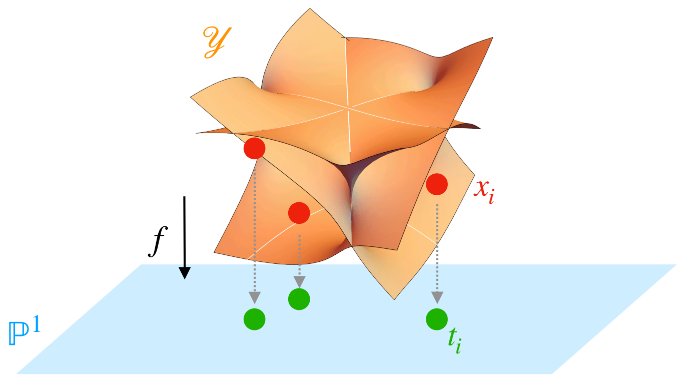
Eric Pichon-Pharabod

Homology and periods of algebraic varieties



Periods of algebraic varieties

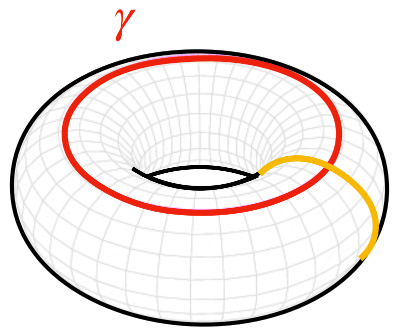
A **period** of an algebraic variety is the integral of a rational form of the variety on a cycle.

$$\int_{T(\gamma)} \frac{A}{P^k}$$

Some integration domain without boundary

A is a polynomial

P defines a smooth variety $V(P) = \{z \mid P(z) = 0\}$



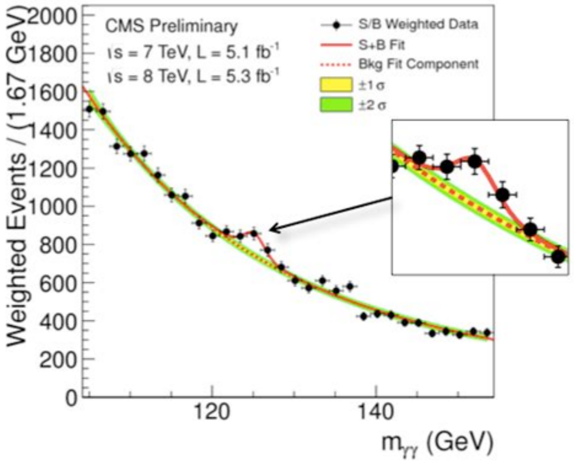
An elliptic curve

Torelli-type theorem for K3 surfaces:
Two K3 surfaces are isomorphic if and only if they have "the same" periods.

They describe the comparison between **topological data** (cycles) and **algebraic data** (algebraic De Rham forms).

$$H_n(S,\mathbb{Z}) \times H^n_{DR}(S) \to \mathbb{C} \qquad \gamma, \omega \mapsto \int_\gamma \omega$$

Motivation and goals

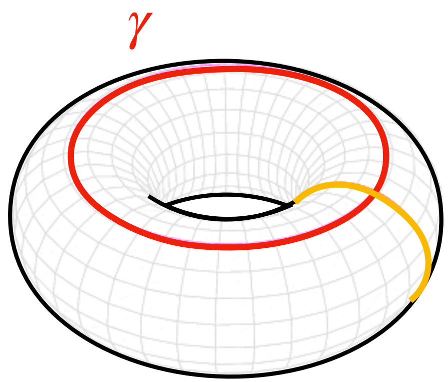


Periods appear in diverse fields of mathematics and physics, such as **Quantum field theory** (Feynman integrals), **Hodge theory**, **motives**, **number theory** (BSD conjecture) …

Goal: compute numerical approximations of these integrals with **large precision.**

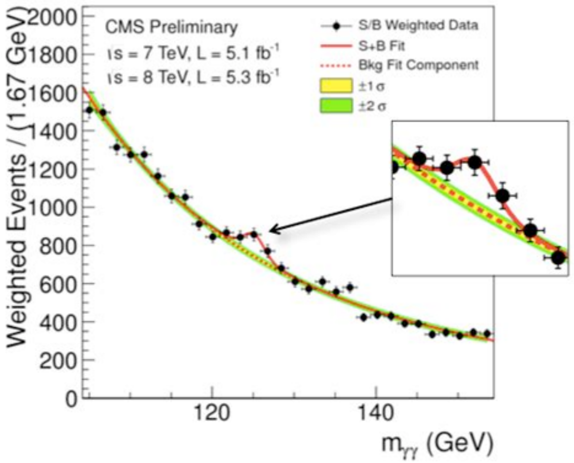
For this, we need an appropriate description of the integrals.

In particular we will focus on **understanding the cycles of integration** (the homology), how to represent them in a way that make integration easy, and how to compute a basis of them.



Furthermore we want this to be **effective** and **efficient**.

Motivation and goals



Periods appear in diverse fields of mathematics and physics, such as **Quantum field theory** (Feynman integrals), **Hodge theory**, **motives**, **number theory** (BSD conjecture) …

Hundreds of digits

Sufficiently many to recover algebraic invariants

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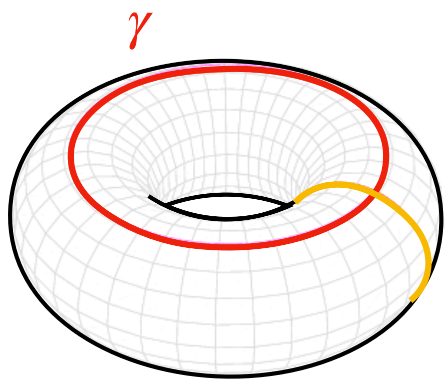

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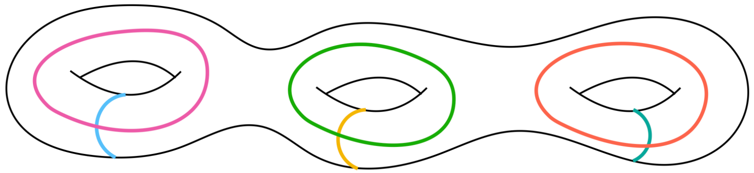
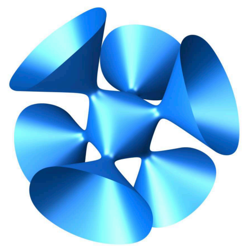
Previous works

[Deconinck, van Hoeij 2001], [Bruin, Sijsling, Zotine 2018], [Molin, Neurohr 2017]:
Algebraic curves (Riemann surfaces)

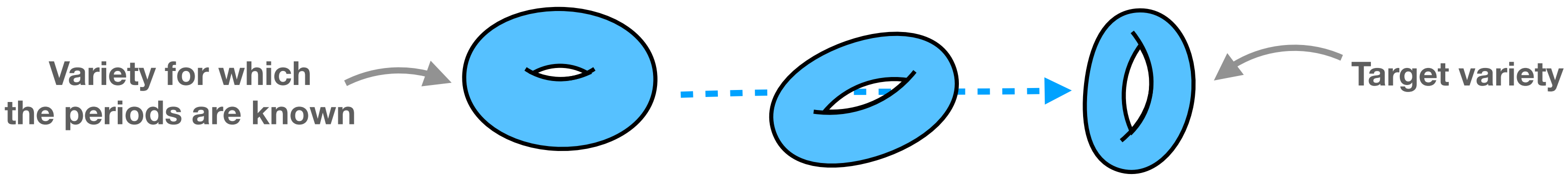


[Elsenhans, Jahnel 2018], [Cynk, van Straten 2019]:
Higher dimensional varieties
(double covers of $\mathbb{P}^2$ ramified along 6 lines / of $\mathbb{P}^3$ ramified along 8 planes)



Picture by Alessandra Sarti

[Sertöz 2019]: compute the period matrix of smooth projective hypersurfaces by **deformation**.

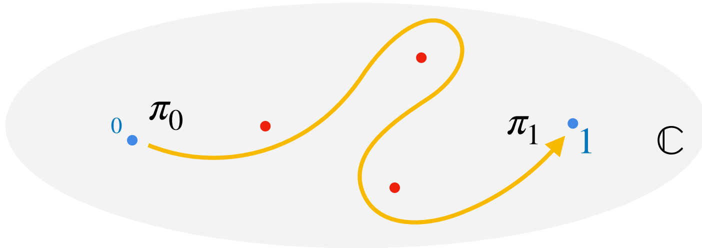


Previous works

[Sertöz 2019]: compute the periods matrix by **deformation**:

We wish to compute $\displaystyle\int_\gamma \frac{\Omega}{X^3+Y^3+Z^3+XYZ}$.

Let us consider instead $\pi_t = \displaystyle\int_{\gamma_t} \frac{\Omega}{X^3+Y^3+Z^3+tXYZ}$,



Exact formulae are known for π_0 **[Pham 65, Sertöz 19]**

Furthermore π_t is a solution to the differential operator
$\mathscr{L} = (t^3+27)\partial_t^2 + 3t^2\partial_t + t$ (Picard-Fuchs equation).

We may numerically compute the analytic continuation of π_0
along a path from 0 to 1. **[Chudnovsky², Van der Hoeven, Mezzarobba]**
This way, we obtain a numerical approximation of π_1.

Previous works

[Sertöz 2019]: compute the periods matrix by **deformation**:

Two drawbacks :

We rely on the knowledge of the periods of some variety.
[Pham 65, Sertöz 19] provide the periods of the Fermat hypersurfaces $V(X_0^d + \ldots + X_n^d)$.
In more general cases (e.g. complete intersections), we do not have this data.

The differential operators that need to be integrated quickly go beyond what current software can manage:

to compute the periods of a smooth quartic surface in $\mathbb{P}^3$, one needs to integrate an operator of order 21 and high degree.

Idea: a more intrinsic description of the cycles of integration should solve both problems.

Contributions

New **effective** method for computing homology and periods
with high precision (hundreds of digits):

→ **implementation** in Sagemath
lefschetz_family

→ applicable to **other types of varieties**
(elliptic surfaces, ramified double covers, …)

→ frontal approach to the the computation of **homology** of complex algebraic varieties

→ sufficiently efficient to compute periods of **previously inaccessible hypersurfaces**
(general smooth quartic surface)

Periods of algebraic curves

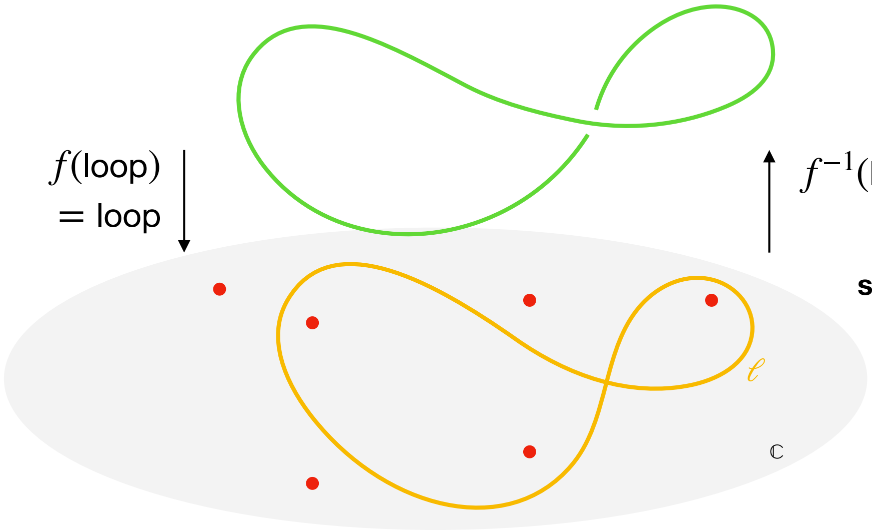
Algorithm from **[Deconinck, van Hoeij 2001]**

First example: algebraic curves

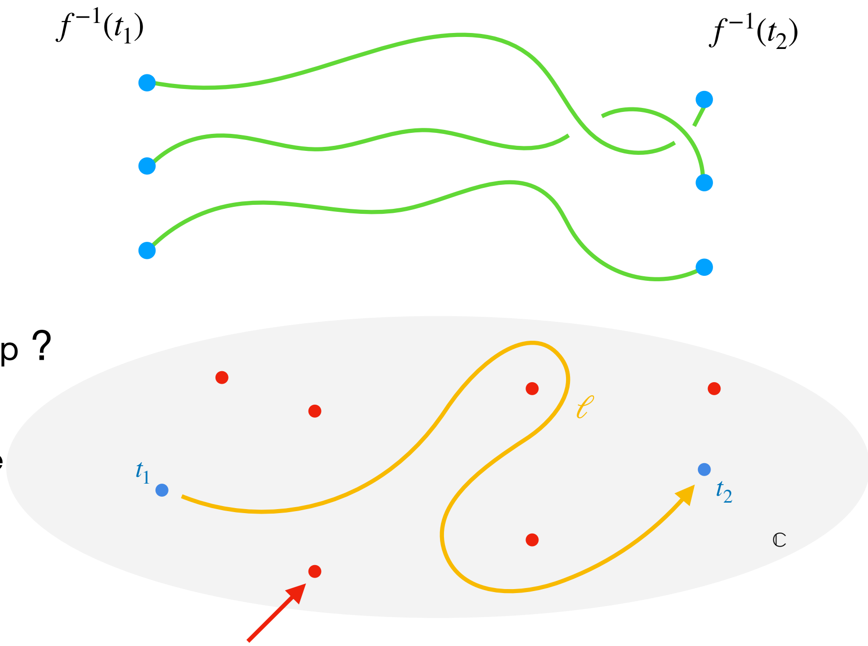
Let $\mathscr{X}$ be the elliptic curve defined by $P = y^3 + x^3 + 1 = 0$ and let $f : (x, y) \mapsto y/(2x+1)$ be a generic projection.

The fibre above $t \in \mathbb{C}$ is $\mathscr{X}_t = f^{-1}(t)$ $= \{(x, t(2x+1)) \mid P\left(x, t(2x+1)\right) = 0\}$. It deforms continuously with respect to t.

In dimension 1, we are looking for closed paths in $\mathscr{X}$, up to deformation (1-cycles).



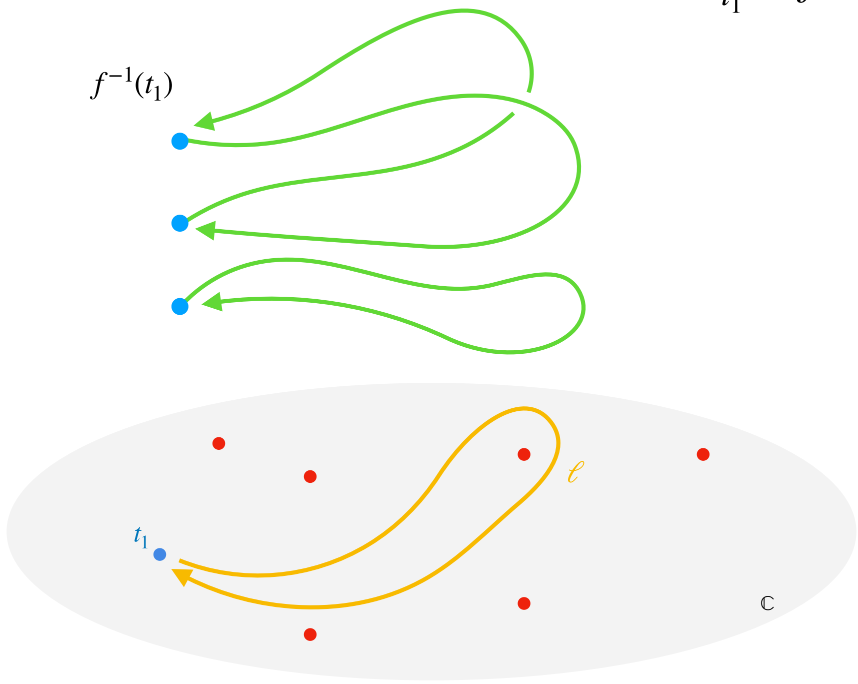
Not always, see next slide

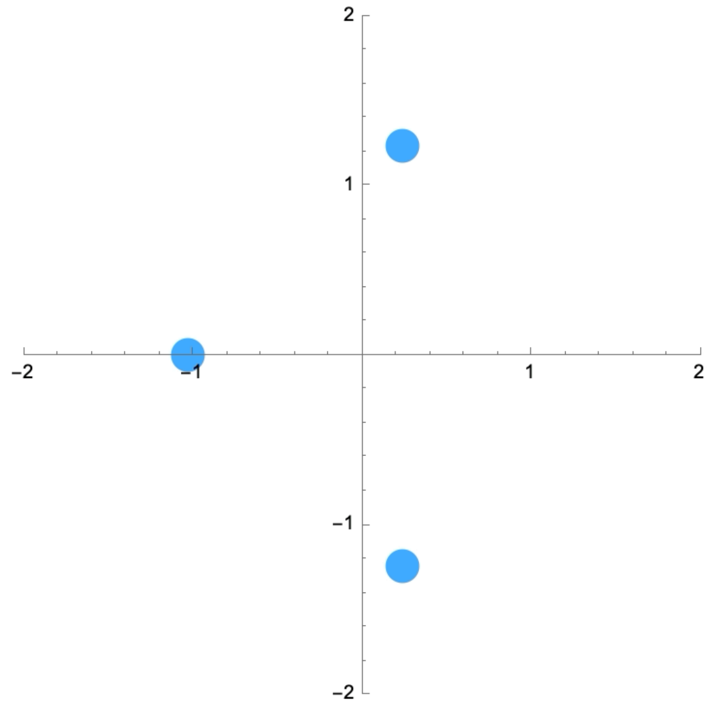


Values of t for which $P(x, t(2x+1)) = t^3(2x+1)^3 + x^3 + 1$ has a double root (critical values)

What happens when you loop around a critical point?

A loop ℓ in $\mathbb{C}$ pointed at t_1 induces a permutation of $\mathcal{X}_{t_1} = f^{-1}(t_1)$.





This permutation is called the **action of monodromy along** ℓ on $\mathcal{X}_{t_1}$.

It is denoted ℓ_*.

If ℓ is a simple loop around a critical value, ℓ_* is a transposition.

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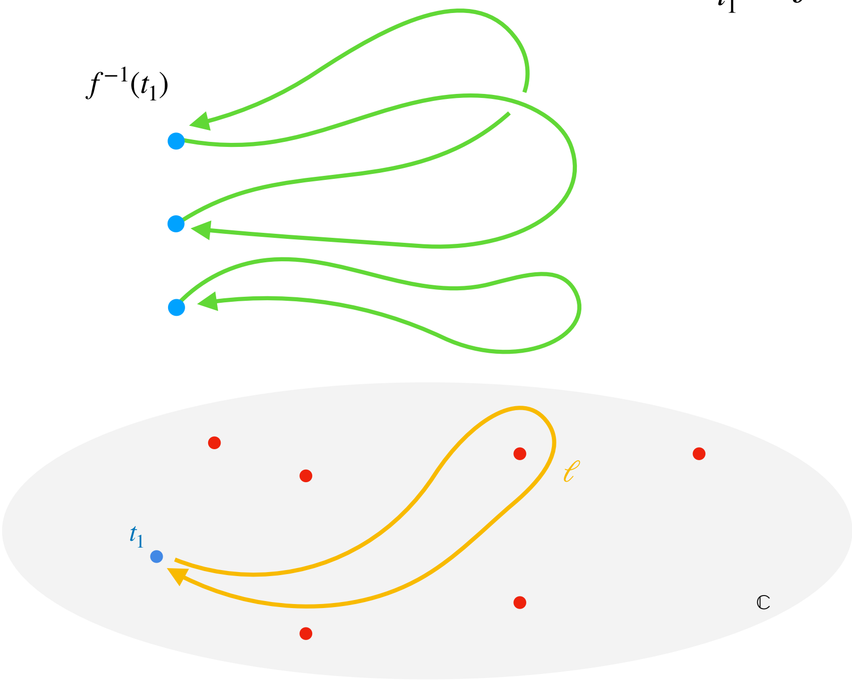

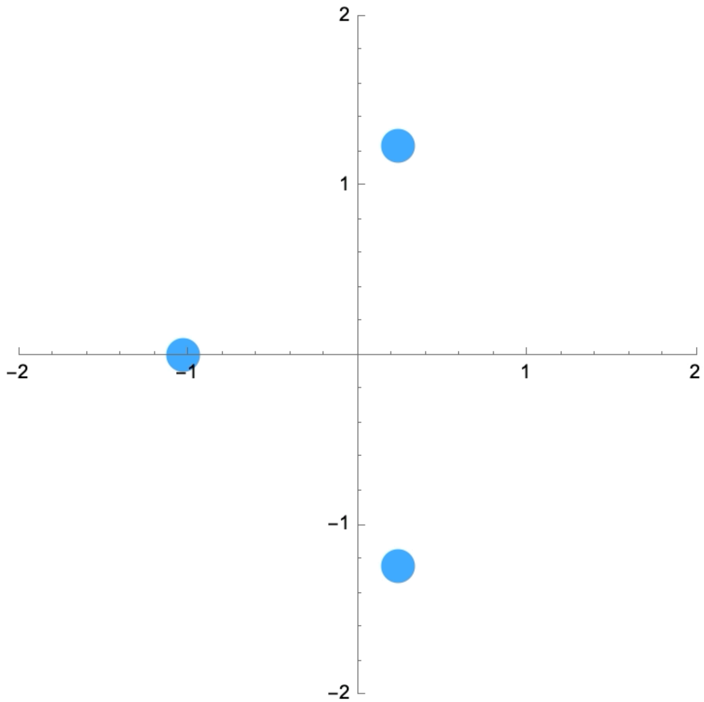

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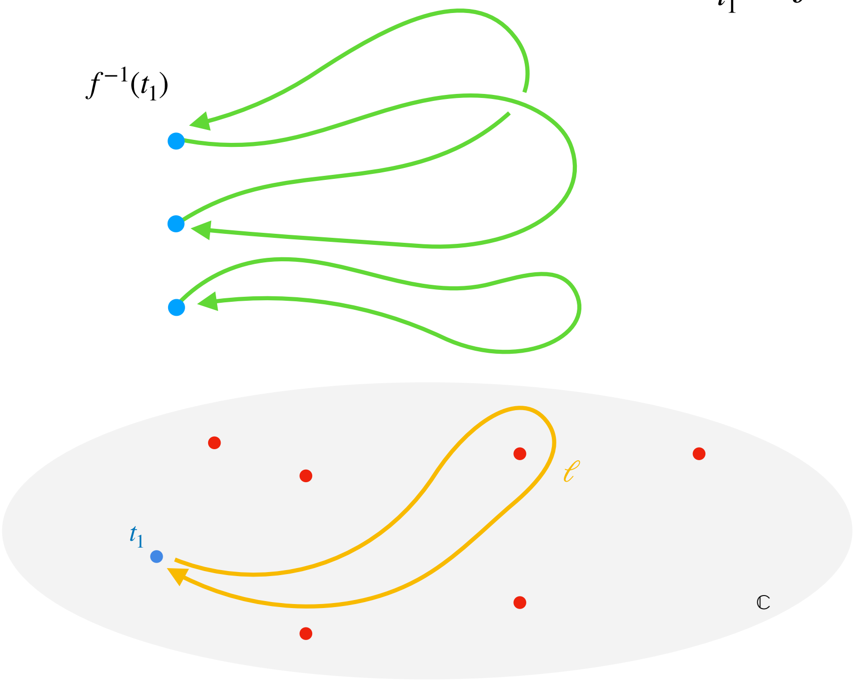

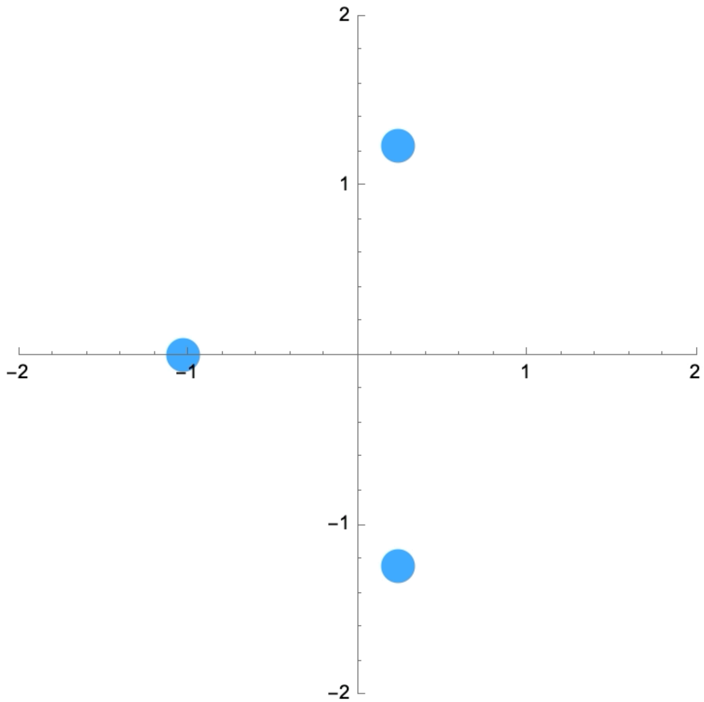

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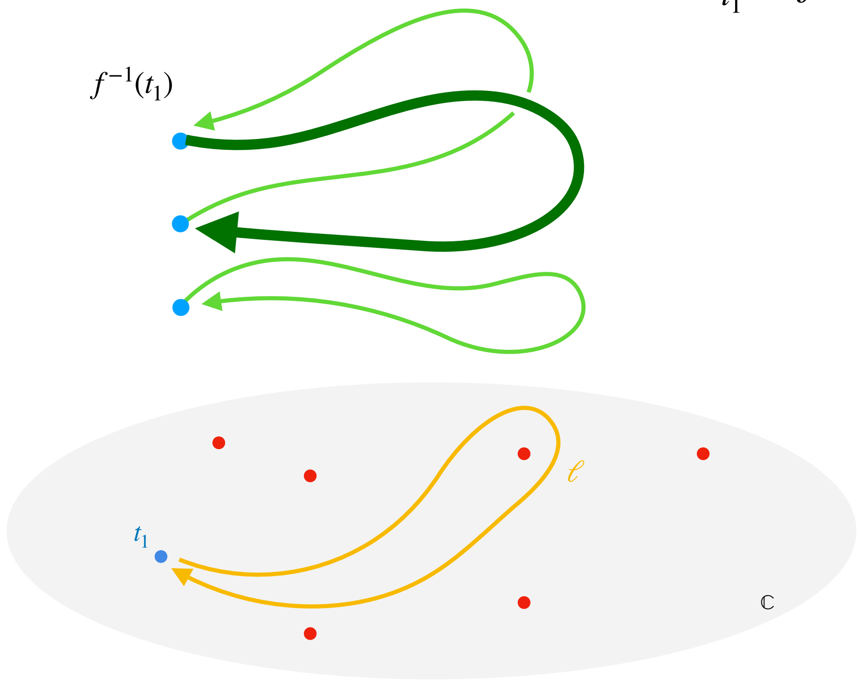

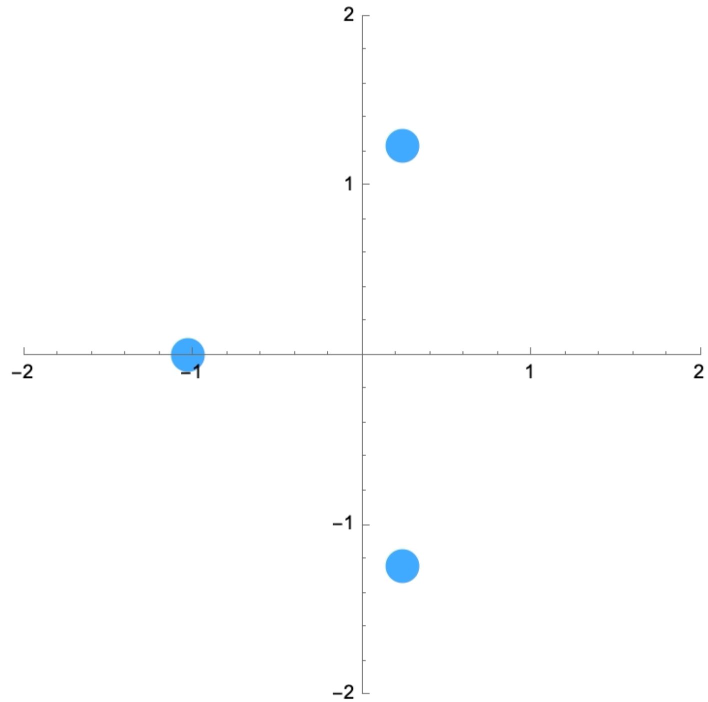

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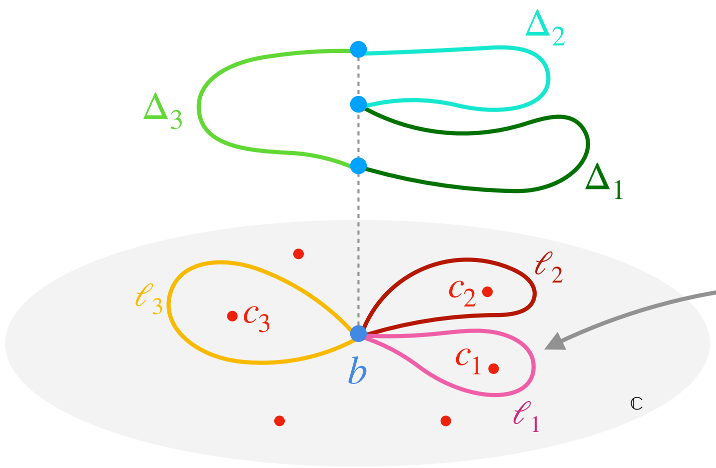
Periods of algebraic curves

The lift of a simple loop ℓ around a critical value c that has a non-trivial boundary in $\mathscr{X}_b$ is called the **thimble** of c. It is an element of $H_1(\mathscr{X}, \mathscr{X}_b)$.



Thimbles serve as building blocks to recover $H_1(\mathscr{X})$.
It is sufficient to glue thimbles together in a way such that their boundaries cancels.

Concretely, we take the kernel of the boundary map

$$\delta : H_1(\mathscr{X}, \mathscr{X}_b) \to H_0(\mathscr{X}_b)$$

Fact: all of $H_1(\mathscr{X})$ can be recovered this way.

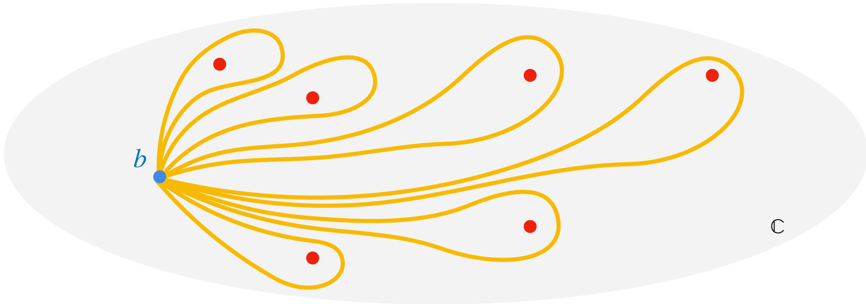
$$0 \to H_1(\mathscr{X}) \to H_1(\mathscr{X}, \mathscr{X}_b) \to H_0(\mathscr{X}_b)$$

Generated by thimbles

Computing periods of algebraic curves

1. Compute simple loops $\ell_1, \ldots, \ell_{\#\text{crit.}}$ around the critical values

— basis of $\pi_1(\mathbb{C} \backslash \{\text{crit. val.}\})$



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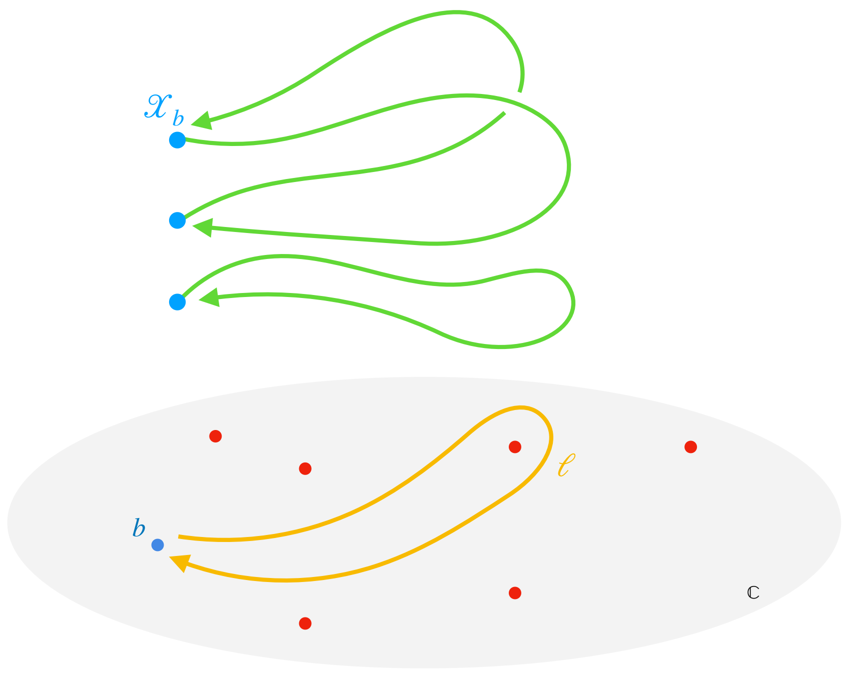

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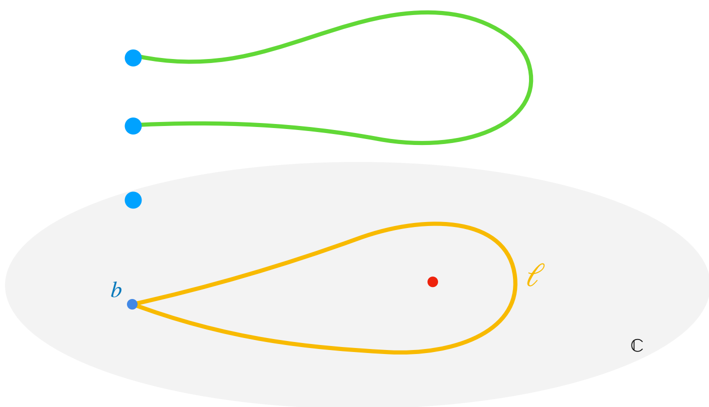
3. This provides the corresponding thimble Δ_i. Its boundary is the difference of the two points of $\mathscr{X}_b$ that are permuted.



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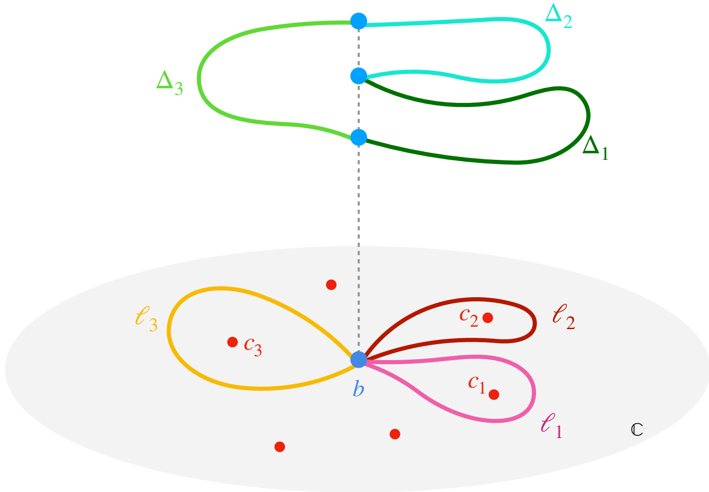
4. Compute sums of thimbles without boundary → basis of $H_1(\mathcal{X})$



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5. Periods are integrals along these loops
→ we have an explicit parametrisation of these paths → numerical integration.

$$\int_\gamma \omega = \int_\ell \omega_t$$

DEMO

Hypersurfaces

An inductive approach

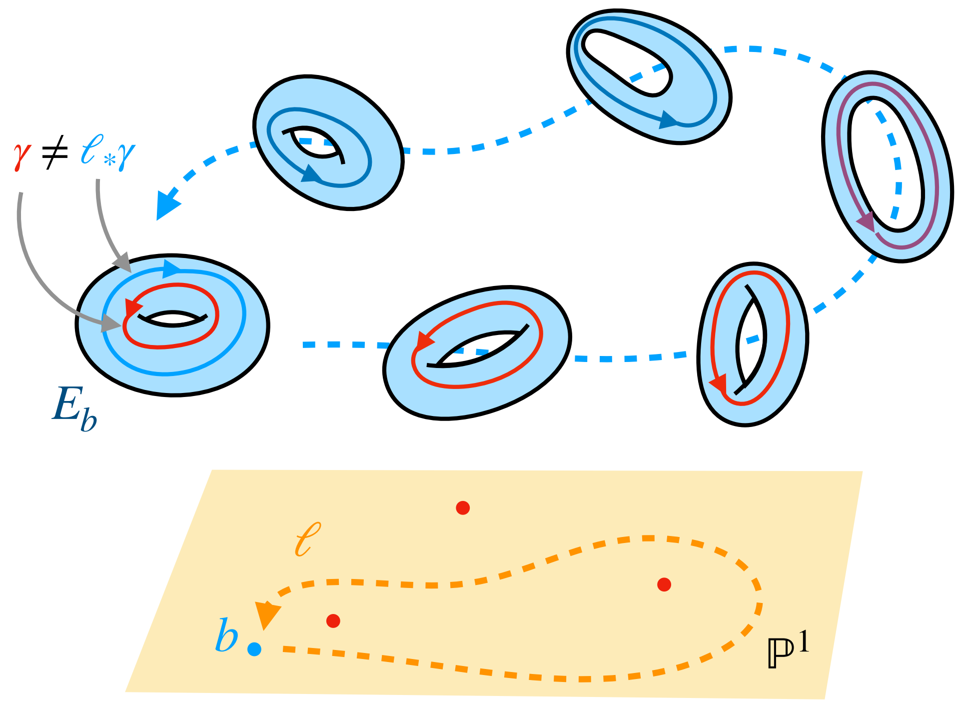
Ideas of **[Lefschetz 1924]**, made effective in **[Lairez, PP, Vanhove 2024]**

Monodromy

Ehresmann's fibration theorem

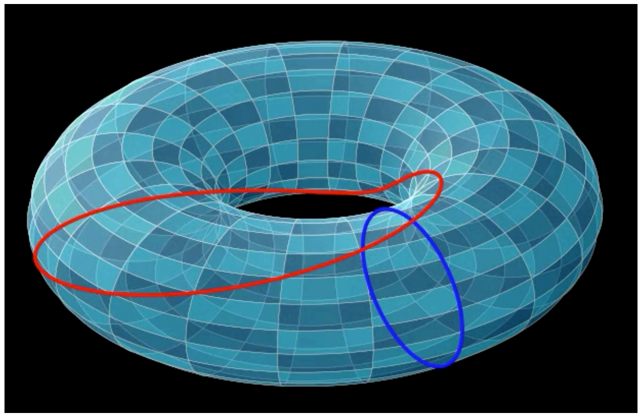
Let $\mathscr{X}$ be a smooth (hyper)surface in $\mathbb{P}^3$. We consider a projection $\mathscr{X} \to \mathbb{P}^1$.
The fibre $\mathscr{X}_t = f^{-1}(t)$ is a curve, which deforms continuously as t moves in $\mathbb{P}^1$.

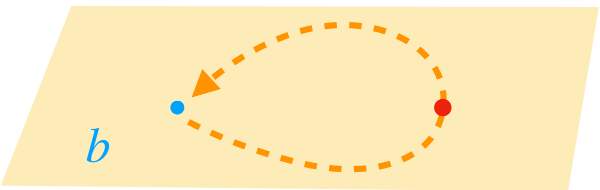
The map $\ell_* : H_1(\mathscr{X}_b) \to H_1(\mathscr{X}_b)$ induced by this deformation along a loop ℓ is called the **monodromy along ℓ**.



The monodromy is encoded in a differential operator: the **Picard-Fuchs equation**.

A Dehn twist





When the monodromy is a Dehn twist, the singular fibre is said to be of **Lefschetz type**.
$\ell_* - \mathrm{id}$ has **rank 1** and its image is **primitive.**

Insight into higher dimensions: surfaces

We are looking for 2-cycles.
The fibre $\mathscr{X}_t$ is a curve which deforms continuously with respect to t.

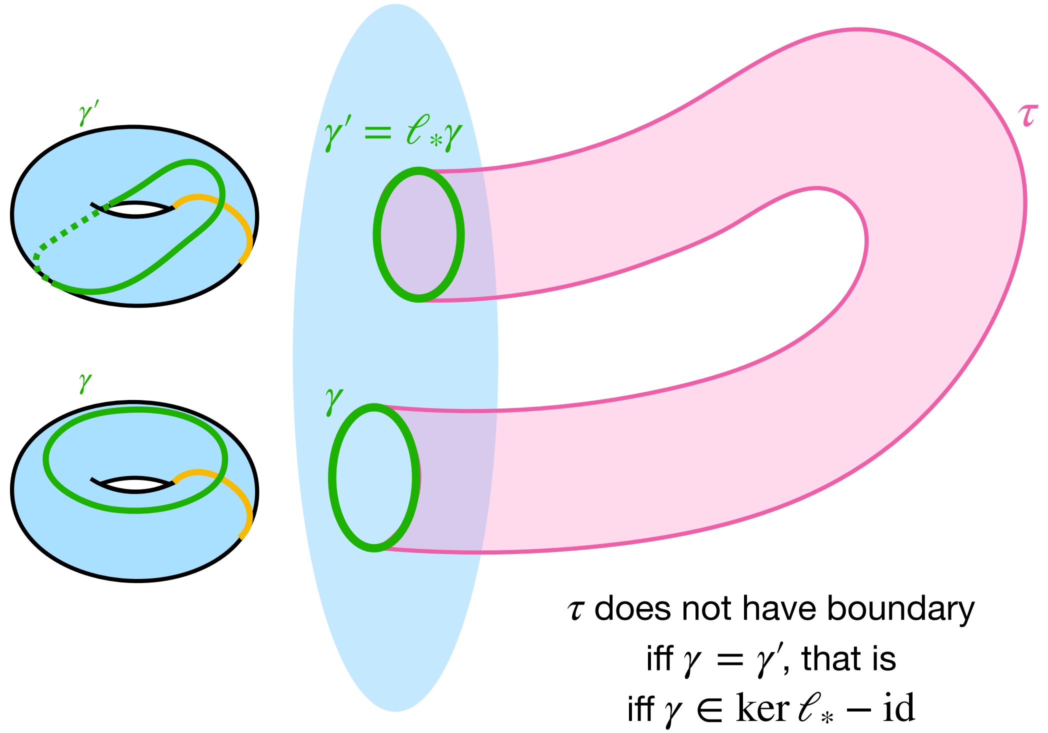
We can recover integration 2-cycles for the periods of elliptic surfaces as **extensions** of 1-cycles of the fibre.

$$\pi_1(\mathbb{P}^1 \backslash \Sigma, b) \times H_1(\mathscr{X}_b) \to H_2(\mathscr{X}, \mathscr{X}_b)$$
$$\ell, \gamma \mapsto \tau_\ell(\gamma)$$

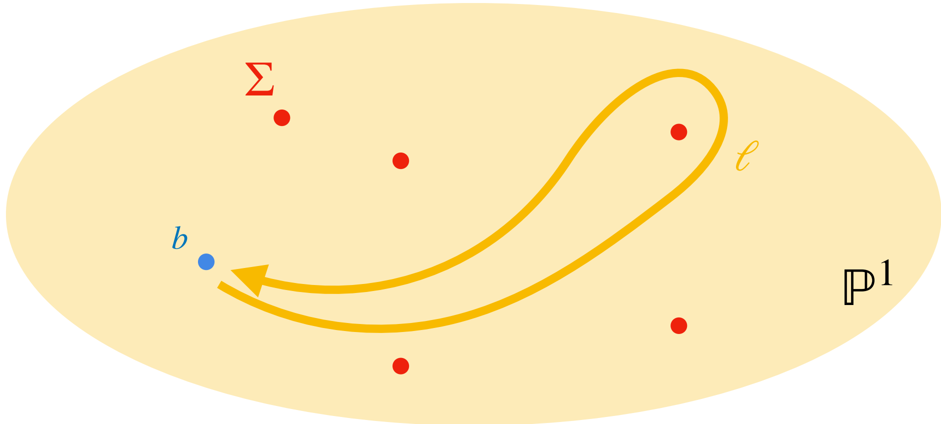
This description of cycles is well-suited for integrating the periods:

$$\int_{\tau_\ell(\gamma)} f(x,y)\mathrm{d}x\mathrm{d}y = \int_\ell \left(\int_\gamma f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

$$\partial\tau_\ell(\gamma) = \gamma' - \gamma$$



τ does not have boundary
iff $\gamma = \gamma'$, that is
iff $\gamma \in \ker \ell_* - \mathrm{id}$



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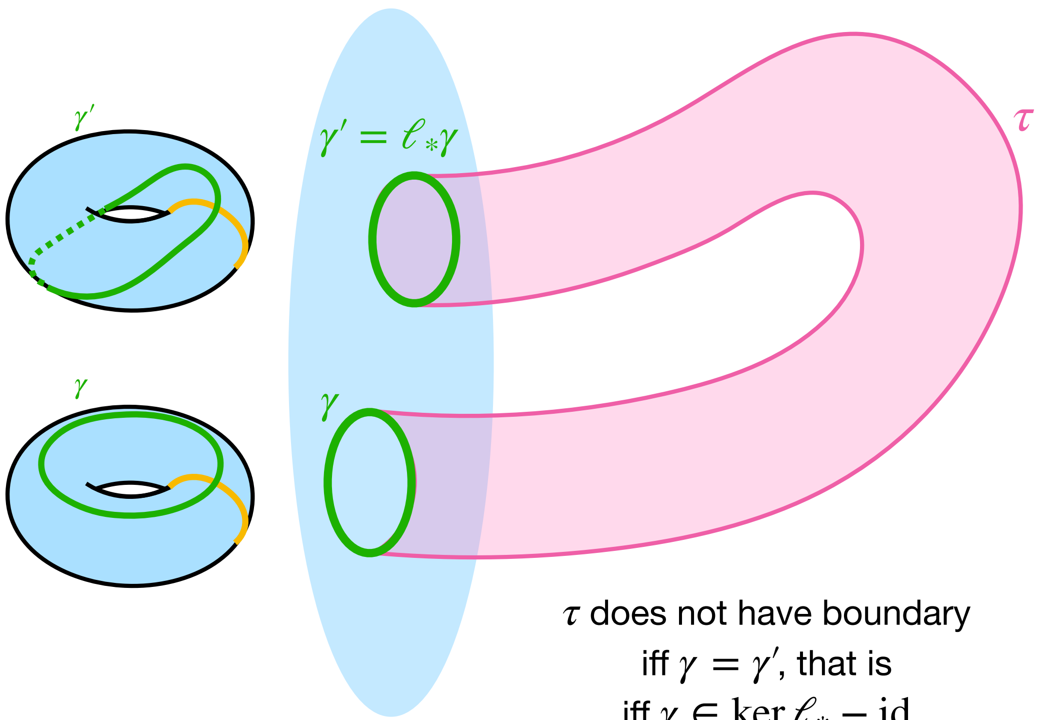

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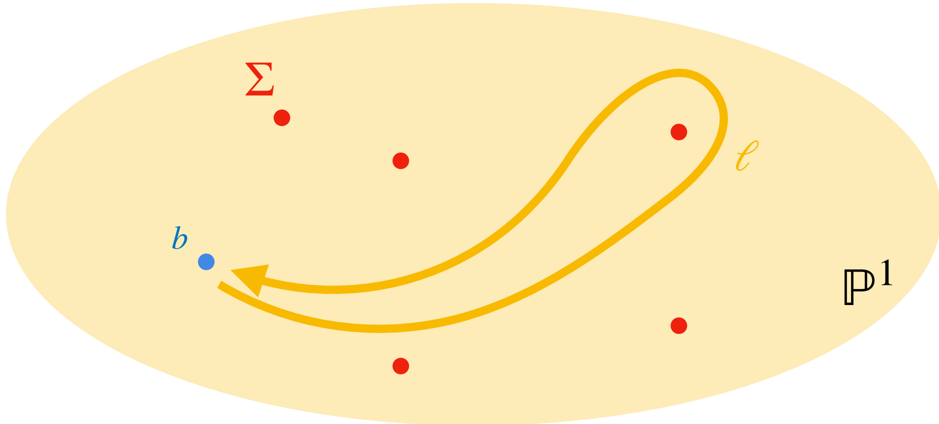
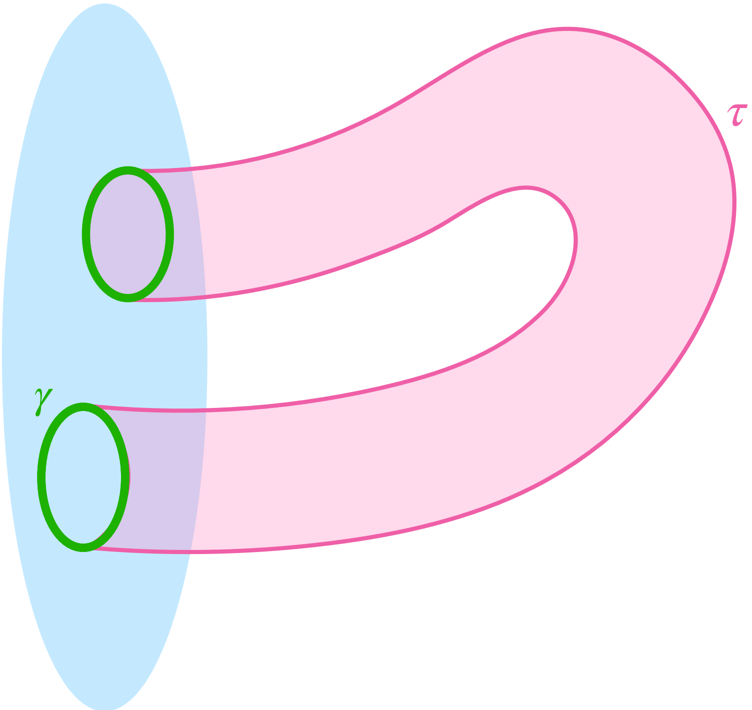
Two line integrals:
we know how to compute these efficiently!
[Chudnovsky², Van der Hoeven, Mezzarobba]

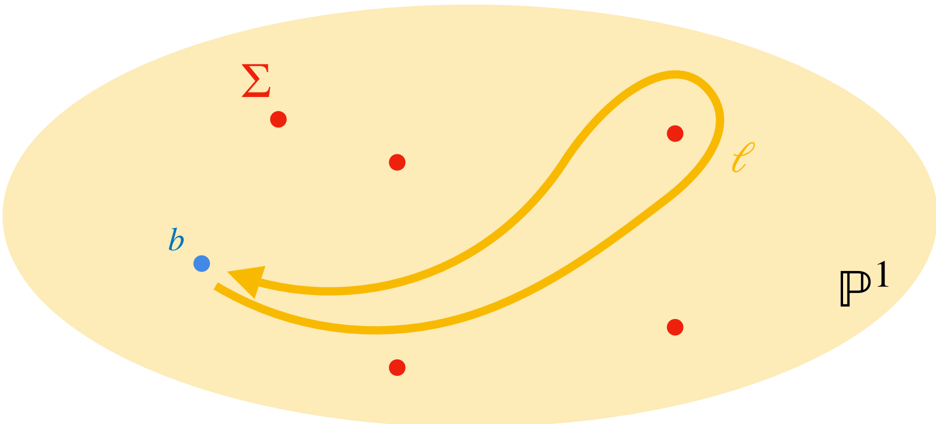


Comparison with dimension 1



Extensions are n**-cycles** obtained by extending $n-1$**-cycles** along loops.

The monodromy along a loop ℓ is an isomorphism of $H_{n-1}(\mathscr{X}_b)$.



If the projection is generic (Lefschetz), singular fibres are simple.

There is a single **thimble** per critical value.

We get *almost* every possible n-cycle by gluing thimbles.

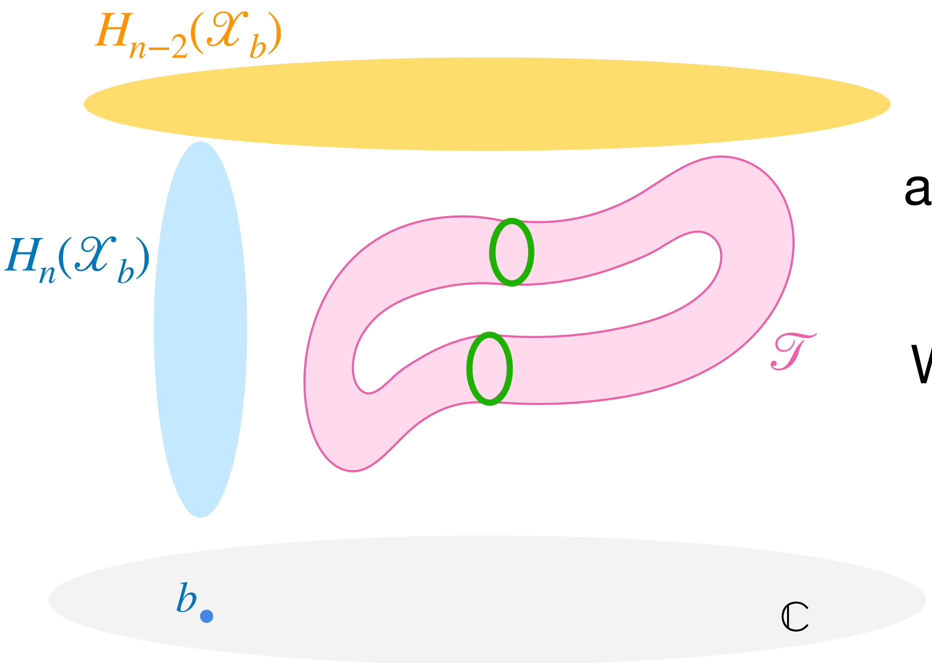
$$H_n(\mathscr{X}_b) \to H_n(\mathscr{X}) \to H_n(\mathscr{X}, \mathscr{X}_b) \to H_{n-1}(\mathscr{X}_b)$$

Possibly nontrivial

Almost generated by thimbles

Some complications

Not all cycles of $H_n(\mathscr{Y})$ are lift of loops, and thus not all are combinations of thimbles.



More precisely, we are missing the homology class of the **fibre** $H_n(\mathscr{X}_b)$ and a **section** (an extension of $H_{n-2}(\mathscr{X}_b)$ to all of $\mathbb{P}^1$).

We have a filtration $\mathscr{F}^0 \subset \mathscr{F}^1 \subset \mathscr{F}^2 = H_n(\mathscr{Y})$ such that

$$\mathscr{F}^0 \simeq H_n(\mathscr{X}_b)$$
$$\mathscr{F}^1/\mathscr{F}^0 \simeq \mathscr{T}$$
$$\mathscr{F}^2/\mathscr{F}^1 \simeq H_{n-2}(X_b)$$

Interesting part

$\mathscr{T}$ is also known as the **parabolic cohomology** of the local system.

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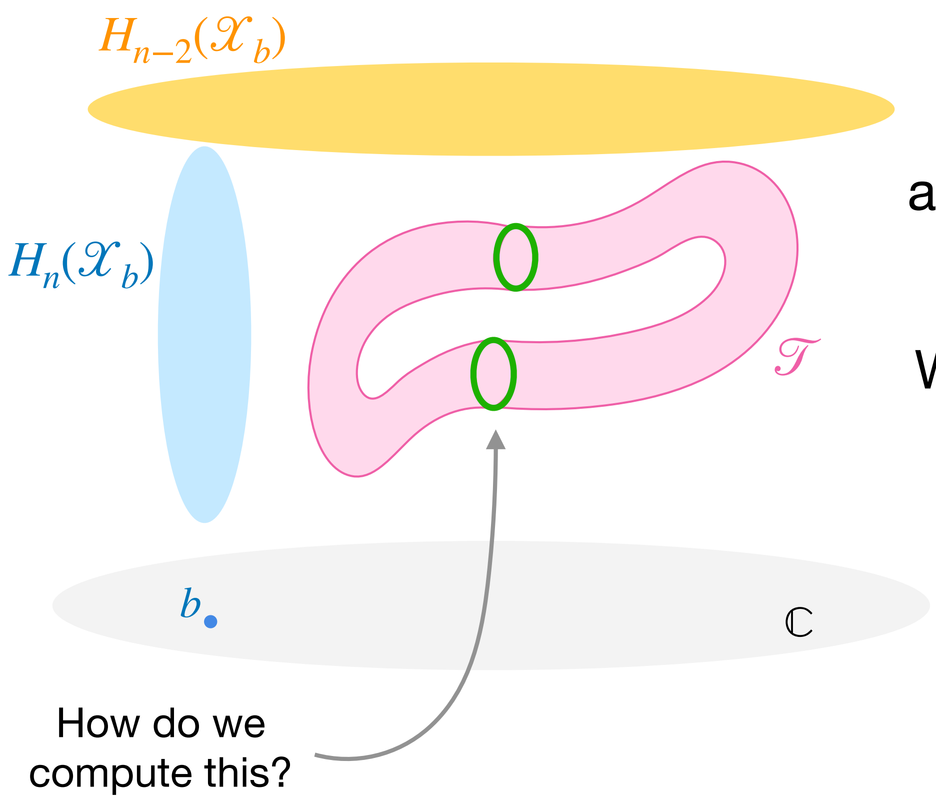

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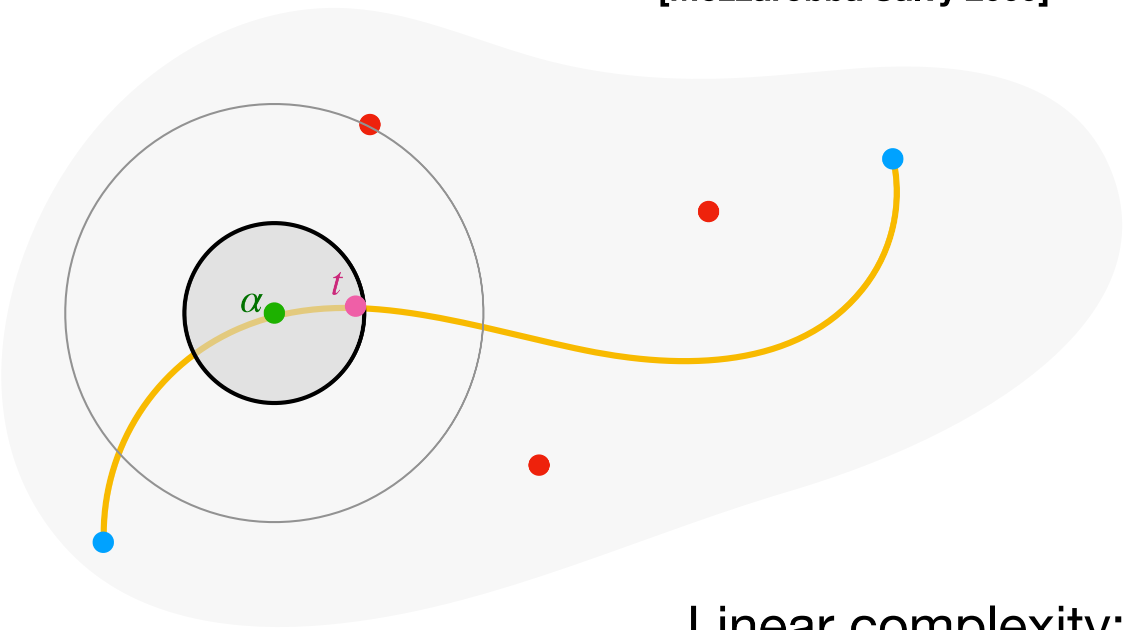
Monodromy of a differential operator

[Chudnovsky[2] 90, Van der Hoeven 99, Mezzarobba 2010]

In a small radius around α:

$$\left| f(t) - \sum_{k=0}^{m} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \right| \leq \mathscr{P}(m)2^{-m}$$

polynomial in m (effective)
[Mezzarobba Salvy 2009]



We compute $f^k(\alpha)$ from $\mathscr{L}$.

In a disk around α, the precision given by the Taylor formula is exponential in its order.

From the derivatives at α, we can recover the derivatives at t.

Linear complexity: Recover m digits in $\mathcal{O}(m)$ operations

(using binary splitting)

Computing monodromy on cycles

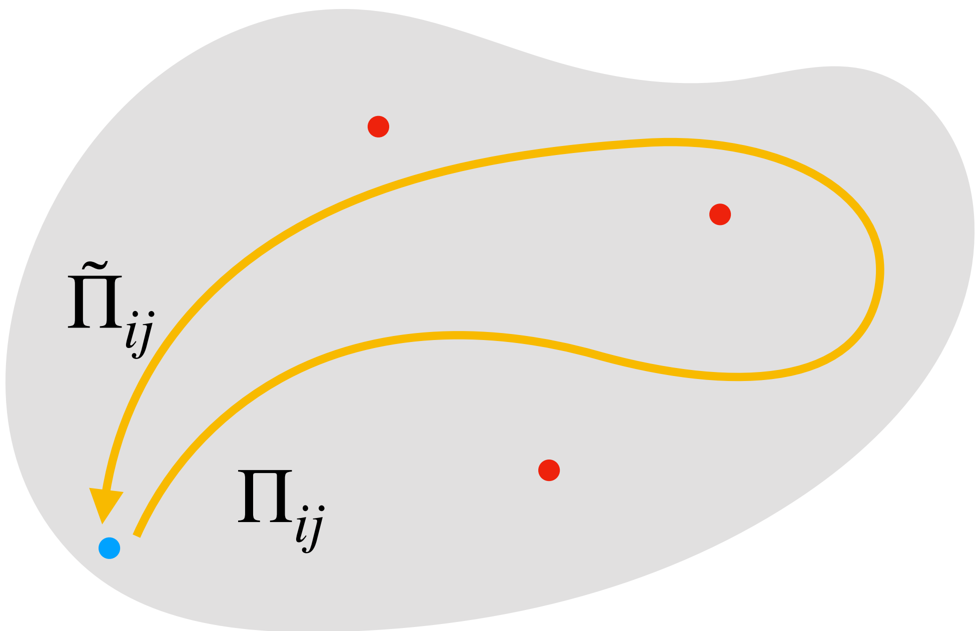
$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t$$

Analytic continuation

[Chudnovsky[2] 90, Van der Hoeven 99, Mezzarobba 2010]

$\tilde{\Pi}_{ij}$

Solution to Picard-Fuchs equation of ω_t



Computing monodromy on cycles

Globally defined
$\Longrightarrow$ no monodromy

$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t$$

Analytic continuation $\longrightarrow$

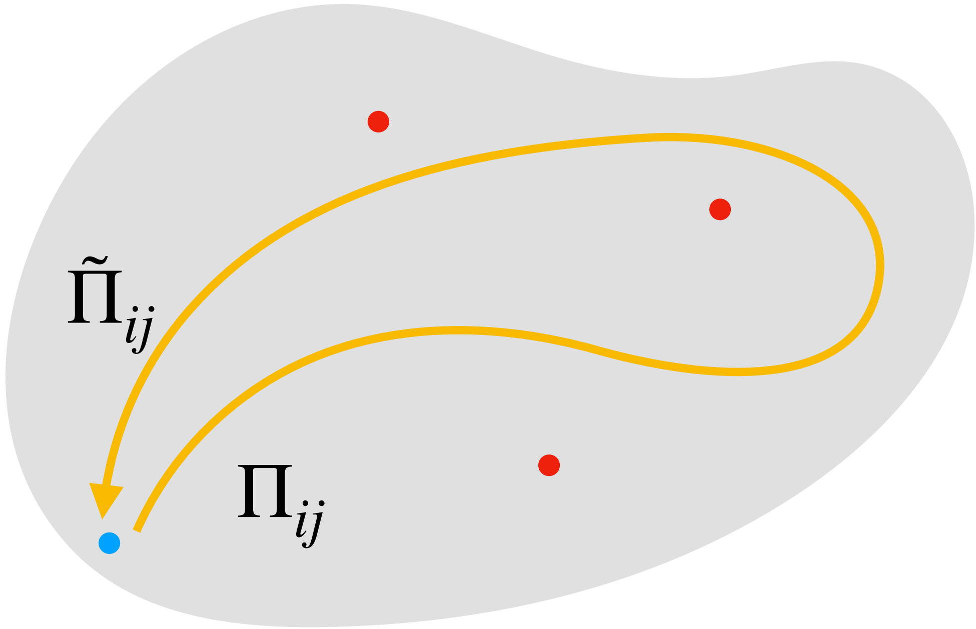
[Chudnovsky² 90, Van der Hoeven 99, Mezzarobba 2010]

$$\tilde{\Pi}_{ij} = \int_{\sum_k c_{kj}\gamma_k} \partial_t^i \omega_t = \sum_k c_{kj} \int_{\gamma_k} \partial_t^i \omega_t$$

Solution to Picard-Fuchs equation of ω_t

$$\tilde{\gamma}_j = \sum_k c_{kj}\gamma_k$$

The c_{kj}'s are integers



Computing monodromy on cycles

Globally defined
$\Longrightarrow$ no monodromy

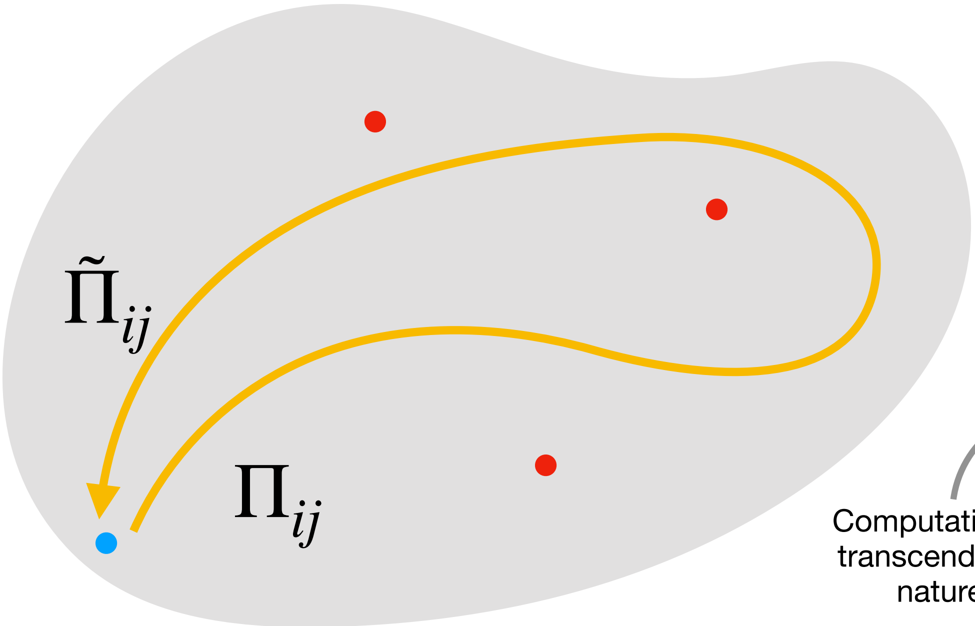
$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t \xrightarrow{\text{Analytic continuation}} \tilde{\Pi}_{ij} = \int_{\textcolor{red}{\sum_k c_{kj}}\gamma_k} \partial_t^i \omega_t = \sum_k c_{kj} \int_{\gamma_k} \partial_t^i \omega_t$$

[Chudnovsky[2] 90, Van der Hoeven 99, Mezzarobba 2010]

Solution to Picard-Fuchs equation of ω_t

$$\tilde{\gamma}_j = \sum_k c_{kj}\gamma_k$$

The c_{kj}'s are integers



Thus $\tilde{\Pi} = \Pi C$ i.e.

$$\Pi^{-1}\tilde{\Pi} = C \in \mathrm{GL}_2(\mathbb{Z})$$

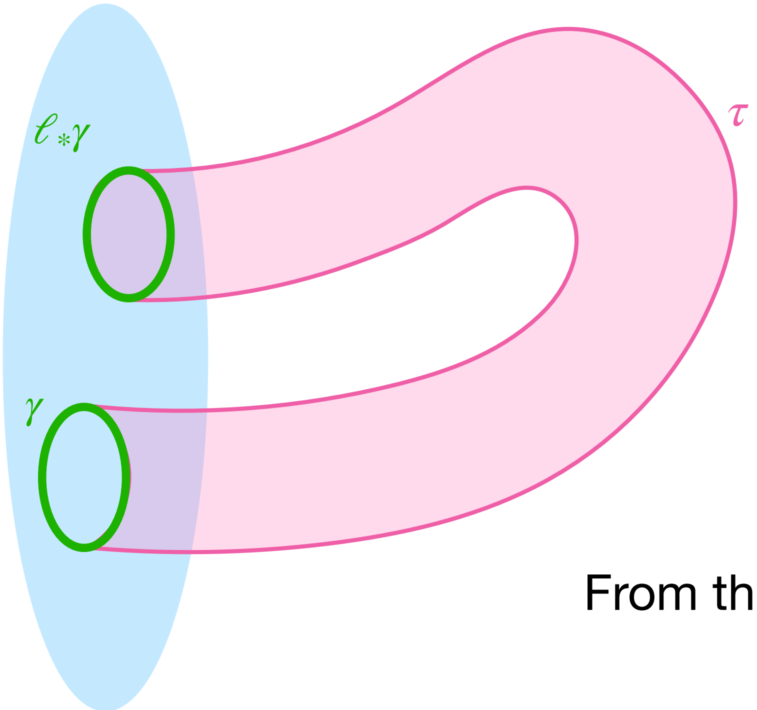
Computation of transcendental nature

It is sufficient to carry out this computation with precision $< 1/2$ to recover C exactly.

Periods of hypersurfaces

From the monodromy we compute the boundary of thimbles, and we can glue them to obtain extensions.

$$\partial\tau_\ell(\gamma) = \ell^*\gamma - \gamma$$



This yields an inductive method for computing the periods of smooth hypersurfaces.

$$\int_{\tau_\ell(\gamma)} \omega = \int_\ell \left(\int_\gamma \omega_t \right) \wedge \mathrm{d}t$$

From the periods, we may recover algebraic invariants.

For example, we can find quartic surfaces with Picard rank 2, 3 and 5, which were missing entries in a search of **[Lairez Sertöz 2019]**.

$$\mathcal{X} = V \begin{pmatrix} x^4 - x^2y^2 - xy^3 - y^4 + x^2yz + xy^2z + x^2z^2 - xyz^2 + xz^3 \\ -x^3w - x^2yw + xy^2w - y^3w + y^2zw - xz^2w + yz^2w - z^3w + xyw^2 \\ +y^2w^2 - xzw^2 - xw^3 + yw^3 + zw^3 + w^4 \end{pmatrix}$$

Periods of hypersurfaces

We thus obtain an algorithm for computing the periods of smooth hypersurfaces, inductive on the dimension.

Because we are working with lower dimensional varieties, this method turns out to be **more efficient** than that of **[Sertöz 2019]**, in particular for the computation of periods of quartic K3 surfaces:

$P - x^4 - w^4 - z^4 - w^4$	*numperiods*	*lefschetz-family*	ord $\mathcal{L}$	deg $\mathcal{L}$
0	< 1 s	384 min.	–	–
$2x^2zw$	4 s	574 min.	3	4
$-2y^3z - 4z^2w^2$	2 min.	510 min.	5	38
$-xyzw + 4xzw^2 - 2y^4$	25 min.	607 min.	7	110
$y^3z + z^4 + y^3w + x^2zw$	346 min.	635 min.	14	591
$4xyz^2 - 5x^2zw - 4xw^3 - 4zw^3$	> 2880 min.	494 min.	21	?
$-2x^2w^2 - 4y^2w^2 - 2yzw^2 + 2yw^3$	> 500 Gb	543 min.	21	?
$x^4 - 4y^2z^2 - 5xz^2w + 2yz^2w + xyw^2$	> 500 Gb	538 min.	14	?

In all cases, *lefschetz-family* integrates an operator of order 7.

We have solved one of the main difficulties:
the direct computation of the homology of hypersurfaces.

The bottleneck for accessing higher dimensions is still the order/degree of the differential operators.

However we are still relying on a generosity assumption.

Beyond hypersurfaces

Non-Lefschetz fibrations

[PP 2024]

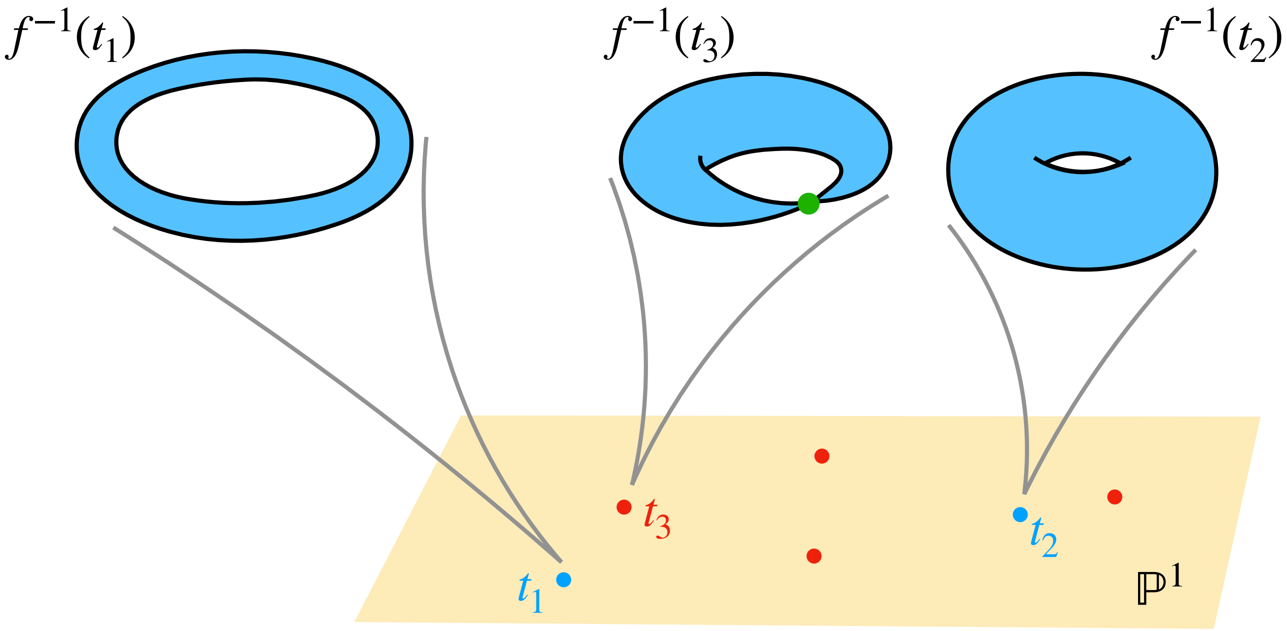
Elliptic surfaces

An **elliptic surface** S is a smooth algebraic surface equipped with a map to the projective line

The fibration is given. We cannot choose it to be generic.

$$f : S \to \mathbb{P}^1$$

such that all but finitely many fibres $f^{-1}(t)$ are elliptic curves.



We will assume the surface has a **section**.

Non-Lefschetz fibrations: an example

The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

Non-Lefschetz fibrations: an example

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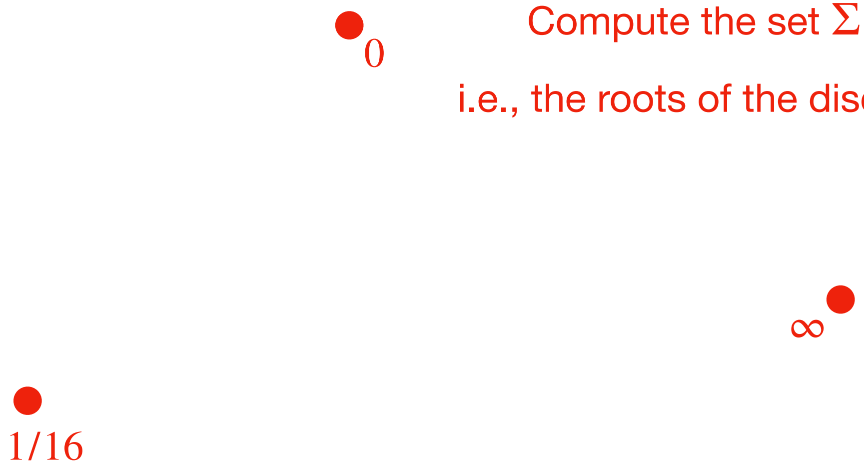
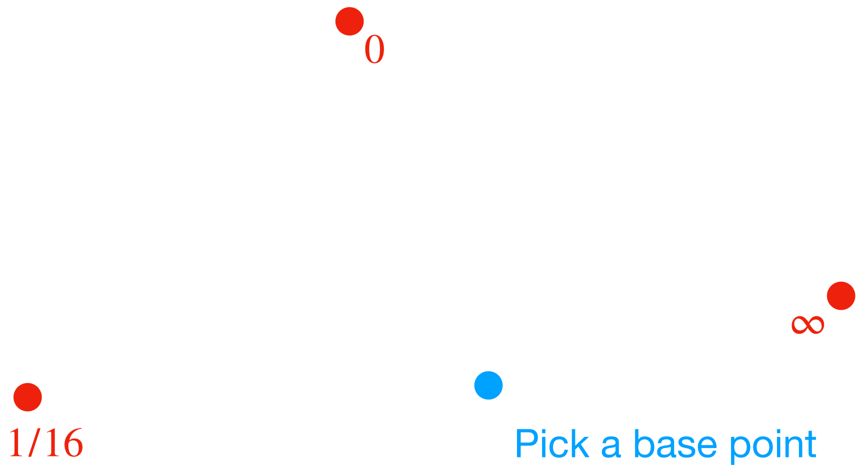


Compute the set Σ of critical values

i.e., the roots of the discriminant $t^4(t - \frac{1}{16})$

Non-Lefschetz fibrations: an example

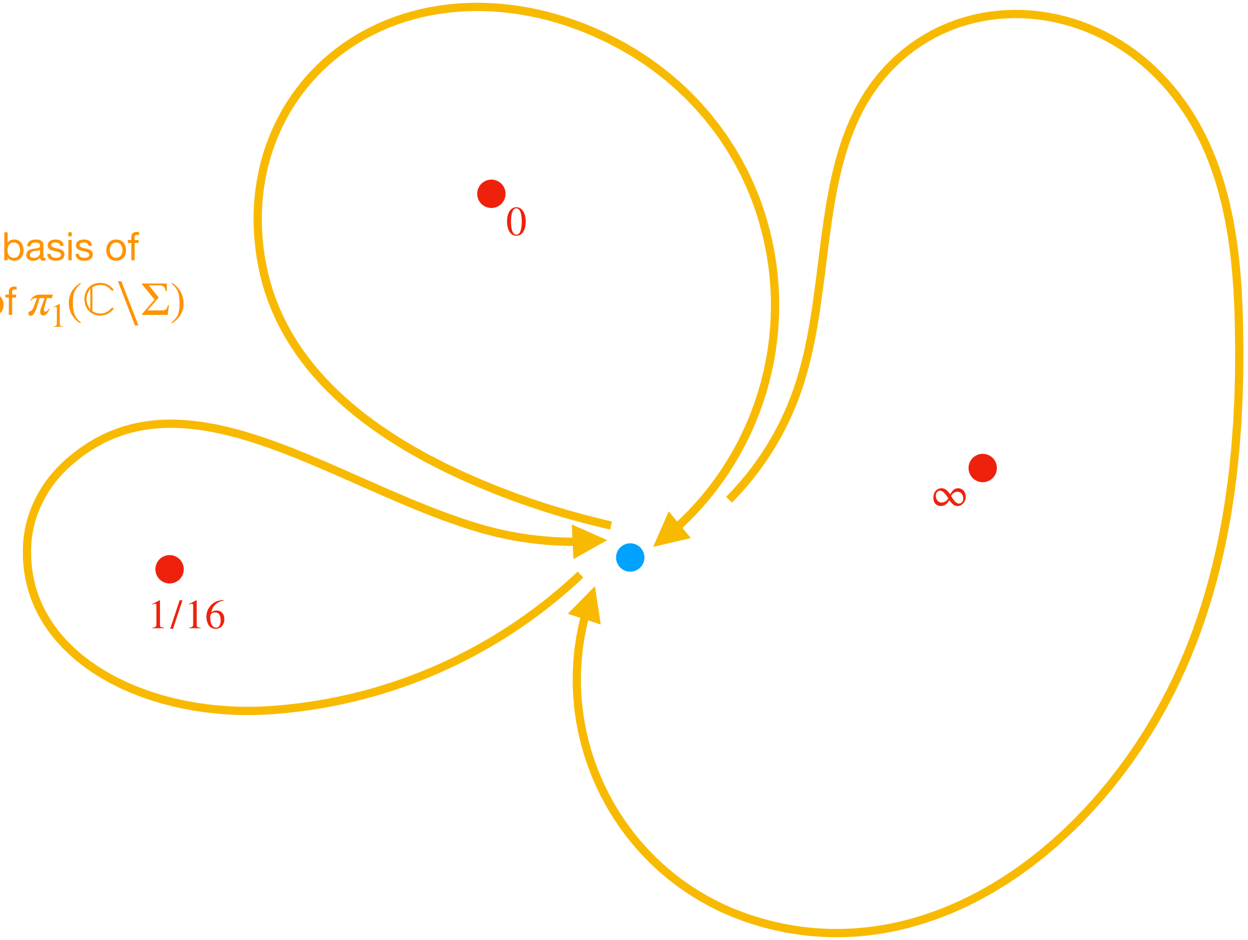
The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.



Non-Lefschetz fibrations: an example

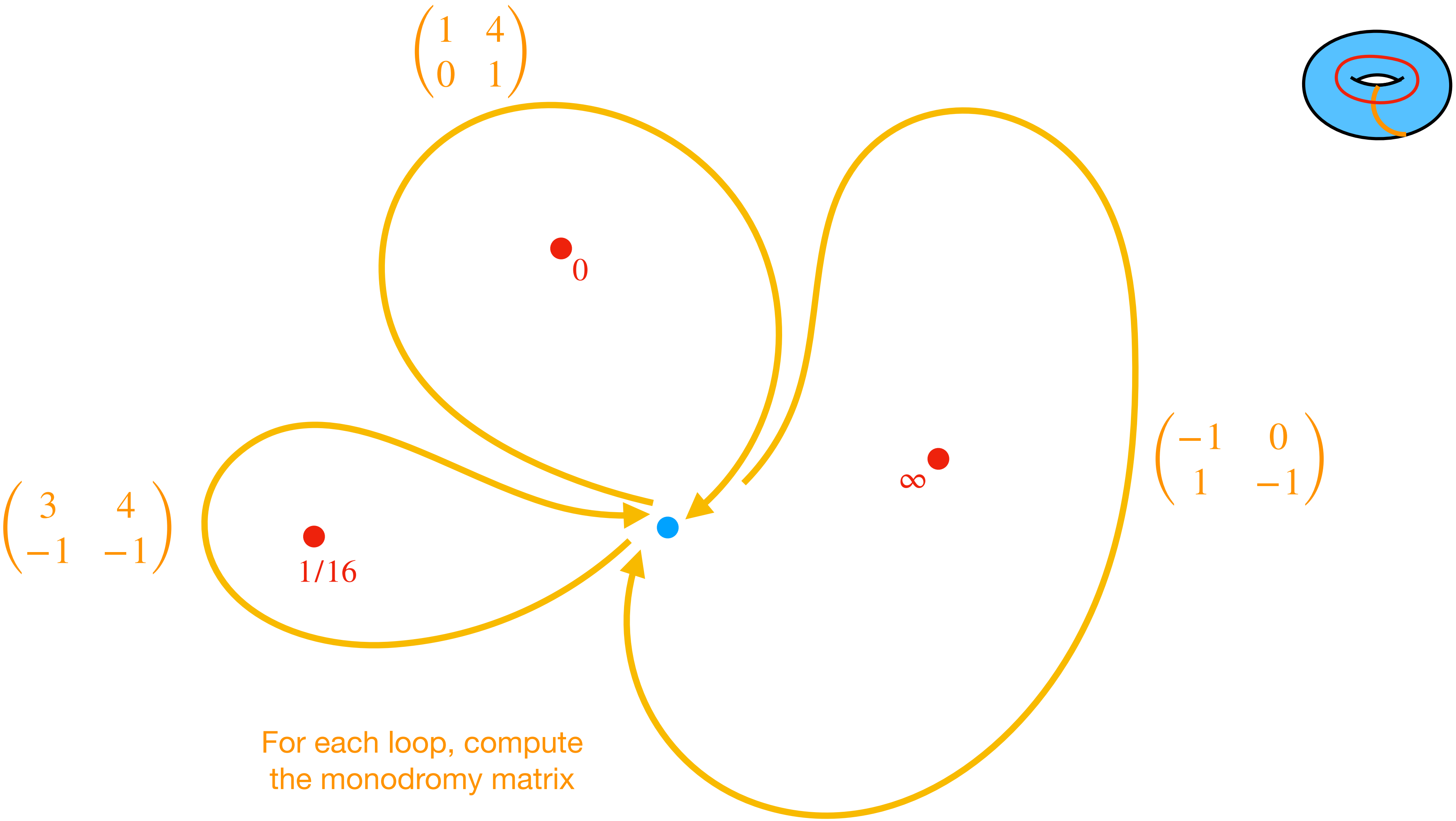
The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

Compute a basis of
simple loops of $\pi_1(\mathbb{C}\backslash\Sigma)$



Non-Lefschetz fibrations: an example

The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.



Non-Lefschetz fibrations: an example

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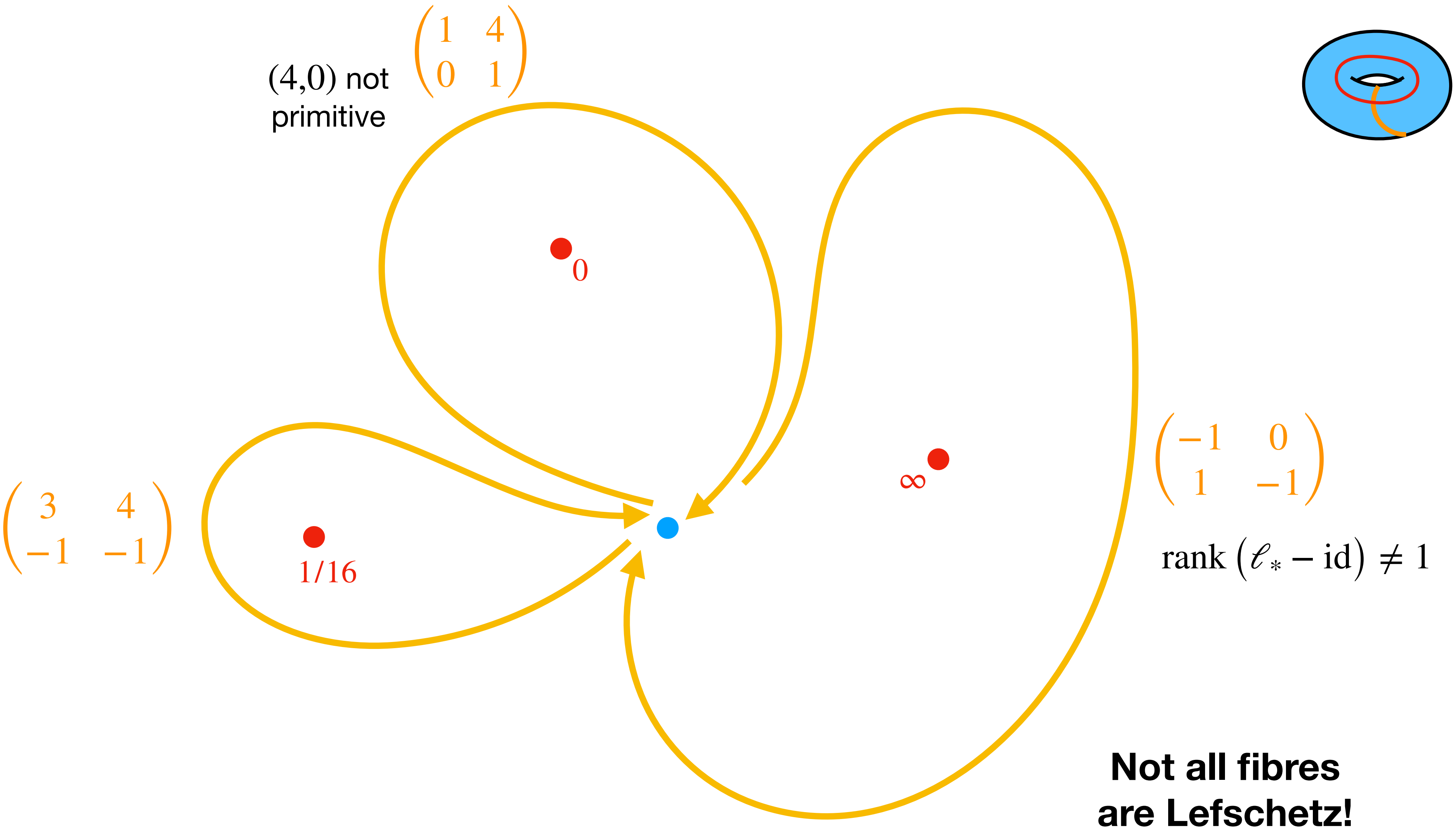
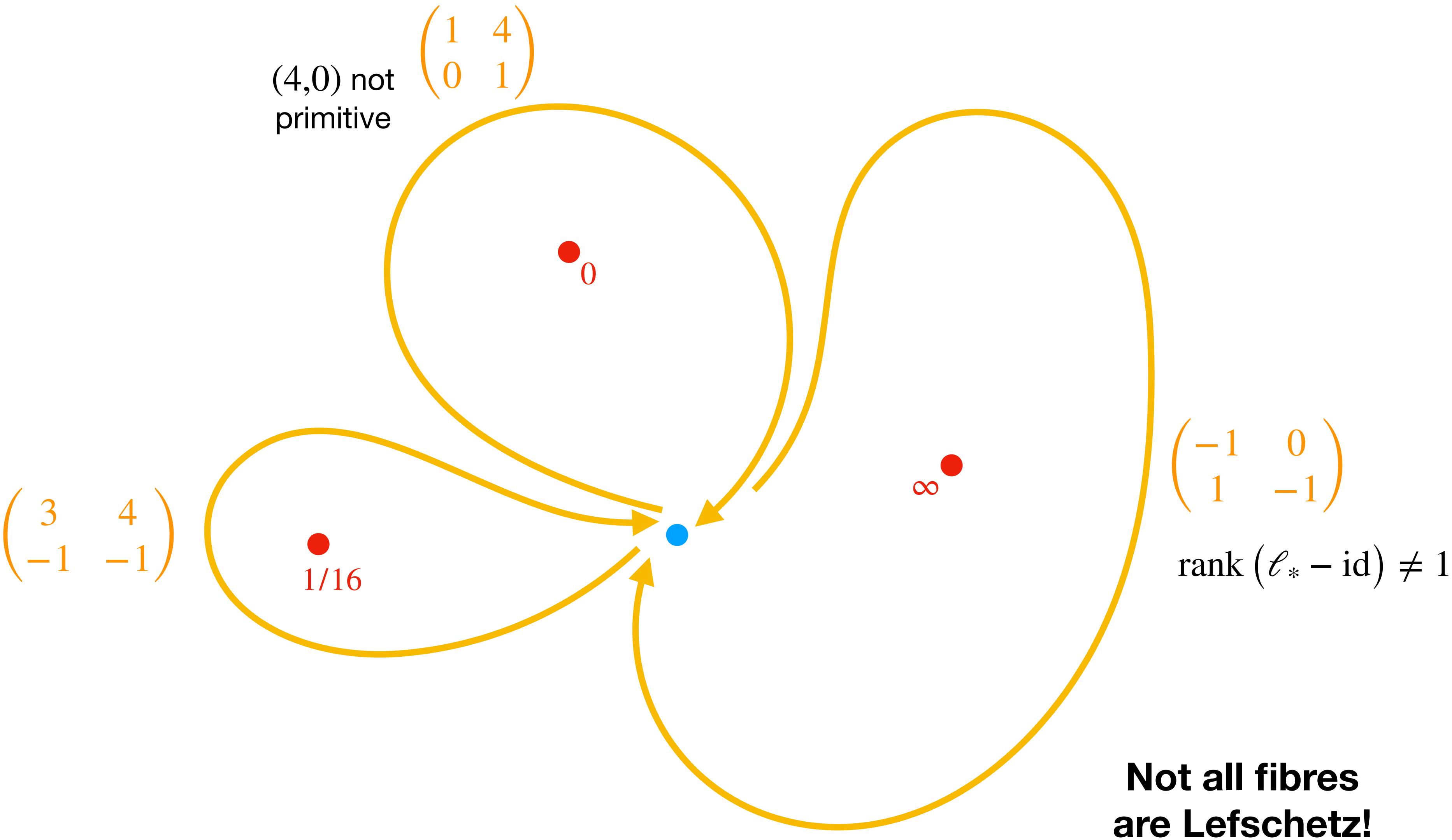


Not all fibres are Lefschetz!

We have to find a workaround …

Non-Lefschetz fibrations: an example

The **Apéry surface** S, defined by $y^2 + (t-1)xy + ty = x^3 - tx^2$.

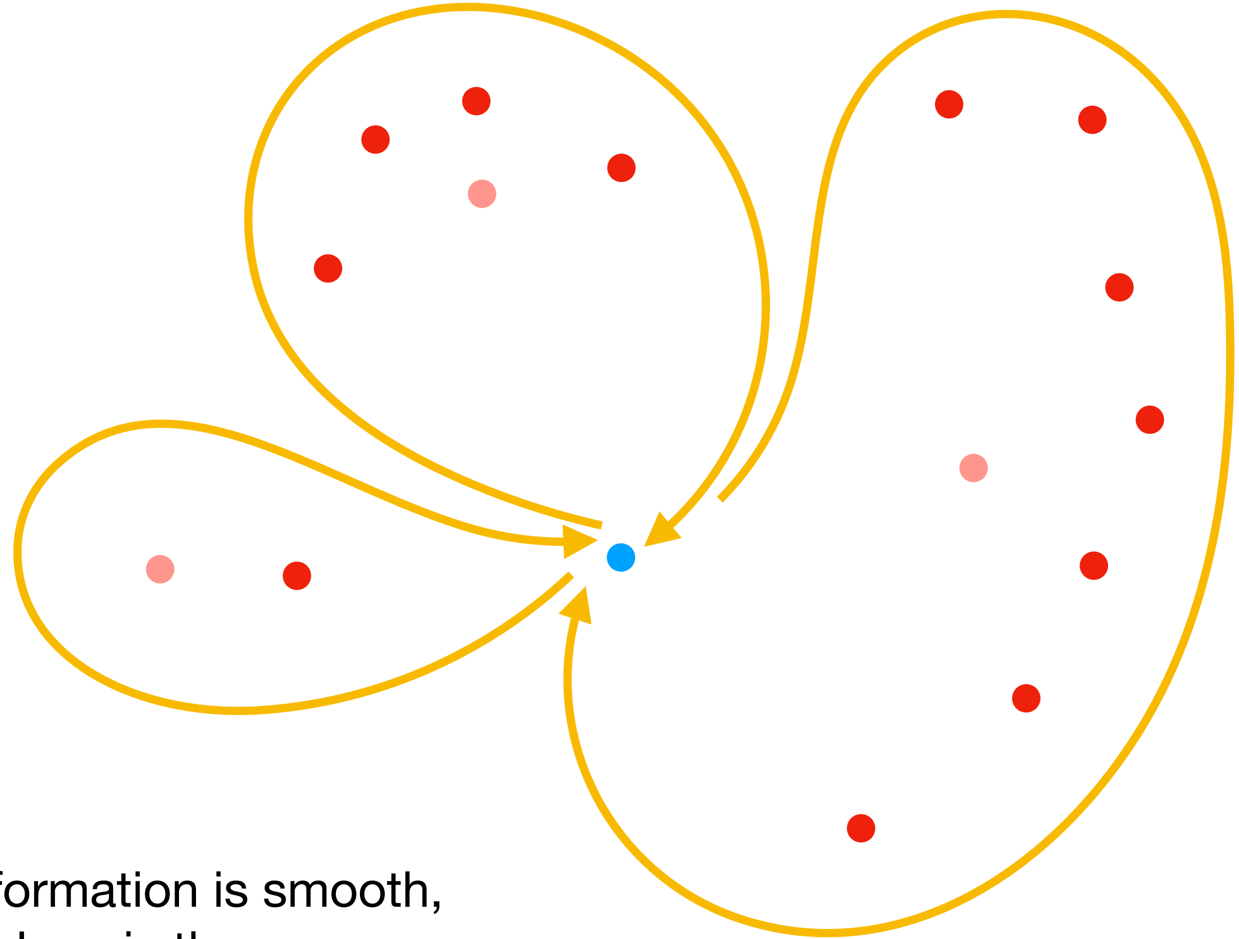


Not all fibres are Lefschetz!

We have to find a workaround …

Non-Lefschetz fibrations: an example

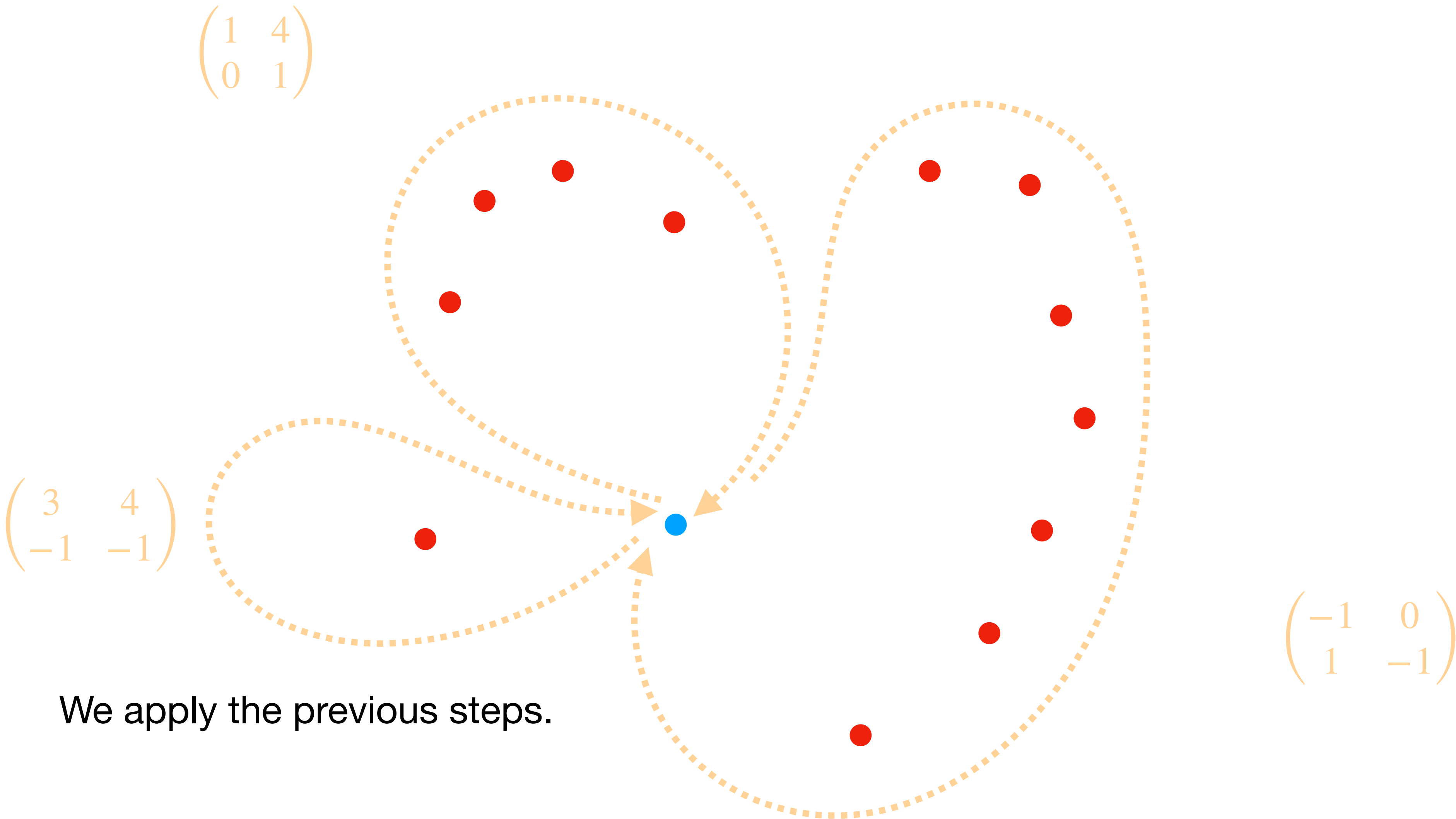
We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



As the deformation is smooth,
the topology is the same:
$H_2(S) \simeq H_2(\tilde{S})$.

Non-Lefschetz fibrations: an example

We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



We apply the previous steps.

Non-Lefschetz fibrations: an example

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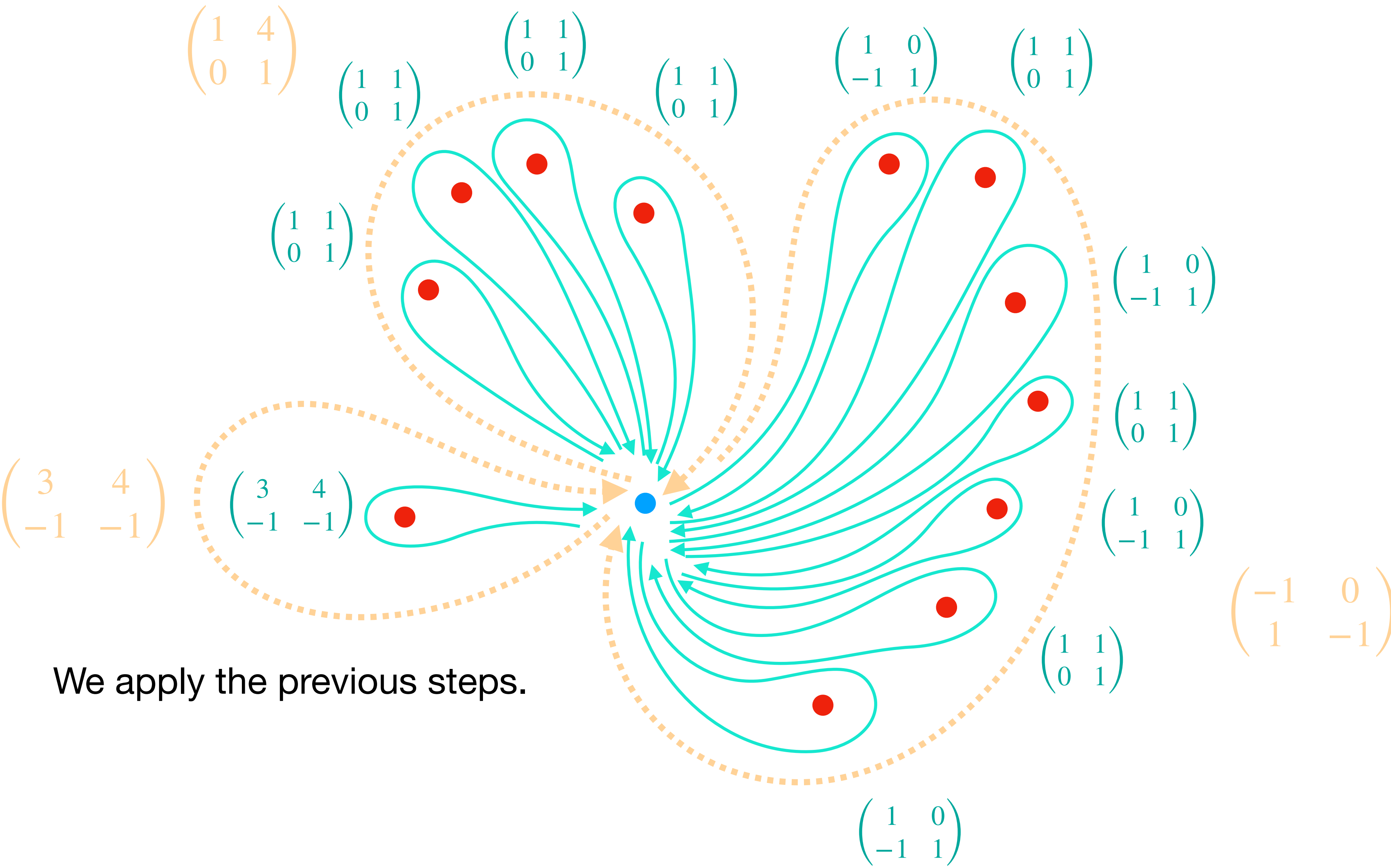

We apply the previous steps.

Non-Lefschetz fibrations: an example

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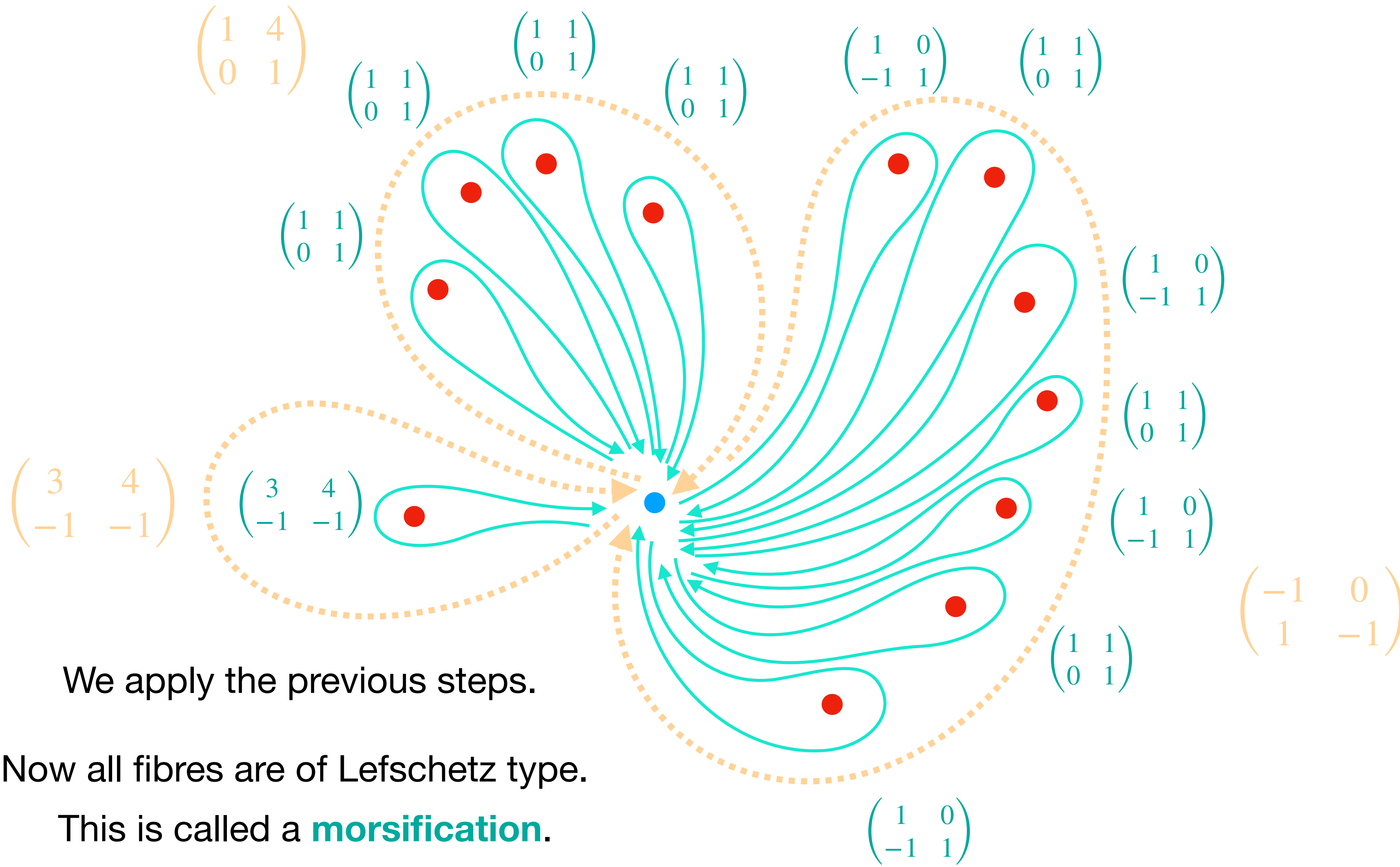


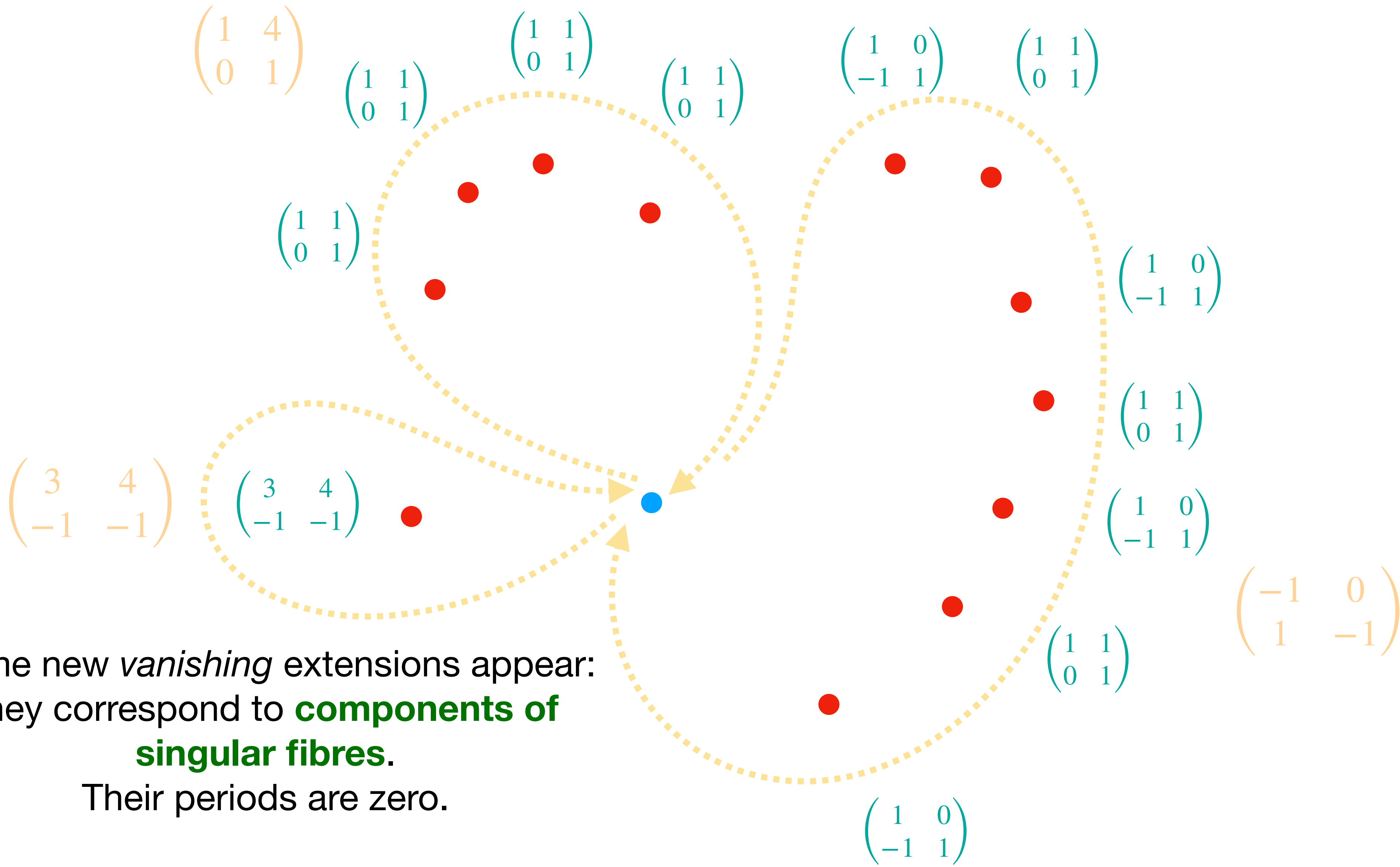
We apply the previous steps.

Now all fibres are of Lefschetz type.

This is called a **morsification**.

Non-Lefschetz fibrations: an example

We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



Some new *vanishing* extensions appear:
they correspond to **components of singular fibres**.
Their periods are zero.

Non-Lefschetz fibrations: an example

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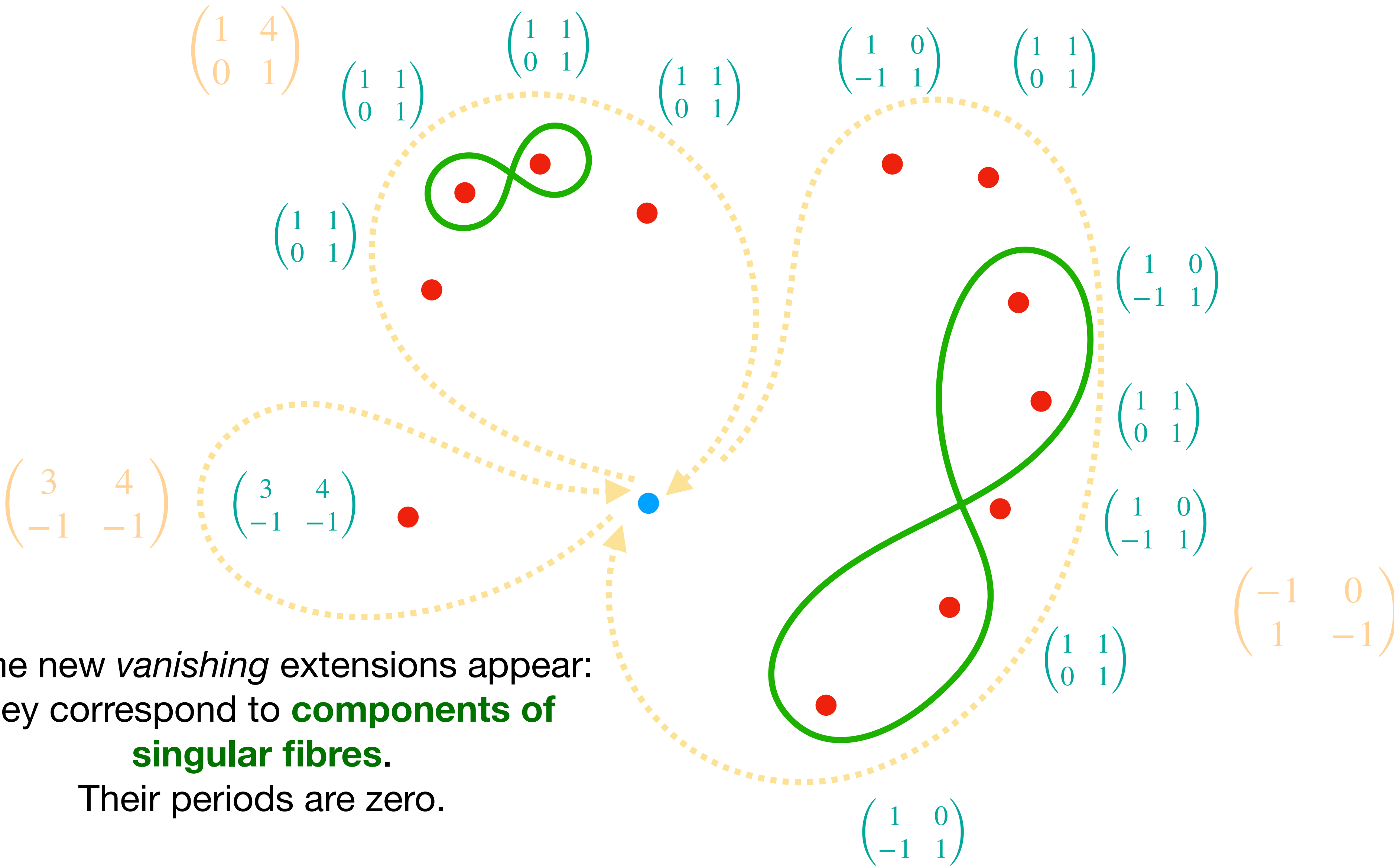

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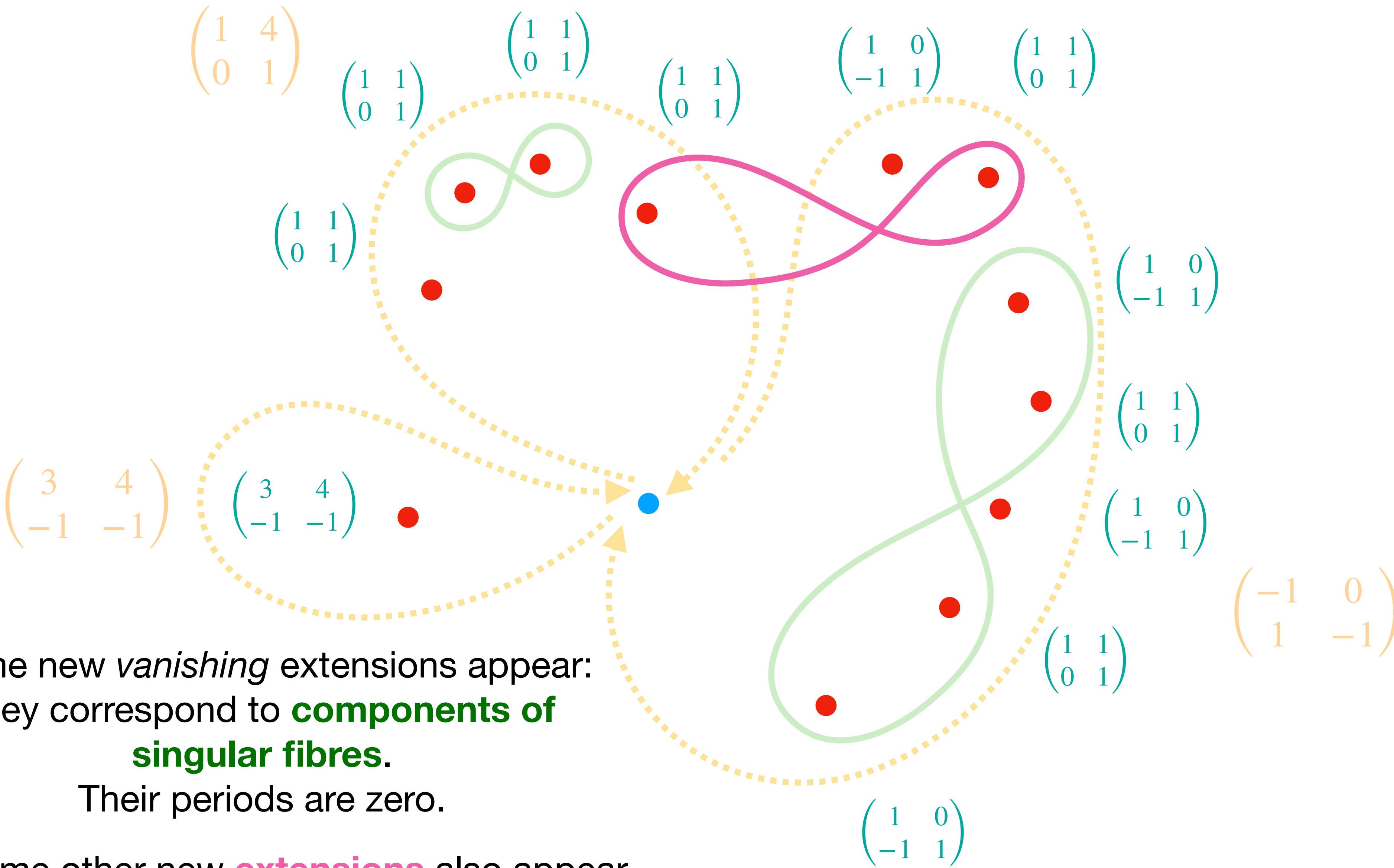
Non-Lefschetz fibrations: an example

We deform the surface to $\tilde{S} : y^2 + (t-1)xy + ty = x^3 - tx^2 + \varepsilon$.



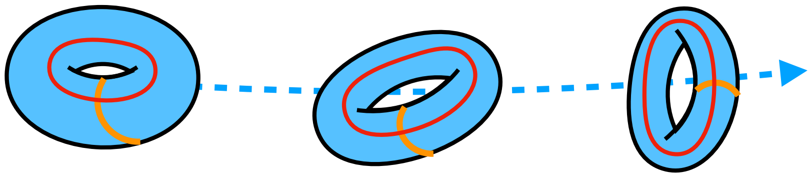
Some new *vanishing* extensions appear: they correspond to **components of singular fibres**. Their periods are zero.

Some other new **extensions** also appear.

Non-Lefschetz fibrations

Theorem [Moishezon 1977]: Morsifications always exist.

Monodromy preserves the intersection product



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\iff ad - bc = 1$$

The monodromy matrix is in $\mathrm{SL}_2(\mathbb{Z})$.

Kodaira classification [1963]

$I_\nu, \nu \geq 1$	$\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$	U^ν
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	VU
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	VUV
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$(VU)^2$

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad V = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

...

Theorem [Cadavid Vélez 2009]:
The monodromy of the morsification is determined by the monodromy of S.

only cycles with nonzero periods

Theorem [PP 2024]: The sublattice of $H_2(S)$ generated by **extensions** of S, the **section**, the **fibre** and **singular components** has full rank.

Non-Lefschetz fibrations

Theorem [Cadavid, Vélez 2009]:
The monodromy of the morsification is determined by the monodromy of S.

ℓ_1 $\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$ $I_1 : U$

ℓ_2 $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ $I_4 : U^4$

ℓ_3 $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ $I_1^* : UVUVUV$

morsification

$\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$ ℓ'_1

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ℓ'_{21}, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ℓ'_{22}, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ℓ'_{23}, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ℓ'_{24}

$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ ℓ'_{31}, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ℓ'_{32}, $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ ℓ'_{33}, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ℓ'_{34}, $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ ℓ'_{35}, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ℓ'_{36}, $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ ℓ'_{37}

In particular **we do not need to find an explicit realisation of the morsification**!

The algorithm for elliptic surfaces

1. Compute a basis of **simple loops** $\ell_1, \ldots, \ell_r$ of $\pi_1(\mathbb{P}^1 \setminus \Sigma, b)$
2. For each $1 \leq i \leq r$, compute the **monodromy map** ℓ_{i*}.
3. Glue thimbles together to obtain **extension cycles** of $H_2(S)$.
4. Integrate the **periods** on these cycles.

5. From the monodromy type of ℓ_{i*}, recover the monodromy matrices of a **morsification** $\tilde{S}$.
6. Glue thimbles together to obtain **extension cycles** of $H_2(\tilde{S})$.
7. Recover the homology $H_2(\tilde{S})$ of the morsification (**extensions** + **fibre** + **section**).
8. Describe the extensions of $H_2(S)$ in terms of the extensions of $H_2(\tilde{S})$.

9. Recover the periods of all of $H_2(S) \simeq H_2(\tilde{S})$.

This allows for the (heuristic) computation of certain algebraic invariants of the elliptic surface (Néron-Severi group, Mordell-Weil group, …)

Implemented in the **lefschetz-family** Sagemath package, available on my webpage.

Further applications

of the methods presented here

[Doran, Harder, PP, Vanhove 2024] and ongoing works

Recovering certain algebraic invariants

Theorem [Doran Harder PP Vanhove 2024]: The Tardigrade hypersurface has the same motivic geometry as a quartic K3 surface with six A_1 singularities.

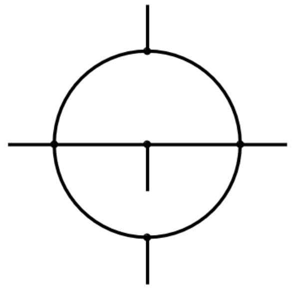


FIGURE 13. The tardigrade graph

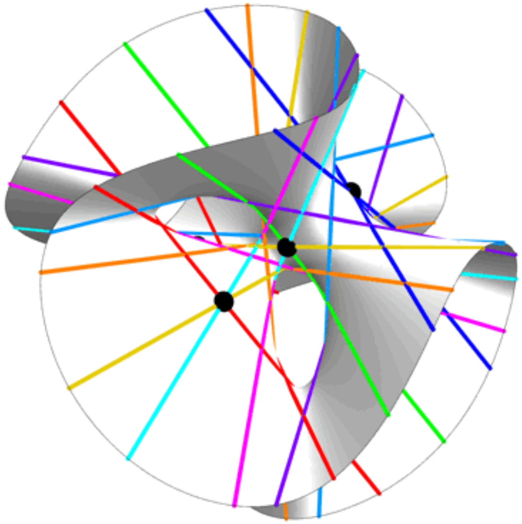
Our methods allow to compute the periods of this quartic K3 surface.

From the periods, we recover numerically that **its Néron-Severi rank is 11** for generic values of the mass parameters.

Lefschetz's theorem on (1,1) classes:
A homology class $\gamma \in H_2(S)$ is in the Néron-Severi group $NS(S)$ iff the periods of holomorphic forms on γ vanish.

Using the LLL algorithm, we can heuristically recover this kernel by finding integer linear relations between the periods.

Lines on a cubic surfaces



Animation by Greg Egan

There are 27 (complex) lines $L_1, L_2, \ldots, L_{27}$ on a cubic surface $\mathcal{X}$.

Such lines are isolated in their linear equivalence class in $H_2(\mathcal{X})$.
These classes are characterised
by the following intersection numbers:

$$\langle L_i, h_{\mathcal{X}} \rangle = 1 \qquad\qquad L_i^2 = -1$$

where $h_{\mathcal{X}}$ is the class of the hyperplane section.

Let $\mathcal{X}_t$ be a one parameter family of cubic surfaces.

We may compute the action of monodromy on homology $\ell_* : H_2(\mathcal{X}_b) \to H_2(\mathcal{X}_b)$.

As ℓ_* preserves the intersection product and $h_{\mathcal{X}}$, we have that

$$\ell_* L_i = L_{\sigma_\ell(i)}$$

for some permutation σ_ℓ of $\{1,2,\ldots,27\}$.

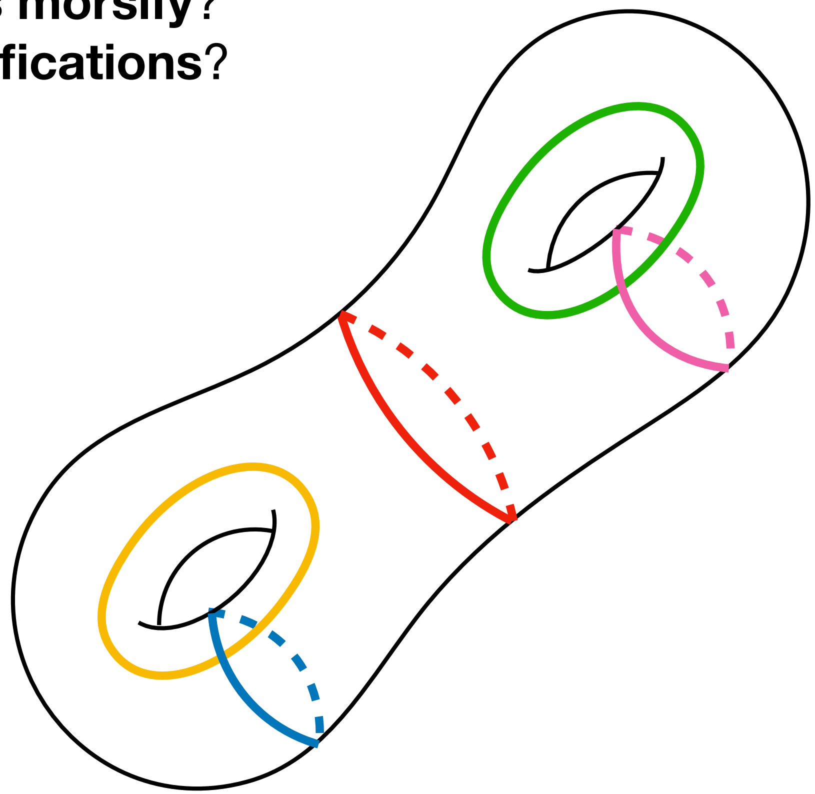
We can compute σ_ℓ!

Genus 2 fibrations

Certain K3 surfaces are given by ramified double covers of $\mathbb{P}^2$.
We may fibre them with **genus 2 curves**.
When the ramification locus is smooth, we may always obtain a Lefschetz fibration.

When it is not, **can we always morsify**?
Can we **bypass explicit morsifications**?

Defining polynomial	Picard number
$w^2 + xy^5 + x^5z + y^3z^3 + xz^5$	1
$w^2 + x^6 + y^5z + xz^5$	2
$w^2 + x^5y + xy^5 + x^3y^2z + z^6$	3
$w^2 + x^5y + y^6 + x^3yz^2 + x^3z^3 + xz^5$	4
$w^2 + x^5y + y^6 + x^4z^2 + x^2yz^3 + xz^5$	5
$w^2 + x^4y^2 + x^5z + y^5z + z^6$	6
–	7
$w^2 + x^5y + y^5z + y^2z^4 + xz^5$	8
$w^2 + x^5y + y^6 + x^2z^4 + z^6$	9
$w^2 + x^6 + y^5z + x^2z^4 + z^6$	10
$w^2 + x^5y + xy^5 + x^3yz^2 + z^6$	11
$w^2 + x^6 + y^6 + z^6 + x^2yz^3$	12
$w^2 + x^6 + y^6 + z^6 + x^2y^4 + x^4z^2$	13
$w^2 + x^6 + y^6 + xz^5$	14
$w^2 + x^6 + y^6 + z^6 + x^4yz + xyz^4$	15
$w^2 + x^6 + y^6 + z^6 + x^4y^2$	16
$w^2 + x^6 + y^6 + z^6 + x^4yz$	17
$w^2 + x^5y + x^3y^3 + xy^5 + z^6$	18
$w^2 + x^6 + y^6 + z^6 + x^3y^3$	18
$w^2 + x^6 + y^6 + z^6 + x^2y^2z^2$	19
$w^2 + x^6 + y^6 + z^6$	20

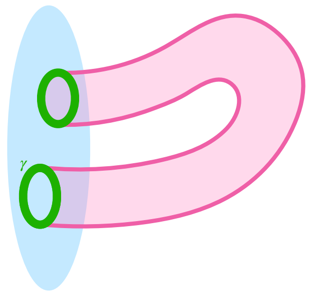


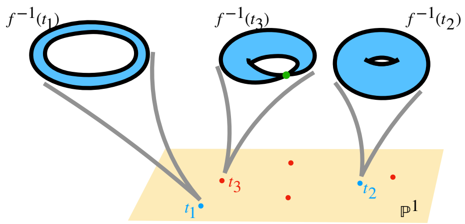
There are **many more singular types** to consider than for elliptic fibrations (118 vs 8).

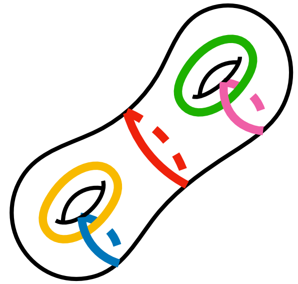
Furthermore the **monodromy matrices alone do not determine the type** of the singular fibres.

Concluding remarks

New methods for computing periods of algebraic varieties, **implemented** for hypersurfaces, elliptic surfaces and Lefschetz genus 2 fibered surfaces.







They are sufficiently **efficient** to recover the periods of examples previously out of reach.

$$\mathscr{X} = V\begin{pmatrix} x^4 - x^2y^2 - xy^3 - y^4 + x^2yz + xy^2z + x^2z^2 - xyz^2 + xz^3 \\ -x^3w - x^2yw + xy^2w - y^3w + y^2zw - xz^2w + yz^2w - z^3w + xyw^2 \\ +y^2w^2 - xzw^2 - xw^3 + yw^3 + zw^3 + w^4 \end{pmatrix}$$

numperiods	*lefschetz-family*
< 1 s	384 min.
4 s	574 min.
2 min.	510 min.
25 min.	607 min.
346 min.	635 min.
> 2880 min.	494 min.
> 500 Gb	543 min.
> 500 Gb	538 min.

From these numerical approximations, we recover **algebraic invariants** of certain varieties arising in other contexts (mirror symmetry, Feynman integrals).

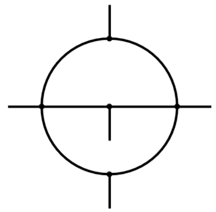
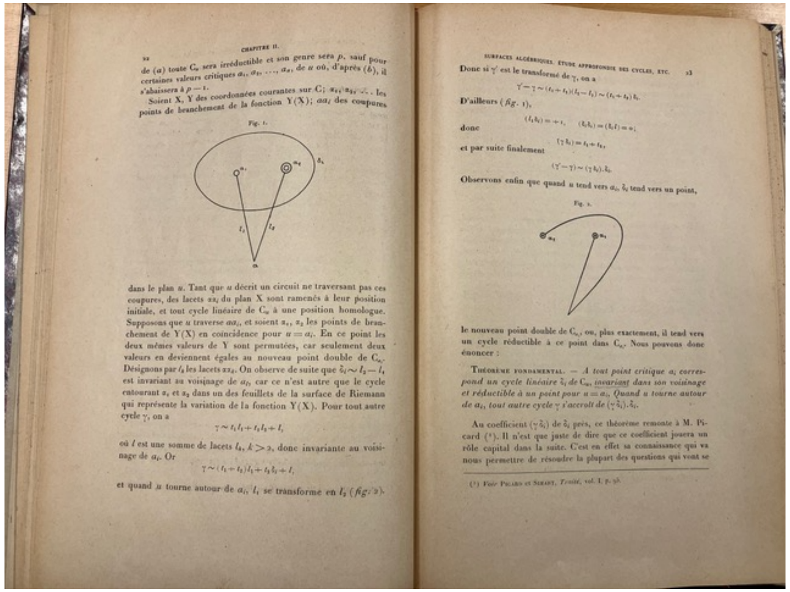


FIGURE 13. The tardigrade graph

Thank you!



L'analysis situs et la géométrie algébrique, 1924, Solomon Lefschetz