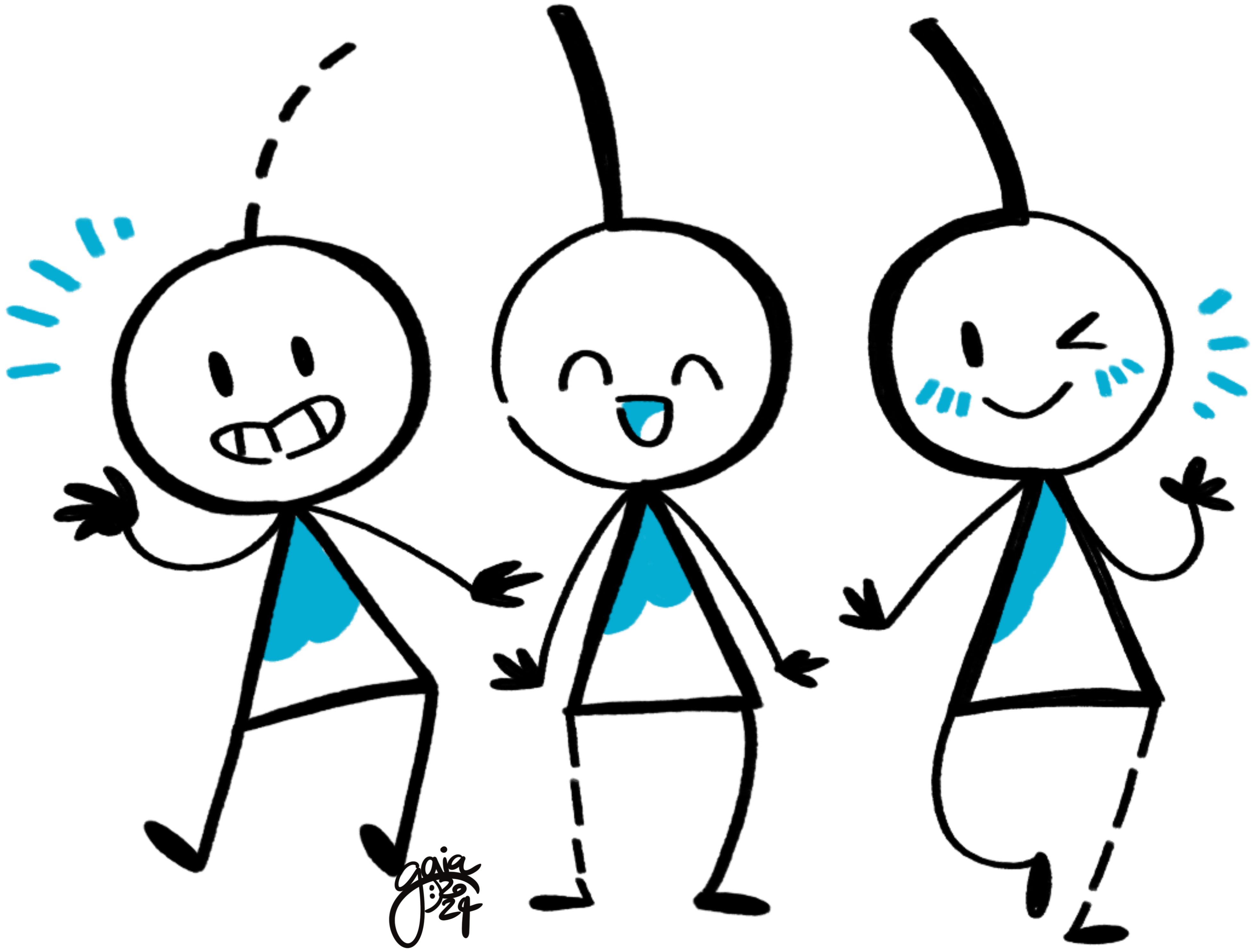
Taming IBPs with Transverse Integration

@ Holonomic techniques for Feynman Integrals



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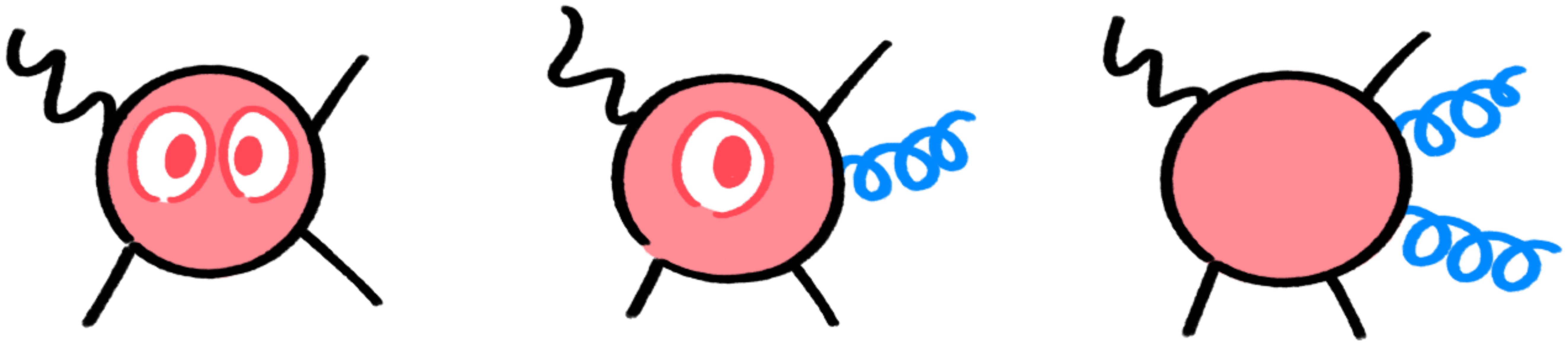
Max Planck Institute for Physics, 14/10/2024

Precision era @ colliders

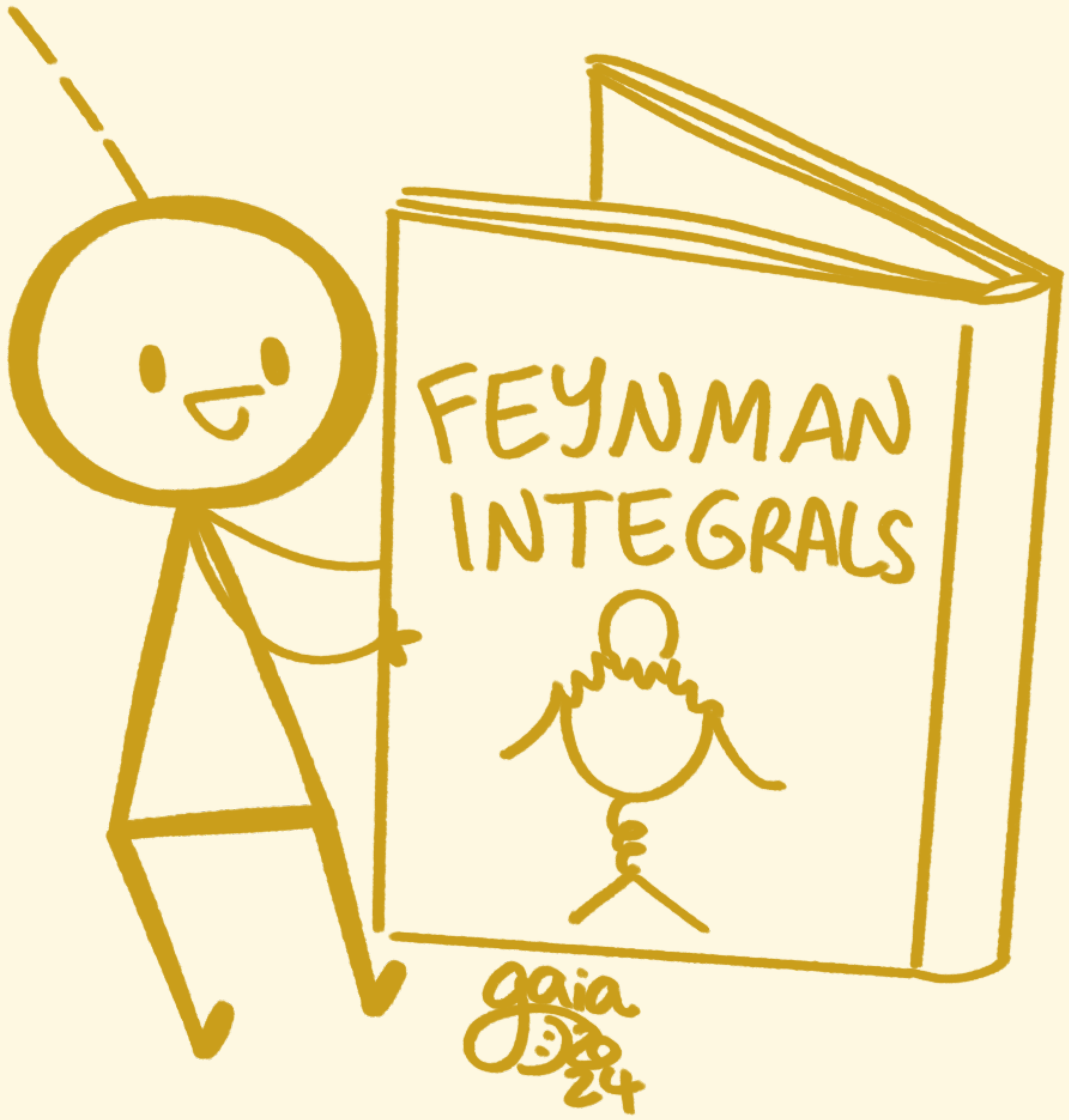
- Precision physics as
  - test of the Standard model
  - gate to new physics

- High-Lumi upgrade of LHC :
- theory and experiments must have comparable uncertainties
  - needed: %-level accuracy:
    - perturbation theory @ NNLO and often N3LO
    - diagrams with increasing no. of loops, legs & mass scales

@ $\mathcal{O}(\alpha_s^2)$:



Part 0: Background



A dictionary for Feynman integrals

- LEGO® blocks of perturbative QFT beyond tree level
- Key ingredient of phenomenological predictions
- Rich and interesting mathematical structures

Integral Family :
defined by a list of generalised denominators $F \leftrightarrow \{D_{F,1}, \ldots, D_{F,n}\}$

- Integral belonging to a family

$$I_{F;\vec{a}}[N] = I_{F;a_1\cdots a_n}[N] = \int \prod_{j=1}^{\ell} d^D k_j \, \frac{N}{\prod_{j=1}^{n} D_{F,j}^{a_j}}, \qquad N = \text{ polynomial in } k_i$$

Numerators are removed via tensor reduction

$\rightarrow N = 1$ for IBPs

$$I_{F;\vec{a}} = \int \prod_{j=1}^{\ell} d^D k_j \, \frac{1}{\prod_{j=1}^{n} D_{F,j}^{a_j}}$$

- Generalised denominators have the form

$$D_{F,j} = l_j^2 - m_j^2$$

$$D_{F,j} = l_j \cdot v_j - m_j^2$$

l_j linear combination of k_j, v_j linear combination of p_j

We distinguish:

- Proper denominators: $D_{F,j}$ such that $a_j > 0$
- Irreducible scalar products (ISPs): $D_{F,j}$ such that $a_j \leq 0$

- Sectors, $S_{F,\vec{a}}$: integrals with the same set of proper denominators
  - Iteratively, one can define also subsectors/parent sectors
- Corner integral of a sector: integral with $a_j \in \{0,1\}$

Integral decomposition

why?

- Extremely large number of integrals contributing to an amplitude
- Properties/symmetries of an amplitude manifest only after the reduction
- Important for the calculation of the integrals

Integral decomposition

Reduction into a basis of linearly independent
master integrals $\{G_j\} \subset \{I_j\}$

$$I_j = \sum_k c_{jk} G_k$$

master integrals

rational coefficients

$\{G_j\}$ = minimal linearly independent set

Laporta algorithm

Feynman integrals in dimensional regularization obey linear relations, e.g. Integration By Parts identities

+ Lorentz Invariance ids, symmetry relations, …

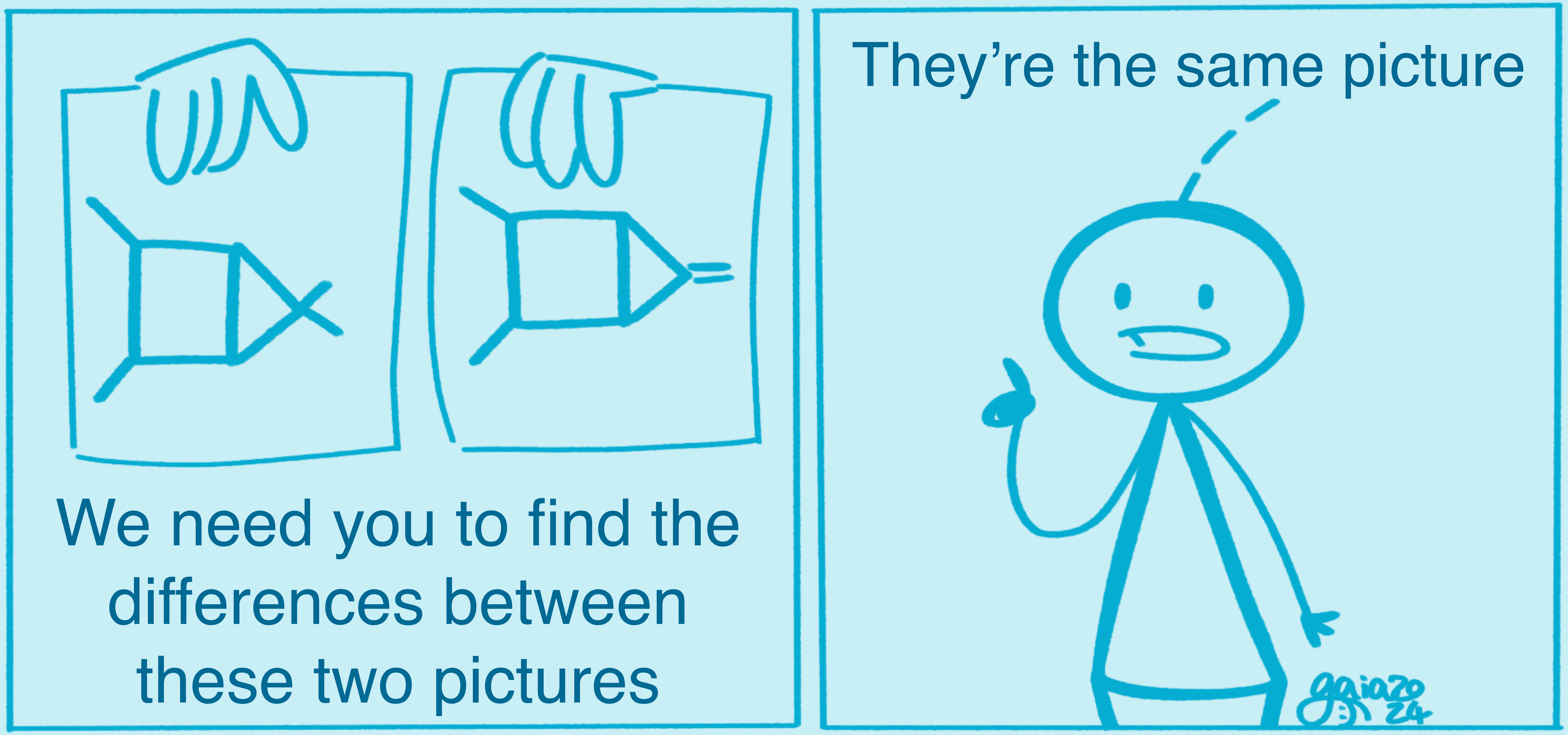
[Chetyrkin, Tkachov (1981), Laporta (2000)]

$$\int \Big(\prod_{i=1}^{\ell} d^{\mathrm{D}} k_i \Big) \frac{\partial}{\partial k_i^{\mu}} \Big(\frac{v_j^{\mu}}{D_1^{a_1} \dots D_n^{a_n}} \Big) = 0, \qquad v^{\mu} = \begin{cases} p_i^{\mu} = \text{external} \\ k_i^{\mu} = \text{loop} \end{cases}$$

reduction as solution of a large and sparse system of identities

Computational bottleneck in state-of-the-art calculations

Part 1:
The main idea



Transverse integration id.s

- A way to simplify the identities in the Laporta system
- Formulation in terms of angular integrations in [Mastrolia, Peraro, Primo 2017]

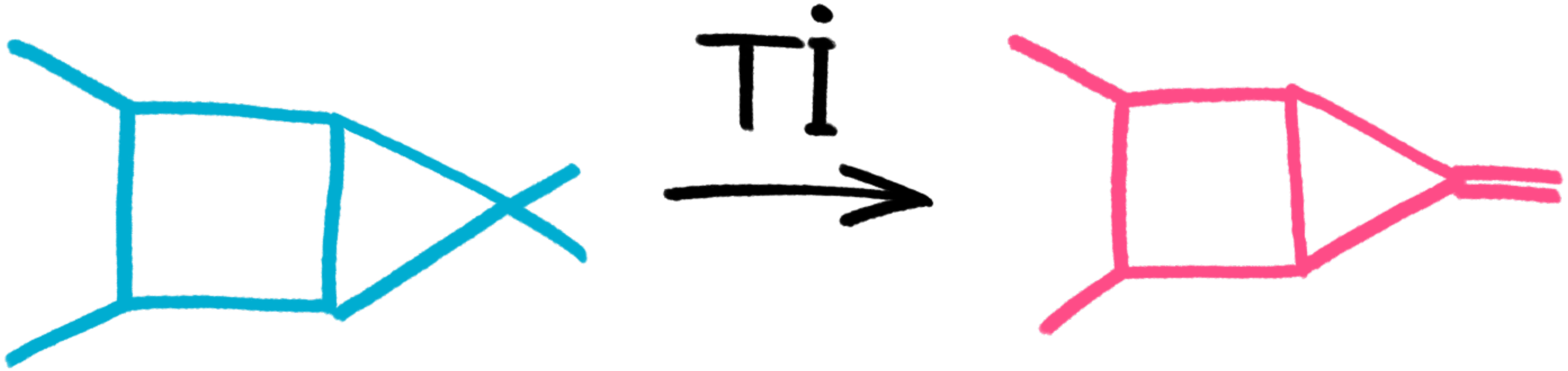
Idea :

Given a family, map its sectors with fewer external legs (or that are factorizable into fewer loops products) to new families having fewer invariants & fewer irreducible scalar products

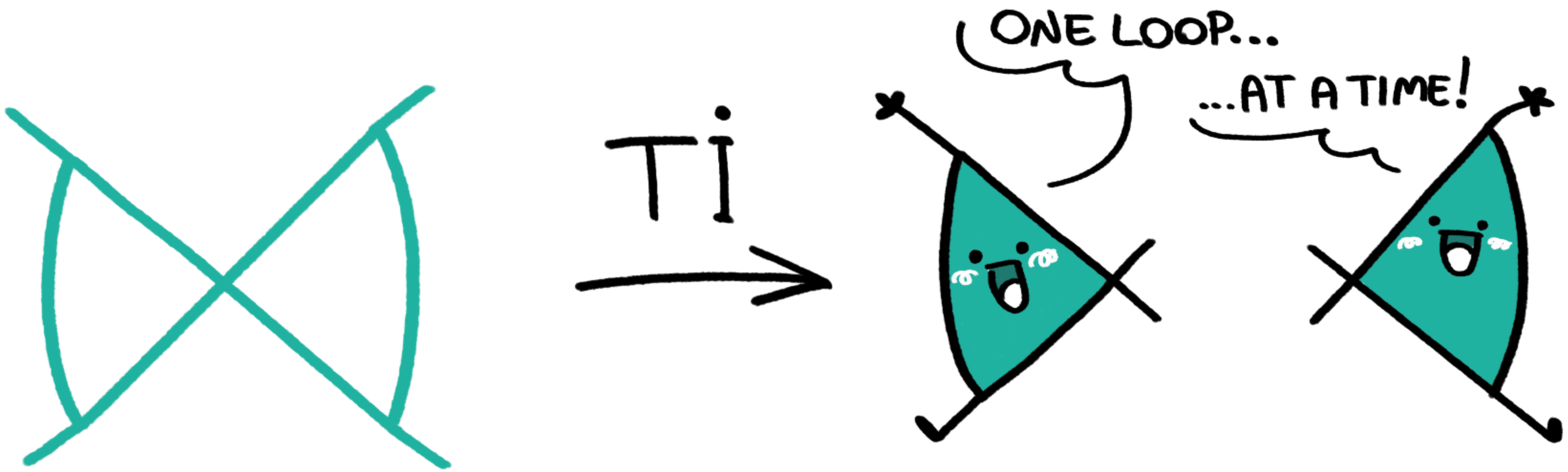
$\Rightarrow$ simpler identities

- Already used in tensor/ integrand reduction and numerical unitarity
- Impact on IBP reduction still unexplored

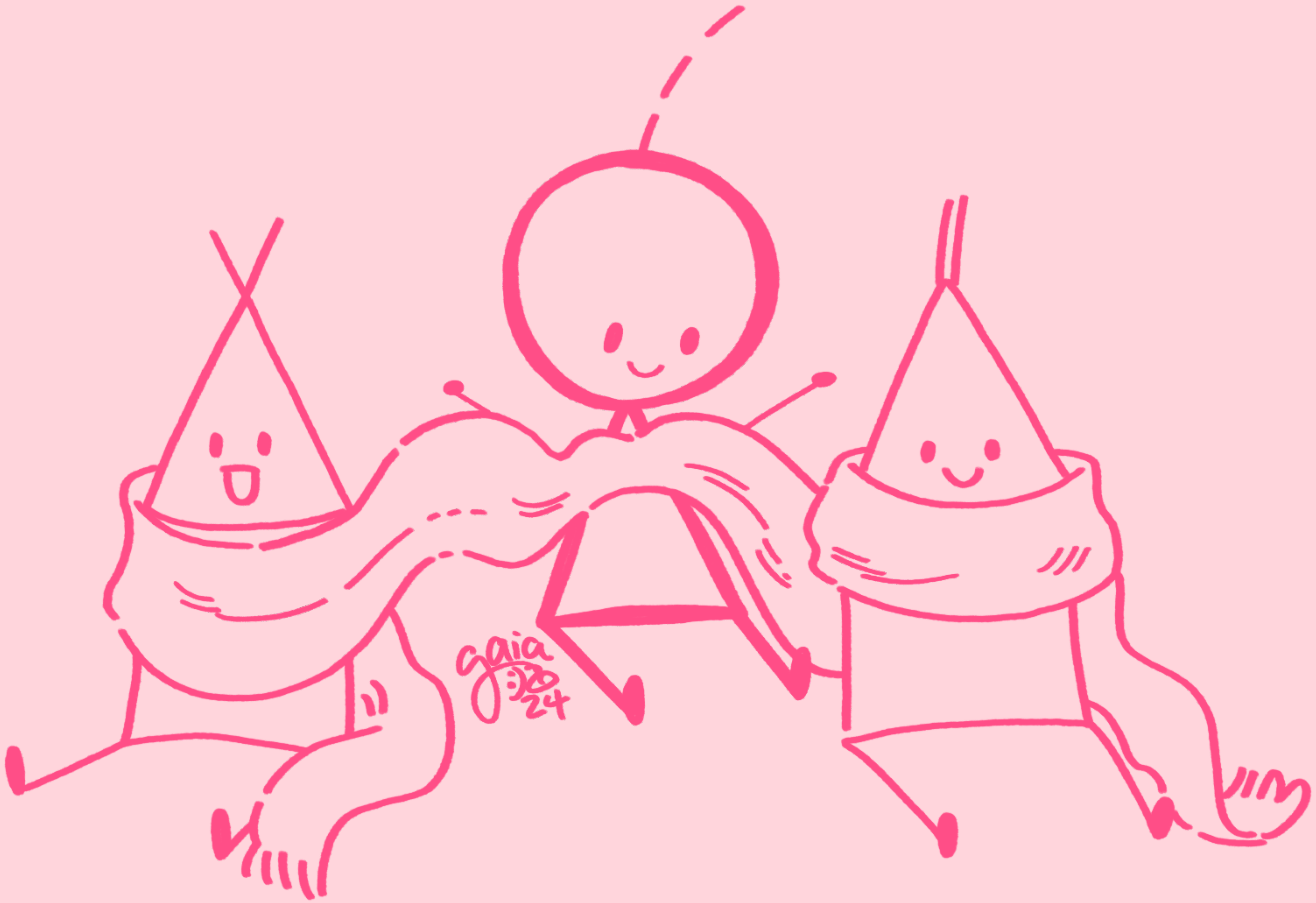
Application/1 → tested on cutting edge examples



Application/2 → only tested in simple cases (for now!)



Part 2:
Practical example

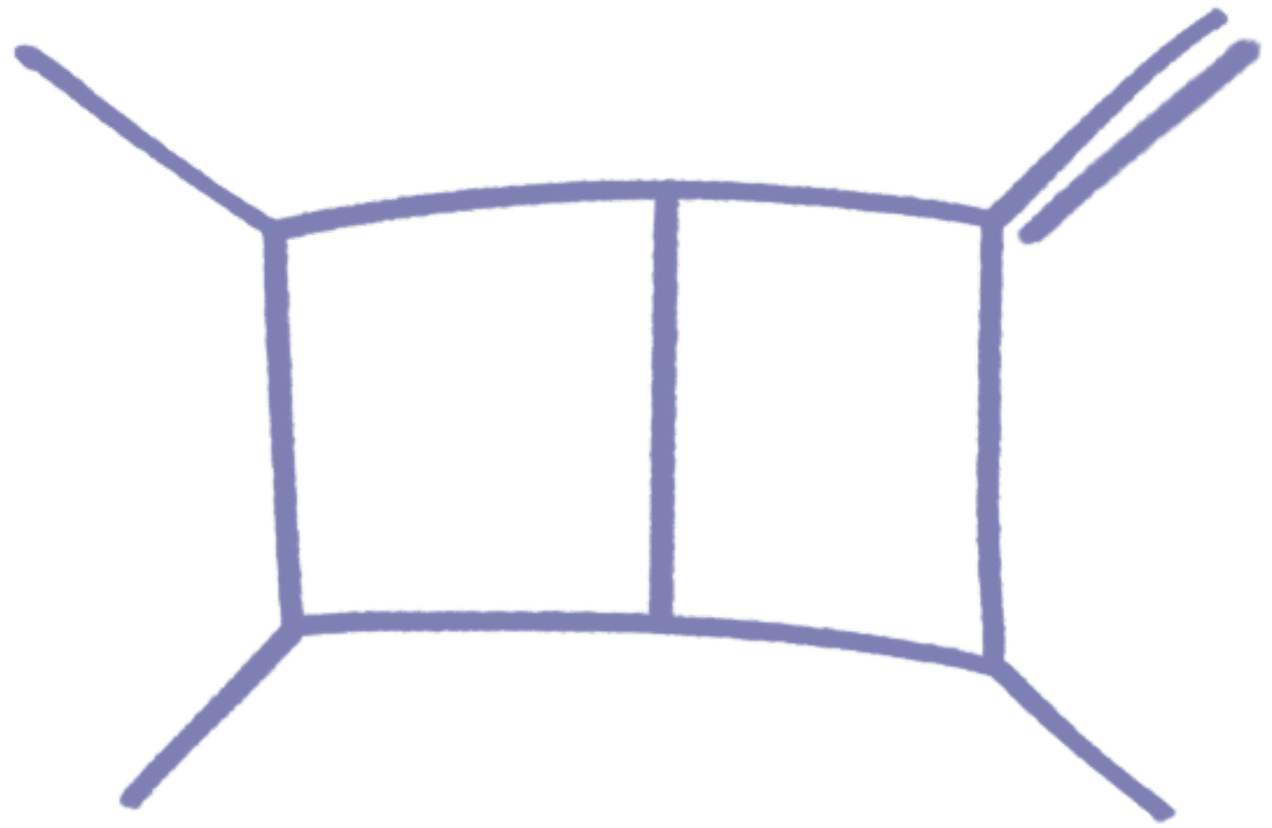


Practical example

Double box family with one external mass integral

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad m^2 = p_4^2, p_1^2 = p_2^2 = p_3^2 = 0$$

Top sector $S_{\text{db};111111100}$



Double box (db)

9 generalised den.s
7 proper denominators
2 ISPs
3 invariants

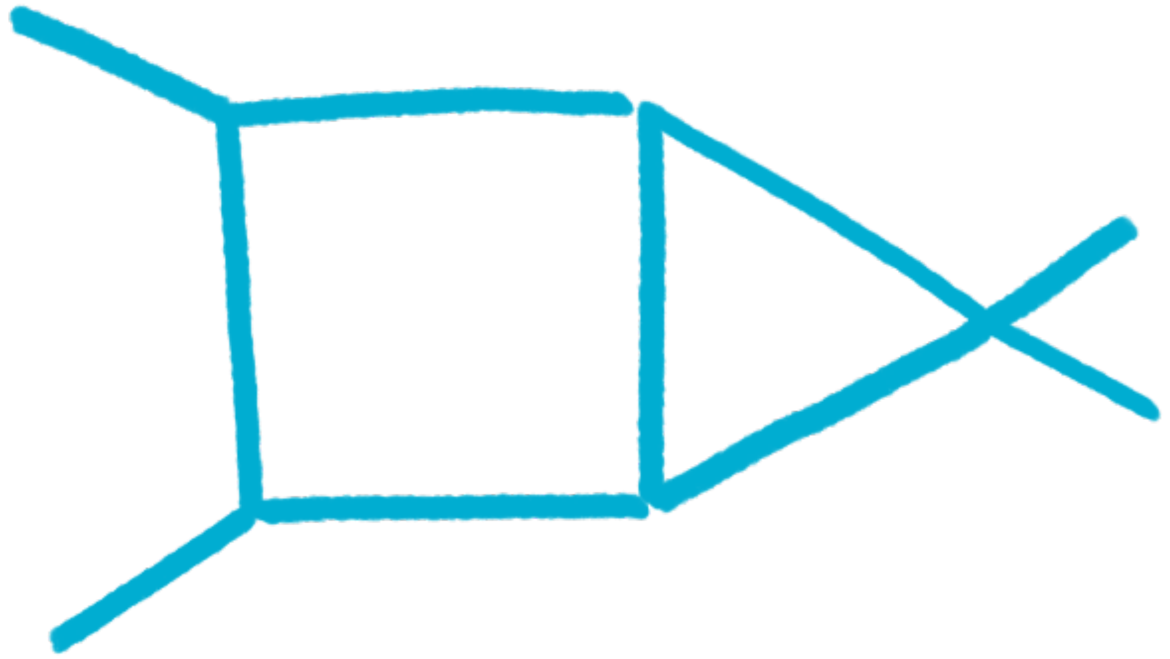
$D_{\text{db},1} = k_1^2$ $\quad D_{\text{db},2} = (k_1 + p_1)^2$ $\quad D_{\text{db},3} = (k_1 + p_1 + p_2)^2$

$D_{\text{db},4} = (k_1 + k_2)^2$ $\quad D_{\text{db},5} = k_2^2$ $\quad D_{\text{db},6} = (k_2 - p_1 - p_2 - p_3)^2$

$D_{\text{db},7} = (k_2 - p_1 - p_2)^2$ $\quad D_{\text{db},8} = k_2 \cdot p_1$ $\quad D_{\text{db},9} = k_1 \cdot (-p_1 - p_2 - p_3)$

Sector with fewer ext legs

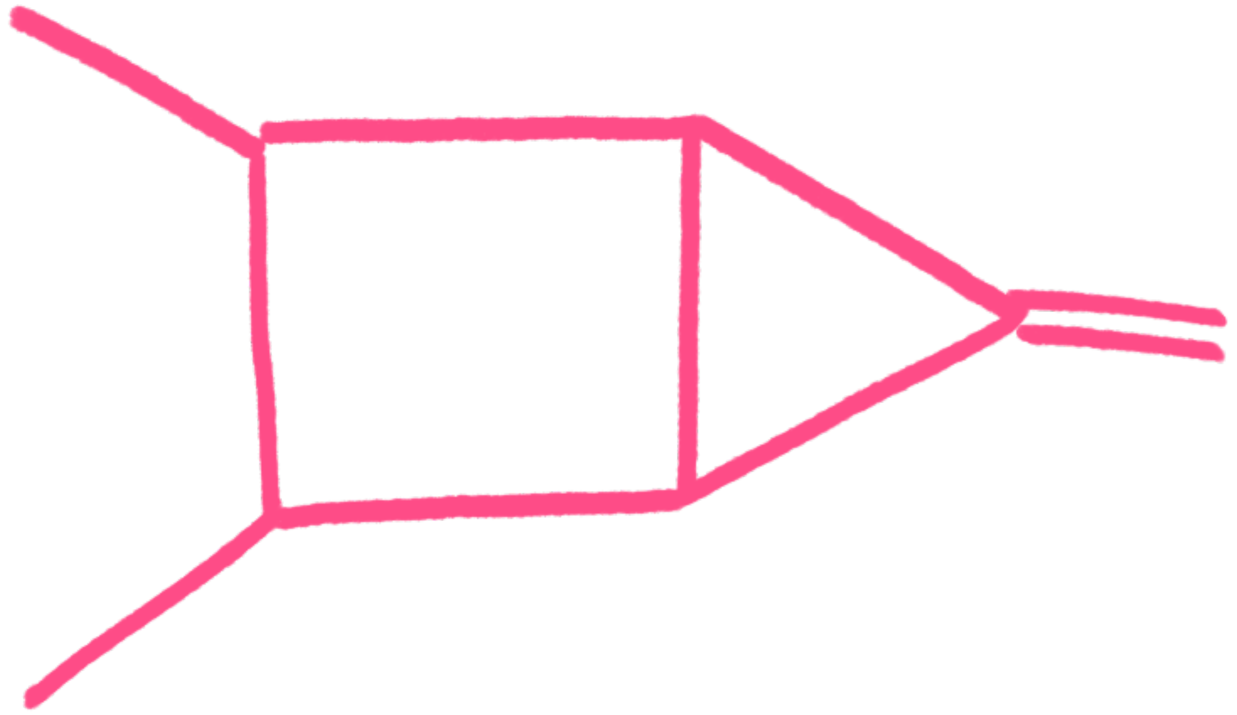
Sector $S_{\text{db};111110100}$



$a_{6,8,9} \leq 0.$

9 generalised den.s
6 proper denominators
3 ISPs
3 invariants

BUT if we consider the boxtriangle as a NEW family = TI family …



7 generalised den.s
6 proper denominators
1 ISPs
1 invariant

Box triangle (bt)

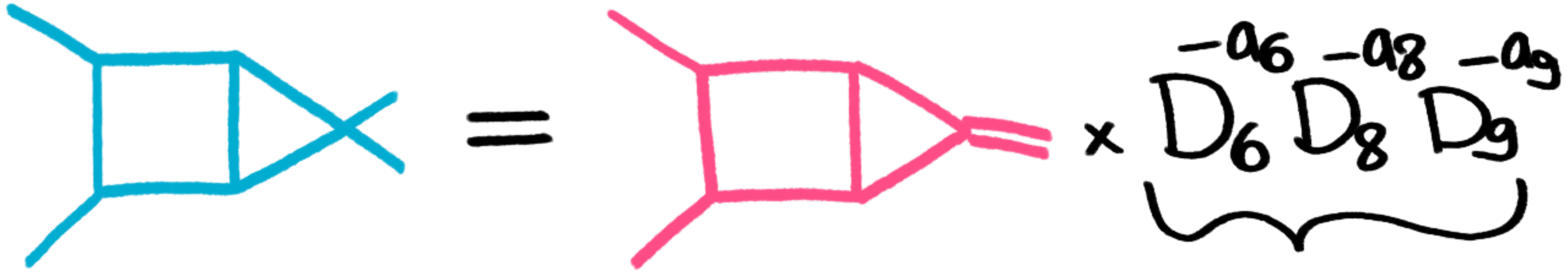
$D_{\text{bt},1} = k_1^2$ $\quad D_{\text{bt},2} = (k_1 + p_1)^2$ $\quad D_{\text{bt},3} = (k_1 + p_1 + p_2)^2$

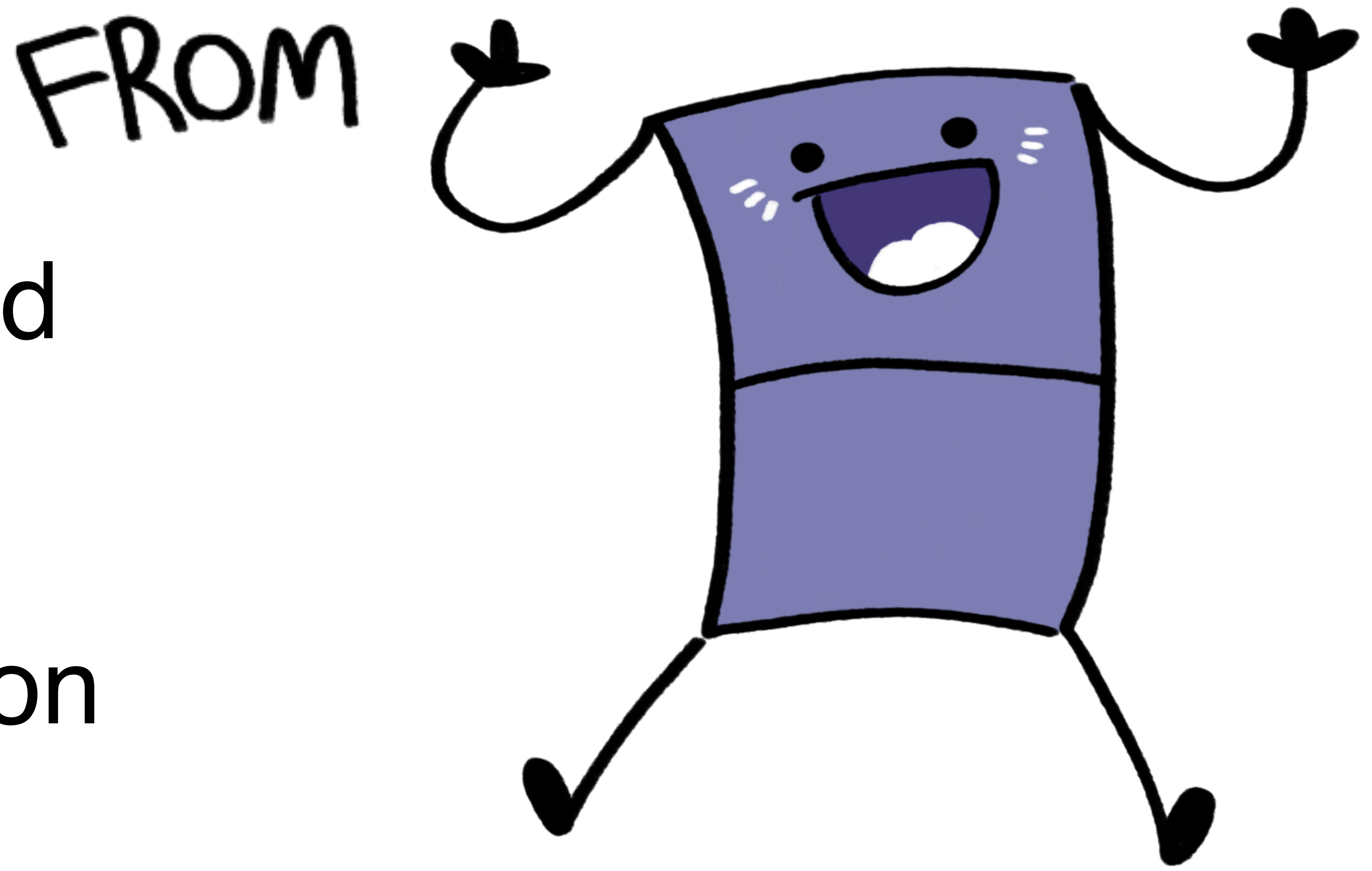
$D_{\text{bt},4} = (k_1 + k_2)^2$ $\quad D_{\text{bt},5} = k_2^2$ $\quad D_{\text{bt},6} = (k_2 - p_1 - p_2)^2$

$D_{\text{bt},7} = k_2 \cdot p_2$

We have the map...

$$I_{\mathrm{db};a_1a_2a_3a_4a_5a_6a_7a_8a_9} = I_{\mathrm{bt};a_1a_2a_3a_4a_5a_70}[D_{\mathrm{db},6}^{-a_6} D_{\mathrm{db},8}^{-a_8} D_{\mathrm{db},9}^{-a_9}]$$





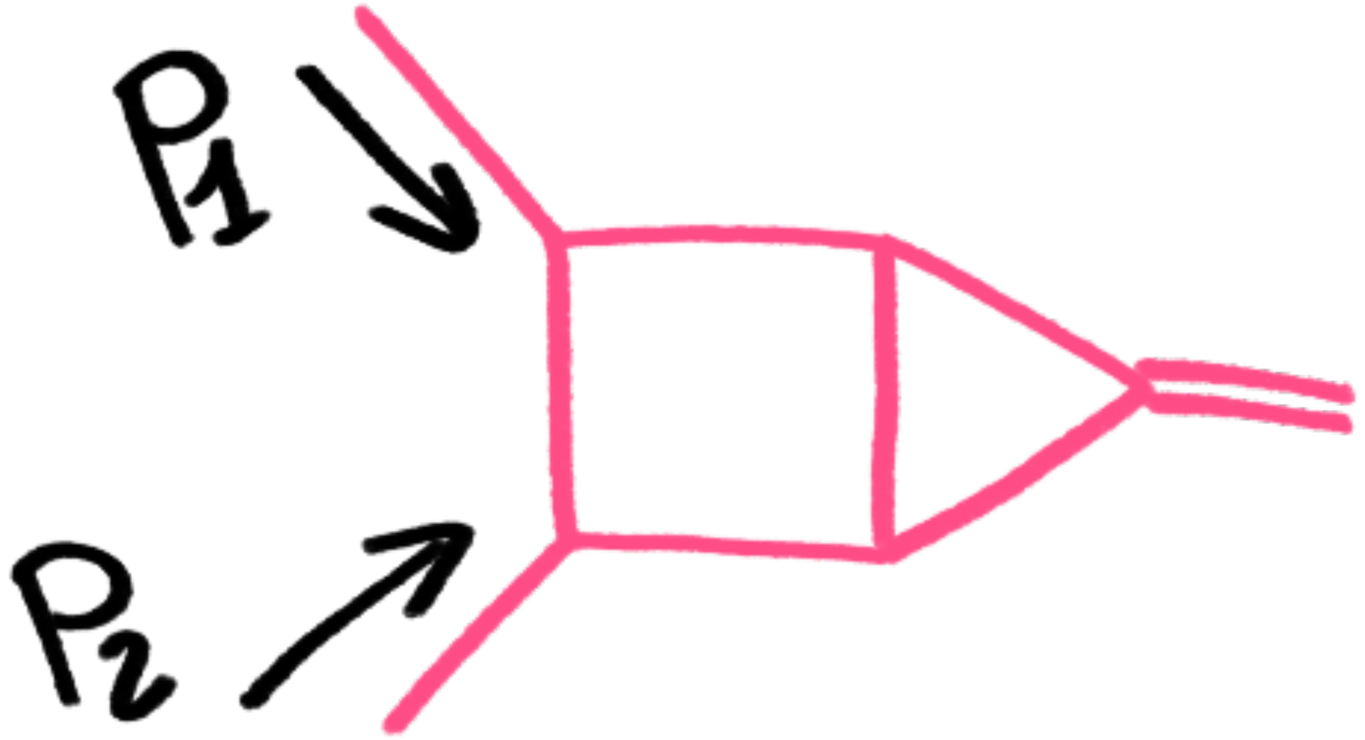
- Numerator needs to be mapped to generalised denominators of new family bt
- Mapping can be done via transverse integration

Blackboard time!

How to do transverse integration

Decomposition of a vector in parallel and transverse component

$$v^\mu = v^\mu_\parallel + v^\mu_\perp, \quad v^\mu_\parallel = c_1 p^\mu_1 + c_2 p^\mu_2$$

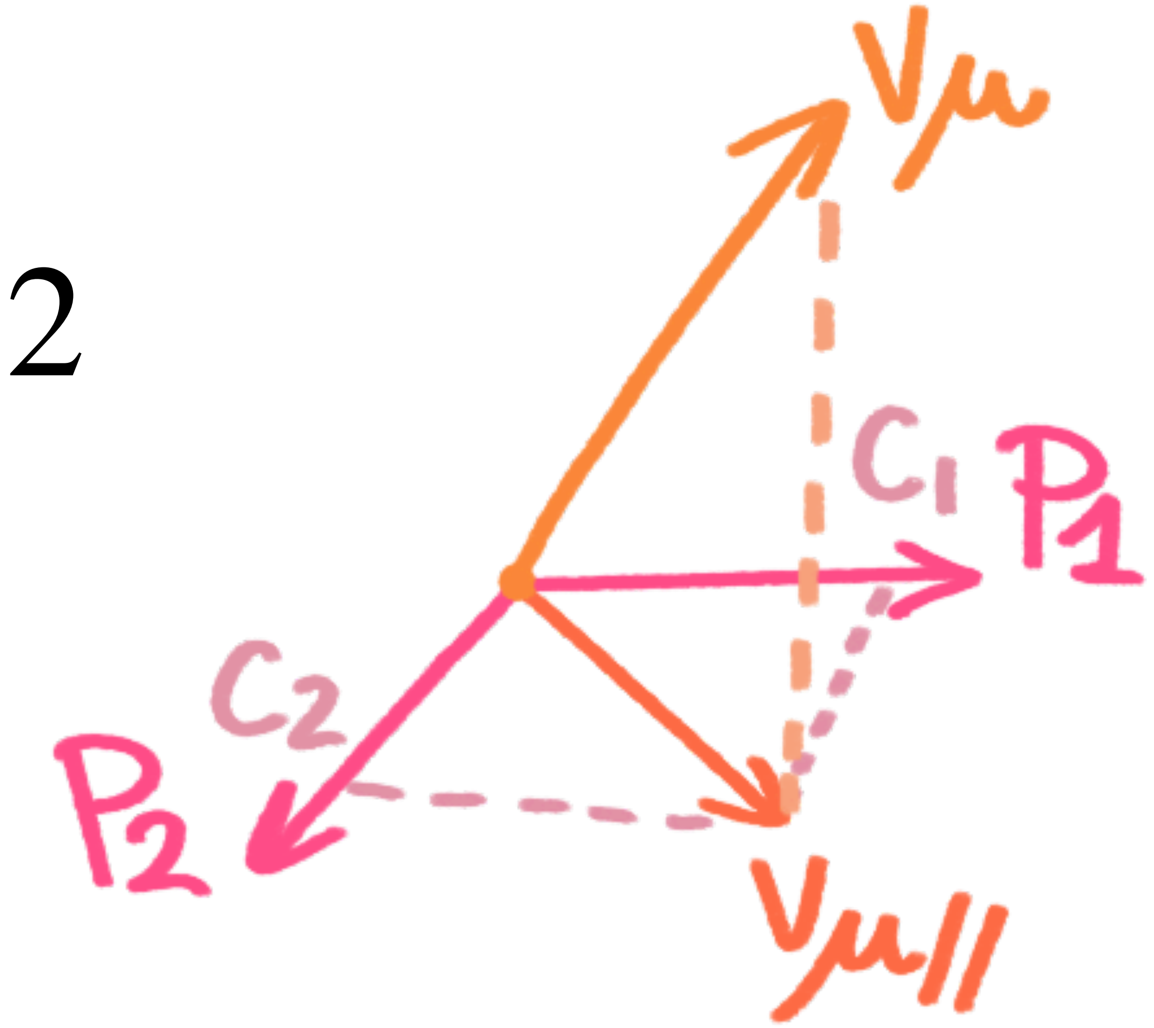


Parallel space spanned by the external legs of the new bt family

With

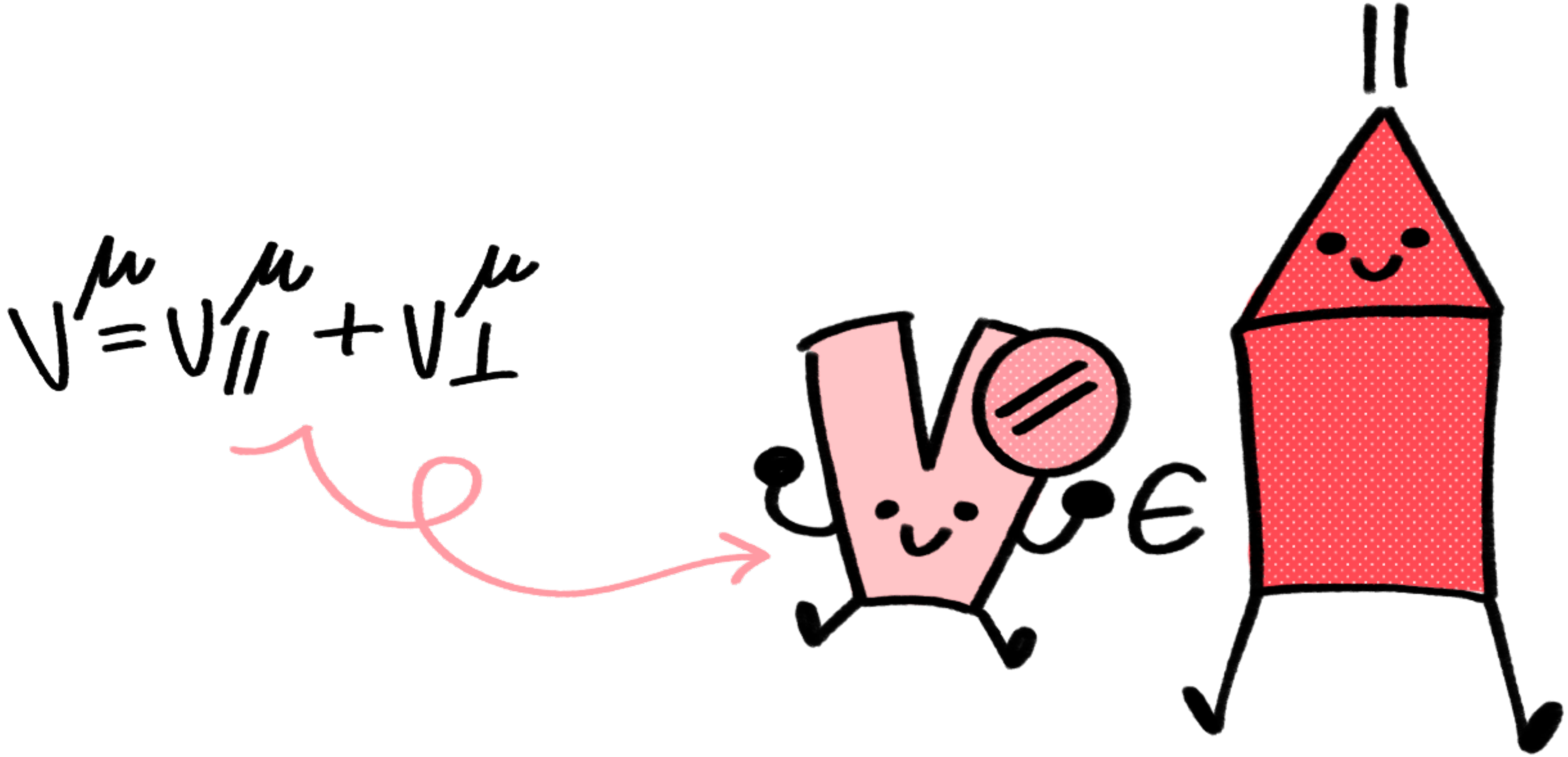
$$v_\perp \cdot p_i = 0, \qquad v \cdot p_i = v_\parallel \cdot p_i \qquad \text{for } i = 1,2$$

$$p^\mu_{1,\perp} = p^\mu_{2,\perp} = 0$$



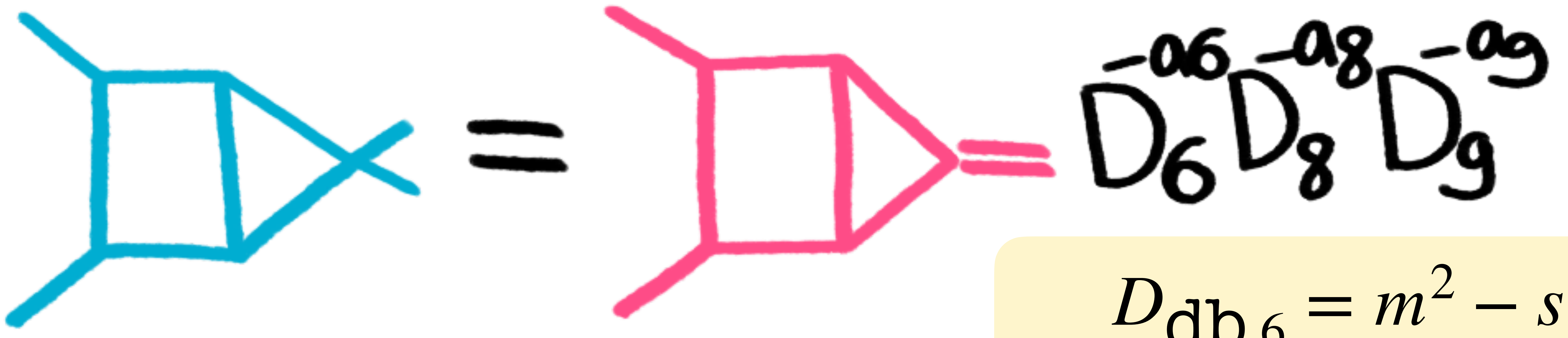
Coefficients of the parallel space decomposition found as

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{2}{s} \begin{pmatrix} p_2 \cdot v \\ p_1 \cdot v \end{pmatrix} \qquad v_{\parallel}^{\mu} = c_1\, p_1^{\mu} + c_2\, p_2^{\mu}$$



In practice

First, rewrite the extra scalar products of db as functions of the ones of bt



$$D_{\text{db},6} = m^2 - s + D_{\text{bt},6} - 2(k_2 \cdot p_3)$$
$$D_{\text{db},8} = s/2 + D_{\text{bt},5}/2 - D_{\text{bt},6}/2 - D_{\text{bt},7}$$
$$D_{\text{db},9} = s/2 + D_{\text{bt},1}/2 - D_{\text{bt},3}/2 - (k_1 \cdot p_3)$$

We are left with integrals of the family bt of the form

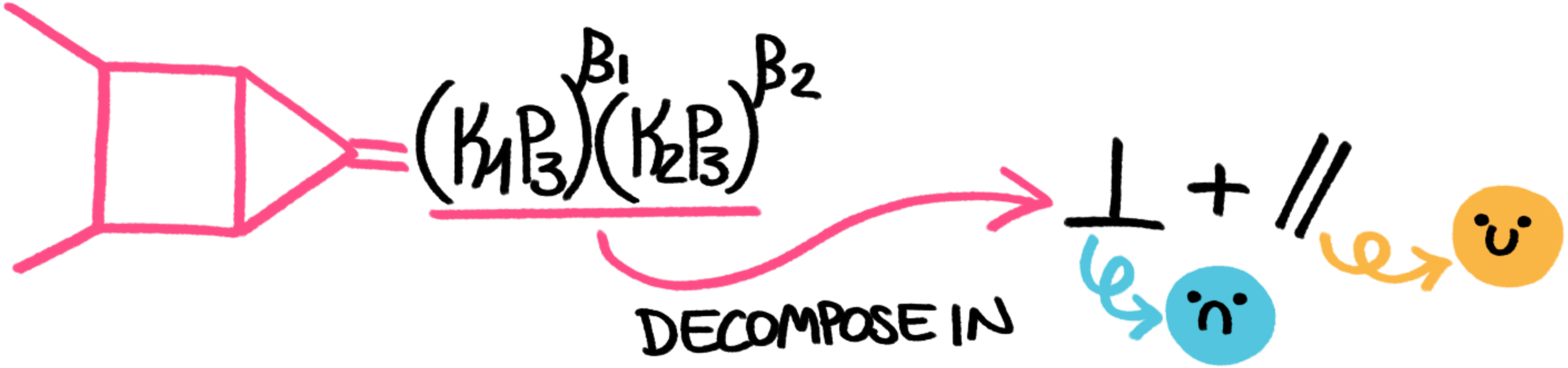
$$I_{\text{bt};\vec{a}}[(k_1 \cdot p_3)^{\beta_1} (k_2 \cdot p_3)^{\beta_2}]$$

Rewrite the scalar products as

$$(k_1 \cdot p_3) = (k_1 \cdot p_{3,\parallel}) + (k_{1,\perp} \cdot p_3)$$
$$(k_2 \cdot p_3) = (k_2 \cdot p_{3,\parallel}) + (k_{2,\perp} \cdot p_3)$$

First RHS term becomes

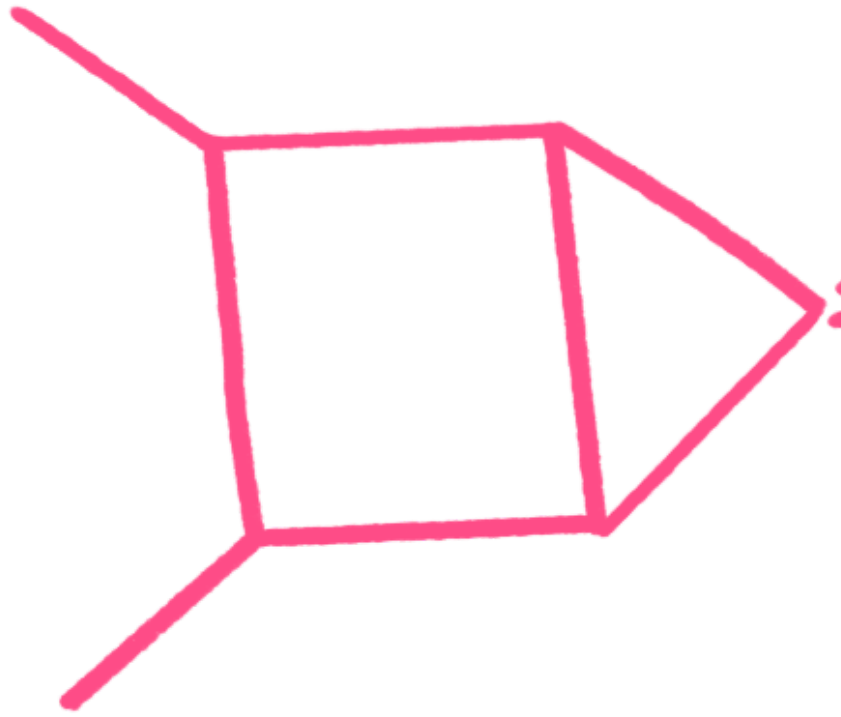
$$(k_i \cdot p_{3,\parallel}) = \frac{2}{s}\Big((k_i \cdot p_1)(p_2 \cdot p_3) + (k_i \cdot p_2)(p_1 \cdot p_3)\Big)$$



Only scalar products remaining are $(k_{1,\perp} \cdot p_3)$ & $(k_{2,\perp} \cdot p_3)$

$$I_{\text{bt};\vec{a}}[(k_{1,\perp} \cdot p_3)^{\beta_1} (k_{2,\perp} \cdot p_3)^{\beta_2}] = p_{3\,\mu_1} \cdots p_{3\,\mu_{\beta_1}} \, p_{3\,\nu_1} \cdots p_{3\,\nu_{\beta_2}} \, I_{\text{bt};\vec{a}}[k_{1,\perp}^{\mu_1} \cdots k_{1,\perp}^{\mu_{\beta_1}} \, k_{2,\perp}^{\nu_2} \cdots k_{2,\perp}^{\nu_{\beta_2}}],$$

Tensor integrals can be decomposed in products of tensors and form factors



$$= [k_i^{\mu} k_j^{\nu} \cdots] = \sum_j C_j T_j^{\mu\nu\ldots}$$

Tensor basis only depends on $g_{\perp}^{\mu\nu}$ and not on ext legs!!

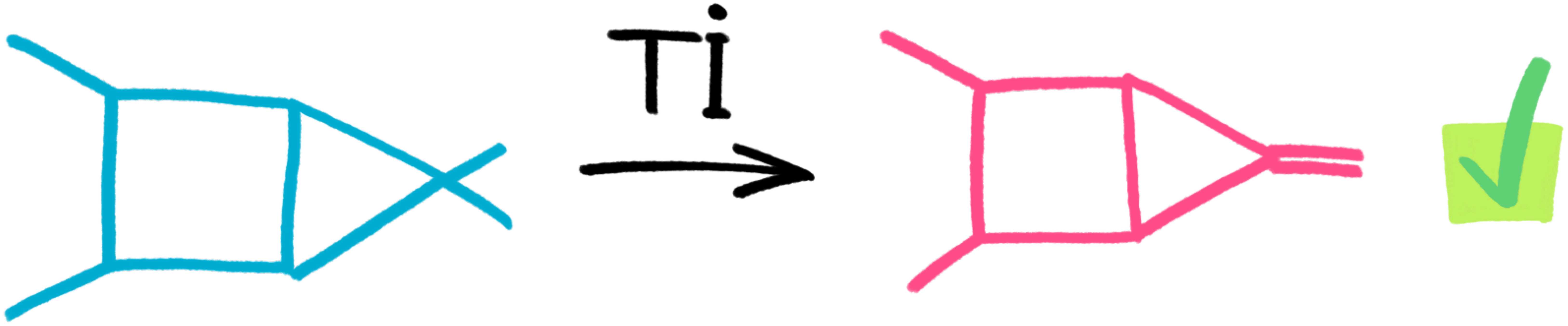
$$I_{\text{bt};\vec{a}}[k_{1,\perp}^{\mu_1} \cdots k_{1,\perp}^{\mu_{\beta_1}} \, k_{2,\perp}^{\nu_1} \cdots k_{2,\perp}^{\nu_{\beta_2}}] = \sum_j C_j \, T_{j,\perp}^{\mu_1 \cdots \mu_{\beta_1} \nu_1 \cdots \nu_{\beta_2}}$$

After this step we have only scalar products of $(k_{i,\perp} \cdot k_{j,\perp})$

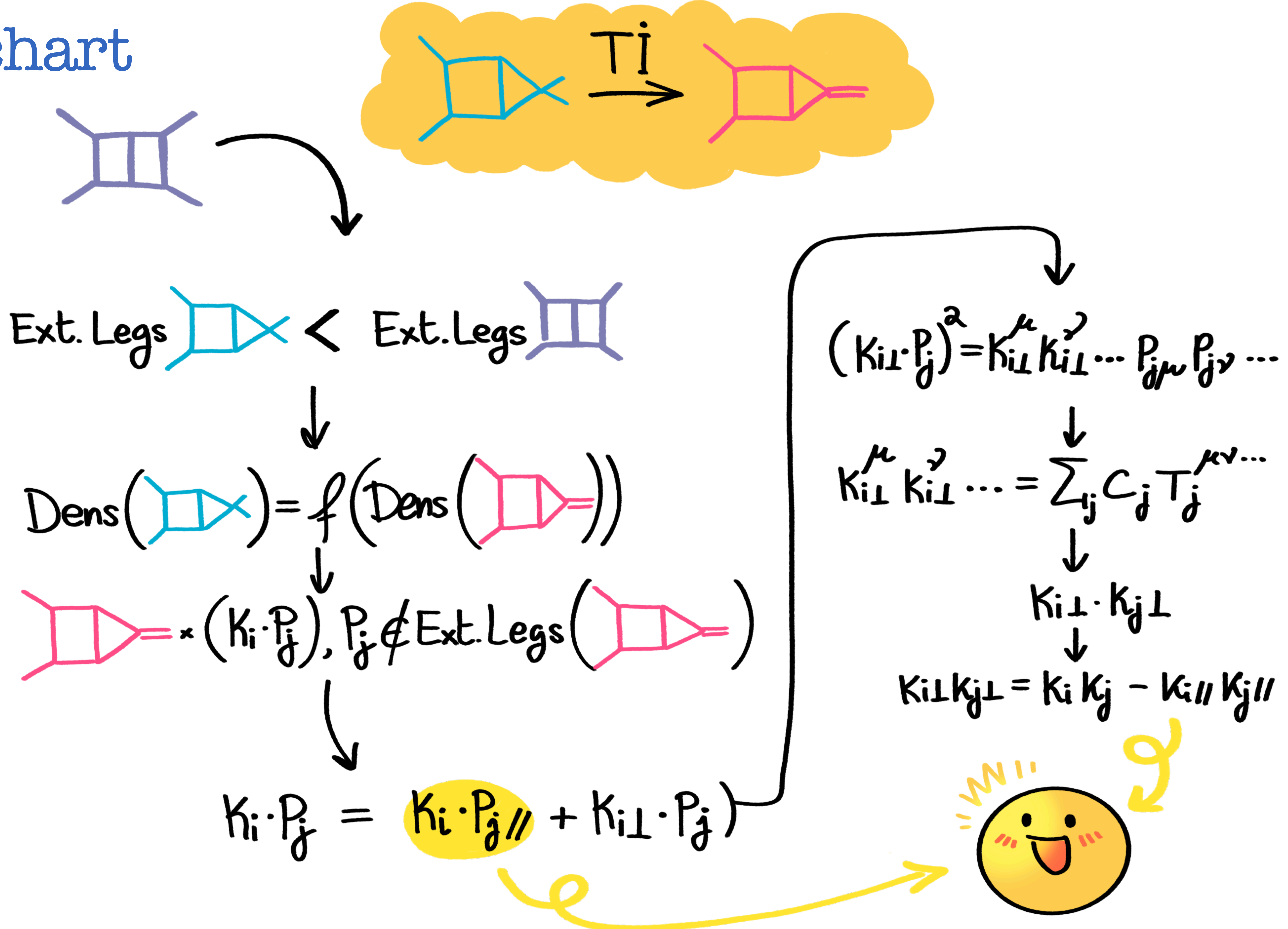
That we can rewrite using $(k_{i,\perp} \cdot k_{j,\perp}) = (k_i \cdot k_j) - (k_{i,\parallel} \cdot k_{j,\parallel})$

with $(k_{i,\parallel} \cdot k_{j,\parallel}) = \dfrac{2}{s}\Big((k_i \cdot p_1)(k_j \cdot p_2) + (k_i \cdot p_2)(k_j \cdot p_1)\Big)$

Successfully mapped db in bt

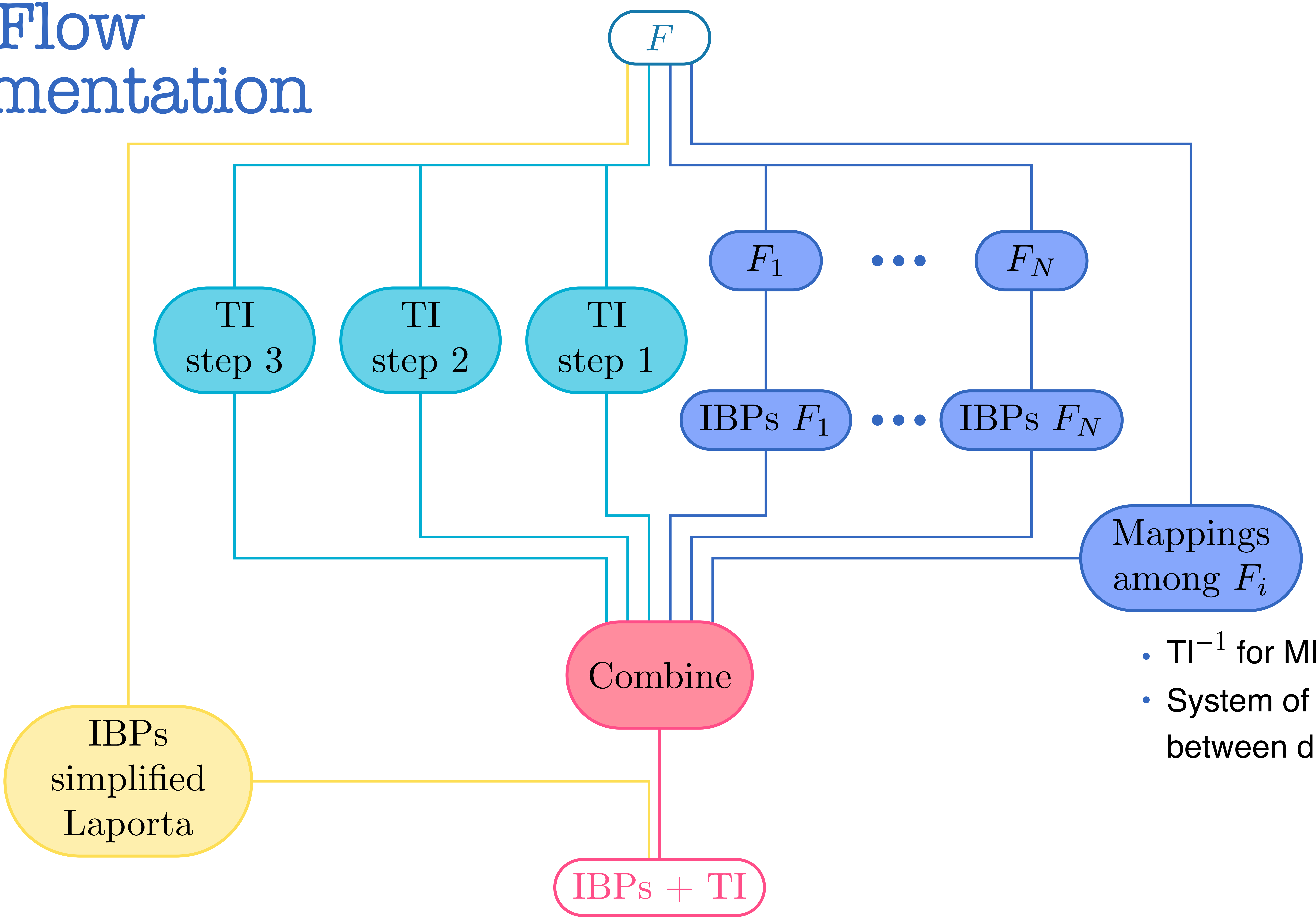


Flowchart


Ti
Ext. Legs < Ext.Legs
Dens( ) = f(Dens( ))
×(Ki·Pj), Pj ∉ Ext.Legs( )
Ki·Pj = Ki·Pj// + Ki⊥·Pj
(Ki⊥·Pj)^2 = Ki⊥^μ Ki⊥^ν ... Pjμ Pjν ...
Ki⊥^μ Ki⊥^ν ... = Σij Cj Tj^μν...
Ki⊥·Kj⊥
Ki⊥Kj⊥ = Ki Kj − Ki// Kj//

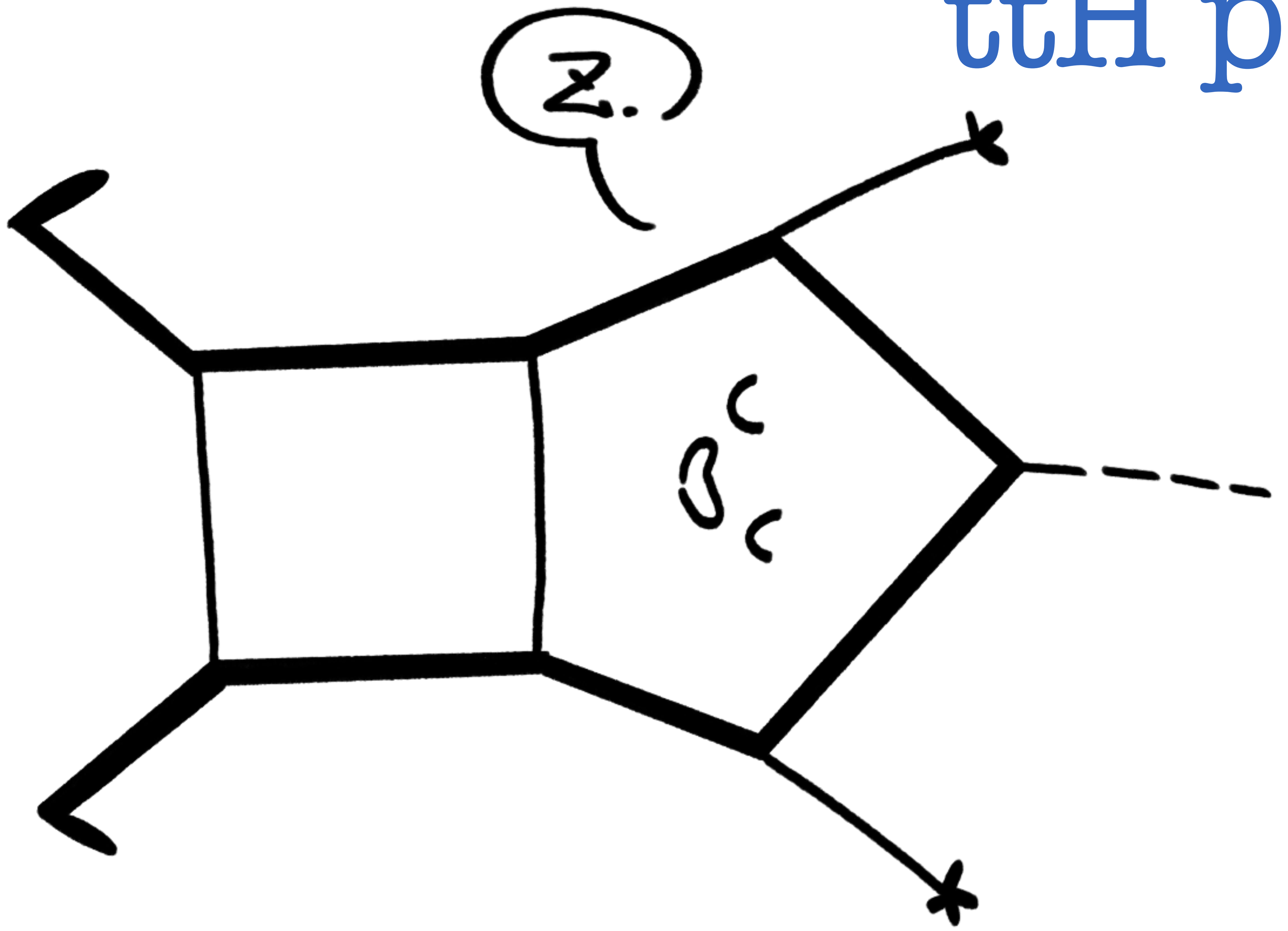
Part 3: Implementation & benchmarks

FiniteFlow implementation



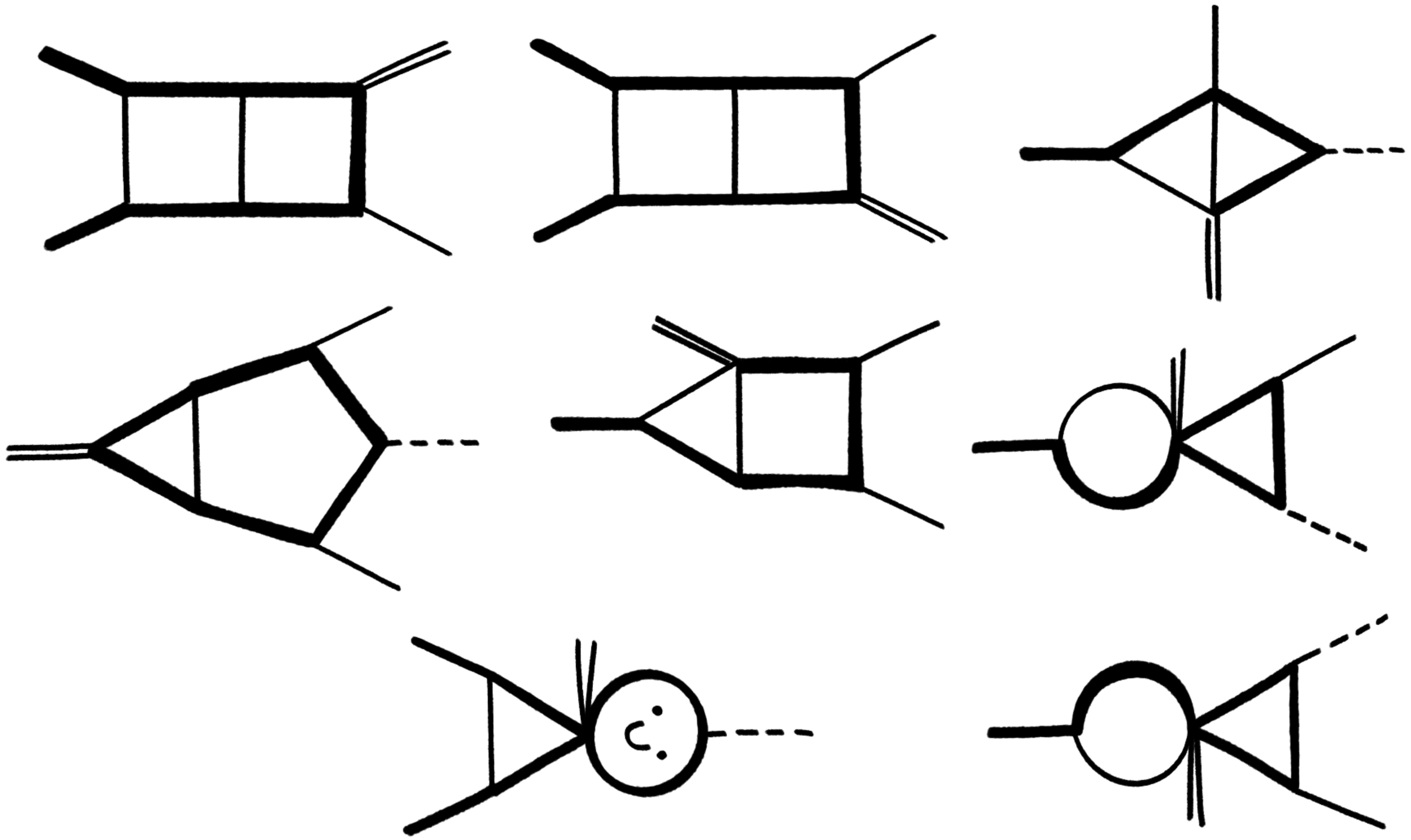
- TI^{-1} for MI of F_i
- System of relations between different F_js

ttH pentabox



All integrals up to rank 5

TI families with 4 external legs



Speedup vs traditional Laporta: 2.3x

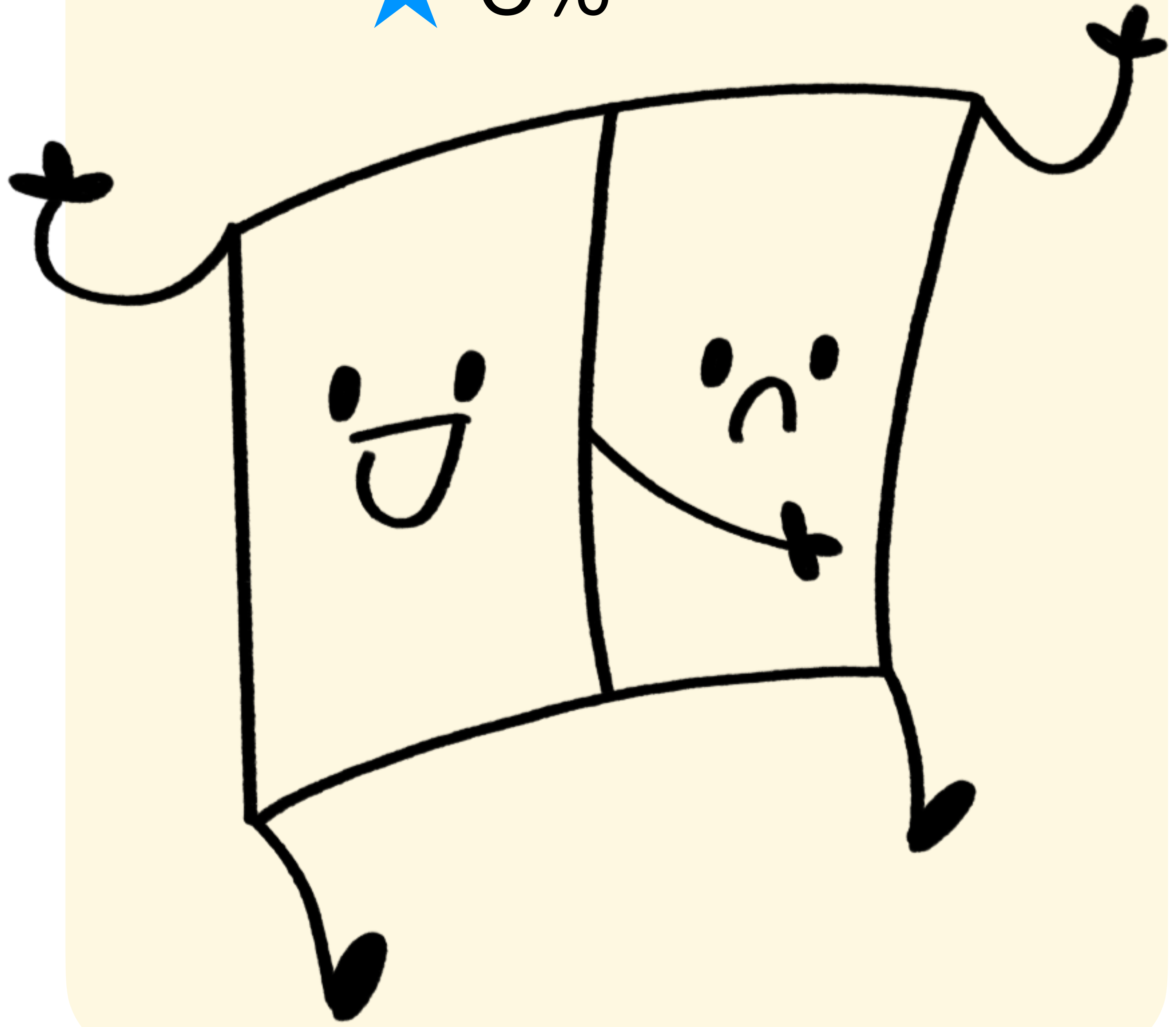
Breakdown:

- ★ solving the simplified Laporta IBP system: 48%
- ★ evaluating the coefficients of the TI identities: 22%
- ★ evaluating the solution of the IBP system for the TI families: 23%

More examples ...

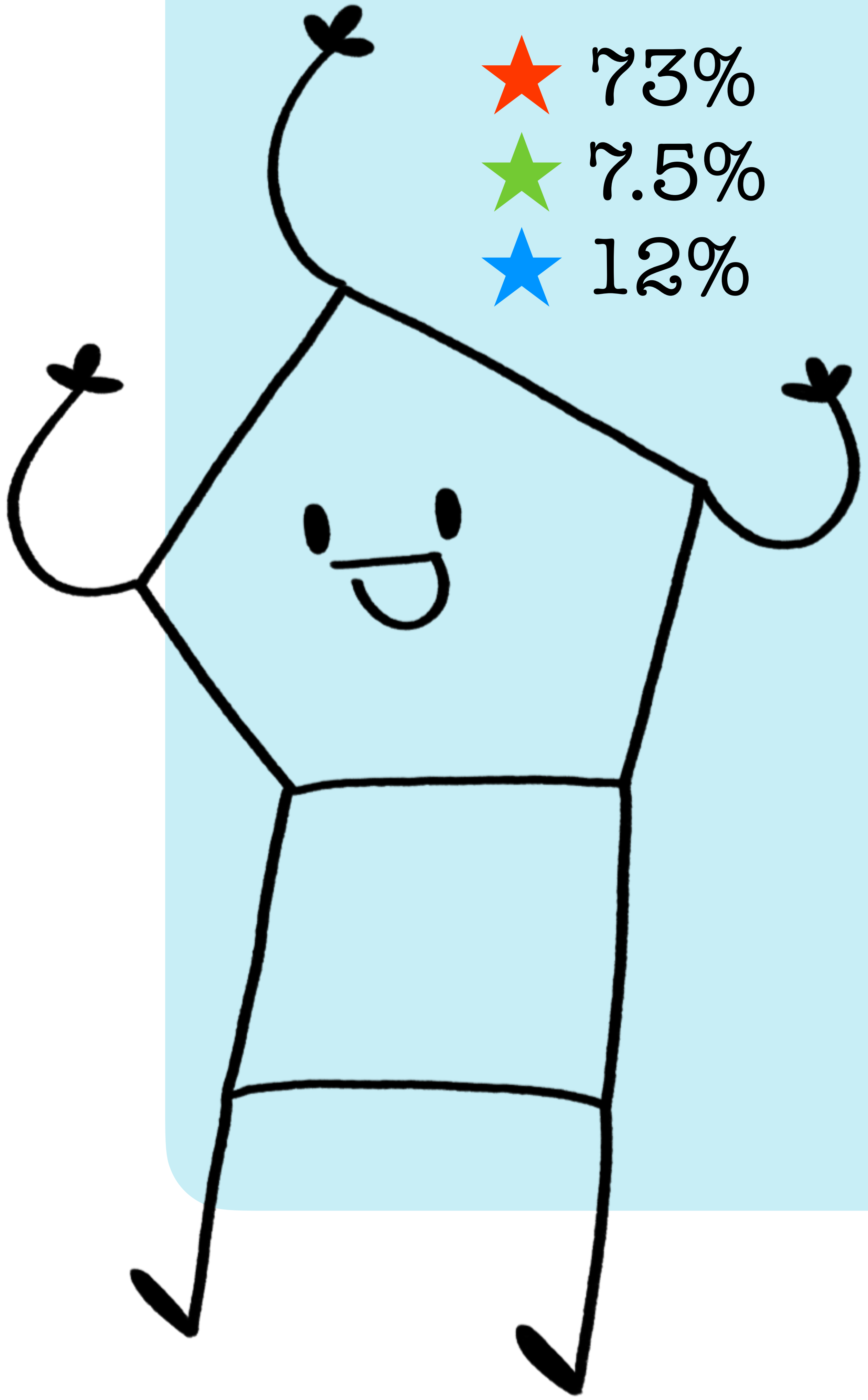
Double pentagon

- Speedup: 3.3x
  - ★ 70%
  - ★ 21%
  - ★ 6%



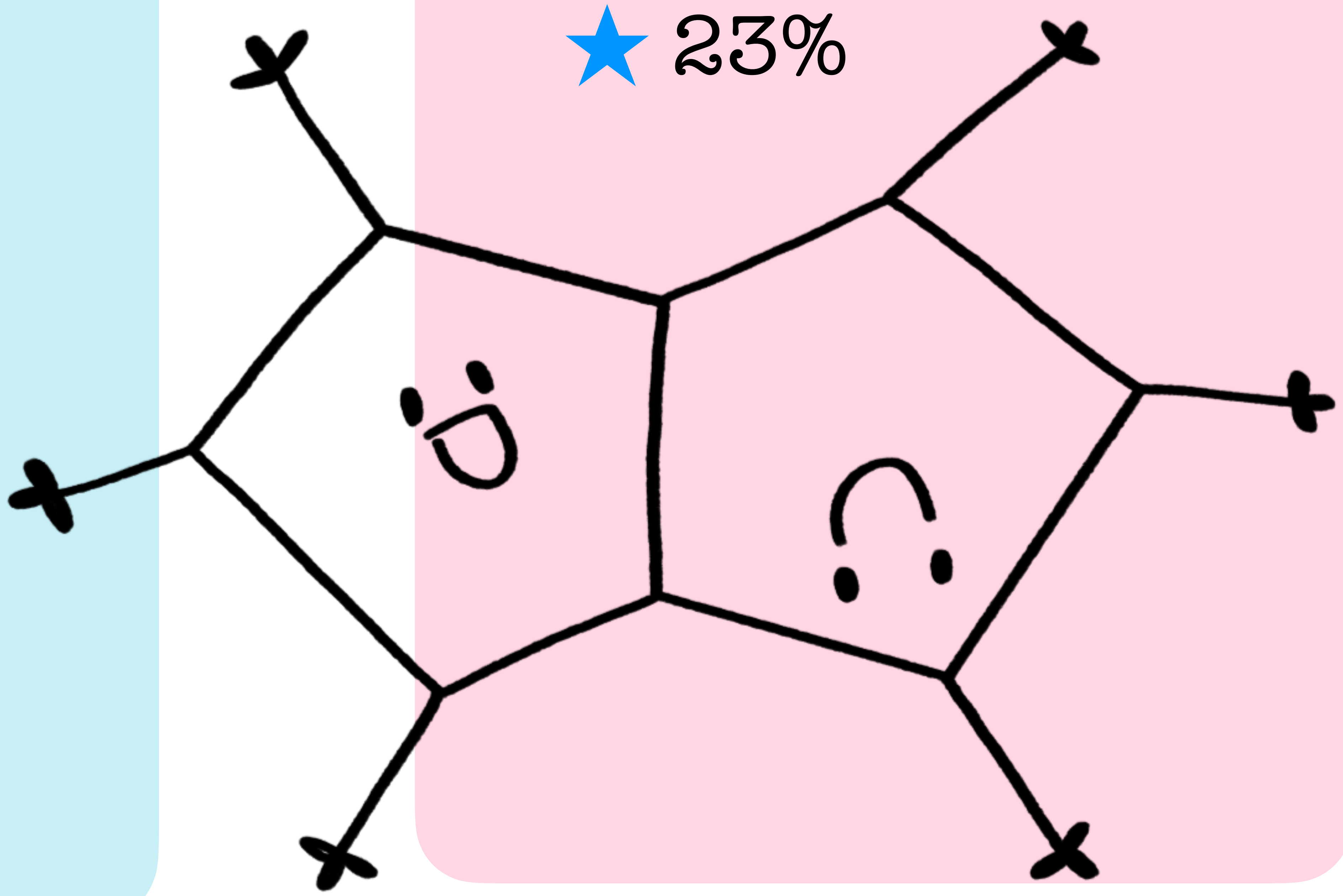
Massless pentabox

- Speedup: 4x
  - ★ 73%
  - ★ 7.5%
  - ★ 12%



Ladybug

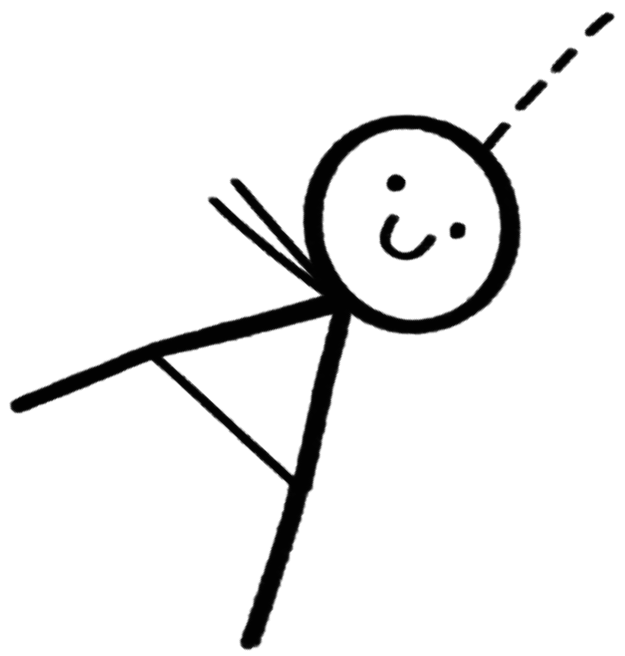
- Speedup: 2.7x
  - ★ 48%
  - ★ 22%
  - ★ 23%

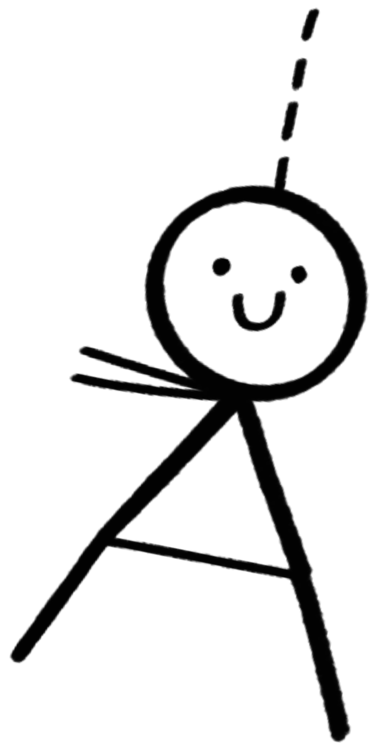


Conclusions…

- IBP reduction: key point ingredient of calculations
- Bottleneck for state-of-the-art precision predictions
- Transverse integration
  - Build simpler identities to feed into the IBP system
  - Map into a new family with fewer invariants and fewer ISPs :

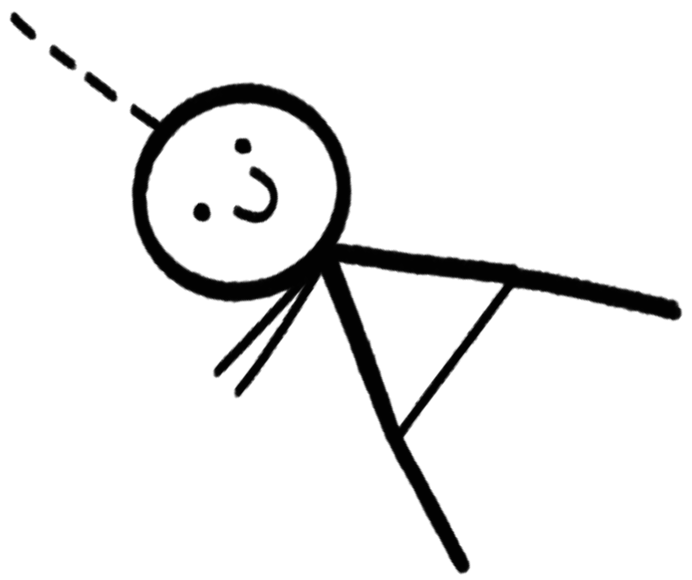
    ⇒easier identities
  - Substantial performance improvements in cutting-edge examples





…& outlook

- Combination with syzygy techniques
- Implementation of factorizable sectors
- Optimizations + release of public package 📦



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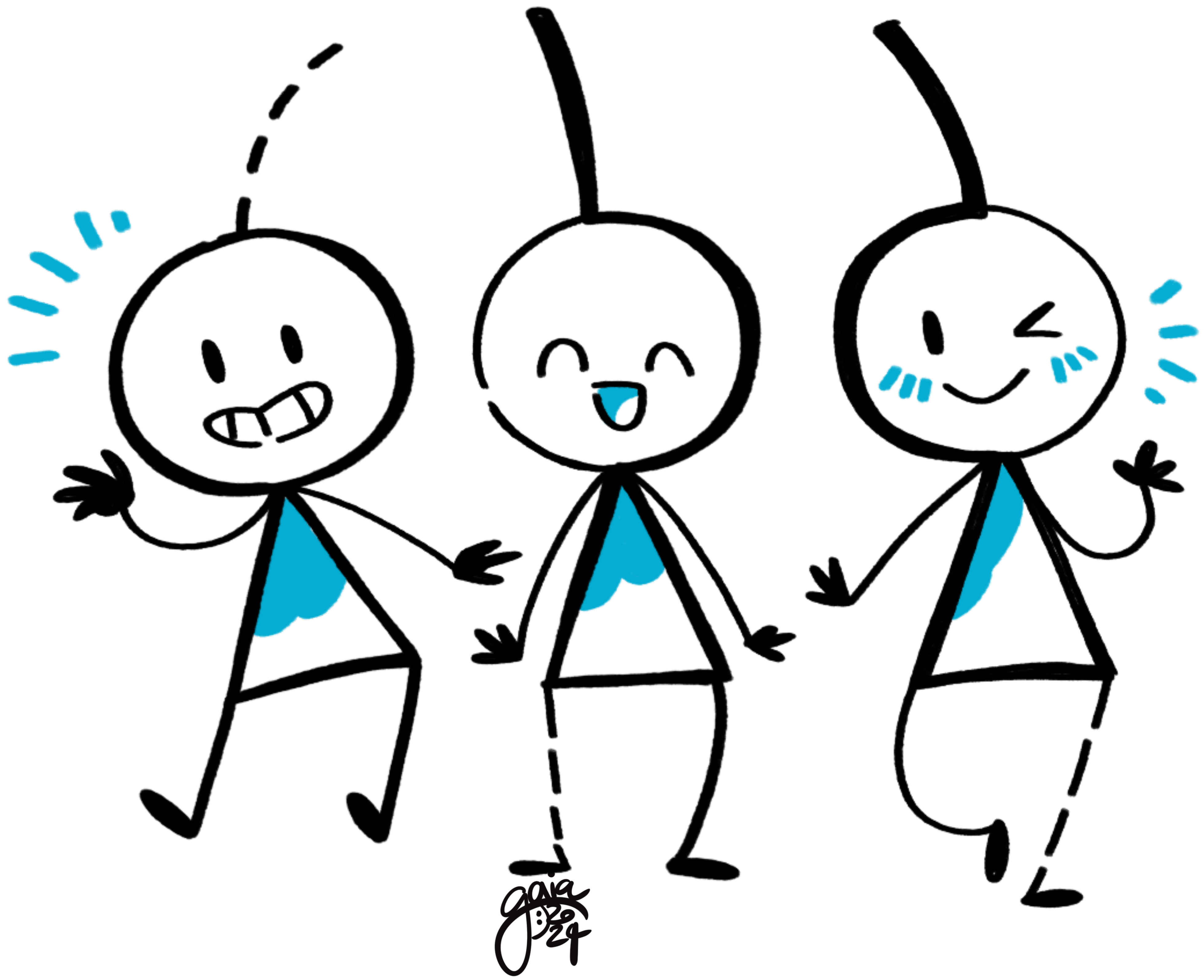

More clearly... let's see what happens on

$$I_{\mathsf{bt};\vec{a}}[(k_{1,\perp}\cdot p_3)\,(k_{2,\perp}\cdot p_3)] = p_{3\,\mu}p_{3\,\nu}\, I_{\mathsf{bt};\vec{a}}[k^{\mu}_{1,\perp}\, k^{\nu}_{2,\perp}]$$

Tensor decomposition: $I_{\mathsf{bt};\vec{a}}[k^{\mu}_{1,\perp}\, k^{\nu}_{2,\perp}] = C_1\, g^{\mu\nu}_{\perp}$

$$C_1 = \frac{1}{D-2} g_{\perp\,\mu\nu}\, I_{\mathsf{bt};\vec{a}}[k^{\mu}_{1,\perp}\, k^{\nu}_{2,\perp}] = \frac{1}{D-2}\, I_{\mathsf{bt};\vec{a}}[(k_{1,\perp}\cdot k_{2,\perp})]$$

Therefore

$$I_{\mathsf{bt};\vec{a}}[(k_{1,\perp}\cdot p_3)\,(k_{2,\perp}\cdot p_3)] = \frac{1}{D-2}\, p^2_{3,\perp}\, I_{\mathsf{bt};\vec{a}}[(k_{1,\perp}\cdot k_{2,\perp})],$$

And $p^{\mu}_{3,\perp} = p^{\mu}_{3} - p^{\mu}_{3,\parallel}$ and $k^{\mu}_{i,\perp} = k^{\mu}_{i} - k^{\mu}_{i,\parallel}$

Everything rewritten in terms of family bt