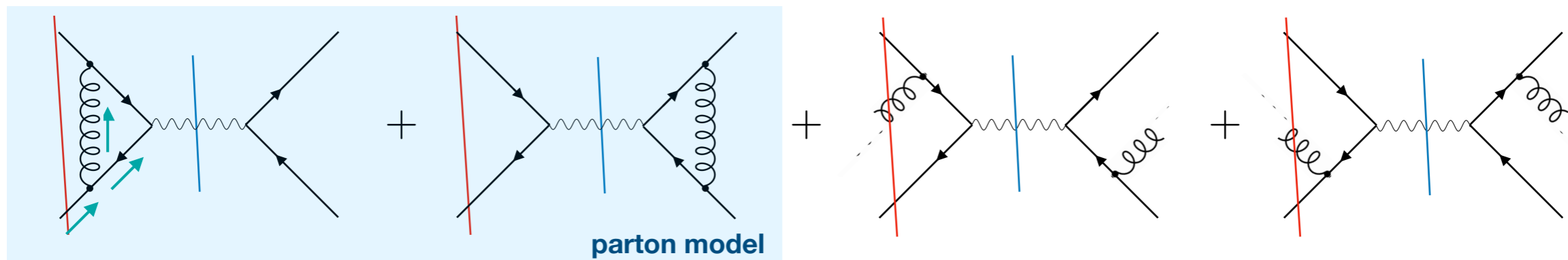
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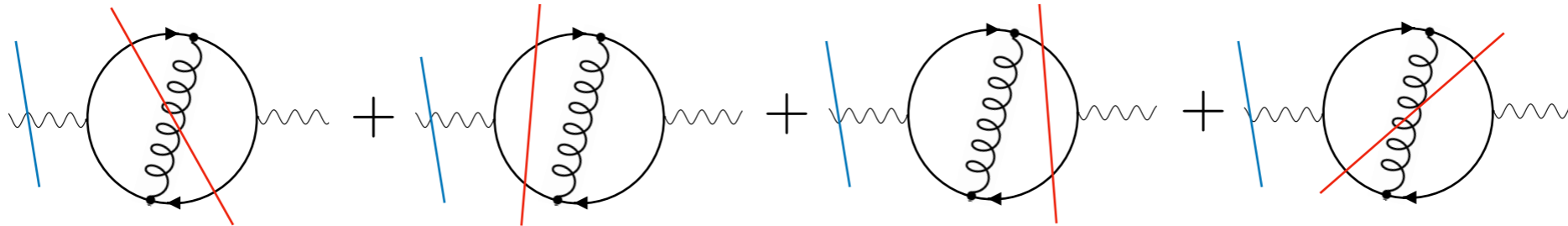
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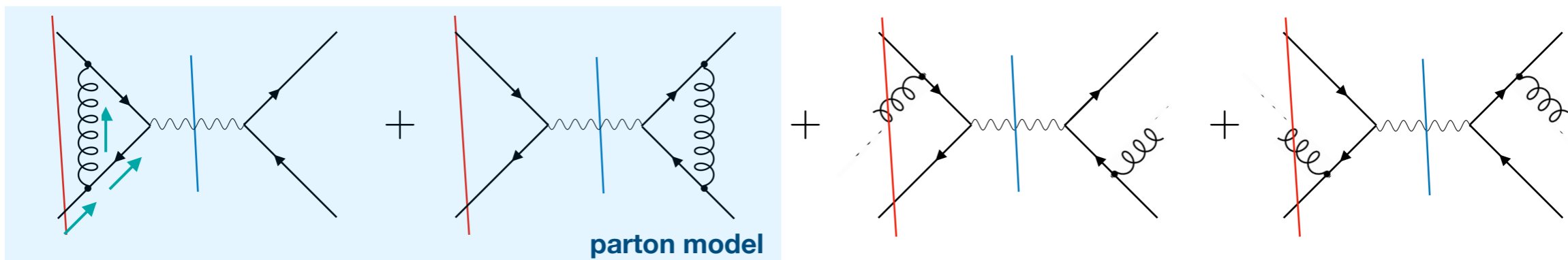
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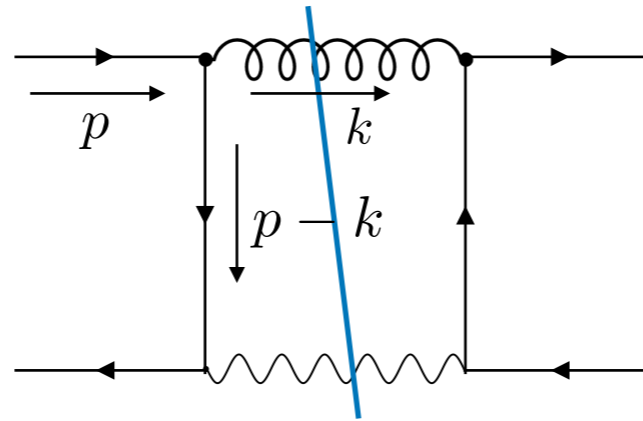
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Include degenerate initial states \rightarrow Higher multiplicity initial states

INITIAL-STATE SINGULARITIES: IDEA

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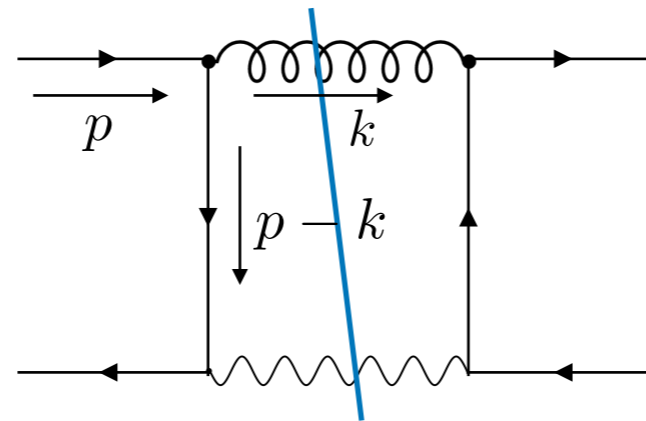
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Also has collinear
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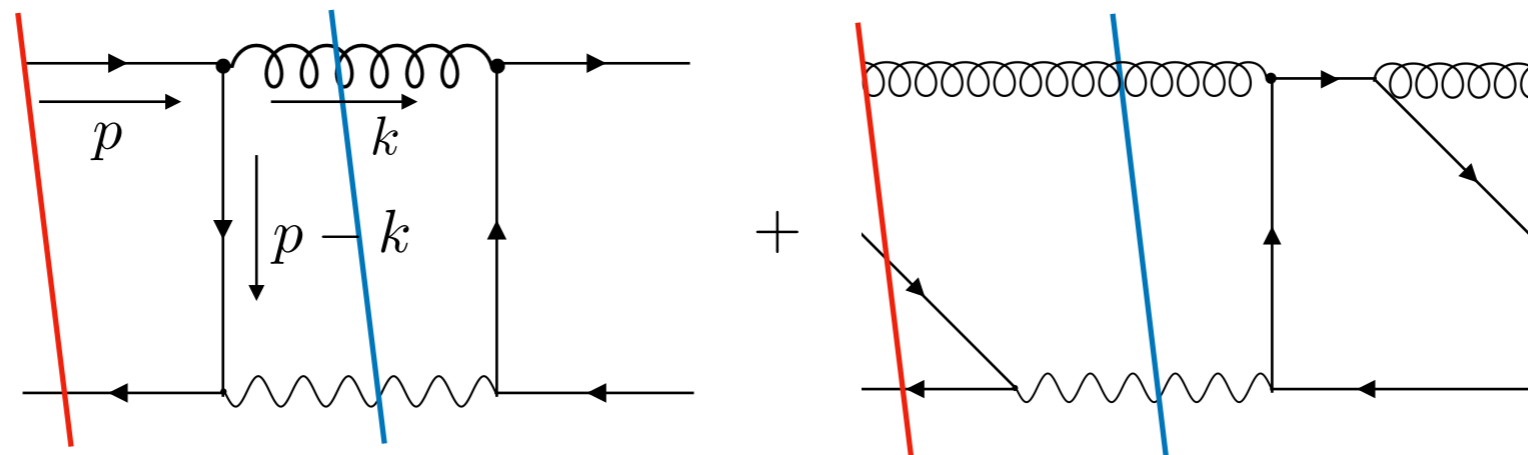
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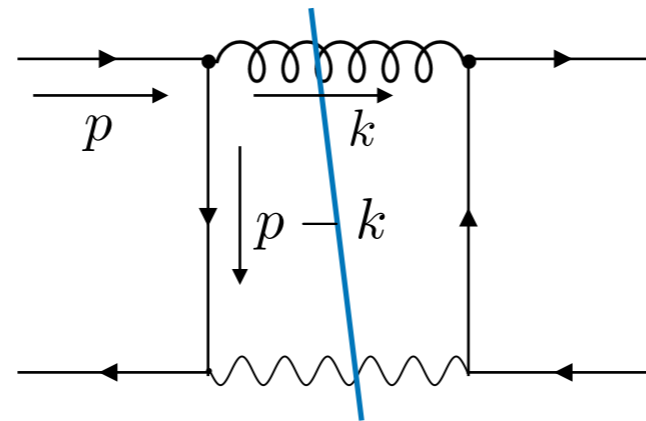
In this case, the cancelling partner is



Higher multiplicity initial states, but also **disconnected!** **Free travelling gluon!**

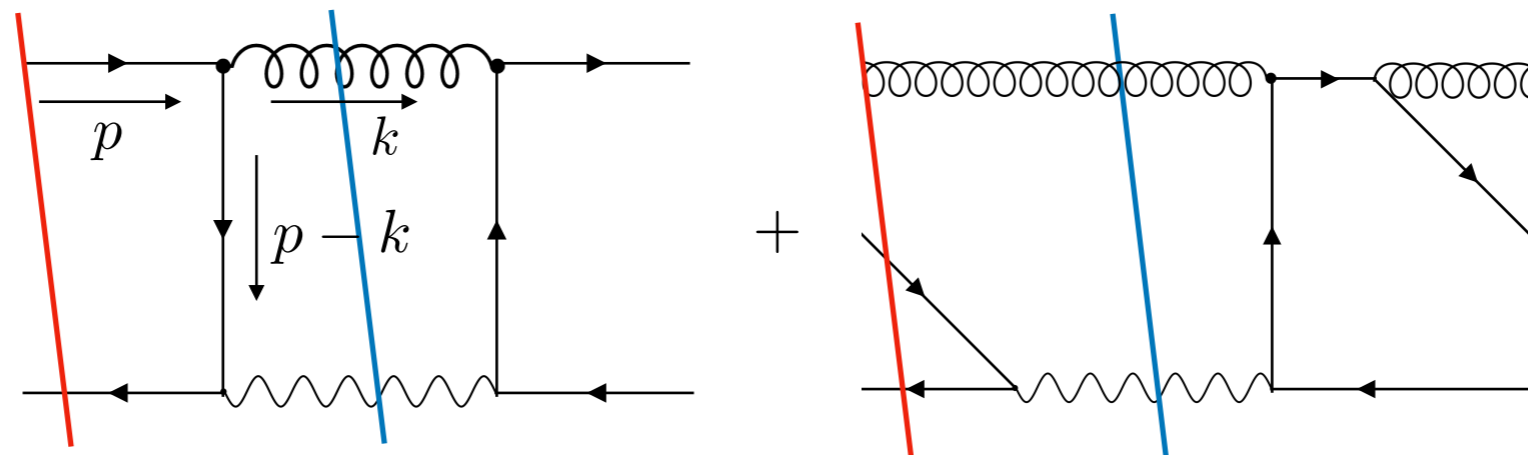
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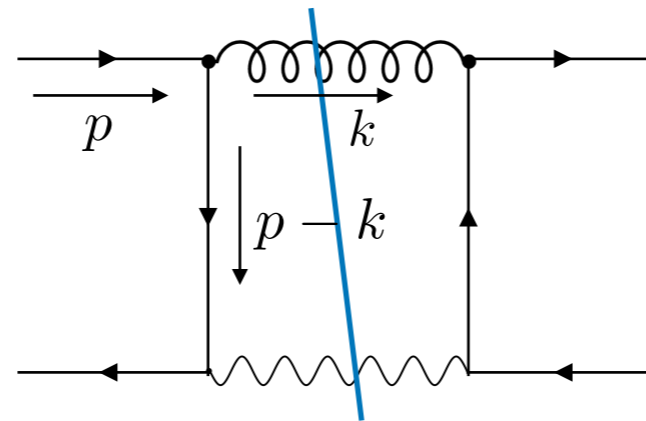


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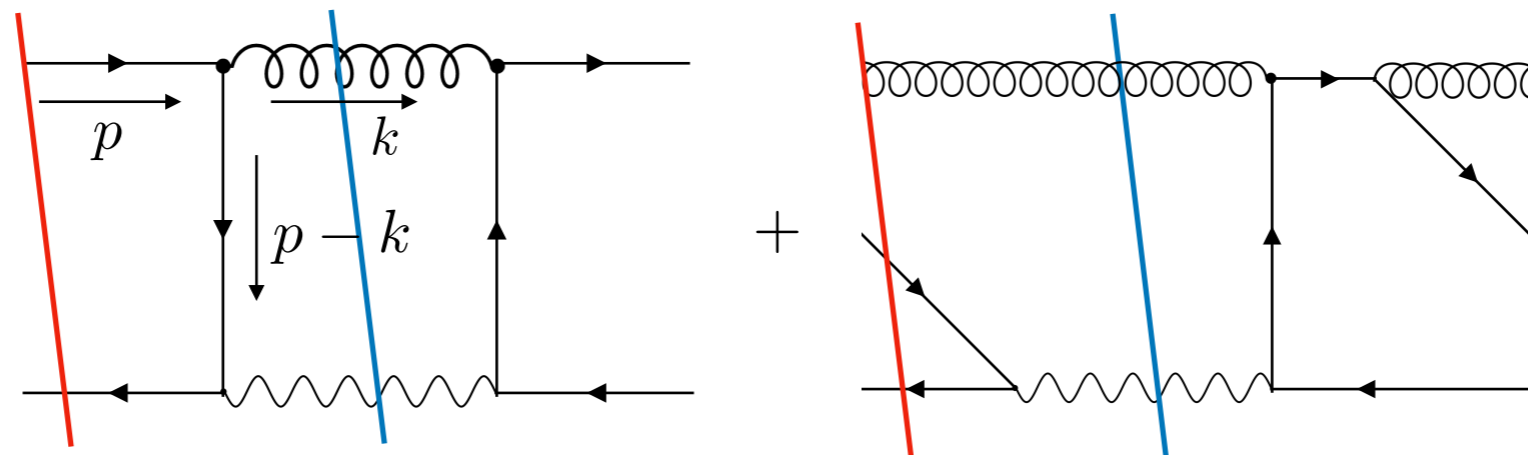
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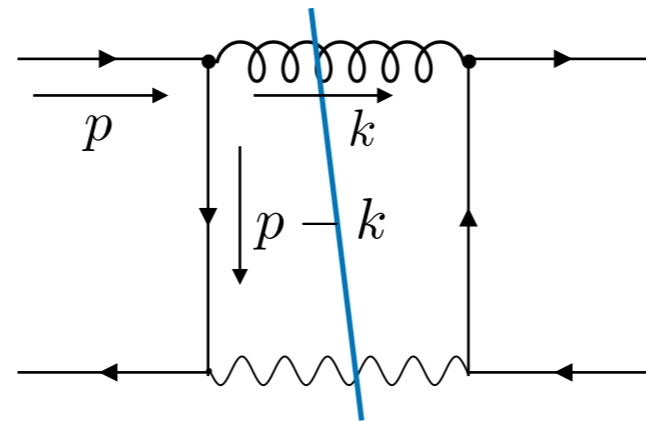
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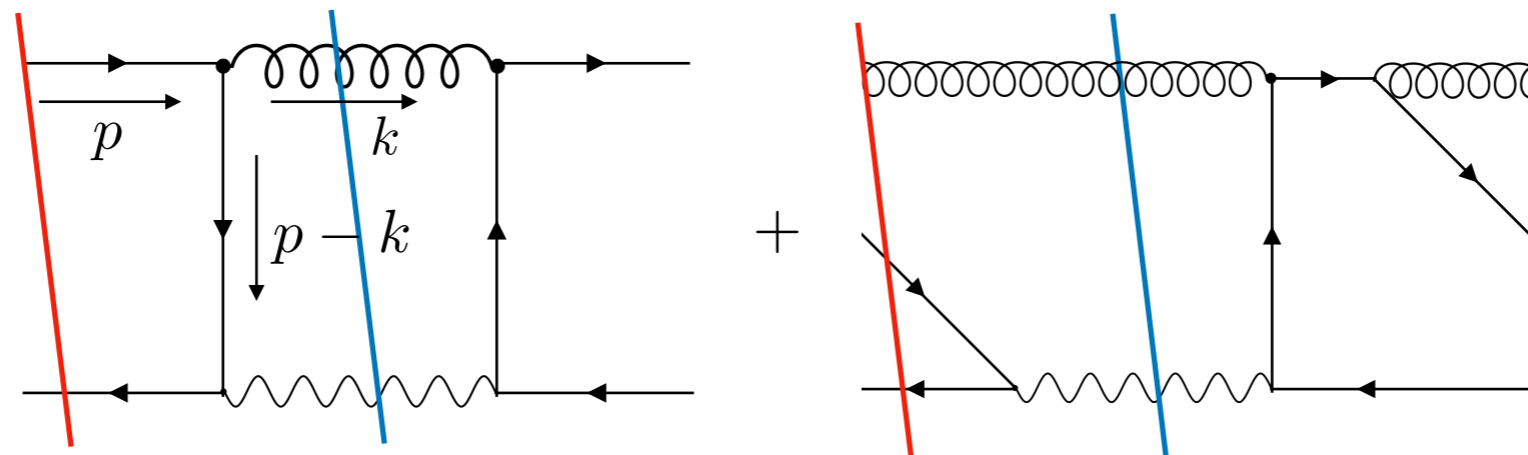
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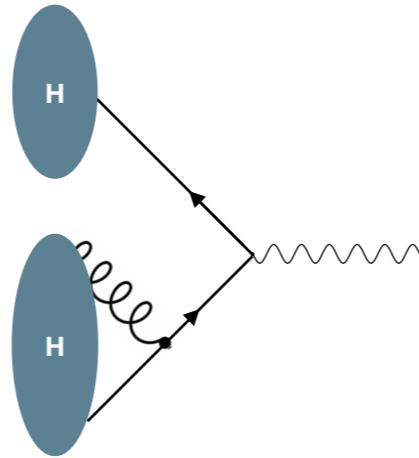
But also more recently, they were studied in:

Frye, Hannesdottir, Paul, Schwartz, Yan
arXiv:1810.10022 (2019)

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

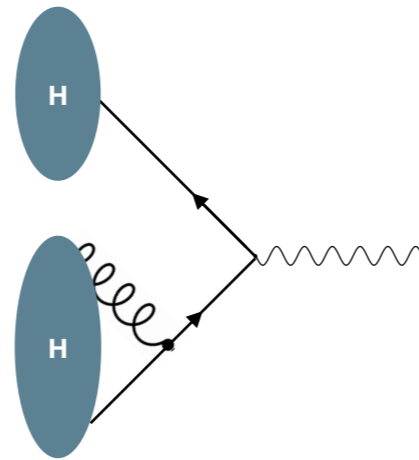
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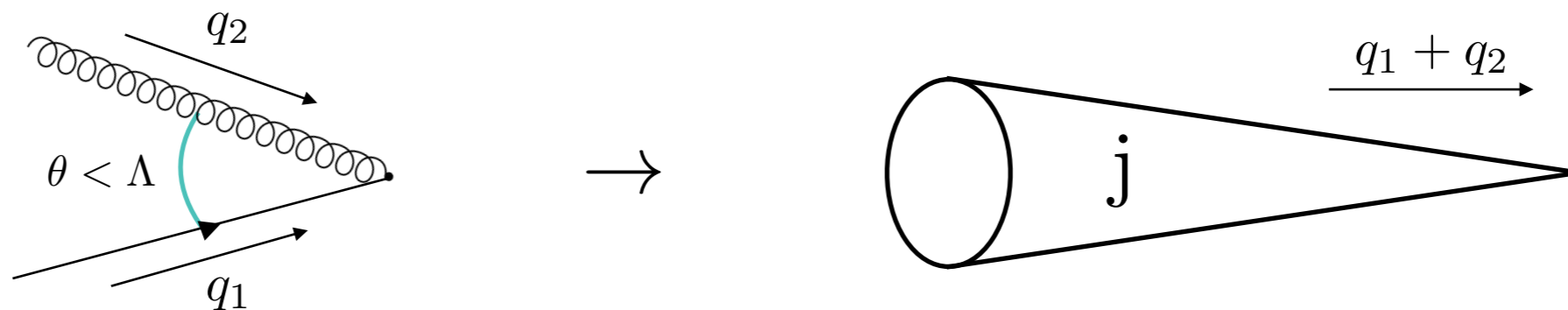


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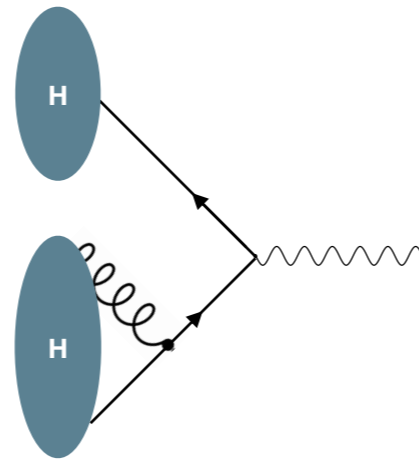
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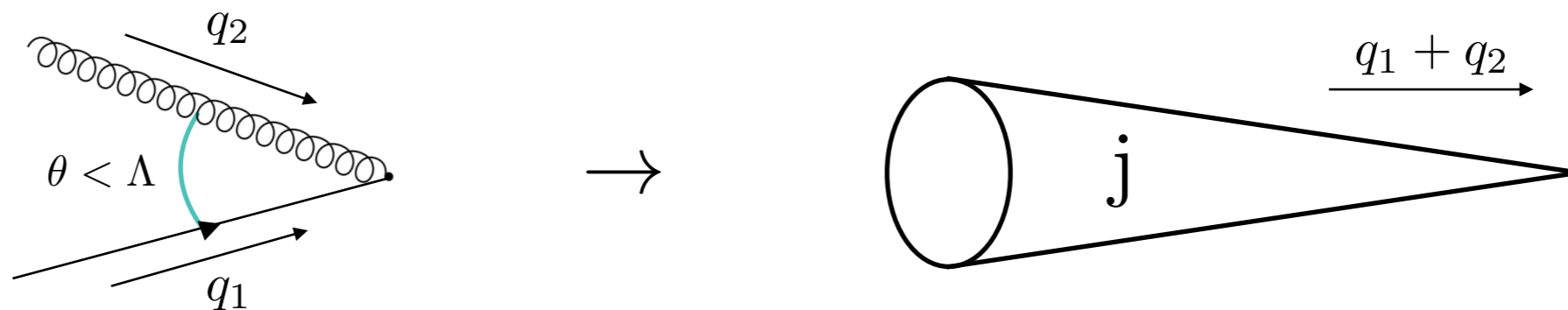
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Cluster initial states analogously to final states: symmetry initial-final state

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There are two relevant scales for the two initial state jets reconstructed:

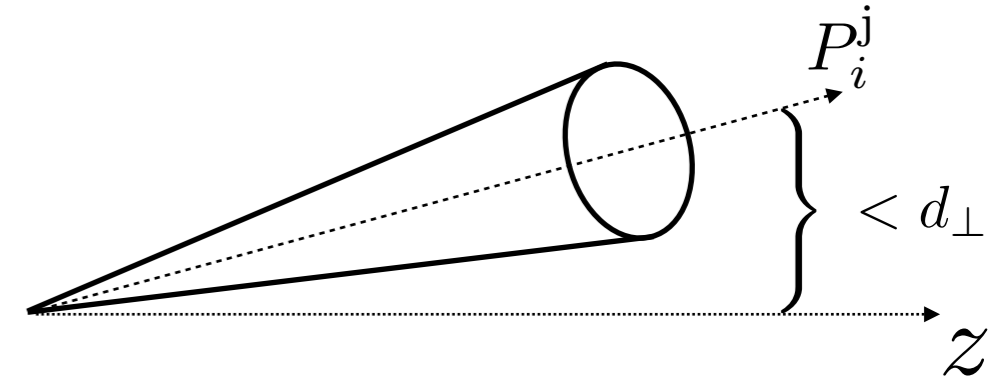
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- One measuring the allowed phase space for the **total momentum** of the jet

$$(P_i^j)_\perp < d_\perp$$

If the scale is zero, the jet lies **exactly** on the z axis



If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

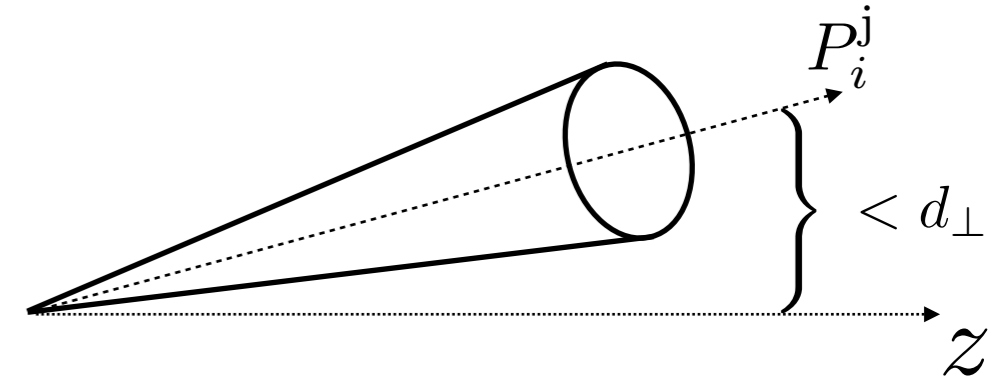
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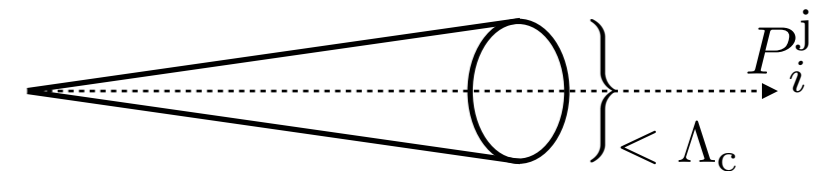


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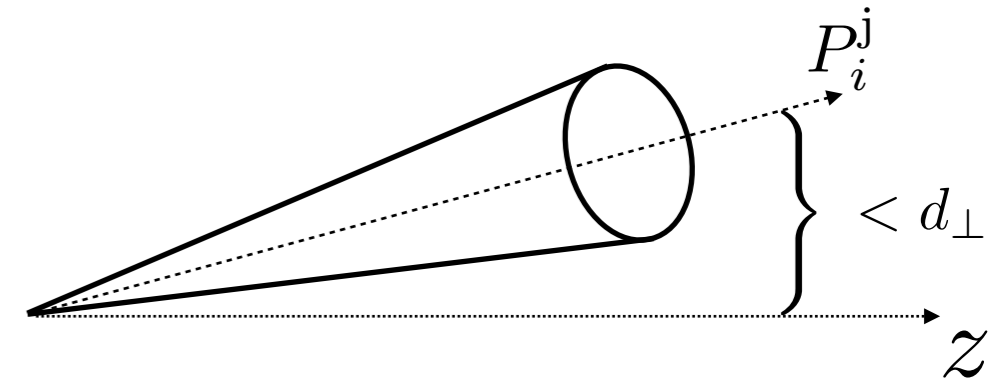
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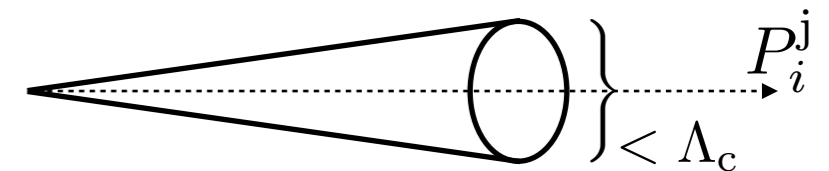


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Λ_c is the equivalent of the factorisation scale! $\approx \log(\Lambda_c)$

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

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(not in $\overline{\text{MS}}$ bar)
- Vary the factorisation scale Λ_c and interpolate the dependence on the factorisation scale **Numerical resummation?** [Banfi, Salam, Zanderighi, arXiv:0407286 \(2004\)](#)

INITIAL-STATE SINGULARITIES: “PDFs”

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We started with a very generic formalism for scattering

$$\sigma(HH \rightarrow X + nj) = \sum_m \int \left[\prod_{i=1}^m d^3 \vec{p}_i \right] f(p_1, \dots, p_m) \frac{d^m \sigma}{dp_1 \dots dp_m}(p_1, \dots, p_m \rightarrow X + nj)$$

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And we “forced” the initial-state observable to reproduce the usual factorised structure:

$$\sigma(HH \rightarrow X + nj) = \int dx_1 dx_2 f(x_1, \Lambda_c) f(x_2, \Lambda_c) \frac{d^2 \sigma_p}{dx_1 dx_2}(2j \rightarrow X + nj, \Lambda_c)$$

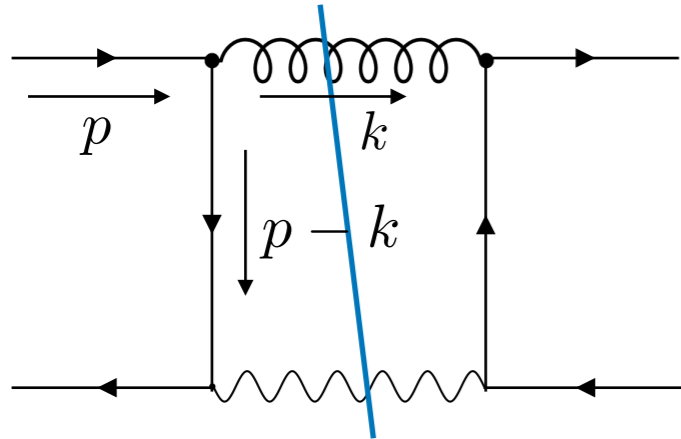
But we did not need to start from this factorised ansatz!

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Numerical example result for this finite sum of two interference diagrams:

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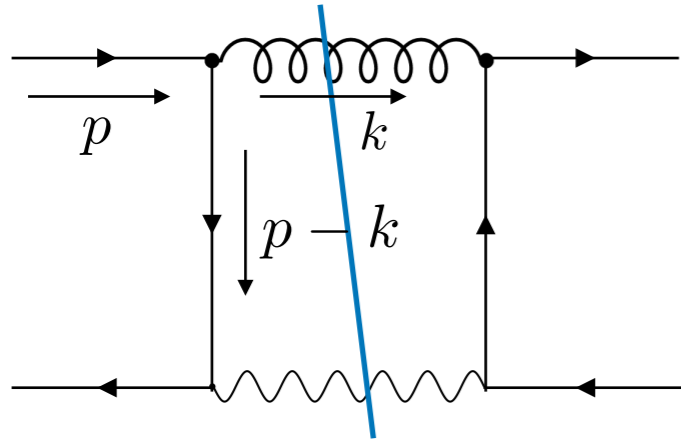


Always included!

This is the usual contribution.

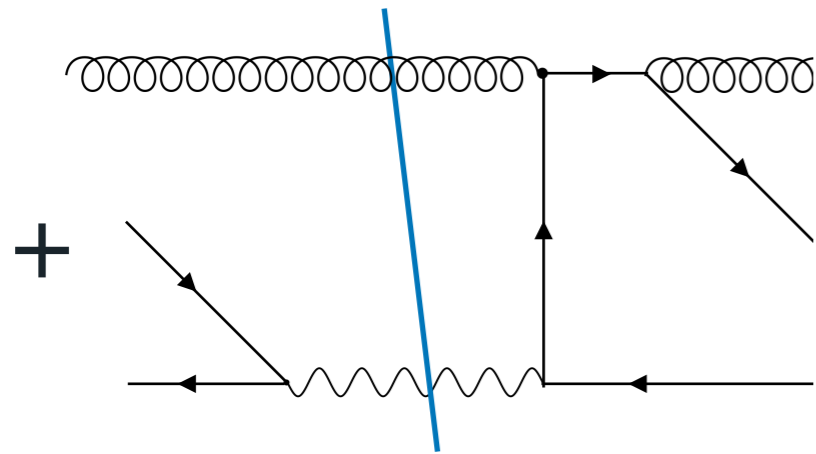
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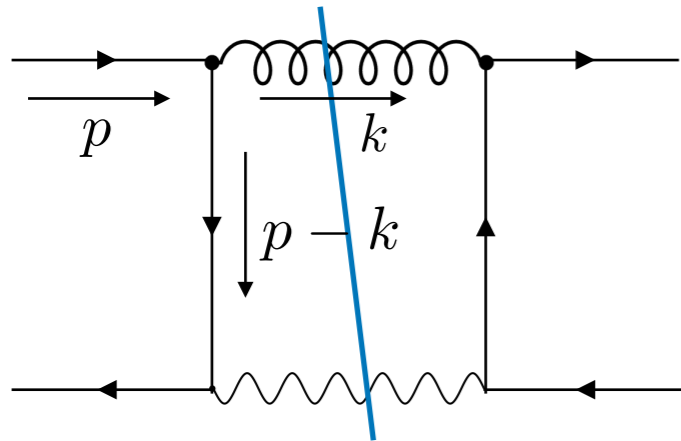
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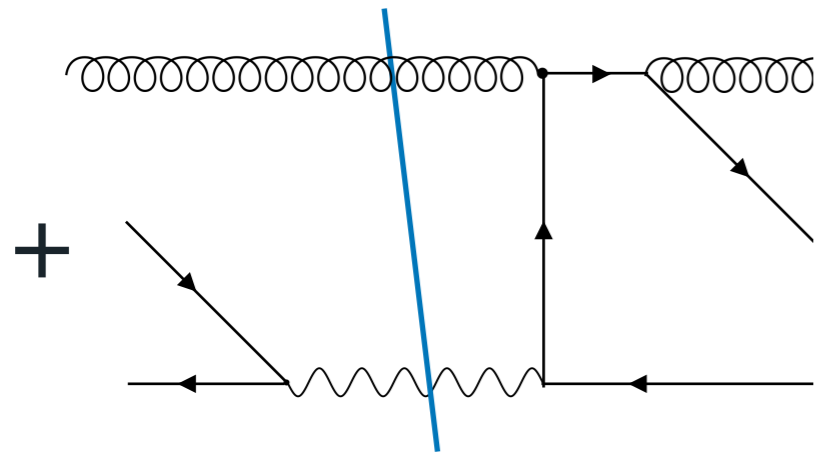
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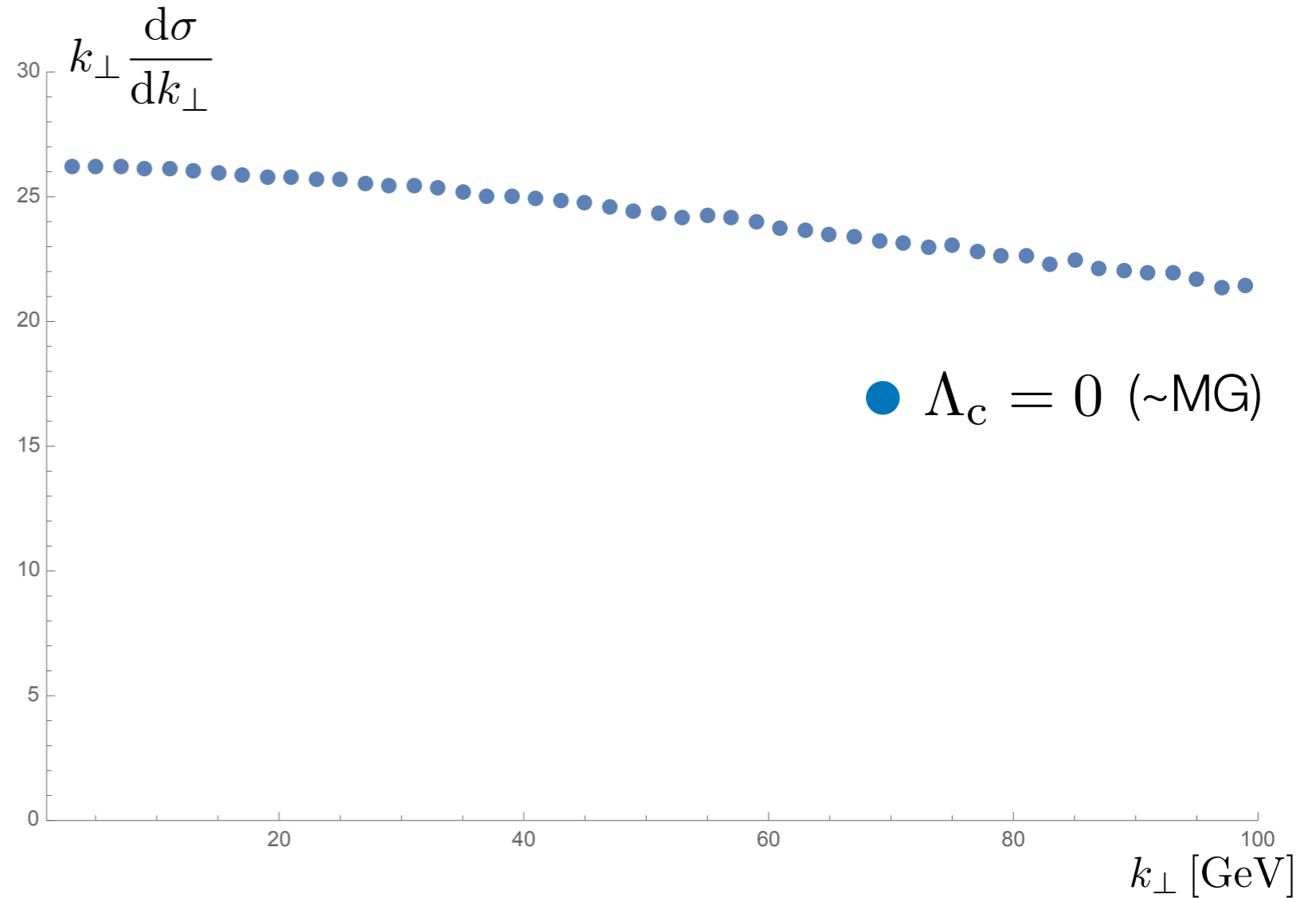
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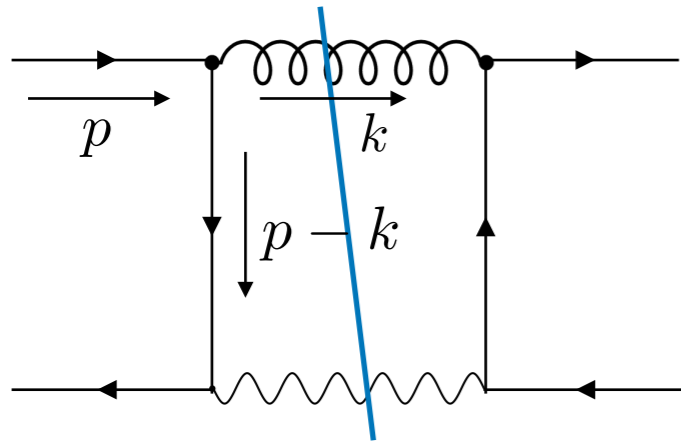


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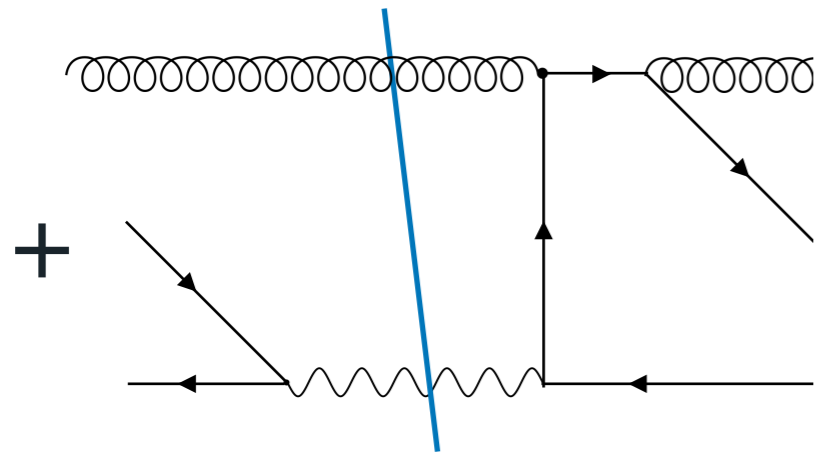


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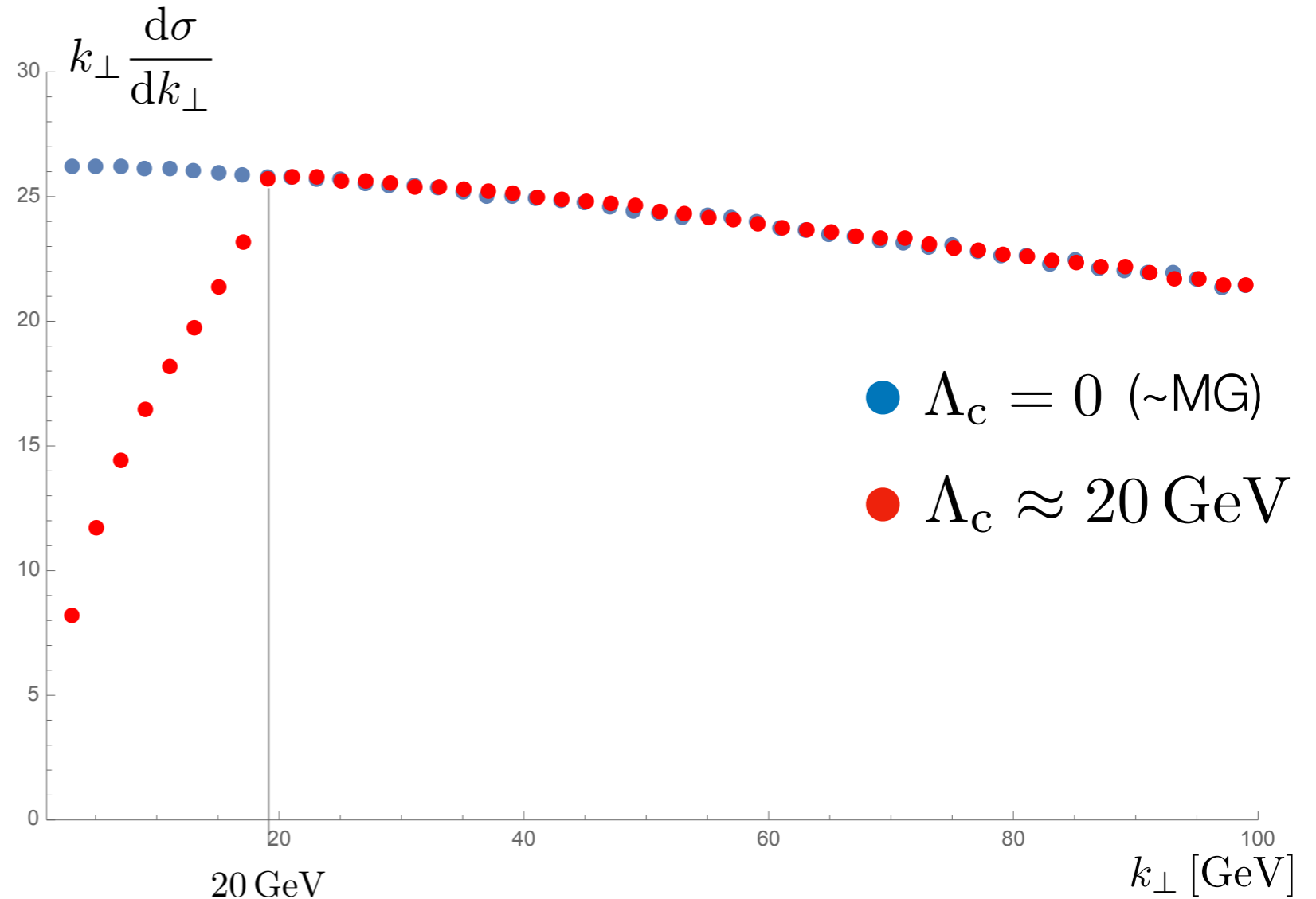
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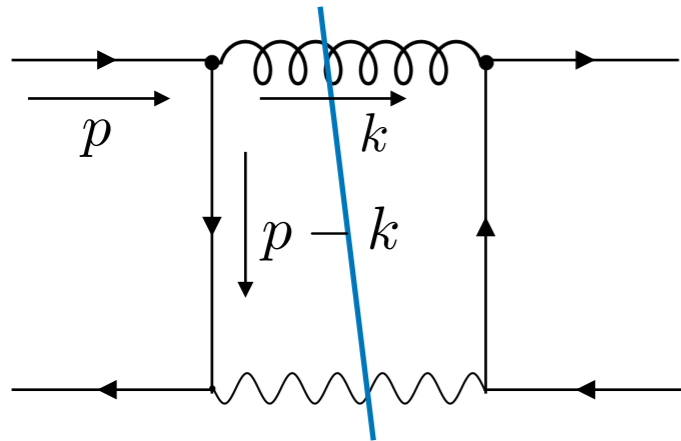


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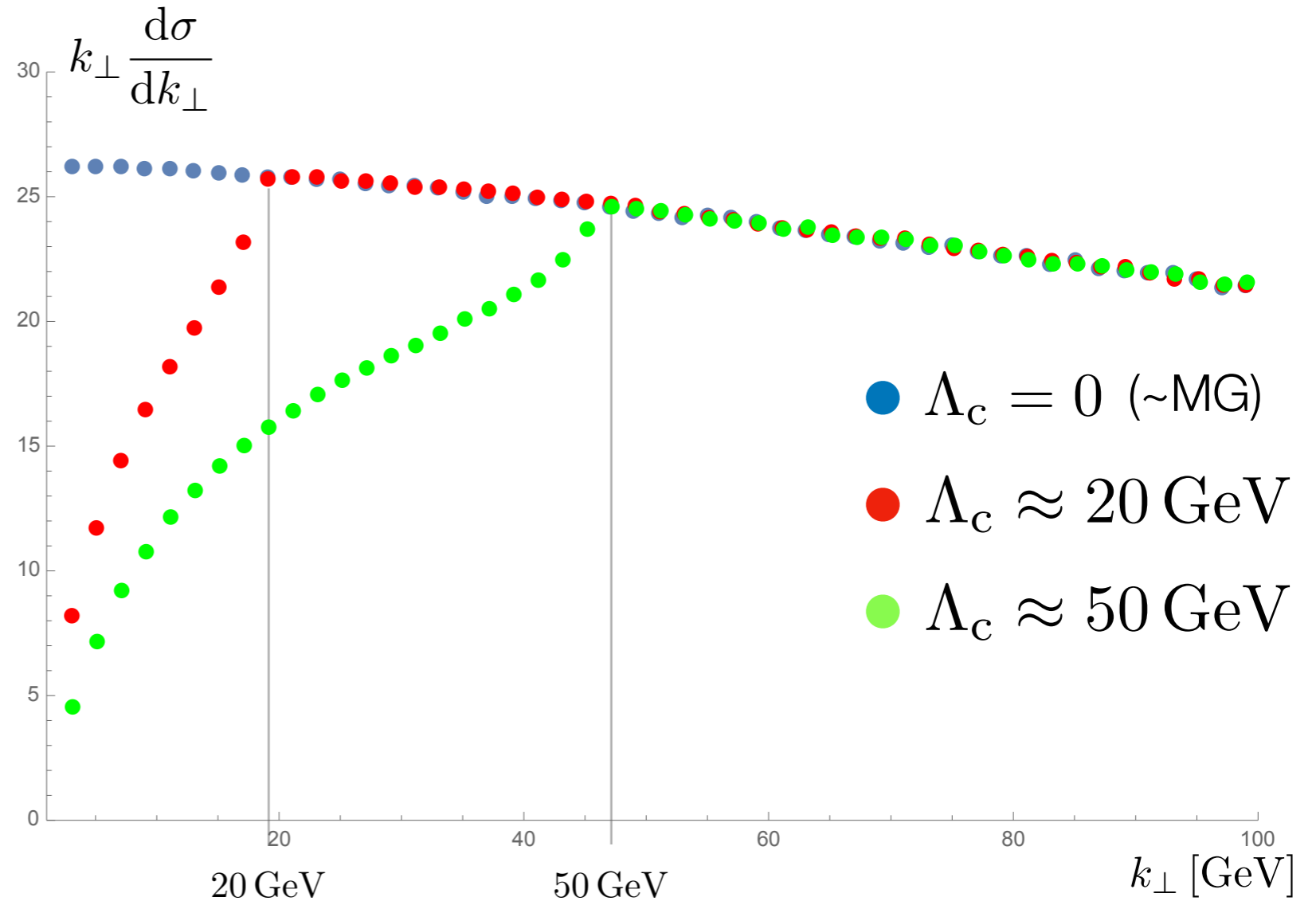
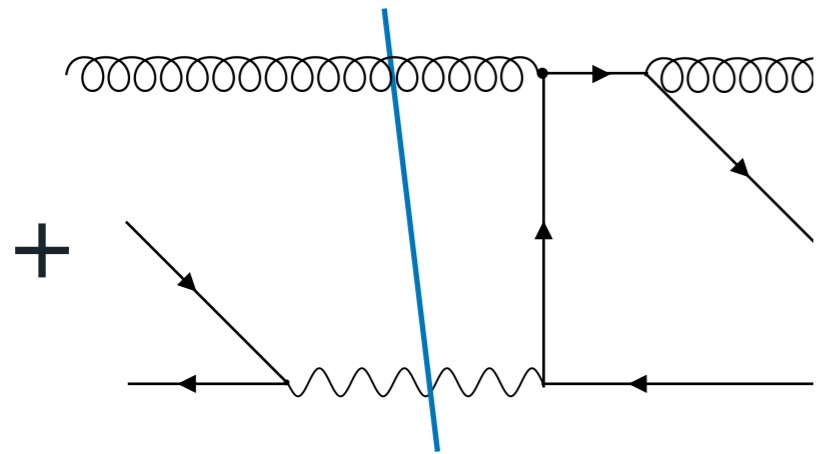


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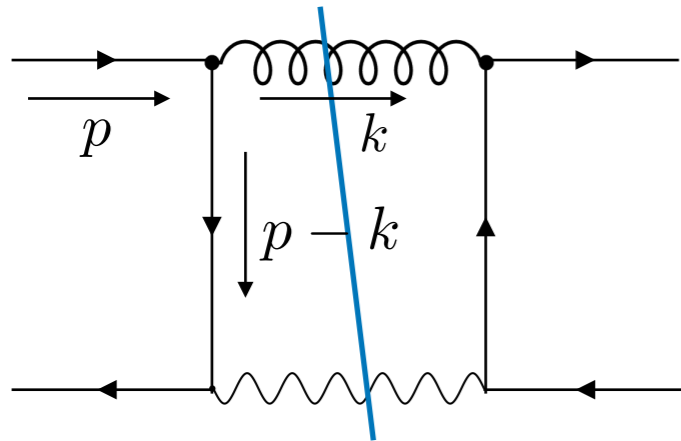
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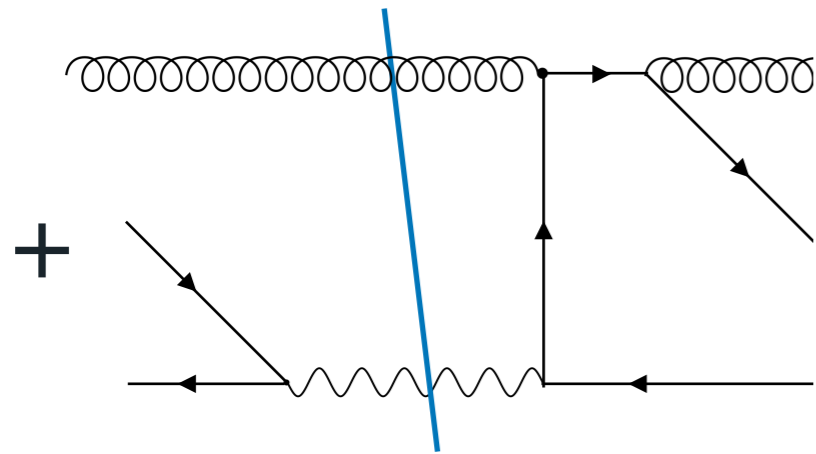
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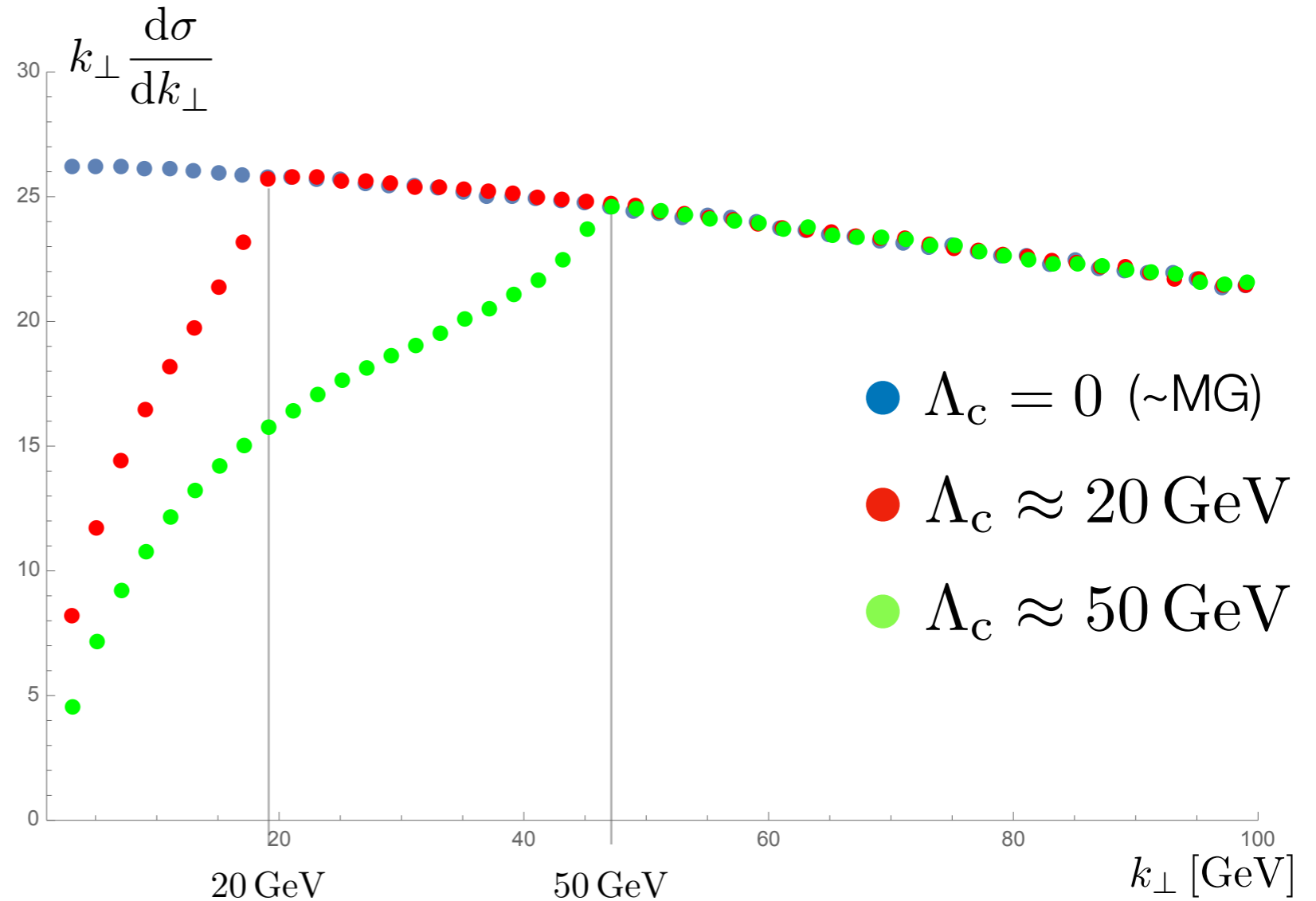
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Note that : for $\Lambda_c > 50$ GeV
the distribution does not change anymore
because highest separation of two partons
in a jet is of **order of Z mass**

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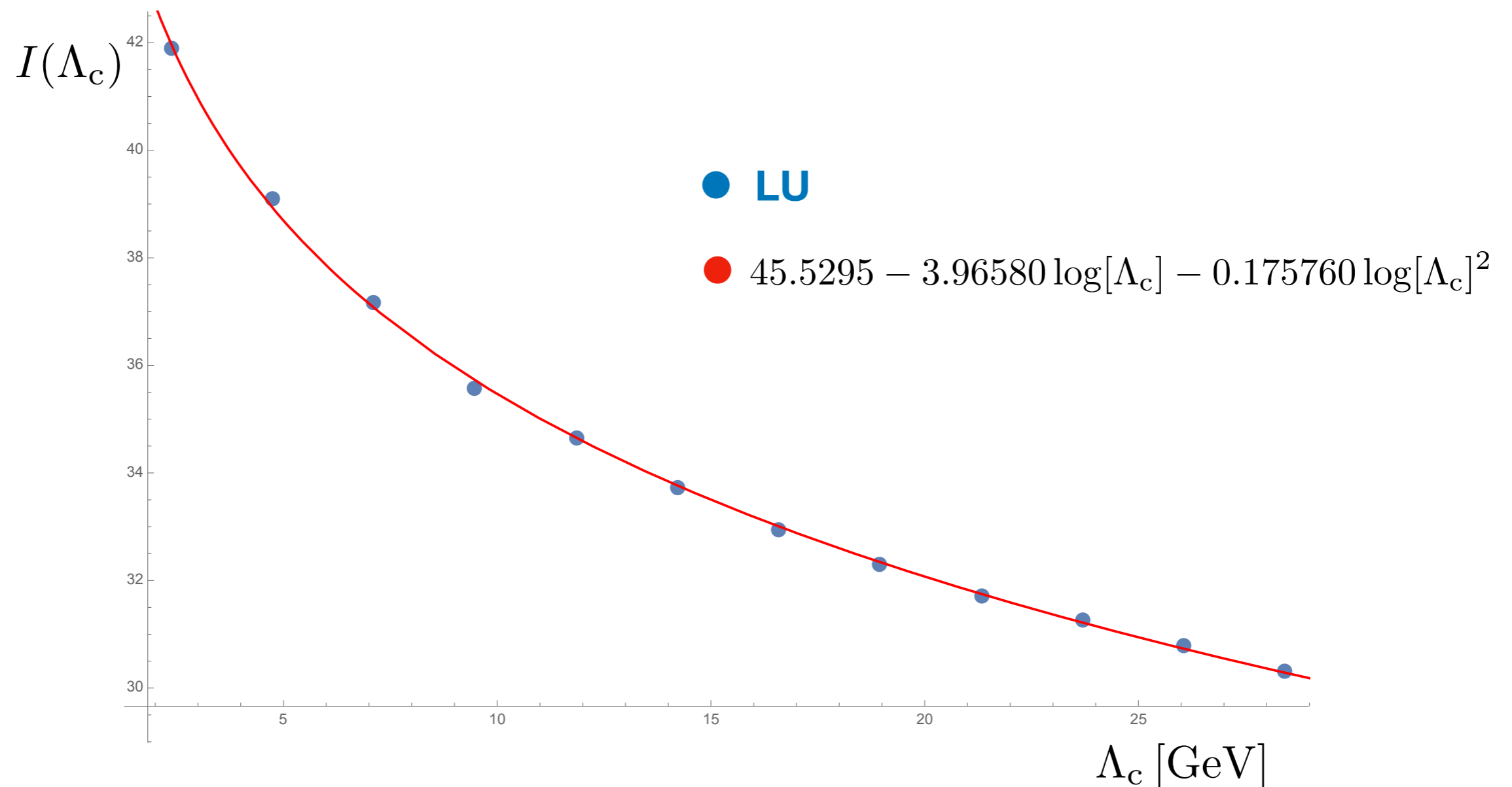
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SUBTRACTION VS LOCAL UNITARITY

- One can introduce the following **local (in x) counterterm**:

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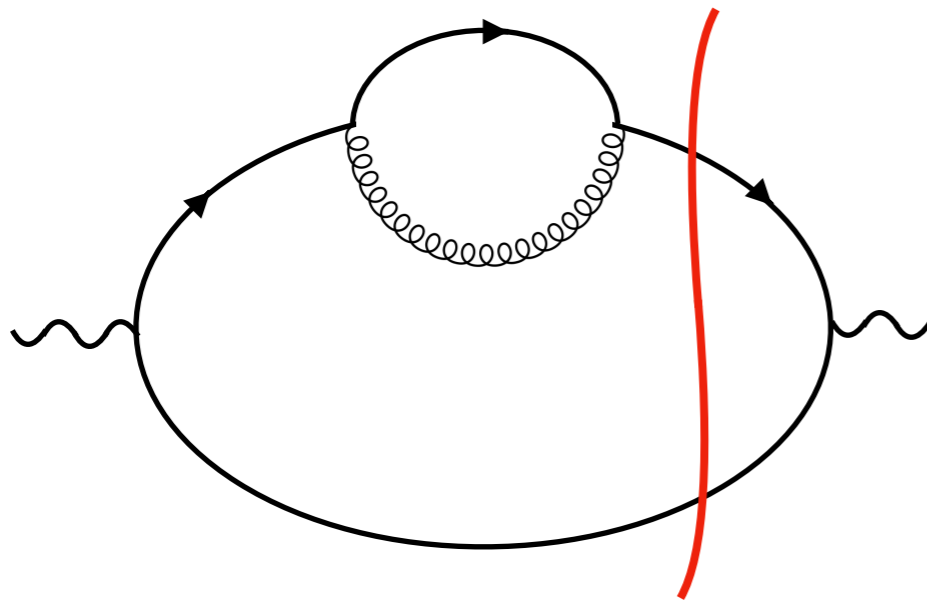
- Local Unitarity** aligns the measure and combines “real and virtual”:

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LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider *truncated* amplitudes only :



Traditional Cutkosky rule

$$\int_{\vec{p}} = -2\pi i \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

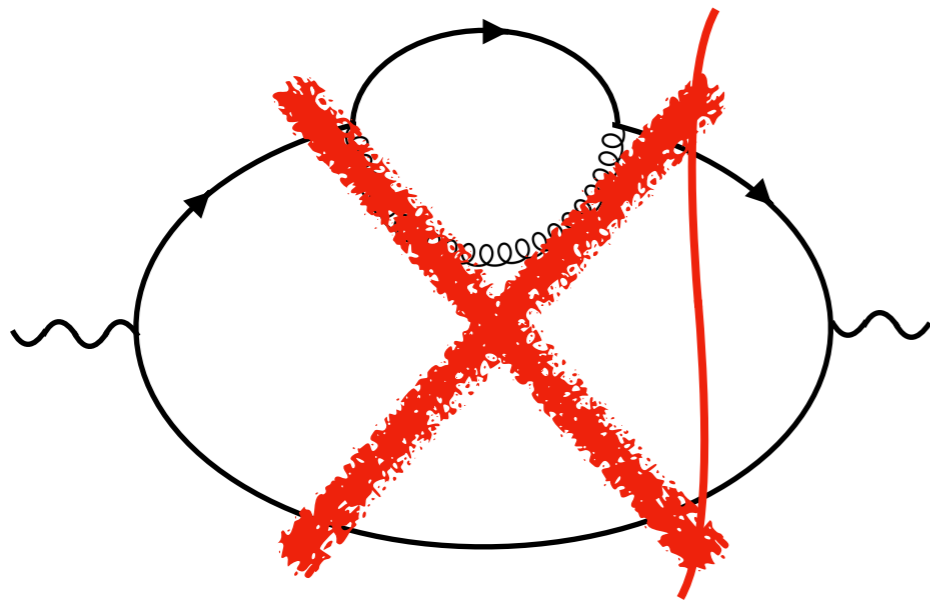
$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

would not apply here !

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider *truncated* amplitudes only :



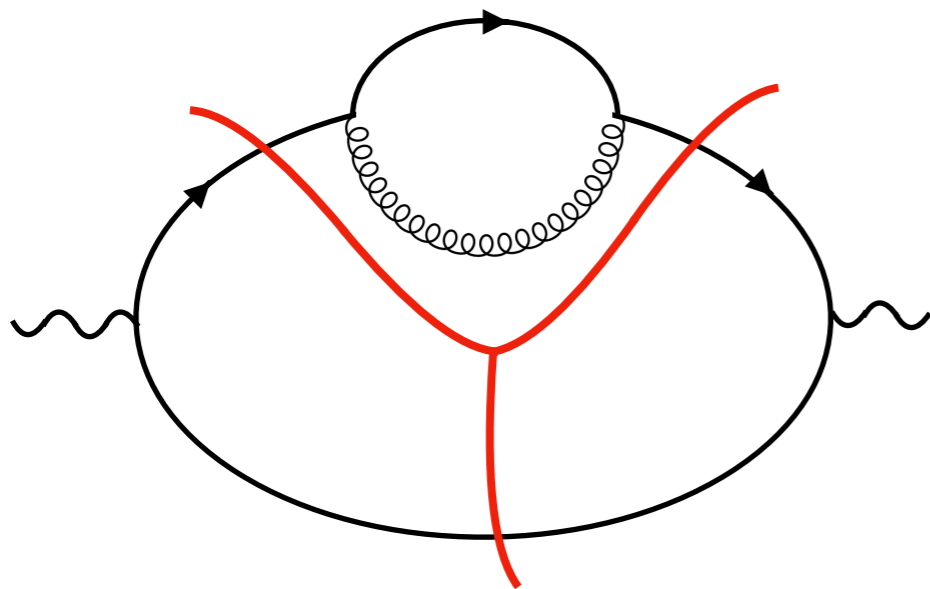
Traditional Cutkosky rule

$$\overrightarrow{p} \int = -2\pi i \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

would not apply here !

So consider this Cutkosky cut as a **higher-order residue** → **Generalised cutting rule**



$$\dots \int \dots = -2\pi i \frac{\delta^{(n)} [p^0 - E(\vec{p})]}{(p^0 + E(\vec{p}))^2}$$

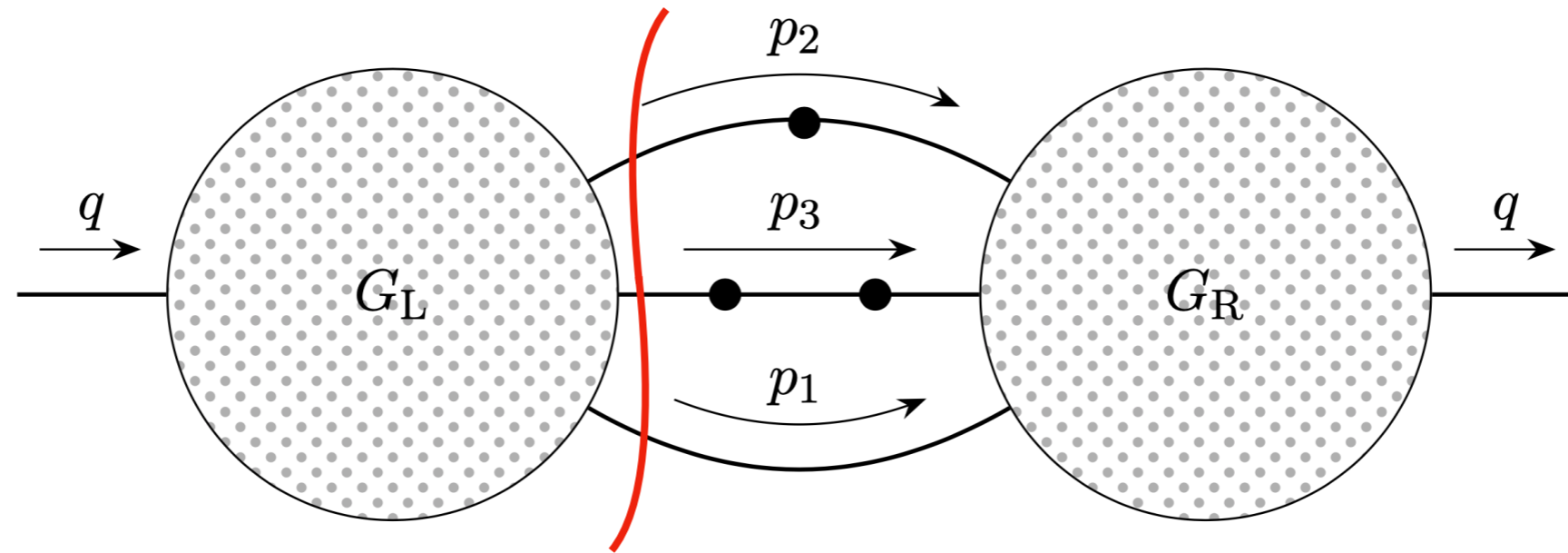
n - times

$$\int dx \delta^{(n+1)} [x] f(x) = \frac{1}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=0}$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :

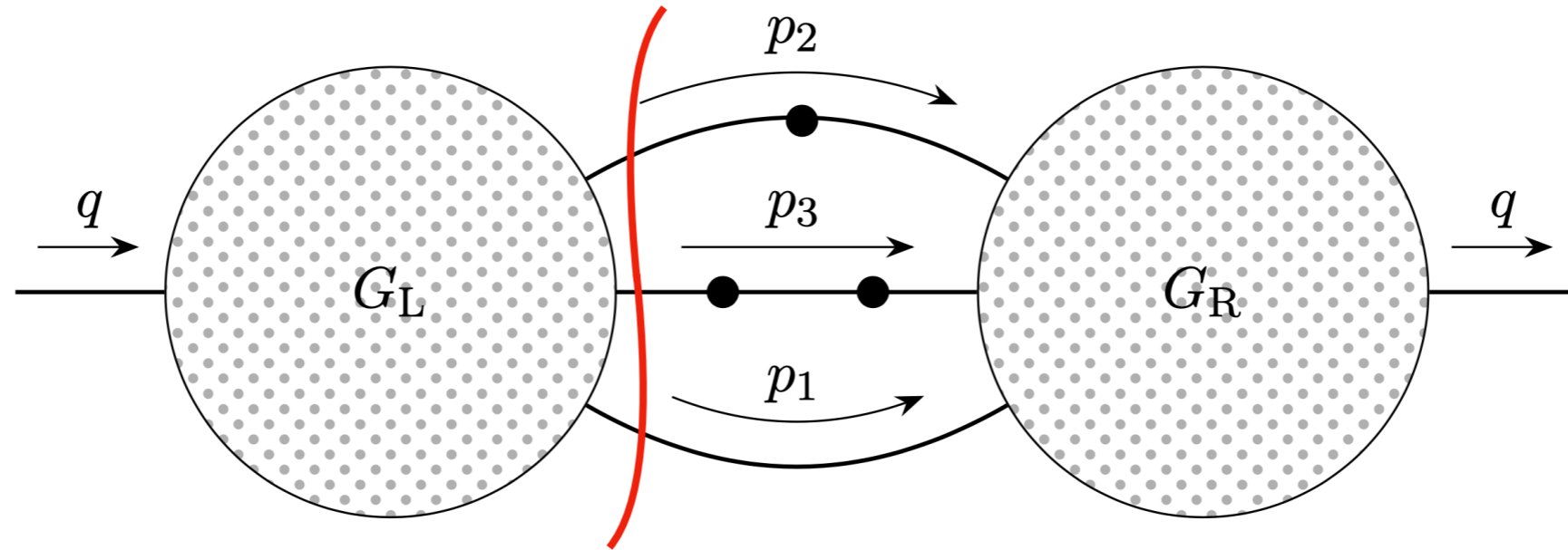


$$\propto \delta^{(1)} [p_1^0 - E(\vec{p}_1)] \delta^{(2)} [p_2^0 - E(\vec{p}_2)] \delta^{(3)} [p_3^0 - E(\vec{p}_3)]$$

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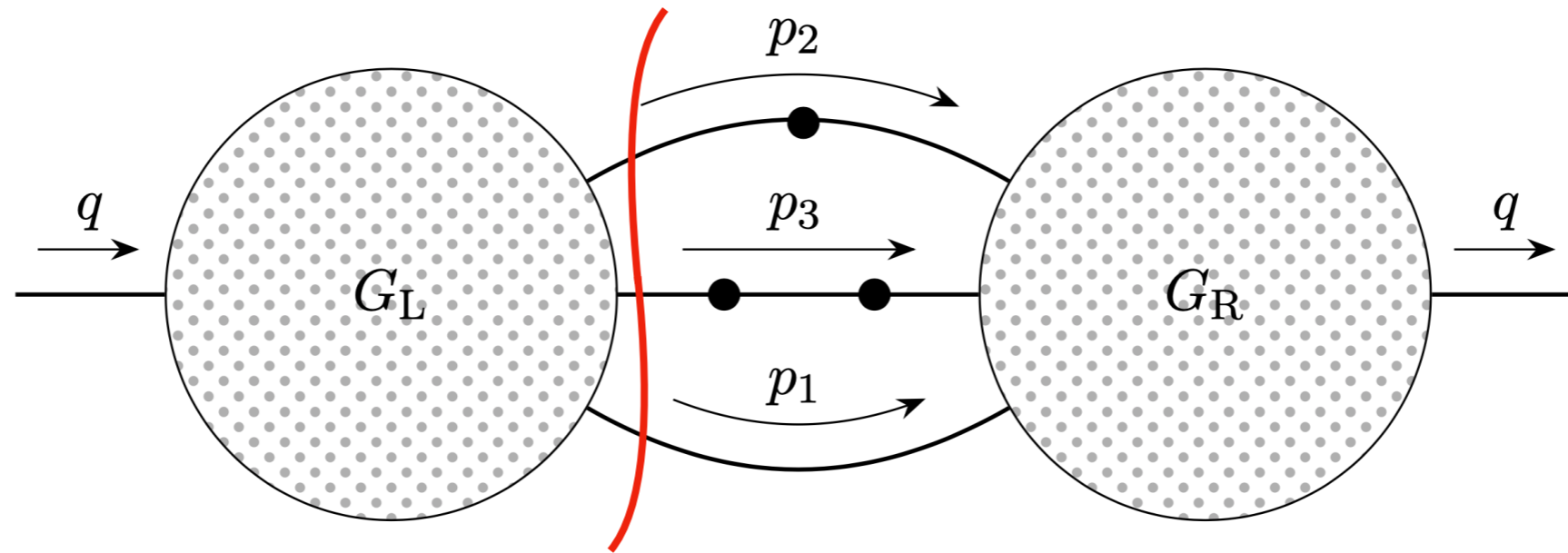
Derivatives can act on each other because: $p_3 = q - p_1 - p_2$

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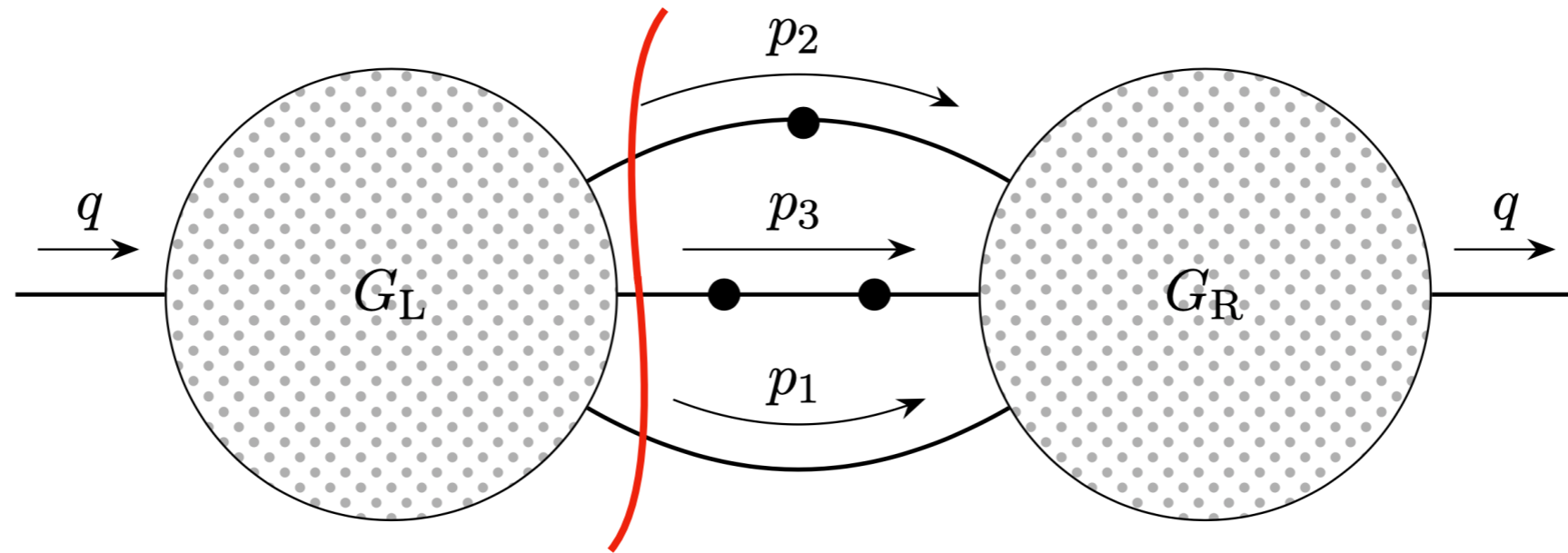
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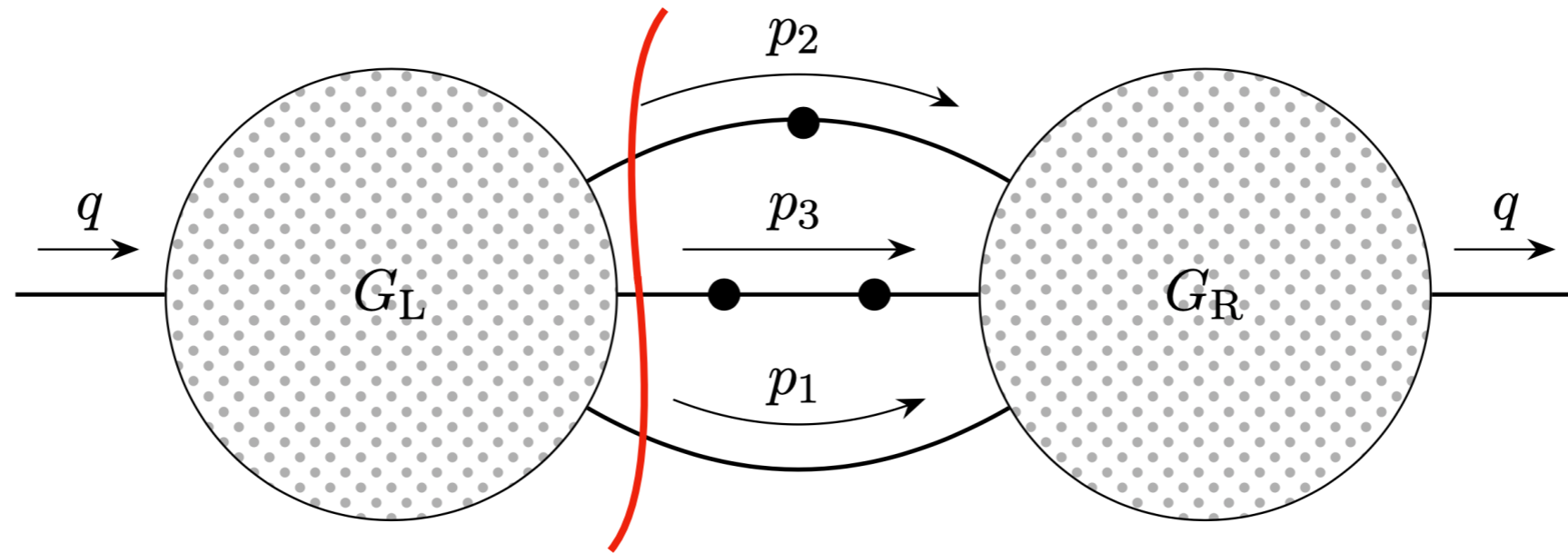
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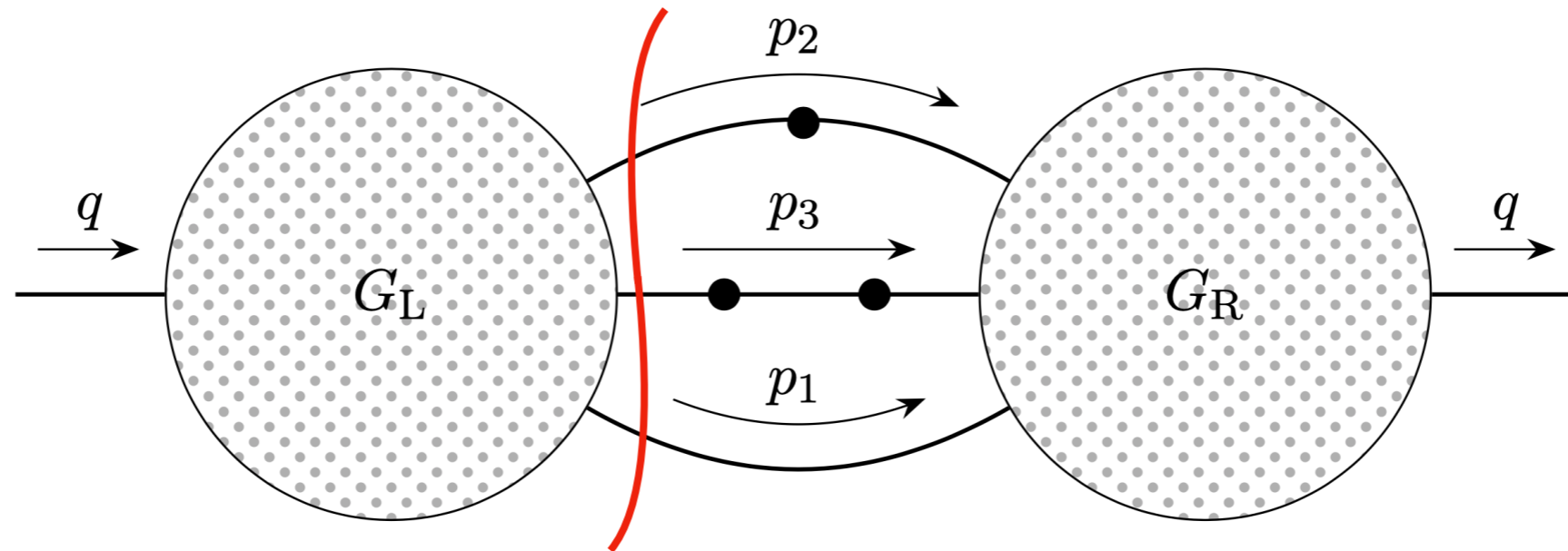
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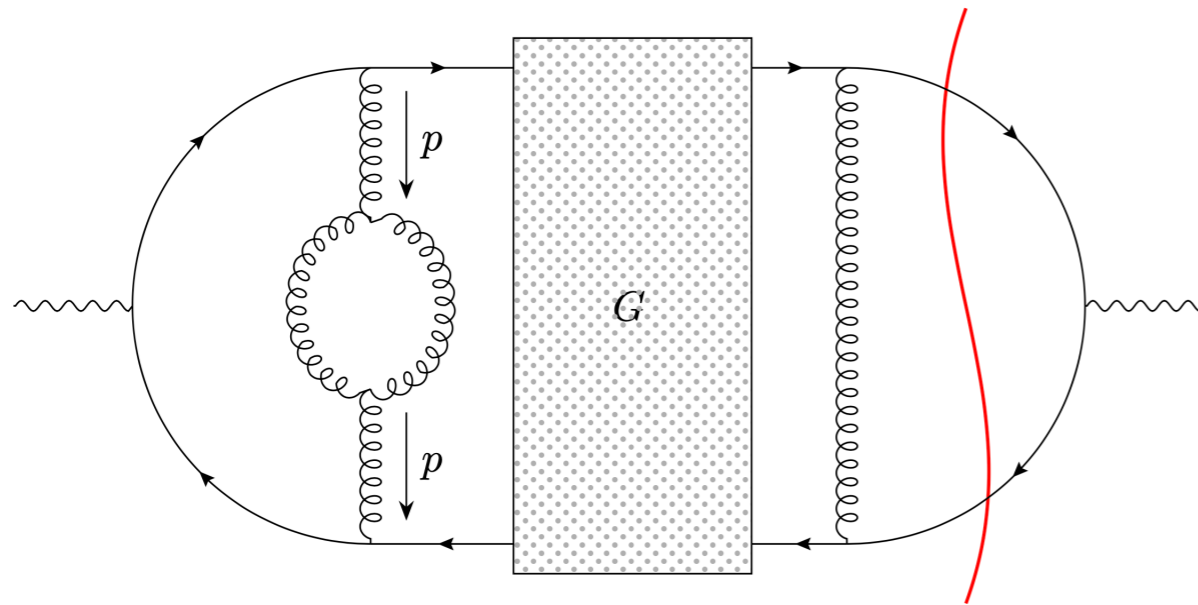
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Use **multivariate dual numbers** (auto-differentiation) in order to **efficiently compute amplitude derivatives** of G_L and G_R in p_2^0 and t (in this example)

SPURIOUS SOFT SINGULARITIES

[Capatti, VH, Ruijl, arxiv : 2203.11038]



$$\propto \frac{1}{(p^2)^2}$$

For $p = 0$

this induces a spurious soft divergence whose cancellation has nothing to do with KLN!

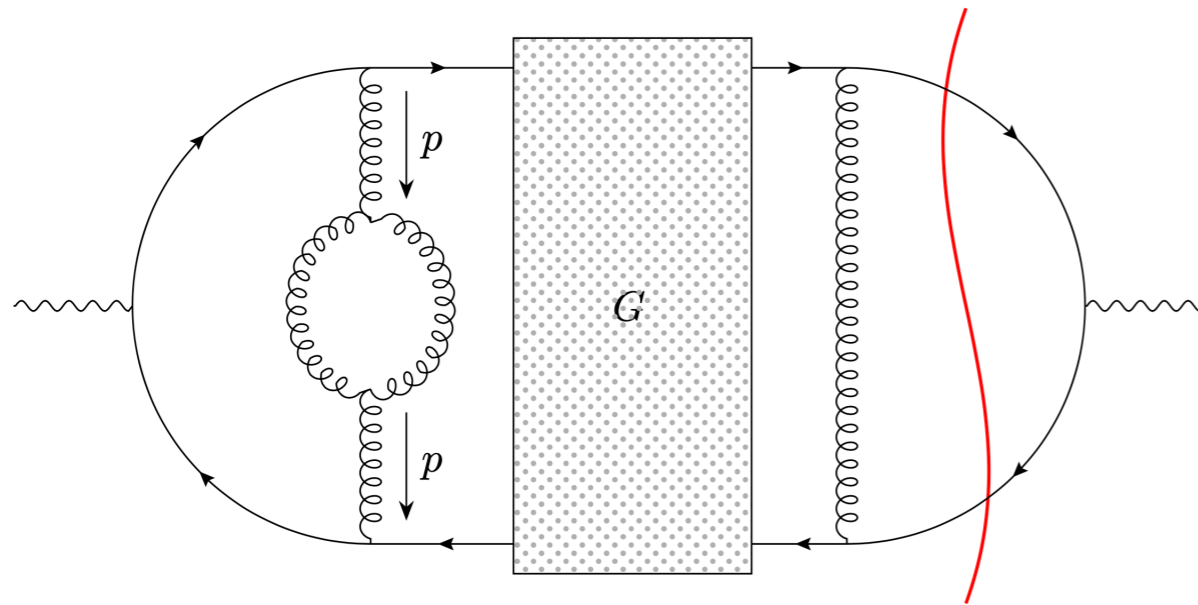
Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

At the integrated level, we have

$$\bullet \text{---} \text{loop} \text{---} \bullet \propto \frac{1}{p^2} (p^2 g^{\mu\nu} - p^\mu p^\nu) \frac{1}{p^2}$$

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but not at the local level; we must introduce **spurious soft counterterms** :

$$\text{loop} - \tilde{T}_1 \left(\text{loop} \right), \quad \tilde{T}_{\text{soft_dod}}(\gamma) = \sum_{j=0}^{\text{soft_dod}(\gamma)} \frac{1}{j!} \frac{d^j}{d\lambda^j} \gamma(\lambda p) \Big|_{\lambda=0}, \quad [\tilde{T}] = 0$$

COMBINED UV AND SPURIOUS IR FOREST

Eureka moment:

- Remarkably, we always have : $\text{soft_dod} = \text{UV_dod} - 1$
- Spurious soft expansion also valid as UV counter term.
- Spurious soft IR forest similar to the one produced by the R-operation

so that we can combine the UV and spurious soft subtraction as one !

$$\hat{T}_{\text{dod}} = T_{\text{dod}} + \tilde{T}_{\text{dod}-1} - T_{\text{dod}}\tilde{T}_{\text{dod}-1}$$

until we realised that we had just re-invented the wheel: [J. H. Lowenstein, 1976]

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Novelty though: **automatic renormalisation** of fermion **masses** in the **OS** scheme:

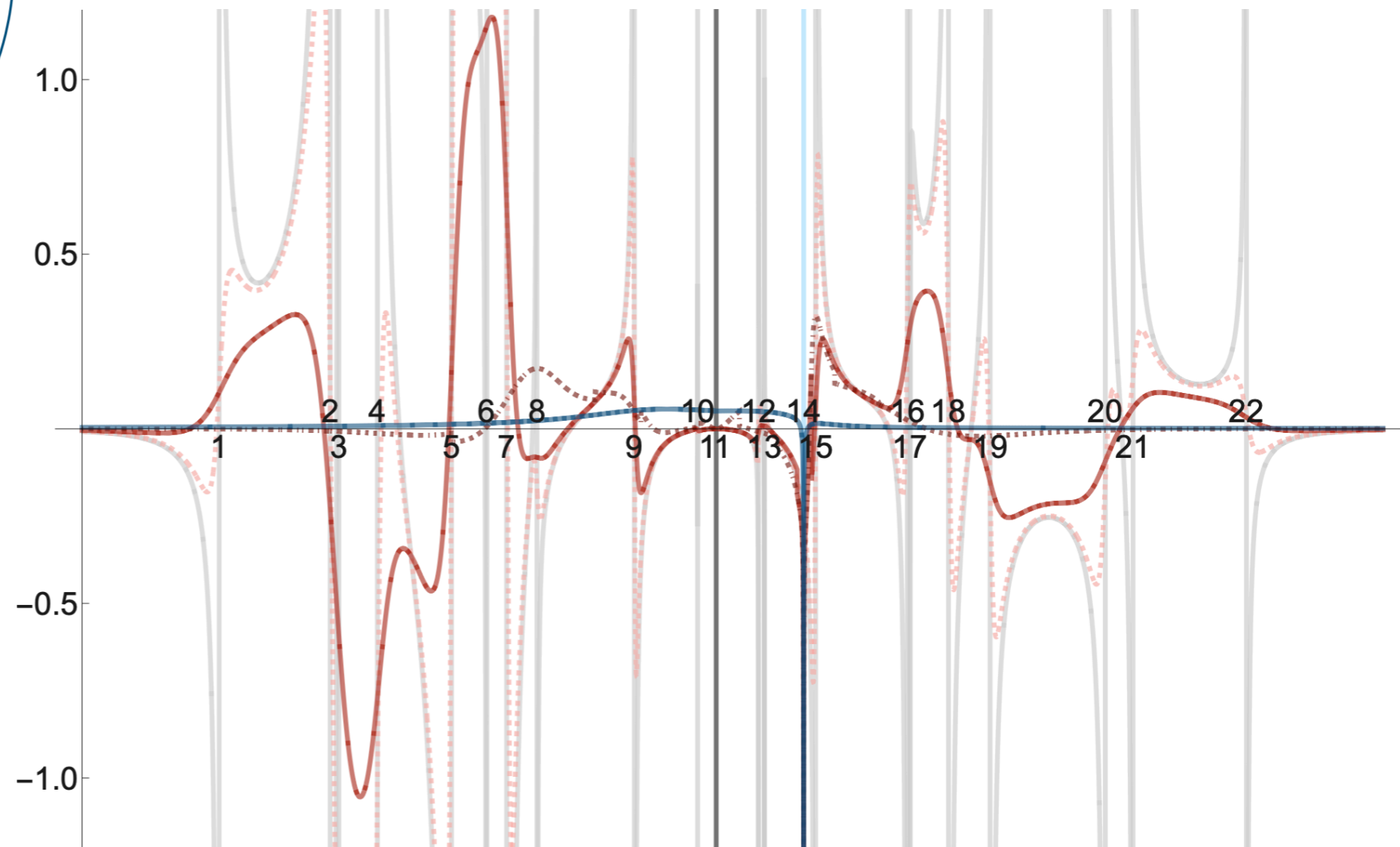
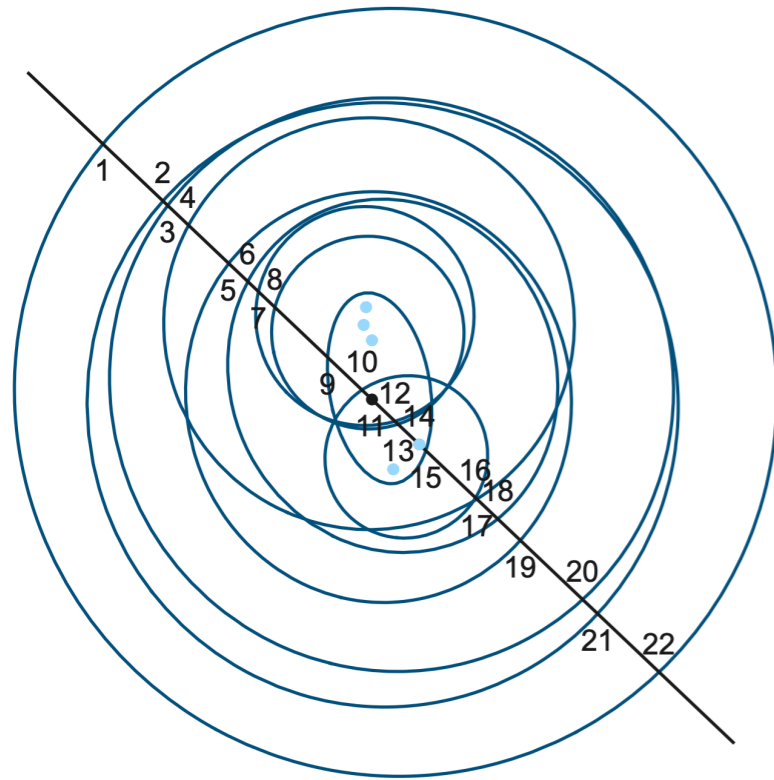
$$T^{\text{os}\pm} \left(\Sigma = \overset{p}{\rightarrow} \bullet \text{---} \right) = (1 \pm \gamma^0) \Sigma(p = \pm p^{\text{os}}), \quad p^{\text{os}} = (m, 0, 0, 0)$$

$$\frac{1}{2} \left([T^{\text{os}+}(\Sigma)] + [T^{\text{os}-}(\Sigma)] \right) = \delta m^{\text{os}}$$

Implying that our local UV counterterm T^{os} automatically generates the OS mass renormalisation counterterm !

THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : [2110.06869](https://arxiv.org/abs/2110.06869)]



— $I_{\epsilon=0}$

— $I_{\text{subtracted}}$

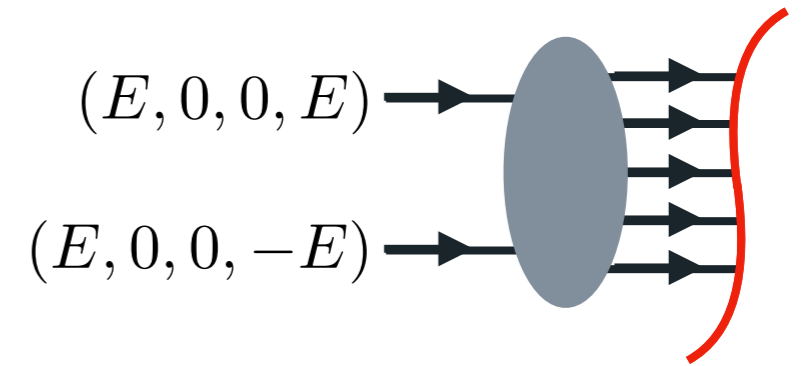
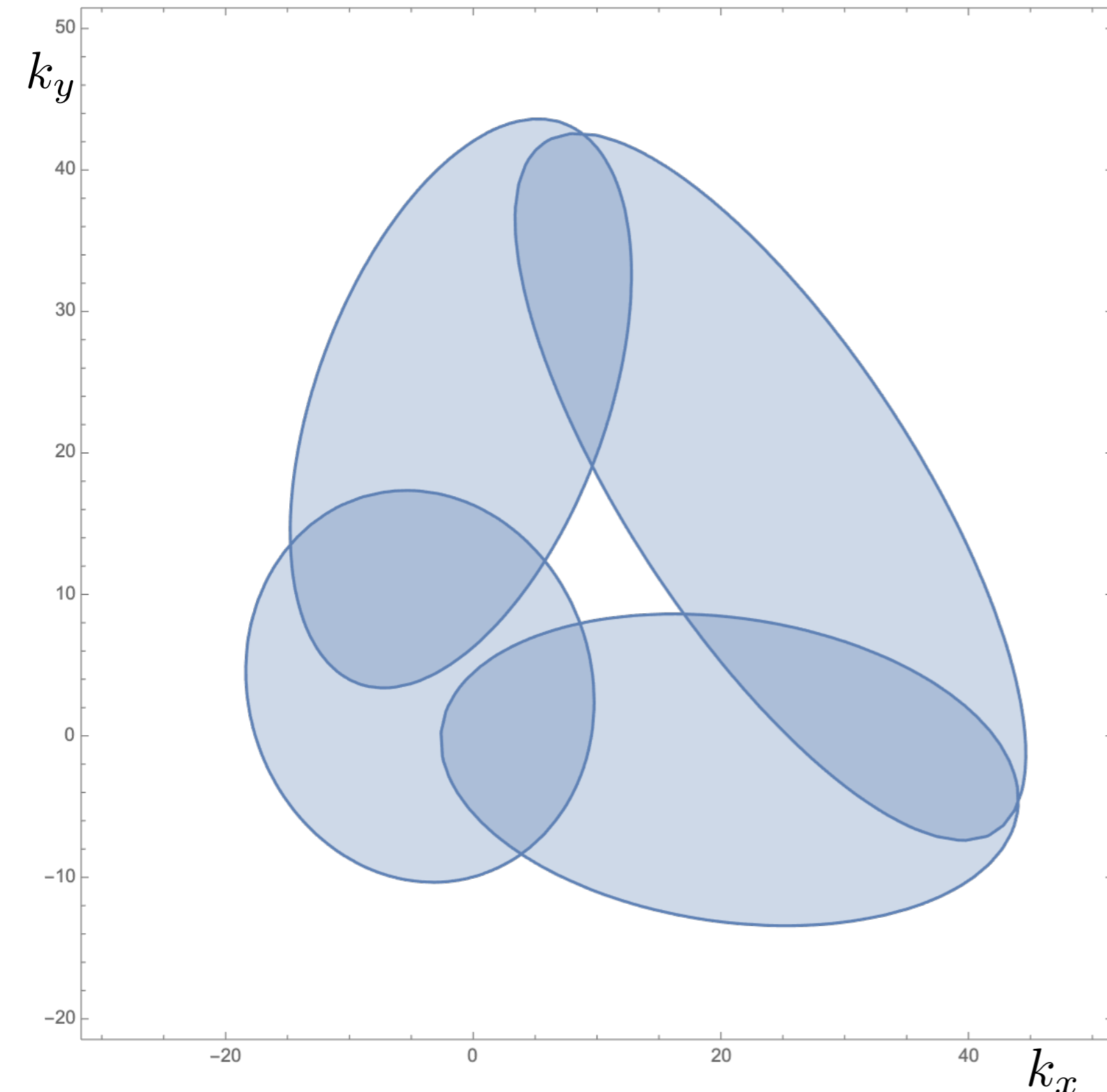
..... $\text{Re } I_{\text{deformed}}(\lambda_{\text{max}} = 3)$

— $\text{Re } I_{\text{deformed}}(\lambda_{\text{max}} = 10)$

..... $\text{Re } I_{\text{deformed}}(\lambda_{\text{max}} = 300)$

LOCALITY UNITARITY: CAUSAL FLOW

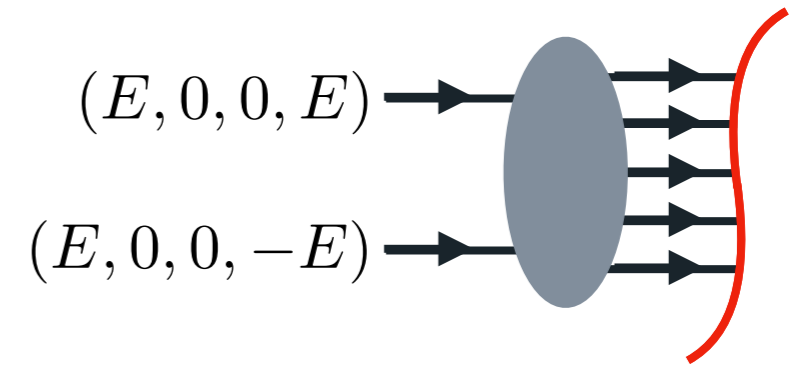
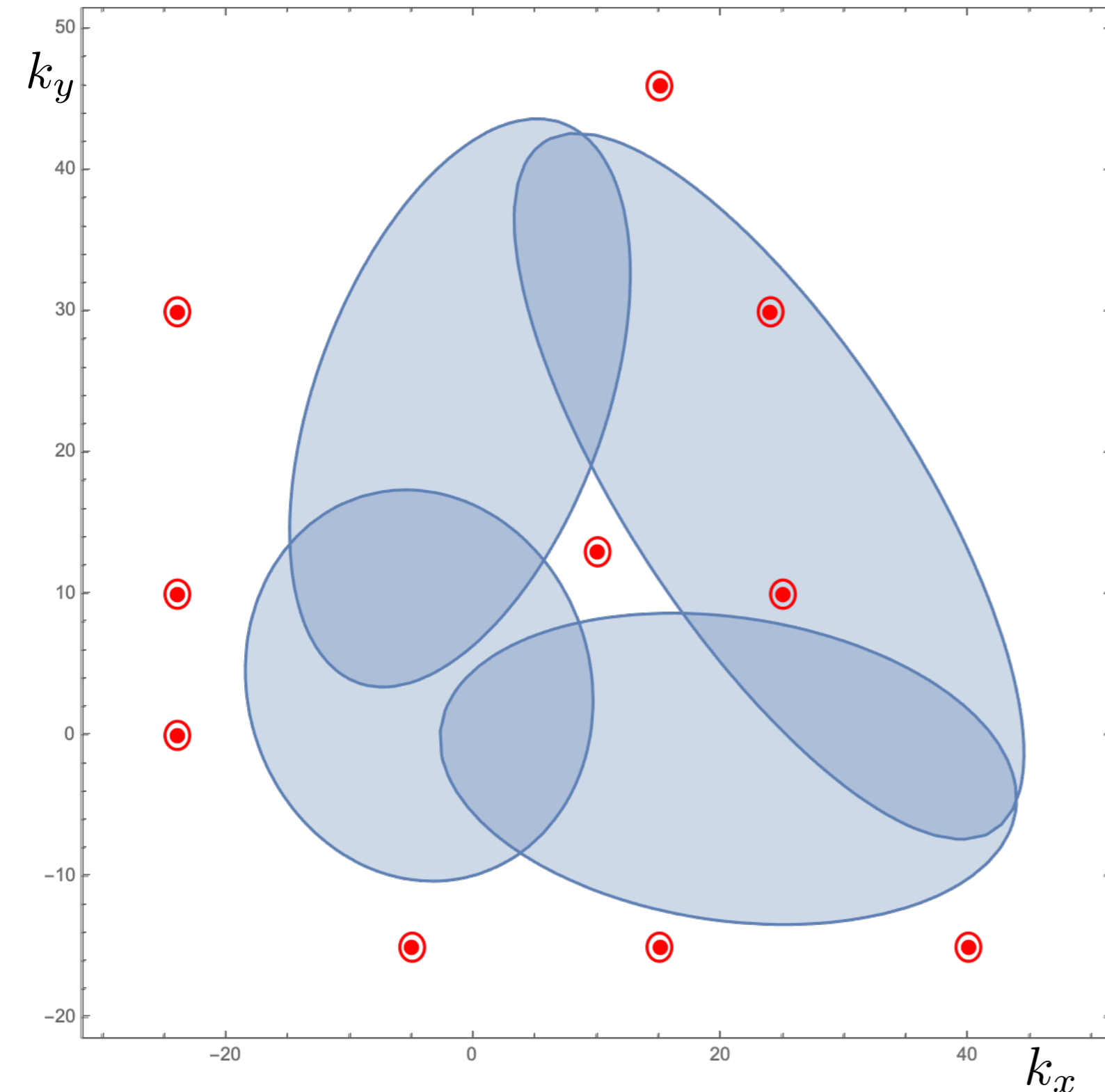
The **rescaling** change of variables is however **not general** : (**but always sufficient in practice!**)



Ex: **Box_4E** from sect. 3.1 of
[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

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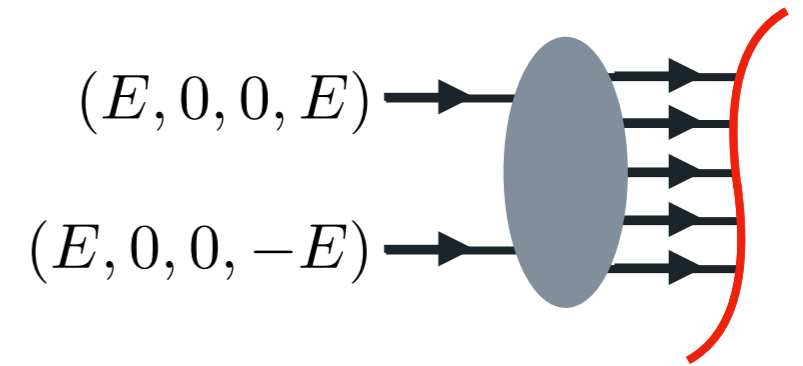
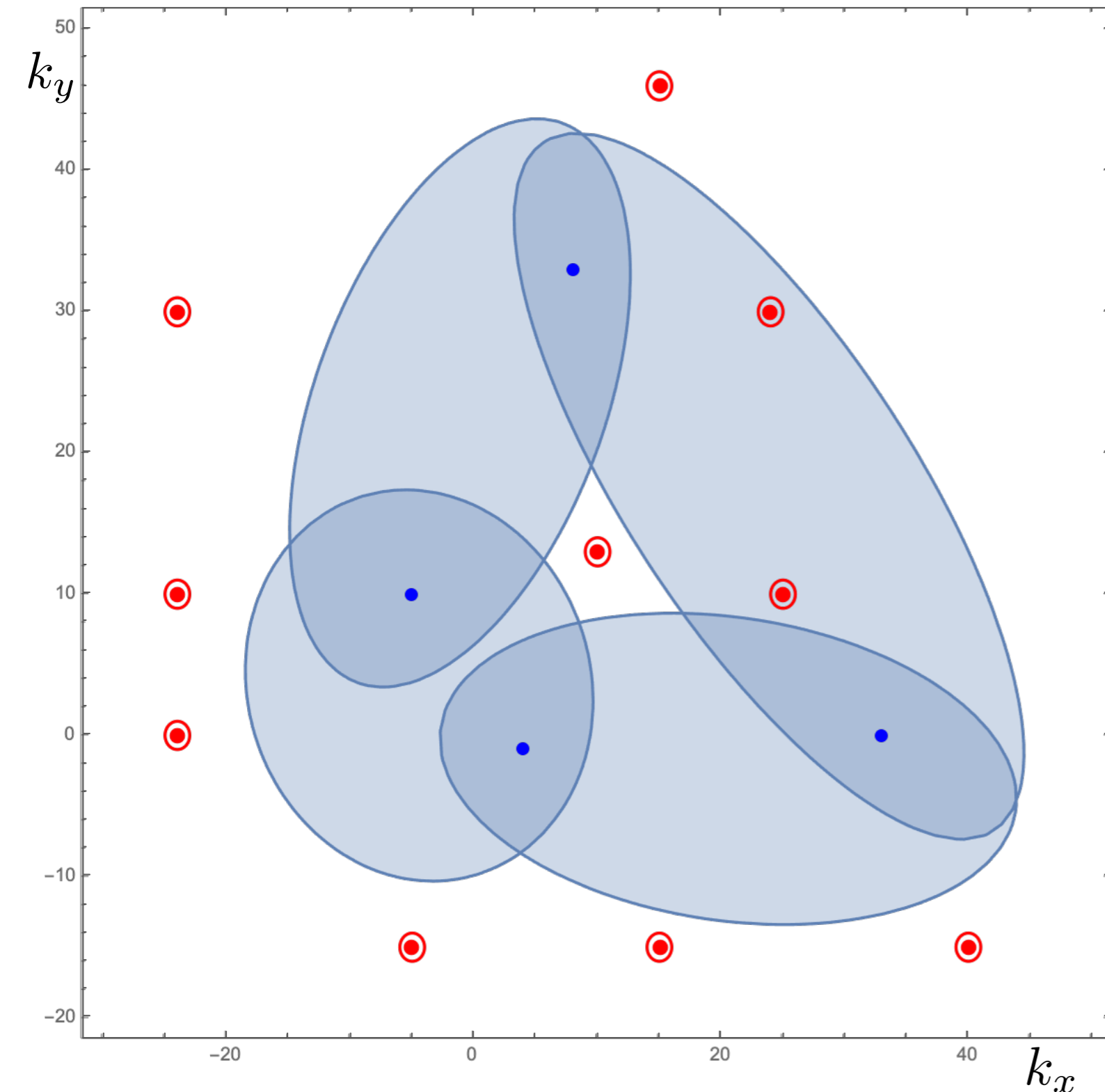
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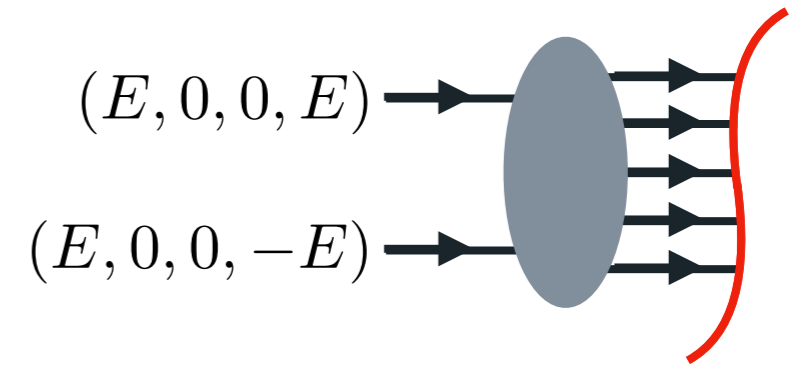
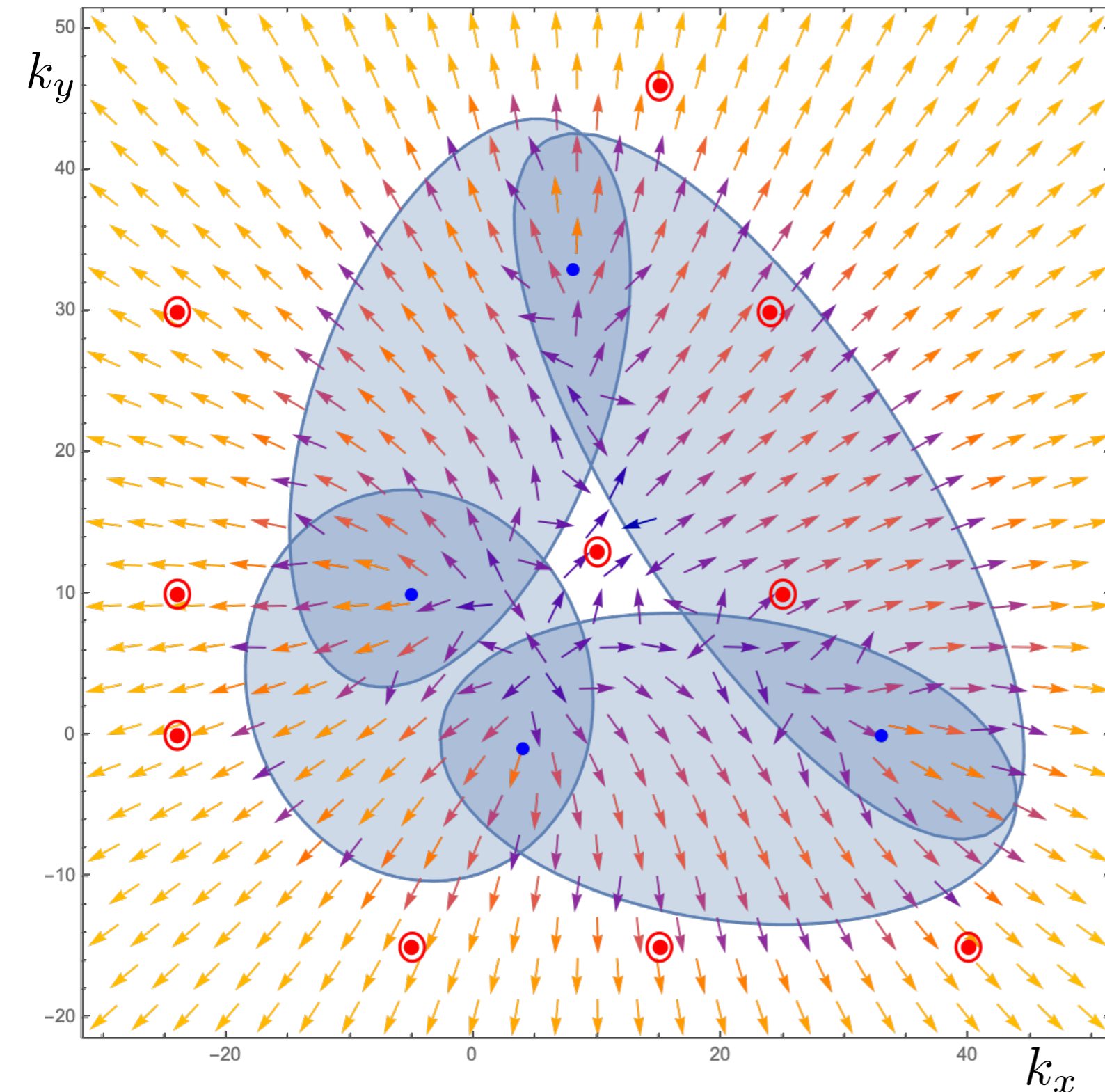
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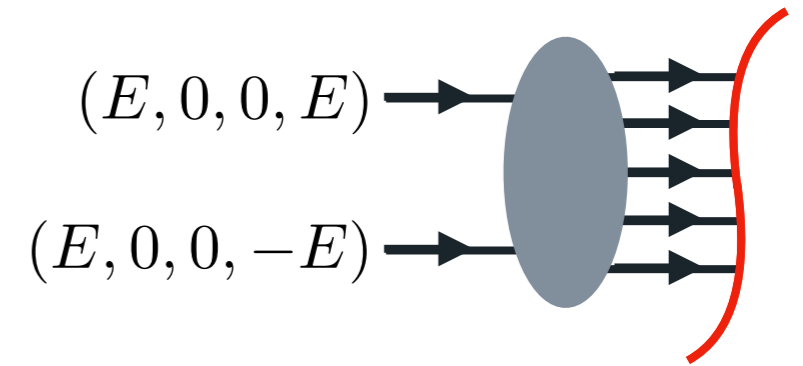
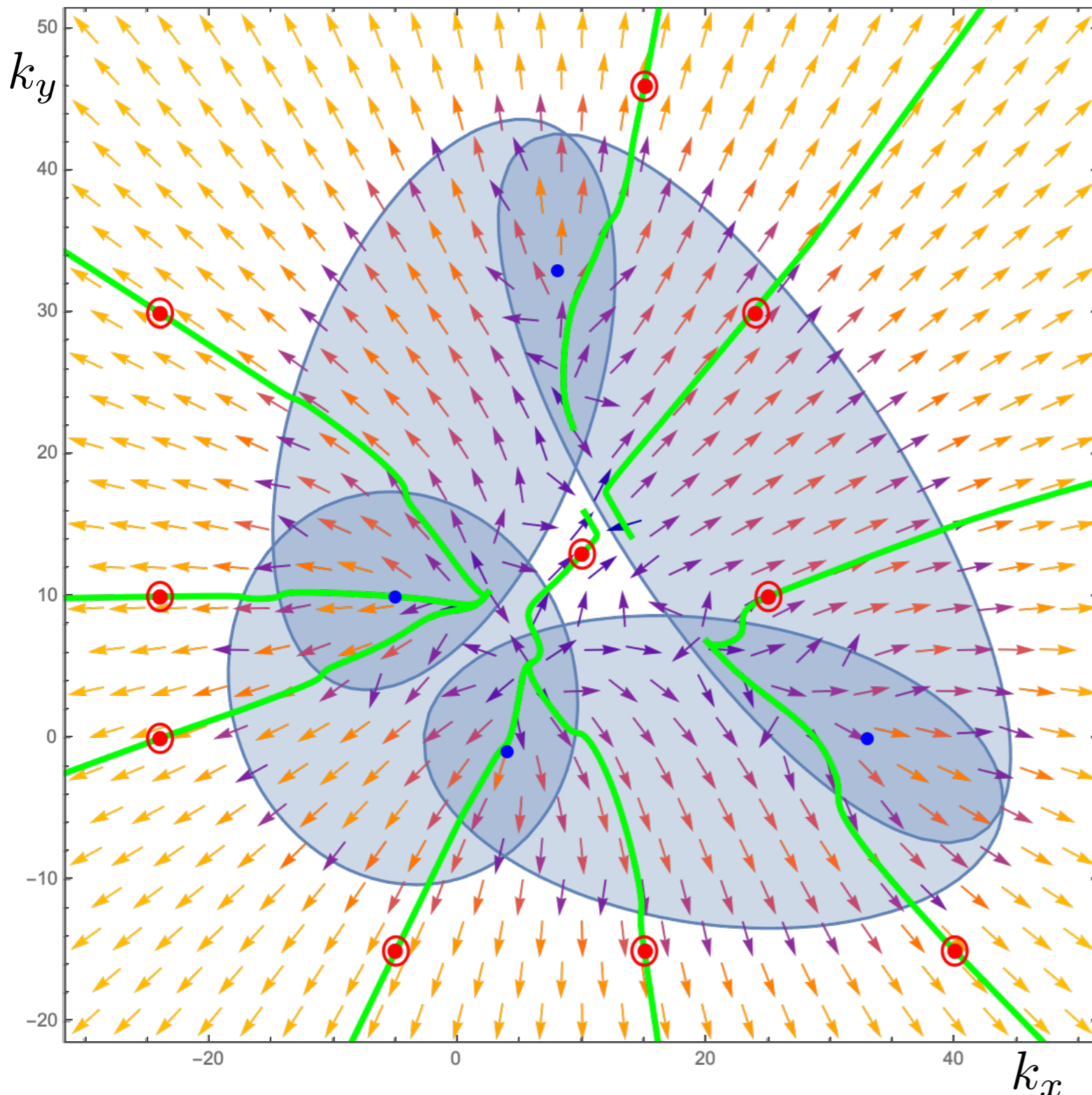
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Compute a **causal flow** $\vec{\phi}$ from our existing construction of a **deformation field** $\vec{\kappa}$:

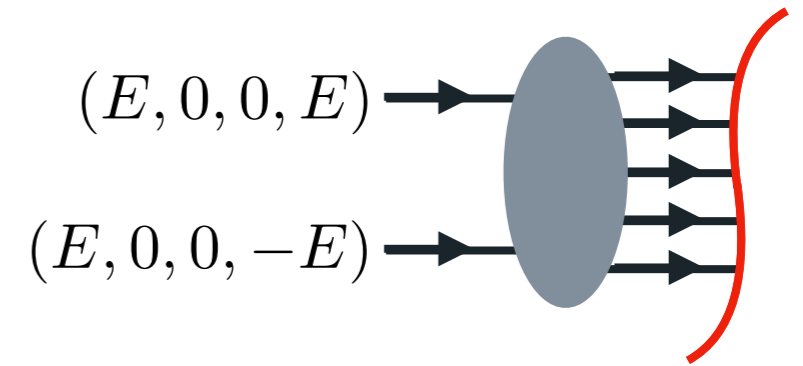
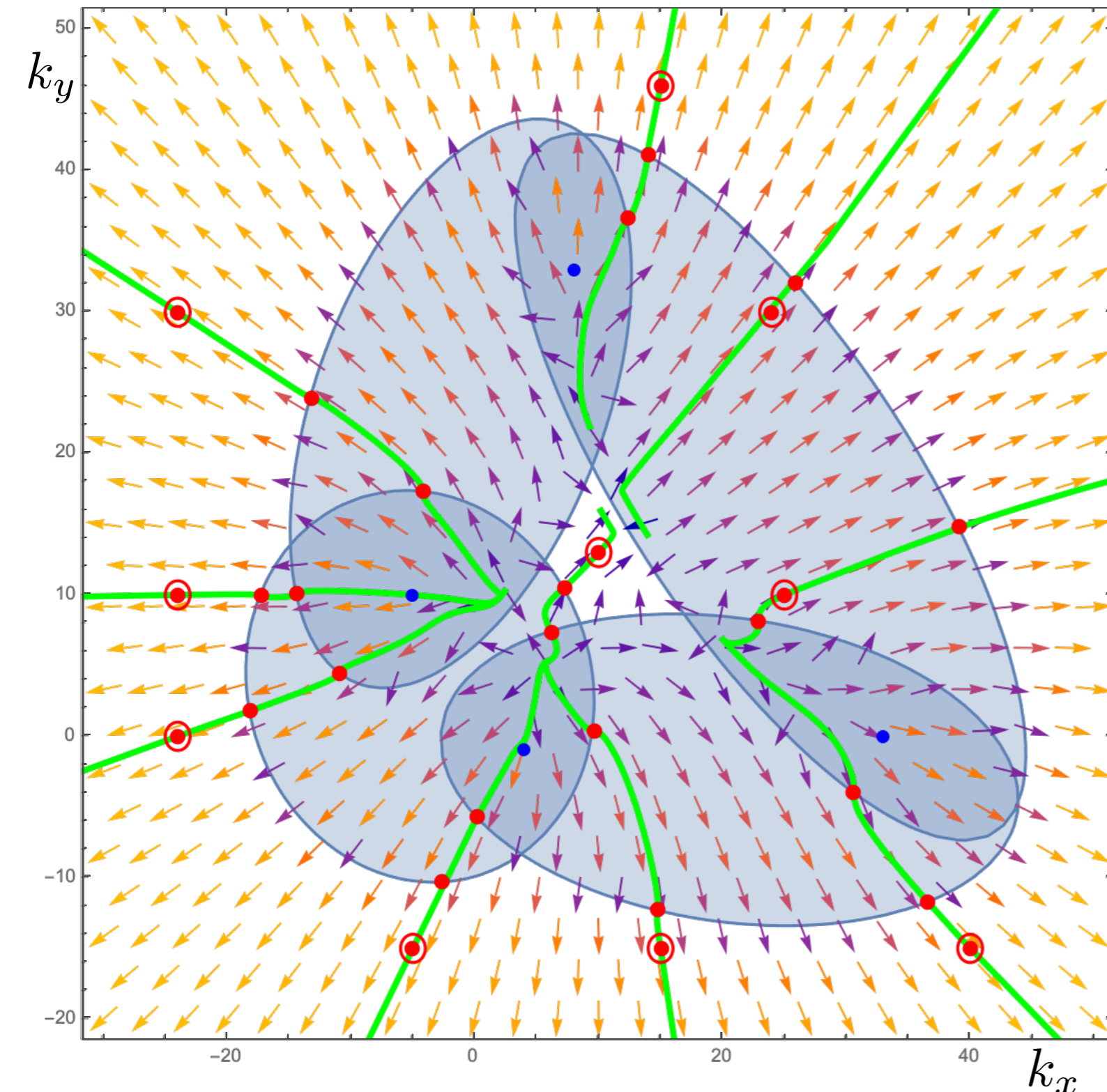
$$\partial_t \vec{\phi}(t, \vec{k}) = \vec{\kappa}(\vec{\phi}(t, \vec{k}))$$

$$\vec{\phi}(0, \vec{k}) = \vec{k}$$

In general, this **ODE** can be **solved numerically**.

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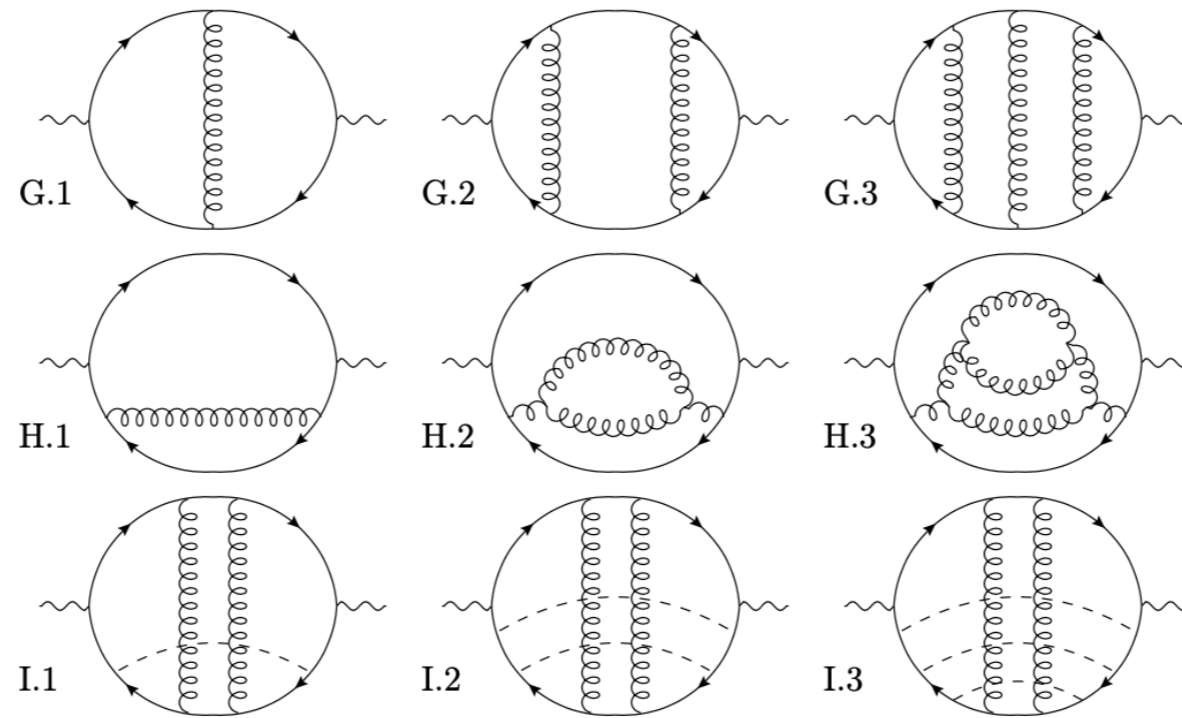
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IMPLEMENTATION RUN-TIME PERFORMANCE



SG	proc.	order	t_{gen} [s]	M_{disk} [MB]	N_{sg} [-]	N_{cuts} [-]	t_{eval} [ms]	$t_{\text{eval}}^{(\text{f128})}$ [ms]
G.1	1 \rightarrow 2	NLO	0.1	0.13	2	4	0.004	0.13
G.2	1 \rightarrow 2	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	1 \rightarrow 2	N3LO	36K	509	220	16	17.6	281
H.1	1 \rightarrow 2	NLO	0.07	0.12	2	2	0.006	0.14
H.2	1 \rightarrow 2	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	1 \rightarrow 2	N3LO	255	43	220	4	2.35	56
I.1	1 \rightarrow 3	NNLO	126	22	266	9	0.32	12.4
I.2	1 \rightarrow 4	NNLO	1.9K	120	4492	9	4.4	67
I.3	1 \rightarrow 5	NNLO	36K	20K	$\mathcal{O}(100\text{K})$	9	3.6K	17.3K

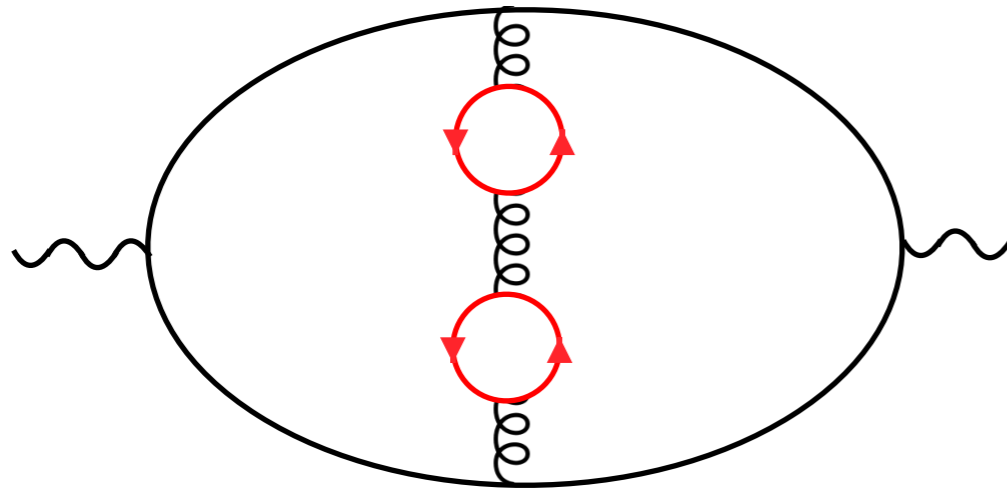
NB: these are **integrand** performance.

(Note: we recently found massive speedup w.r.t the above)

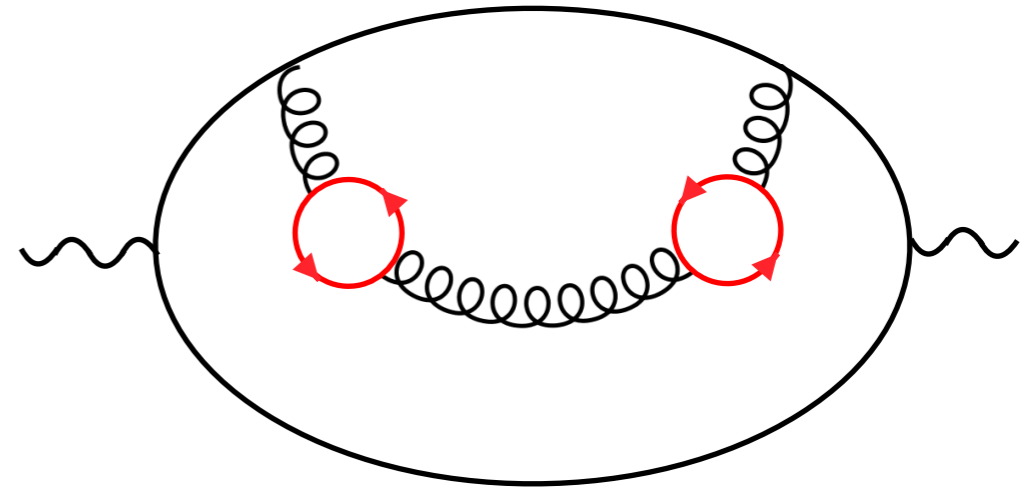
Integration (sampling) not optimised yet.
so we do not report quantitatively on it yet.

PARTIAL N³LO RESULTS

n_f^2 contributions :



$$K_{jj}^{(\text{MC LU}) \text{ I}} = 24.45(10)$$



$$K_{jj}^{(\text{MC LU}) \text{ II}} = -24.80(22)$$

(Large accidental cancellation between the two graphs, but validation otherwise successful)

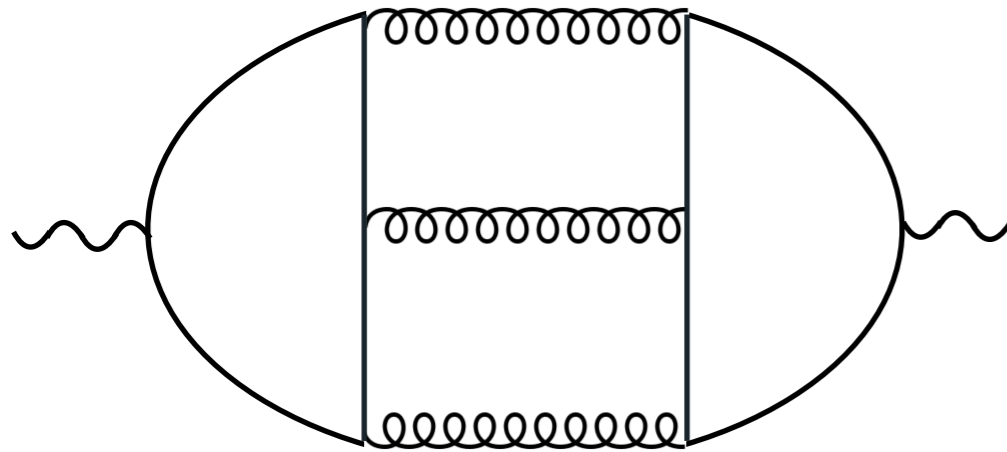
$$K_{jj}^{(\text{MC LU}) \text{ I+II}} = -0.35(24)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f^2)} = C_F \left(\frac{1208}{27} - \frac{8}{3} \zeta_2 - \frac{304}{9} \zeta_3 \right) = -0.331415$$

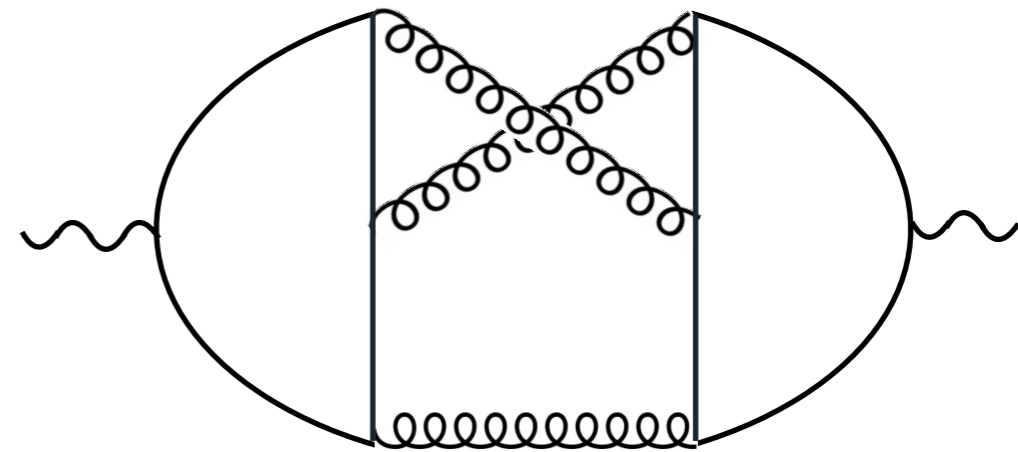
[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

PARTIAL N3LO RESULTS

Singlet contributions : (Results for low Monte-Carlo statistics here)



$$K_{jj}^{(\text{MC LU})\text{I}} = 48.4(1.0)$$



$$K_{jj}^{(\text{MC LU})\text{II}} = -74.0(1.1)$$

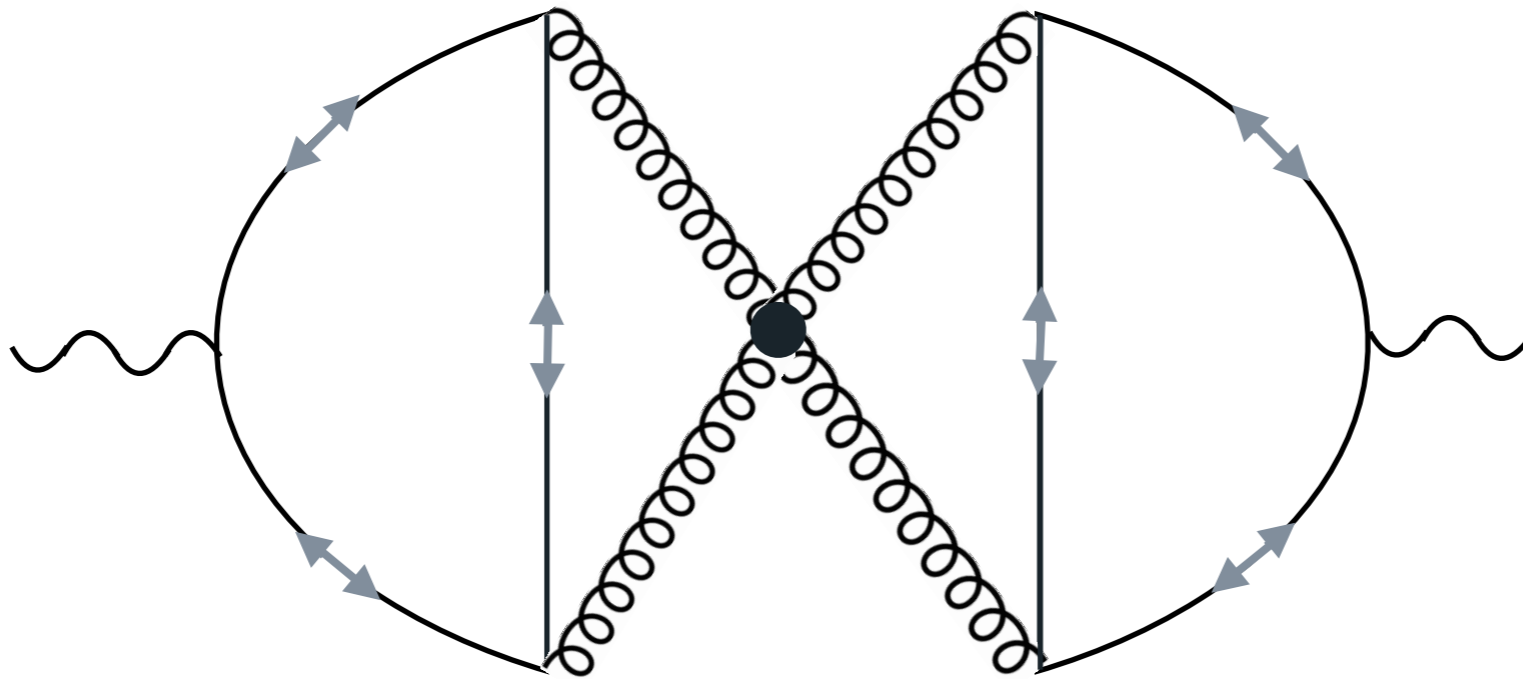
$$K_{jj}^{(\text{MC LU})\text{I+II}} = -25.6(1.5)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3), \text{singlet}} = \frac{d_F^{abc} d_F^{abc}}{N_R} \left(\frac{176}{3} - 128\zeta_3 \right) = -26.4435$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

CURIOSITY

Singlet contributions : A non-obvious zero contribution...

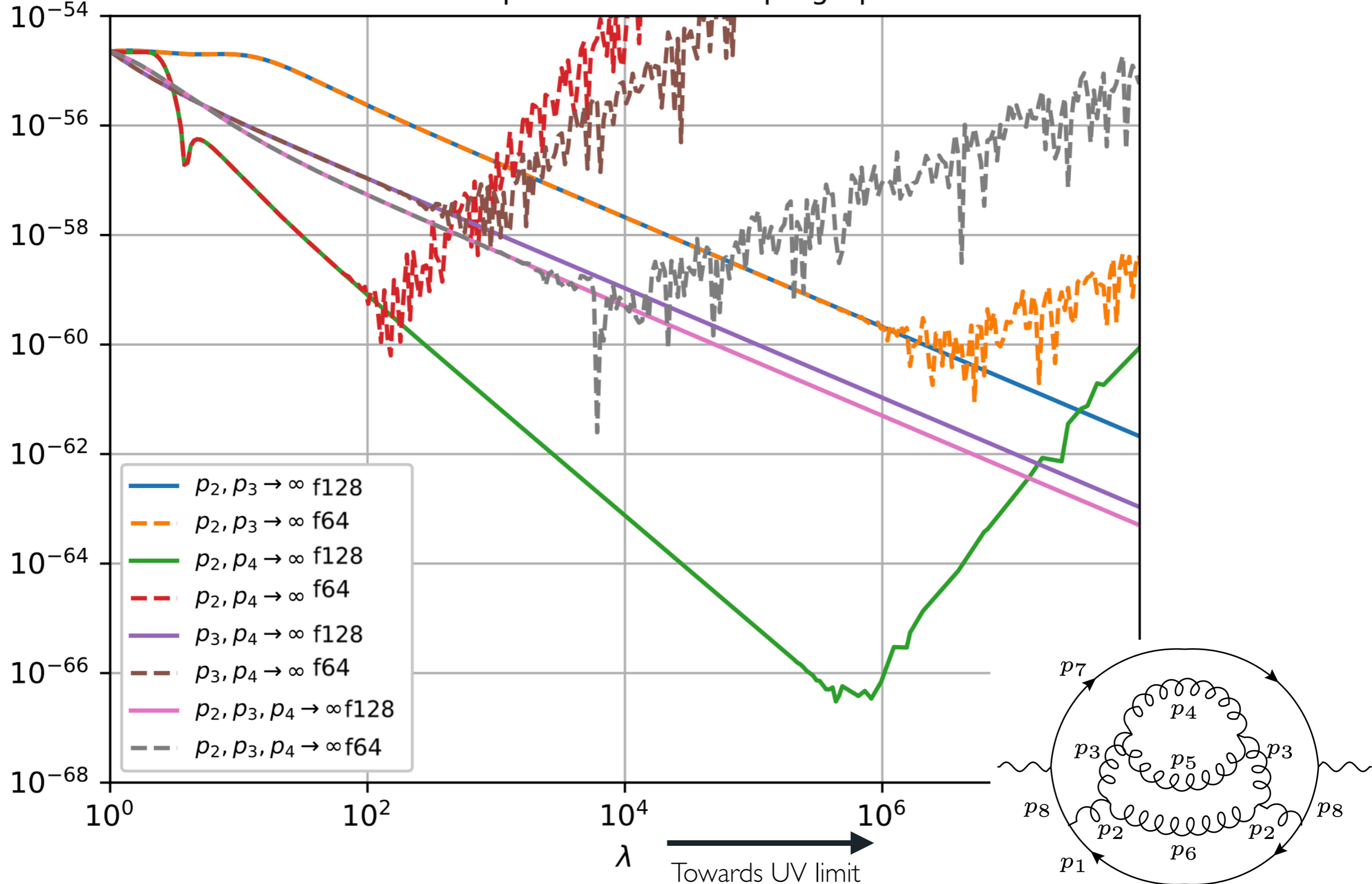


This graph gives no contribution inclusively.
I have not looked into it with any depth, but I don't see
an obvious reason as to why it should be zero...

TESTING N3LO UV LIMITS

[arbitrary units]

Double and triple UV limits of supergraph H.3

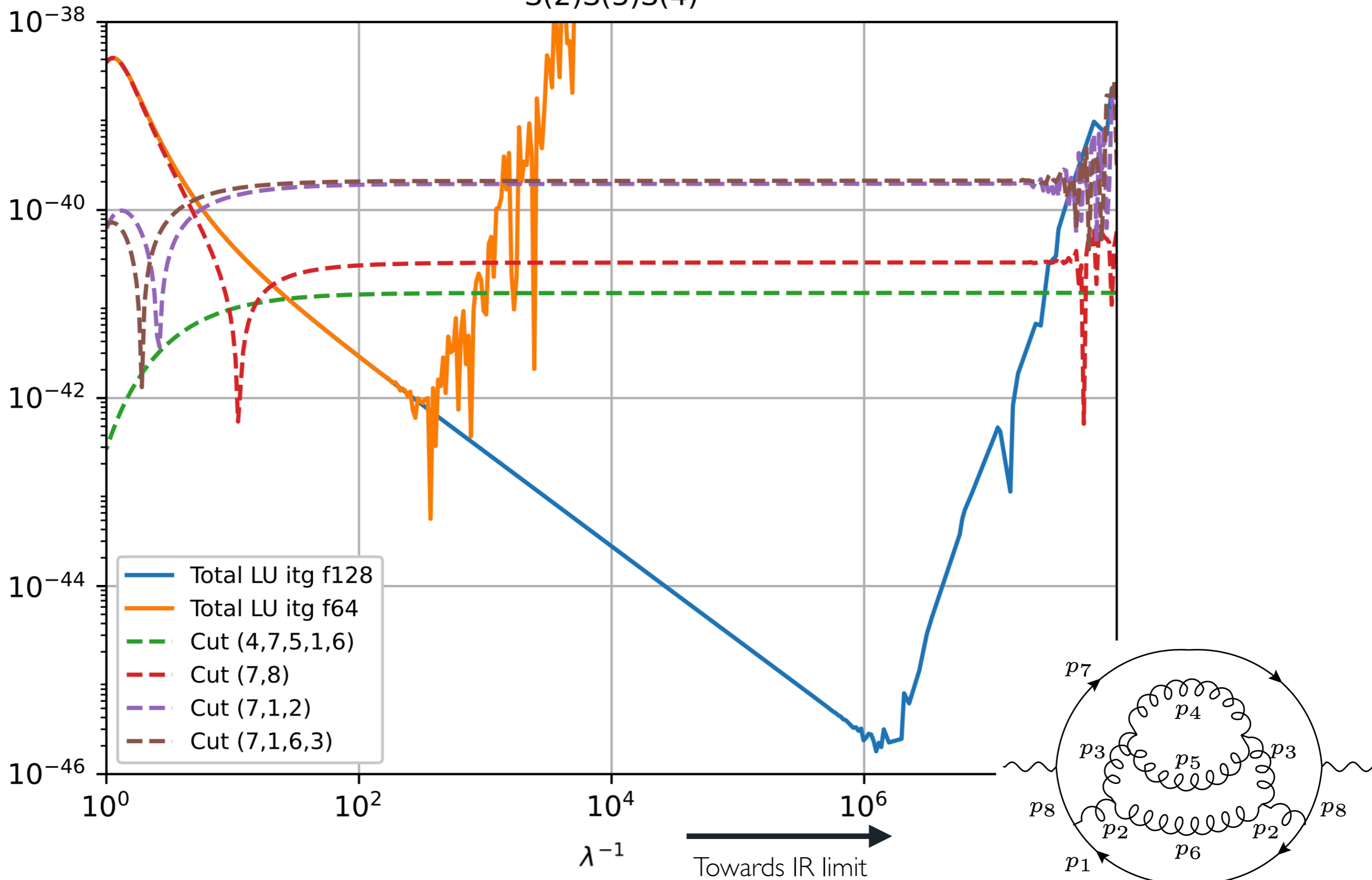


- $p_2, p_3 \rightarrow \infty$ f128
- - $p_2, p_3 \rightarrow \infty$ f64
- $p_2, p_4 \rightarrow \infty$ f128
- - $p_2, p_4 \rightarrow \infty$ f64
- $p_3, p_4 \rightarrow \infty$ f128
- - $p_3, p_4 \rightarrow \infty$ f64
- $p_2, p_3, p_4 \rightarrow \infty$ f128
- - $p_2, p_3, p_4 \rightarrow \infty$ f64

TESTING IR SOFT LIMITS

[arbitrary units]

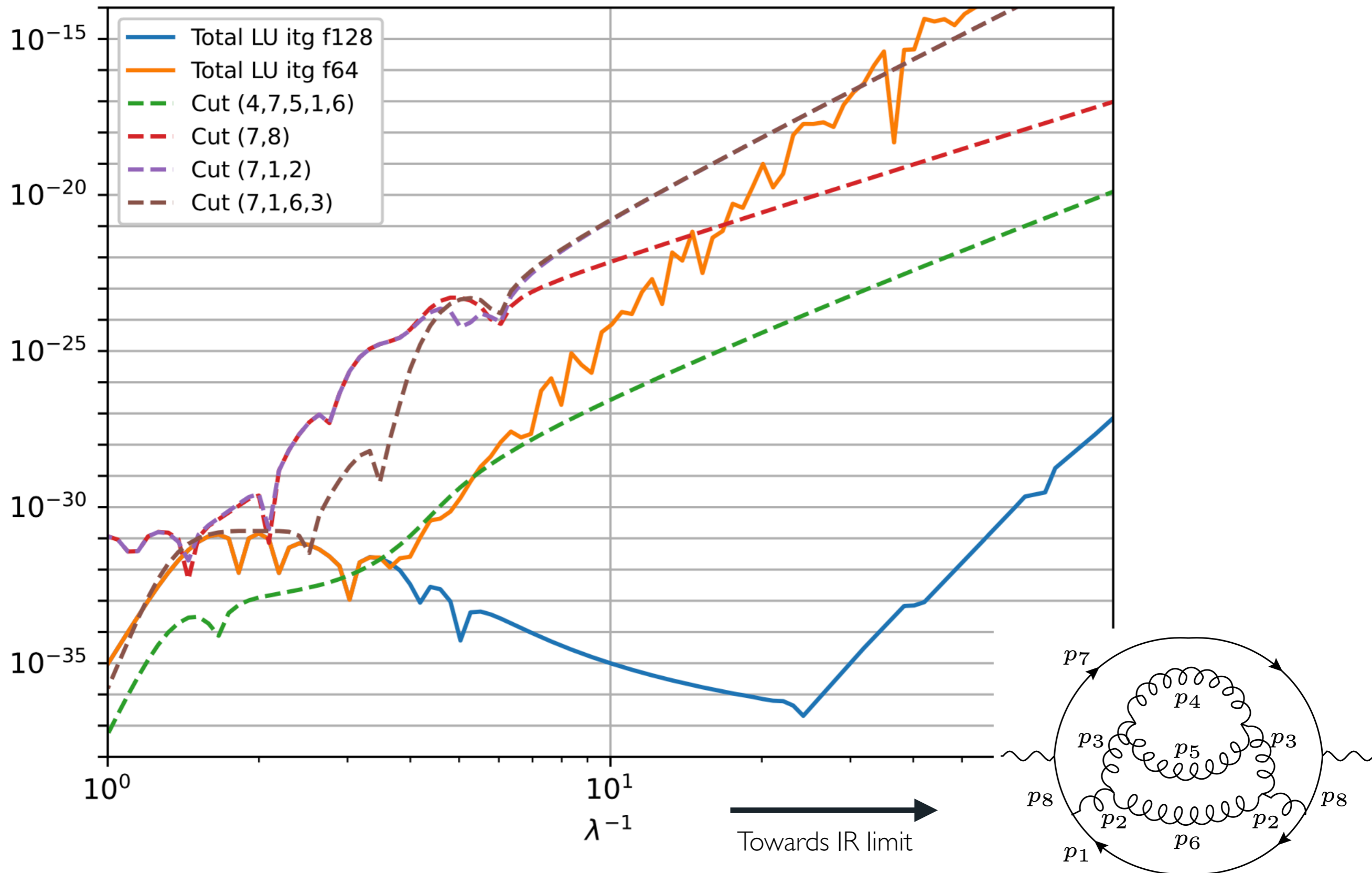
S(2)S(3)S(4)



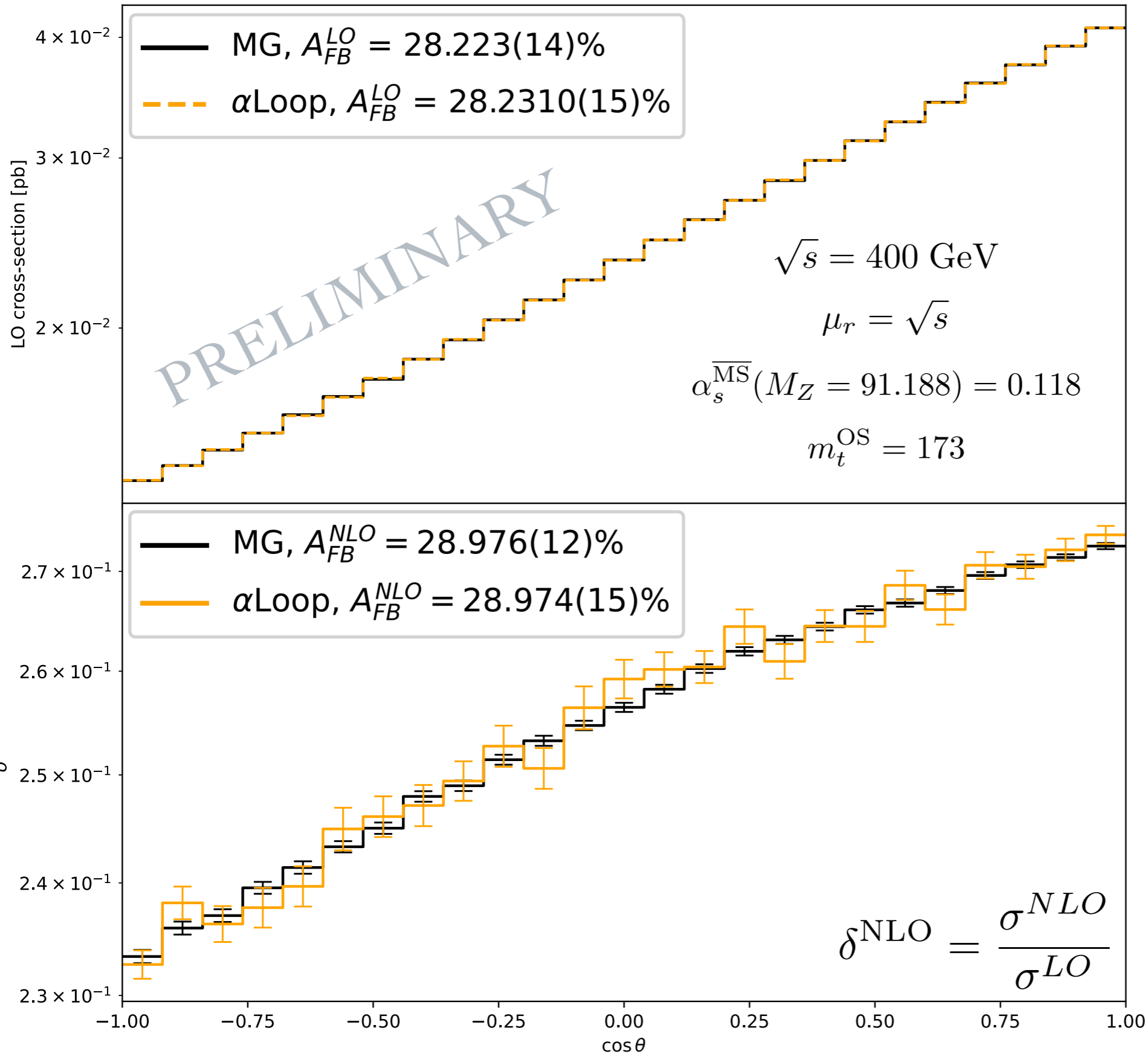
TESTING IR SOFT-COLLINEAR LIMITS

[arbitrary units]

C[1,2,S(3),S(4)]



EXAMPLE II : NLO AFB FOR $e^+e^- \rightarrow \gamma^*/Z \rightarrow t\bar{t}$



First result in LU with γ^5 and EW-boson

Contour deformation well-behaved in this case

Credits to ETHZ student

Max Hofer