Positive geometries and canonical forms via mixed Hodge theory

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Plan

1. Overview
2. Logarithmic forms
3. Basics of mixed Hodge theory
4. Genus and combinatorial rank
5. Examples

Overview

In certain situations there is a map

$$\text{domain } \sigma \quad \rightsquigarrow \quad \text{logarithmic differential form } \varpi_\sigma.$$

$$\text{boundary structure of } \sigma \quad \longleftrightarrow \quad \text{residue structure of } \varpi_\sigma.$$

Terminology in the physics literature

$$\sigma = \text{a “positive geometry”} \quad \rightsquigarrow \quad \varpi_\sigma = \text{its “canonical form”}.$$

(Motivation: Arkani-Hamed–Trnka’s amplituhedra.)

Example: an interval

$$\sigma = [a, b] \subset \mathbb{R} \quad \rightsquigarrow \quad \varpi_\sigma = \mathrm{dlog}\left(\frac{x-b}{x-a}\right) = \frac{(b-a)\,\mathrm{d}x}{(x-a)(x-b)}.$$

Our message

This is a byproduct of Deligne’s mixed Hodge theory.

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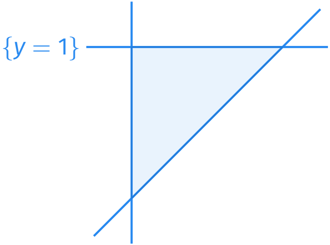
"The residue of the canonical form is the canonical form of the boundary".

Example: a triangle

$$\sigma = \{0 \leqslant x \leqslant y \leqslant 1\} \subset \mathbb{R}^2 \qquad \rightsquigarrow \qquad \varpi_\sigma = \frac{\mathrm{d}x \wedge \mathrm{d}y}{x(y-x)(y-1)}.$$

On the horizontal boundary component $\{y = 1\}$, we have

$$\mathrm{Res}_{\{y=1\}}(\varpi_\sigma) = \frac{\mathrm{d}x}{x(1-x)} = \varpi_{\sigma \cap \{y=1\}}.$$



Definition (Arkani-Hamed–Bai–Lam)

A *positive geometry* of dimension n is a pair (X, σ) where

- X is a projective real algebraic variety of dimension n;
- σ is an n-dimensional oriented semi-algebraic domain in X;

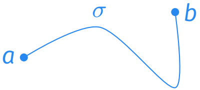
such that *there exists a unique* algebraic n-form ϖ_σ on X with logarithmic singularities, satisfying:

- every boundary component $(D, D \cap \sigma)$ is a positive geometry of dimension $n-1$ and
$$\mathrm{Res}_D(\varpi_\sigma) = \varpi_{D\cap\sigma}.$$

The base case ($n = 0$) is:

- if $X = \sigma = \text{point}$, then $\varpi_\sigma = 1$.

- "Positivity" is a red herring. No need for the domain σ to be real.



$$\rightsquigarrow \qquad \varpi_\sigma = \mathrm{dlog}\left(\frac{z-b}{z-a}\right).$$

- *Existence* of a canonical form is always guaranteed, but *uniqueness* is really the key issue.

The 1-dimensional case

Let X be a compact Riemann surface, $a, b \in X$ distinct. There always exists a logarithmic form ω on $X \setminus \{a, b\}$ with $\mathrm{Res}_b(\omega) = 1$ and $\mathrm{Res}_a(\omega) = -1$. It is unique if and only if X has genus zero.

- Mixed Hodge theory provides a natural non-recursive definition of "positive geometries" (*genus zero pairs*) and their canonical forms. The properties of canonical forms (e.g., recursion) are consequences of the definition.
- Separates the tasks of proving that something is a "positive geometry" and of *computing* the canonical form + gives tools for computations.

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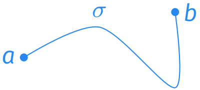

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Main messages

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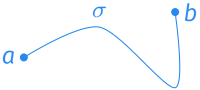

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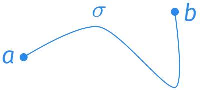

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Let X be a compact complex algebraic variety of dimension n, and $Y \subset X$ a closed subvariety such that $X \setminus Y$ is smooth.

Theorem (Brown–D.)

Assume that (X, Y) has *genus zero*. We construct a linear map

$$\begin{array}{ccc} \mathrm{H}_n(X,Y) & \longrightarrow & \Omega^n_{\log}(X \setminus Y) \\ \sigma & \longmapsto & \varpi_\sigma \end{array}$$

It associates to an n-chain σ with $\partial\sigma \subset Y$ a logarithmic n-form ϖ_σ, called its *canonical form*. It satisfies:

- recursion;
- invariance under triangulation;
- invariance under modification (e.g., blow-up);
- functoriality;
- etc.

First ingredient: Poincaré duality

- Relative homology: $\mathrm{H}_n(X, Y) \ni [\sigma]$ with $\partial\sigma \subset Y$.
- Related to locally finite homology / compactly supported cohomology:

$$\mathrm{H}_n(X, Y) \simeq \mathrm{H}_n^{\mathrm{lf}}(X \setminus Y) \simeq \mathrm{H}_{\mathrm{c}}^n(X \setminus Y)^\vee.$$

- Therefore:

Poincaré duality

$$\mathrm{H}_n(X, Y) \simeq \mathrm{H}^n(X \setminus Y).$$

Logarithmic forms

Normal crossing divisors

Definition

Let X be a smooth complex variety. A hypersurface $D \subset X$ is called a *normal crossing divisor* if around every point of D one can find local holomorphic coordinates $(z_1, \ldots, z_n)$ on X such that D is defined by the vanishing locus $\{z_1 \cdots z_r = 0\}$ for some $r \in \{1, \ldots, n\}$.

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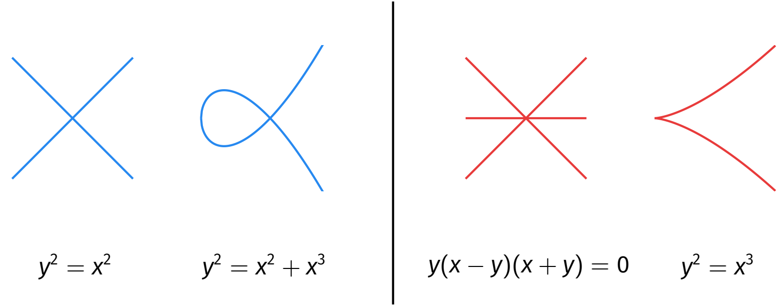

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Let X be a smooth complex variety, and $D \subset X$ a normal crossing divisor.

Definition

A form on $X \setminus D$ has *logarithmic poles along D* if in local coordinates $(z_1, \ldots, z_n)$ where $D = \{z_1 \cdots z_r = 0\}$ it can be expressed as a linear combination of forms

$$\alpha \wedge \frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}z_{i_s}}{z_{i_s}}$$

where α is holomorphic on X and $1 \leqslant i_1 < \cdots < i_s \leqslant r$.

- Residues of logarithmic forms:

$$\mathrm{Res}_{\{z_r=0\}}\left(\eta \wedge \frac{\mathrm{d}z_r}{z_r}\right) = \eta|_{\{z_r=0\}}$$

(for η a logarithmic form that does not involve $\frac{\mathrm{d}z_r}{z_r}$).

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Let U be a smooth complex variety, e.g., $U = \mathbb{P}^n_{\mathbb{C}} \setminus \{f = 0\}$.

- One can view $U \subset \overline{U}$ where $\overline{U}$ is a *compact* smooth complex variety and $D := \overline{U} \setminus U$ is a normal crossing divisor. (Hironaka's resolution of singularities.)

- The space

$$\Omega^k_{\log}(U) := \{k\text{-forms on } U \text{ with logarithmic poles along } D\}$$

  is a finite dimensional vector space, independent of the choice of $\overline{U}$.
- A form $\omega \in \Omega^k_{\log}(U)$ is said to be *logarithmic at infinity*.
- It is automatically closed: $\mathrm{d}\omega = 0$.

- These facts are consequences of mixed Hodge theory.

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Basics of mixed Hodge theory

Theorem (Hodge, 1941)

Let X be a smooth projective complex variety. Then

$$\mathrm{H}^n(X;\mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}(X)$$

where $\mathrm{H}^{p,q}(X)$ is the space of cohomology classes which can be represented by forms of type (p,q), i.e., with p $\mathrm{d}z$'s and q $\mathrm{d}\bar{z}$'s.

Example: compact Riemann surfaces

Let X be a compact Riemann surface of genus g. Then

$$\mathrm{H}^1(X;\mathbb{C}) = \mathrm{H}^{1,0}(X) \oplus \mathrm{H}^{0,1}(X)$$

where a basis $\omega_1,\ldots,\omega_g$ of $\mathrm{H}^{1,0}(X)$ is given by global forms on X and a basis of $\mathrm{H}^{0,1}(X)$ by their conjugates $\overline{\omega_1},\ldots,\overline{\omega_g}$.

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Definition (pure Hodge structure)

A *pure Hodge structure* of weight n is a finite dimensional $\mathbb{Q}$-vector space H together with a decomposition

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q} \qquad \text{such that} \qquad \overline{H^{p,q}} = H^{q,p}.$$

- The *Hodge numbers*:

$$h^{p,q} = \dim \mathrm{H}^{p,q} = h^{q,p}.$$

- The *Hodge filtration*:

$$\mathrm{F}^k H_{\mathbb{C}} = \bigoplus_{\substack{p+q=n \\ p \geqslant k}} \mathrm{H}^{p,q}.$$

Definition (mixed Hodge structure)

A *mixed Hodge structure* is a finite dimensional $\mathbb{Q}$-vector space H together with

- an increasing filtration W on H called the *weight filtration*;
- a decreasing filtration F on $H_{\mathbb{C}}$ called the *Hodge filtration*;

such that for each n, F induces on $\mathrm{gr}_n^{\mathrm{W}} H := \mathrm{W}_n H / \mathrm{W}_{n-1} H$ a pure Hodge structure of weight n.

- As a first approximation, a mixed Hodge structure is a collection of pure Hodge structures $\mathrm{gr}_n^{\mathrm{W}} H$ of different weights.
- It has Hodge numbers $h^{p,q}$ for all p, q.

Theorem (Deligne, 1970s)

There is a canonical mixed Hodge structure on all (relative) (co)homology groups of all (pairs of) complex varieties.

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Let U be a smooth complex variety, compactified by $\overline{U}$ such that $D := \overline{U} \setminus U$ is a normal crossing divisor.

- The filtrations W and F on $\mathrm{H}^\bullet(U;\mathbb{C})$ come from filtrations of the de Rham complex of forms on U with logarithmic singularities along D.
- A consequence of the general formalism: all global logarithmic forms are closed, the morphism

$$\begin{array}{ccc} \Omega^k_{\log}(U) & \longrightarrow & \mathrm{H}^k(U;\mathbb{C}) \\ \omega & \longmapsto & [\omega] \end{array}$$

  is injective, and its image is identified with $\mathrm{F}^k\,\mathrm{H}^k(U;\mathbb{C})$.

Genus and combinatorial rank

Consider the mixed Hodge structure on $\mathrm{H}^n(X, Y)$, where $n = \dim(X)$.

Definition (genus)

The *genus* of the pair (X, Y) is the sum of Hodge numbers

$$g(X, Y) = h^{1,0} + h^{2,0} + \cdots + h^{n,0}.$$

- Counts obstructions for the uniqueness of a canonical form.

Definition (combinatorial rank)

The *combinatorial rank* of the pair (X, Y) is the Hodge number

$$cr(X, Y) = h^{0,0}.$$

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How to compute the genus?

- For X a smooth compact variety,

$$g(X) = h^{n,0} = \dim\{n\text{-forms on } X\}$$

is usually called the "geometric genus".

- The genus is a birational invariant of *smooth* varieties.

- For a smooth hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ of degree d, we have

$$g(X) = \binom{d-1}{n},$$

and in particular

$$g(X) = 0 \iff d \leqslant n.$$

- Bound on the genus of a pair:

$$g(X, Y) \leqslant g(X) + \sum_{I \neq \varnothing} g(Y_I)$$

where $Y = Y_1 \cup \cdots \cup Y_N$ and $Y_I = \bigcap_{i \in I} Y_i$.

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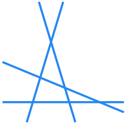
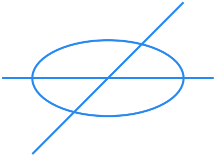
Examples of genus zero pairs

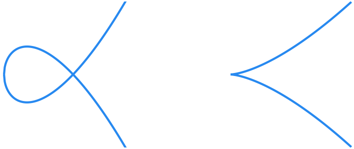
- For $H_i \subset \mathbb{P}^n_{\mathbb{C}}$ hyperplanes and $H = \bigcup_i H_i$, the pair $(\mathbb{P}^n_{\mathbb{C}}, H)$ has genus zero.



- In dimension 2, the same holds for unions of hyperplanes *and quadrics*.



- The nodal and cuspidal cubics have genus zero.



Let X be a compact complex variety of dimension n, and $Y \subset X$ be a closed subvariety such that $X \setminus Y$ is smooth.

Definition (corner residue map)

The composition

$$R : \Omega^n_{\log}(X \setminus Y) \simeq \mathrm{F}^n \mathrm{H}^n(X \setminus Y; \mathbb{C}) \overset{\mathrm{PD}}{\simeq} \mathrm{F}^0 \mathrm{H}_n(X, Y; \mathbb{C}) \twoheadrightarrow \mathrm{H}^{0,0}_n(X, Y; \mathbb{C})$$

is called the *corner residue map*.

- If Y is a normal crossing divisor, it is computed by n-fold residues in the "corners" of Y.
- It is a *surjective* linear map.

Easy lemma

The kernel of R has dimension $g(X, Y)$.

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Construction of canonical forms

If (X, Y) has genus zero then one can consider the composition

$$\mathrm{H}_n(X,Y) \twoheadrightarrow \mathrm{H}_n^{0,0}(X,Y) \quad \text{and} \quad R^{-1} : \mathrm{H}_n^{0,0}(X,Y;\mathbb{C}) \xrightarrow{\sim} \Omega^n_{\log}(X \setminus Y).$$

It produces canonical forms

$$\begin{array}{ccc} \mathrm{H}_n(X,Y) & \longrightarrow & \Omega^n_{\log}(X \setminus Y) \\ \sigma & \longmapsto & \varpi_\sigma \end{array}$$

- In general, for $g(X,Y) = g$, canonical forms are well-defined modulo the g-dimensional vector space $\ker(R)$.

The simple normal crossing case

Let $D = D_1 \cup \cdots \cup D_N$ be a simple normal crossing divisor. Then for every $\sigma \in \mathrm{H}_n(X,D)$, the canonical form ϖ_σ is characterized by the equalities

$$\mathrm{Res}_{i_1,\ldots,i_n}(\varpi_\sigma) = \partial_{i_1,\ldots,i_n}(\sigma)$$

for all $i_1, \ldots, i_n \in \{1, \ldots, N\}$.

Construction of canonical forms

If (X, Y) has genus zero then one can consider the composition

$$\mathrm{H}_n(X,Y) \twoheadrightarrow \mathrm{H}_n^{0,0}(X,Y) \quad \text{and} \quad R^{-1} : \mathrm{H}_n^{0,0}(X,Y;\mathbb{C}) \xrightarrow{\sim} \Omega^n_{\log}(X \setminus Y).$$

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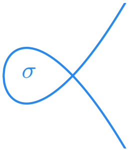
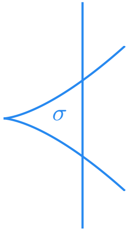
Examples



- Take $X = \mathbb{P}^2_{\mathbb{C}}$ and $Y = \{y^2 = x^2 + x^3\}$.
- It is a genus zero pair with $cr = 1$.
- The canonical form of σ is

$$\varpi_\sigma = -\frac{2x\,\mathrm{d}x \wedge \mathrm{d}y}{y^2 - x^2 - x^3}.$$



- Take $X = \mathbb{P}^2_{\mathbb{C}}$ and $Y = \{y^2 = x^3\} \cup \{x = 1\}$.
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A non-recursive example

- Take $X = E \times \mathbb{P}^1_{\mathbb{C}}$ where E is the elliptic curve

$$E: \quad y^2 = x^3 - 4x.$$

- Consider the isogenous curve $\pi: E' \xrightarrow{2:1} E$

$$E': \quad y^2 = x^3 + x$$

  and let Y be the Zariski closure of the image of $E' \to X$ given by (π, x).

- Then (X, Y) has genus 0 with $cr = 2$.

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... because $g(Y) = 1$. It is not a "positive geometry" according to the current definition.

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Hyperplane arrangements and convex polytopes

- For hyperplanes H_i in $\mathbb{P}^n_{\mathbb{C}}$, the pair $(\mathbb{P}^n_{\mathbb{C}}, H_0 \cup \cdots \cup H_N)$ has genus zero.
- Work on $\mathbb{C}^n = \mathbb{P}^n_{\mathbb{C}} \setminus H_0$ with affine hyperplanes $H_1, \ldots, H_N$.
- Hyperplanes defined over $\mathbb{R} \rightsquigarrow$ a basis of $\mathrm{H}_n(\mathbb{P}^n_{\mathbb{C}}, H_0 \cup \cdots \cup H_N)$ is given by bounded regions (polytopes).
- The space of logarithmic forms is well-understood (Orlik–Solomon). It is spanned by the forms

$$\omega_I = \bigwedge_{i \in I} \mathrm{dlog}(f_i) \quad \text{where } H_i = \{f_i = 0\}.$$

Theorem (Brown–D.)

For $P \in \mathrm{H}_n(\mathbb{P}^n_{\mathbb{C}}, H_0 \cup \cdots \cup H_N)$, we have the formula

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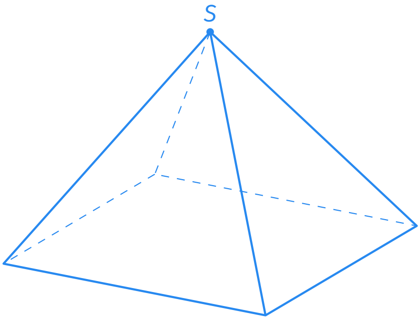
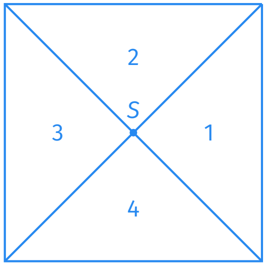
Example: the square pyramid



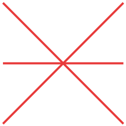


The canonical form of the square pyramid

$$\varpi_P = -\frac{1}{2}\mathrm{Cyc}_{\{1,2,3,4\}}(\omega_1 \wedge \omega_2 \wedge \omega_3) + \mathrm{Cyc}_{\{1,2,3,4\}}(\omega_1 \wedge \omega_2) \wedge \omega_5$$

Thank you!

How to tell whether a form is logarithmic?



$$y(x-y)(x+y)=0$$

Warning

The form $\omega = \dfrac{\mathrm{d}x \wedge \mathrm{d}y}{y(x-y)(x+y)}$ is NOT logarithmic.

- Proof: one resolves the singularities by the change of coordinates

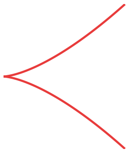
$$(u,v) = \left(x, \frac{y}{x}\right)$$

for which

$$\omega = \frac{\mathrm{d}u \wedge \mathrm{d}v}{u^2 v(1-v)(1+v)}$$

has a double pole at $u=0$.

How to tell whether a form is logarithmic? (2)



$$y^2 = x^3$$

Warning

$\omega = \dfrac{\mathrm{d}x \wedge \mathrm{d}y}{y^2 - x^3}$ is NOT logarithmic.

$\omega' = \dfrac{x\,\mathrm{d}x \wedge \mathrm{d}y}{y^2 - x^3}$ is logarithmic.

- Proof: one resolves singularities by working in coordinates

$$(u, v) = \left(\frac{x^2}{y}, \frac{y^2}{x^3}\right)$$

for which

$$\omega = \frac{\mathrm{d}u \wedge \mathrm{d}v}{u^2 v(v-1)} \quad \text{and} \quad \omega' = \frac{\mathrm{d}u \wedge \mathrm{d}v}{v-1}.$$