Moduli spaces of graphs

MPI Garching

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I study moduli spaces of finite metric graphs, originally motivated by geometric group theory

The combinatorics of these objects seem related to various aspects of Feynman integrals

1. n-loop contribution $\leftrightarrow$ integral over moduli space (Berghoff)

2. Renormalization $\leftrightarrow$ structure at ∞ of moduli space (Berghoff)

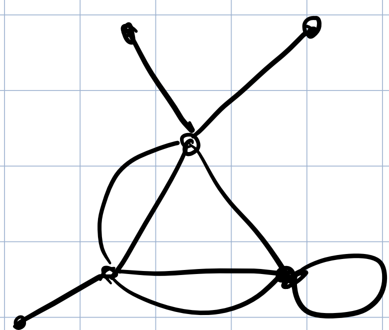
3. Cutkosky rules $\leftrightarrow$ cubical structure of spine of mod. space (Bloch-Kreimer, Kreimer)

$\mathcal{MG}_{n,s}$ = moduli space of graphs with n loops and s leaves

Here a graph is a 1-dimensional cell complex

All my graphs will be connected

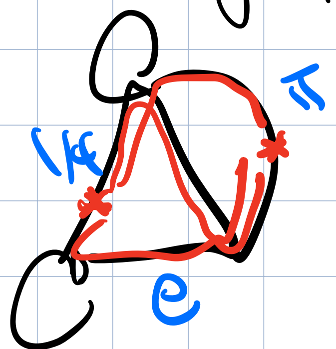
They may have self-loops and multiple edges



I want to make <u>spaces</u> of graphs.

If graphs are endowed with metrics, can use the Gromov-Hausdorff topology

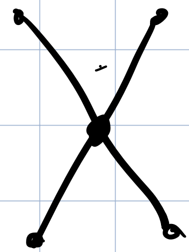
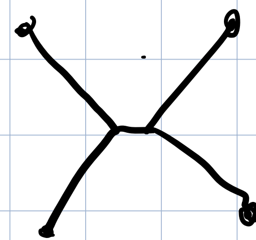
Metric graphs: Edges have positive real lengths



– (interior isometric to an open interval of $\mathbb{R}$)

$\rightsquigarrow$ path metric on graph.

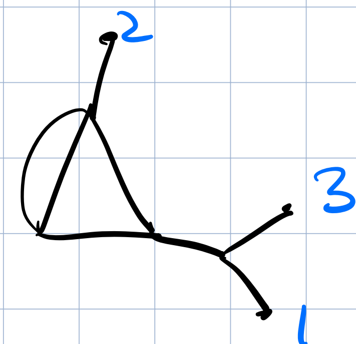
Bivalent vertices are not detected by the metric, so we won't allow them in our combinatorial graphs

$\rightsquigarrow$ Neighborhood of G: vary the edge lengths subtlety:  is close to 

(nearby points may not be homeomorphic)

$\mathcal{MG}_{n,s}$ will consist of connected metric graphs with

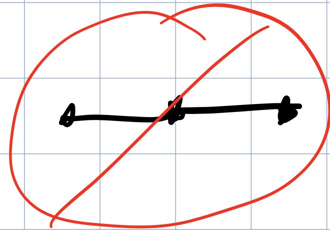
- $\chi(G) = 1 - n$ $(n = \text{rank}(\pi_1(G)))$
- s = number of univalent vertices

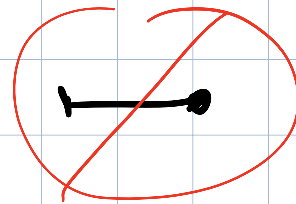


We will label the univalent vertices $1, \ldots, s$

- $G_1 = G_2$ if $\exists$ label-preserving isometry
- $\mathcal{MG}_{n,s}$ comes with an action of Σ_s

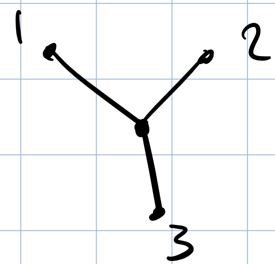
$\boxed{\mathcal{M}_{0,s}:}$





Let $\mathcal{T}_s$ = space of metric trees with s labeled leaves ($s \geq 3$)

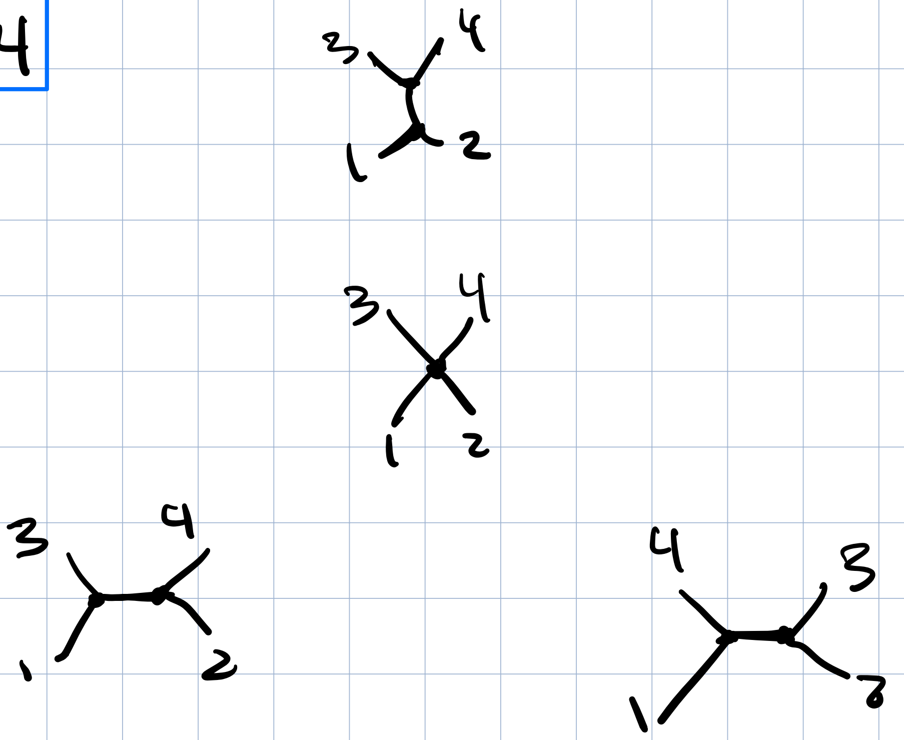
$\boxed{s=3}$



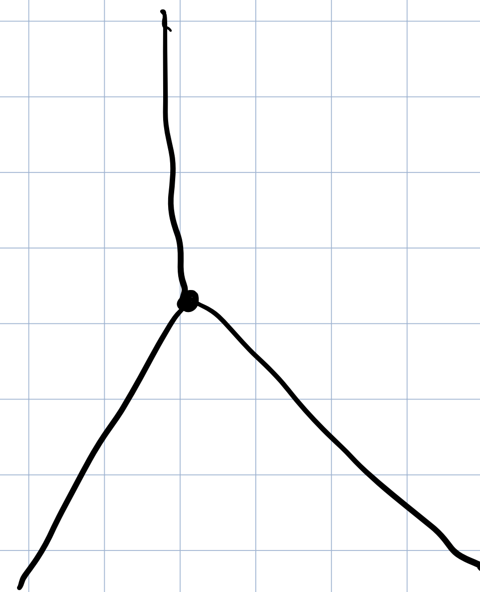
$\mathcal{T}_3 = \bullet \times \mathbb{R}^3_{>0}$

length of leaves always give factor of $\mathbb{R}^s_{>0}$ — we understand this so will ignore leaf lengths

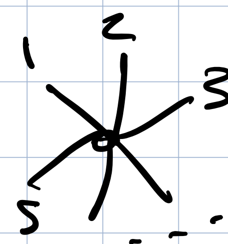
$\boxed{s=4}$



$\mathcal{T}_4 =$

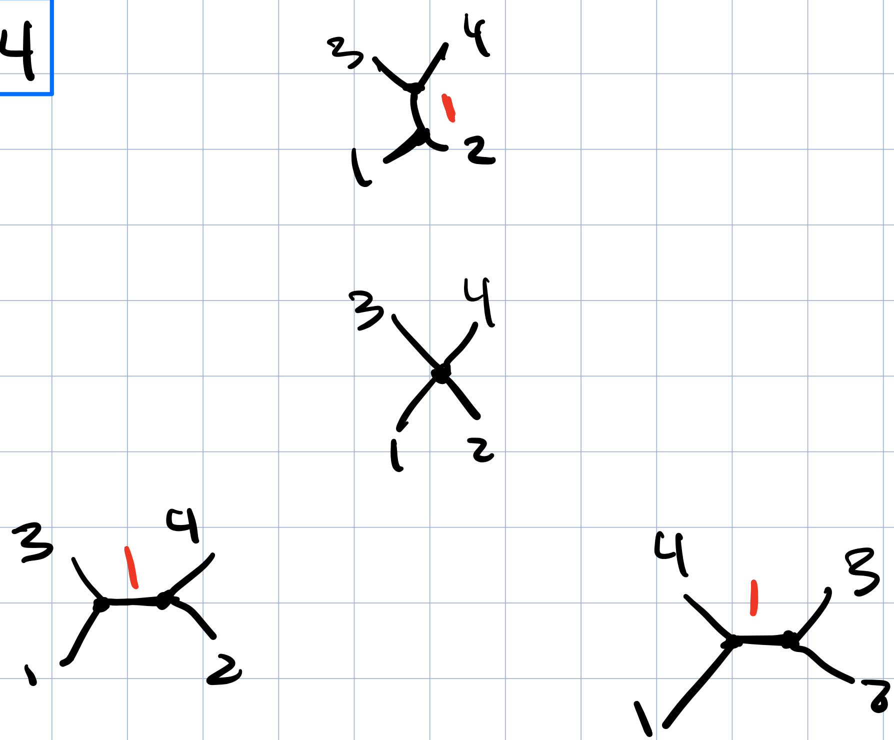


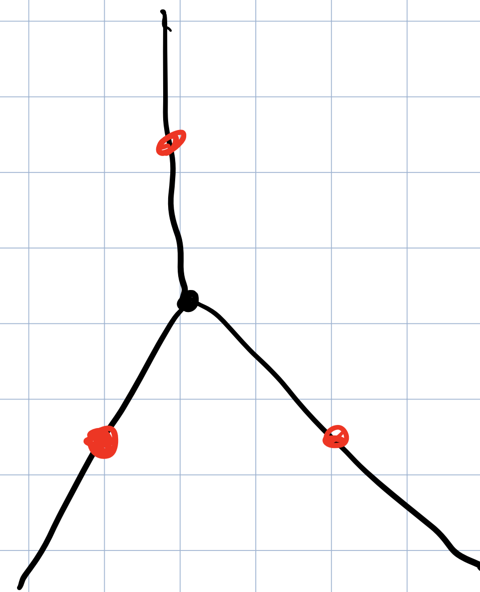
$\mathcal{T}_s$ is always a <u>cone</u> w/ cone point



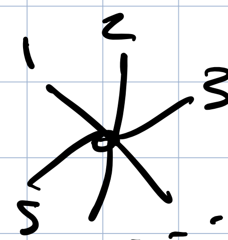
The interesting structure is in its link

$\boxed{s=4}$



$T_4 =$ 

$\mathcal{M}_{0,4}$

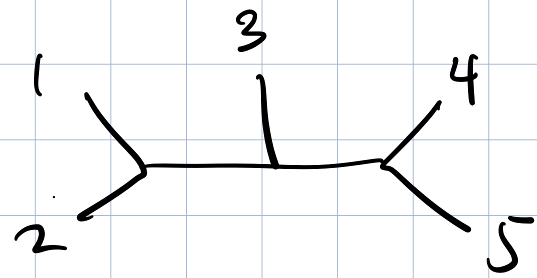
T_s is always a <u>cone</u> w/ cone point 

The interesting structure is in its link:

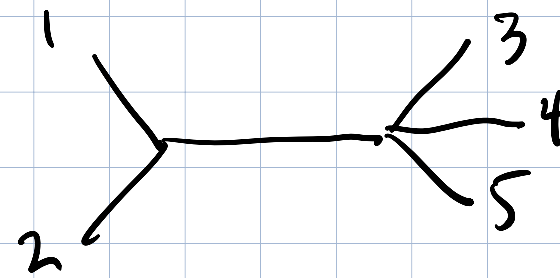
Normalize so $\sum$ internal edges $= 1$

to get $\mathcal{M}_{0,s}$

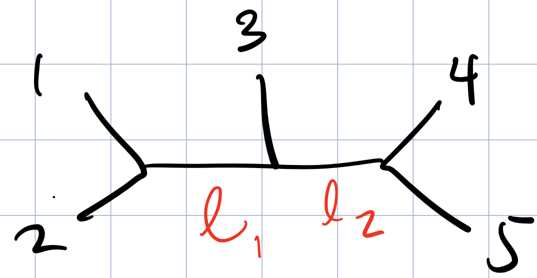
$\mathcal{T}_5$:



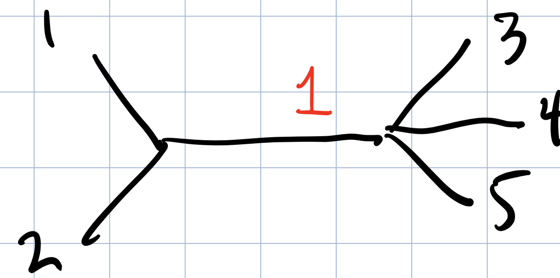
or

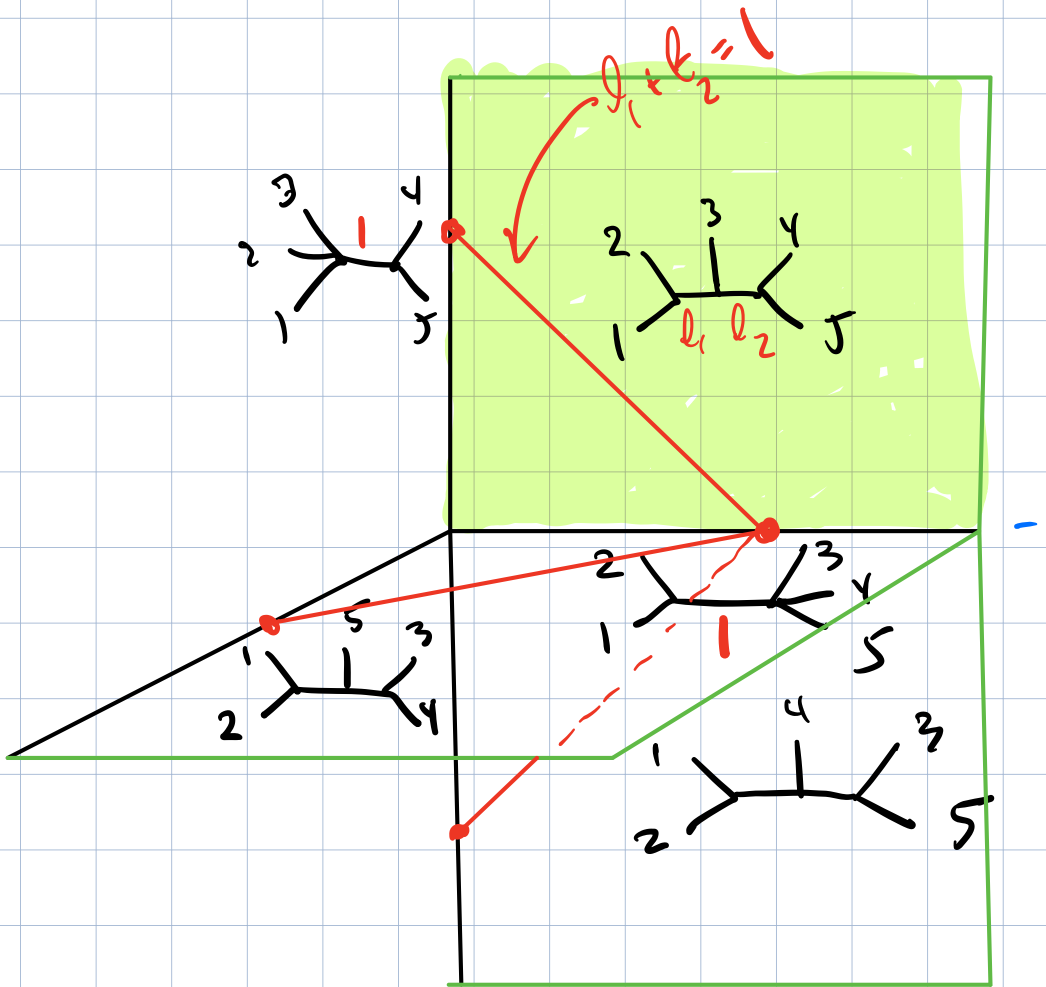


$\mathcal{T}_5$ =

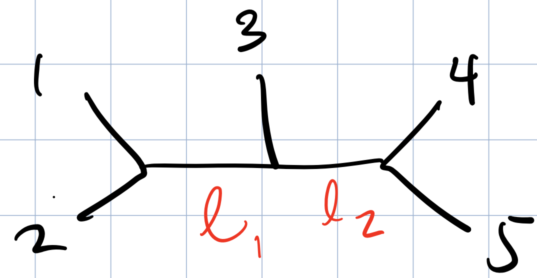


or

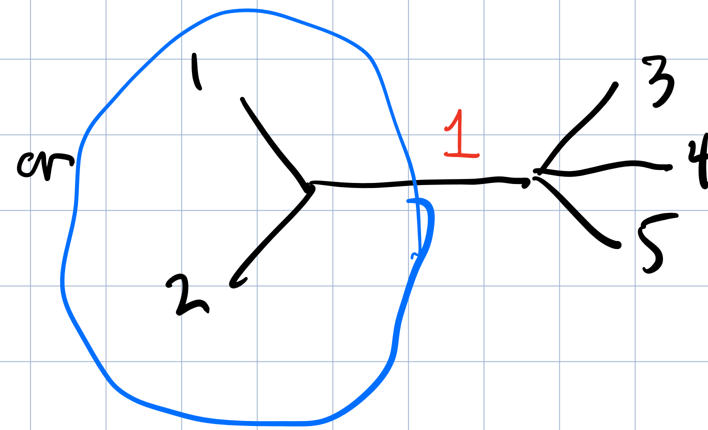


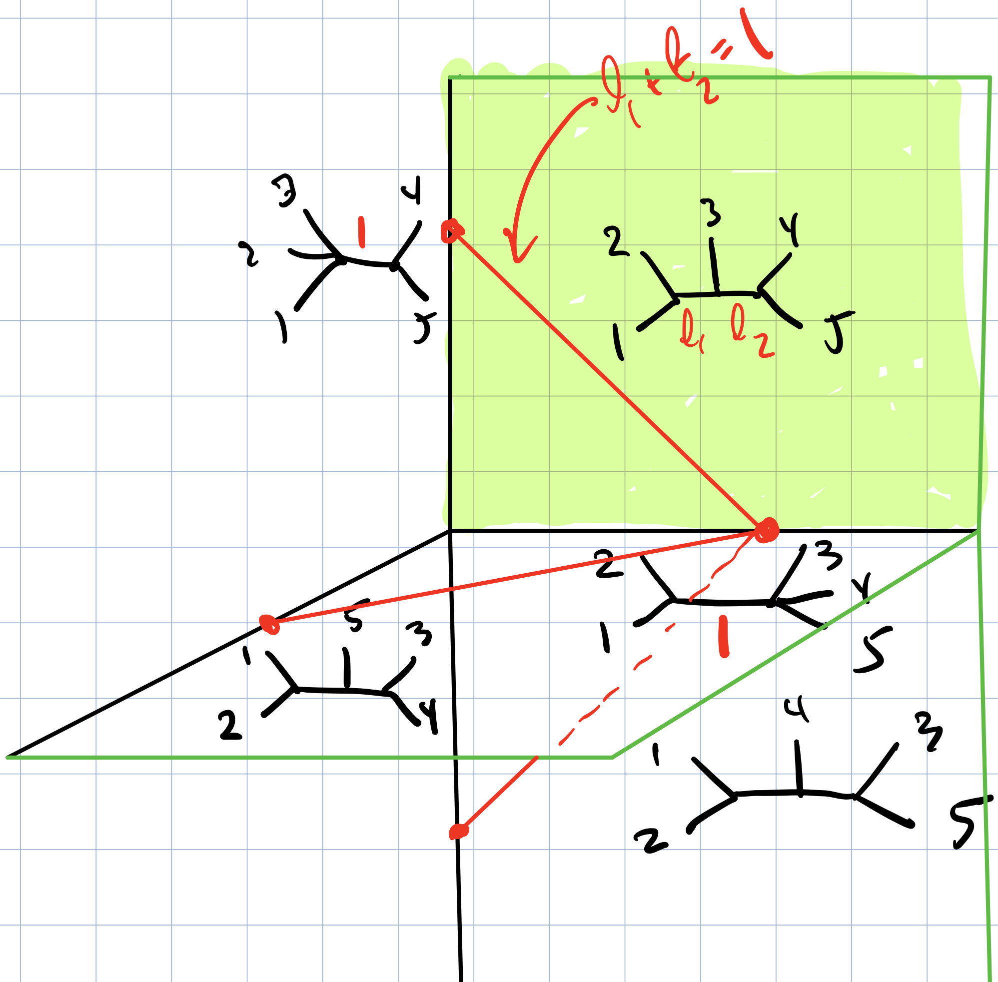


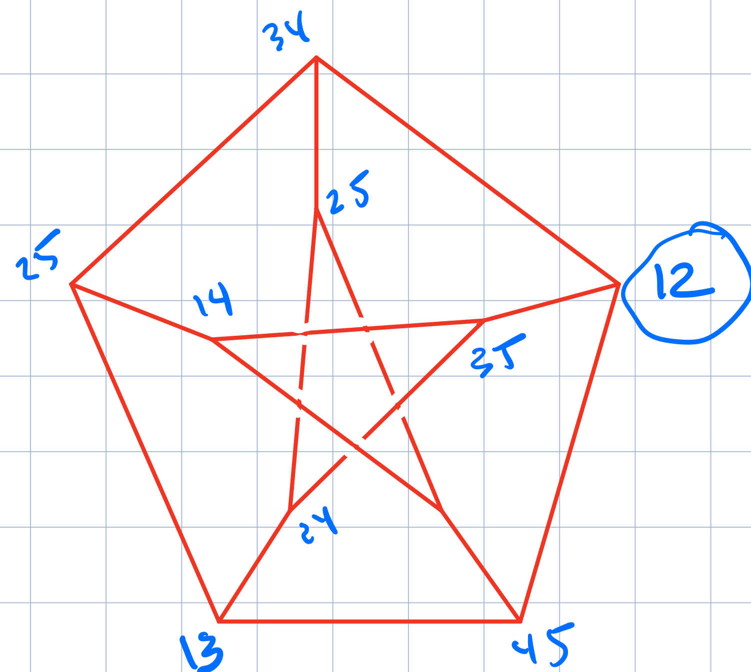
T_5 =



or







$\mathcal{M}_{0,5}$

Other appearances of $\mathcal{MG}_{0,s}$

= Curve complex for 2-sphere with s punctures

= Realization of poset of thick partitions of $\{1, \ldots, s\}$

$\mathcal{T}_n$ = Tropical Grassmannian

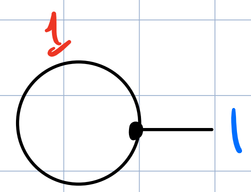
$\mathcal{T}_n$ (with appropriate CAT(0) metric)

= BHV-Space of phylogenetic trees

$$\boxed{\mathcal{M}_{1,s}}$$

No bivalent vertices $\Rightarrow s \geq 1$

$$\boxed{s=1}$$

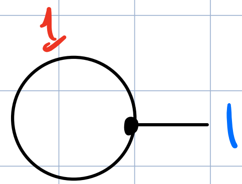


$\mathcal{M}_{1,1}$ = point

$\boxed{\mathcal{M}_{1,s}}$

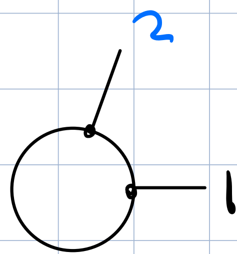
No bivalent vertices $\Rightarrow s \geq 1$

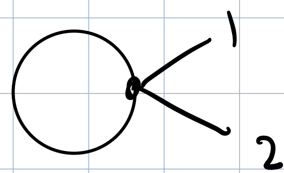
$\boxed{s = 1}$

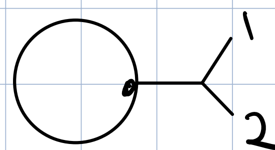


$\mathcal{M}_{1,1}$ = point

$\boxed{s = 2}$



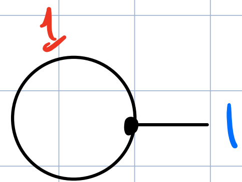




$\boxed{Mg_{1,s}}$

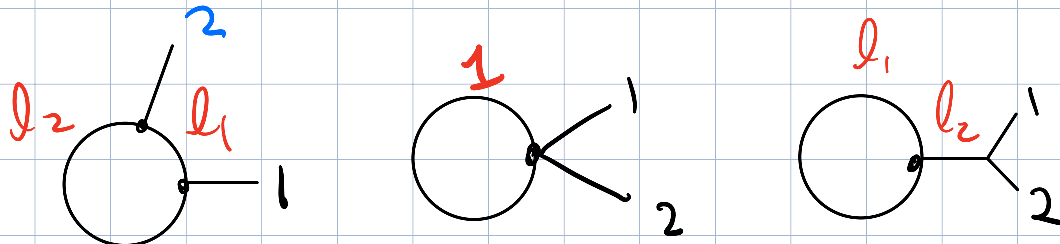
No bivalent vertices $\Rightarrow s \geq 1$

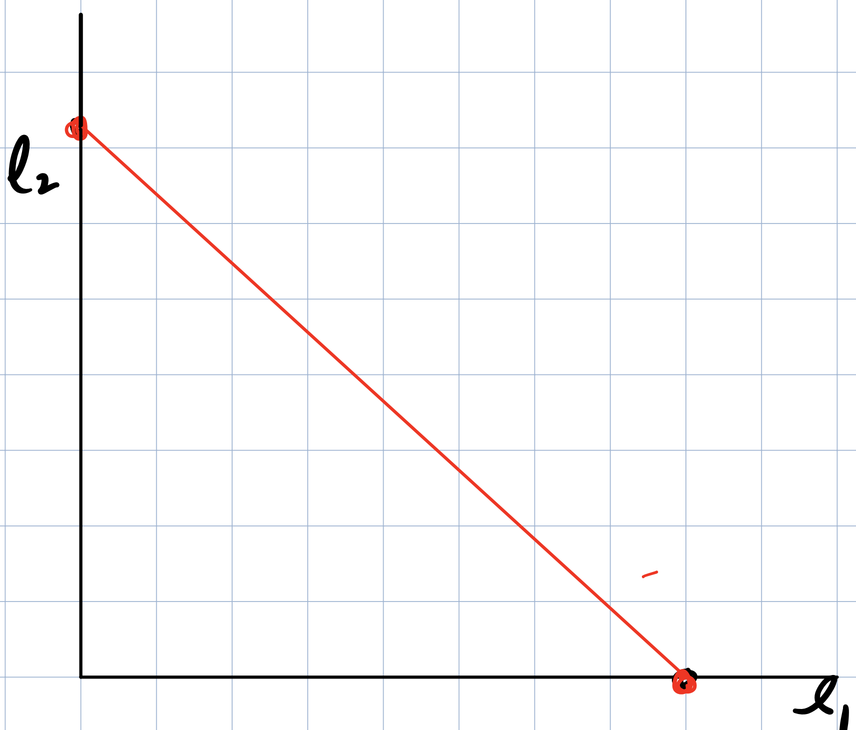
$\boxed{s = 1}$

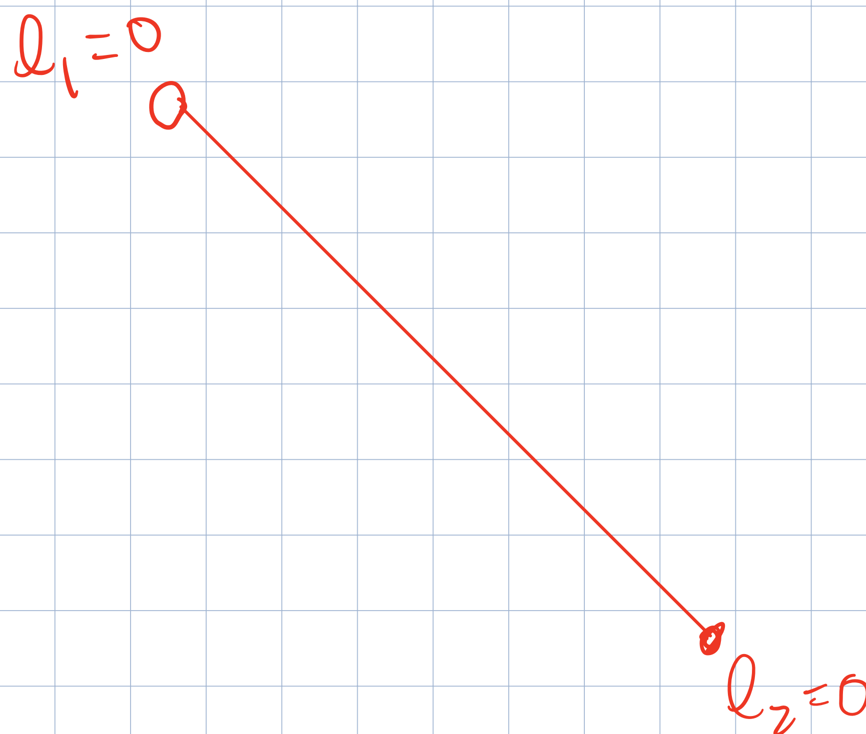


$Mg_{1,1}$ = point

$\boxed{s = 2}$



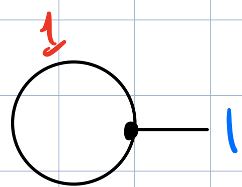




$\boxed{\mathcal{M}g_{1,s}}$

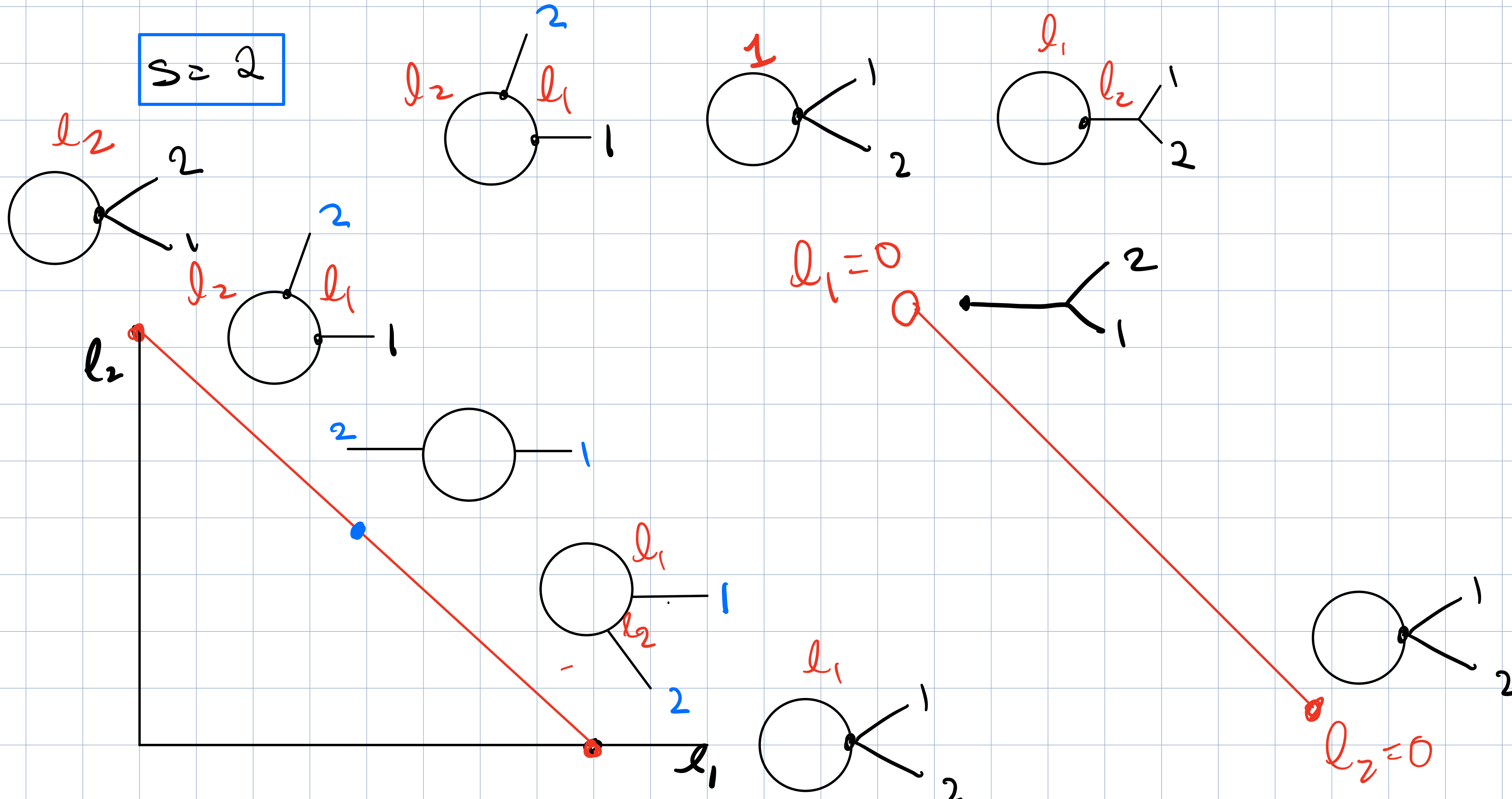
No bivalent vertices $\Rightarrow s \geq 1$

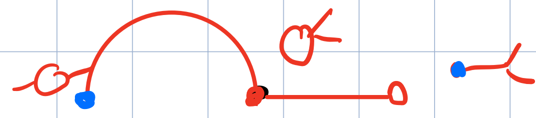
$\boxed{s = 1}$

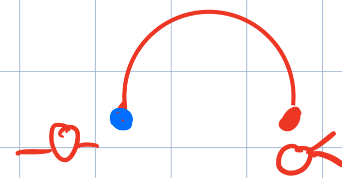


$\mathcal{M}g_{1,1}$ = point

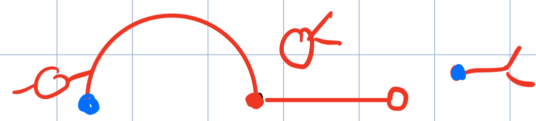
$\boxed{s = 2}$

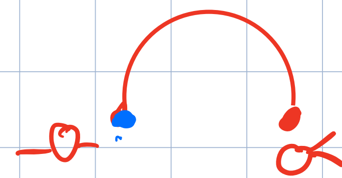


So $\mathcal{MG}_{1,2} =$  (think of $S^1 \subset \mathbb{C}$, $z \sim \bar{z}$.)

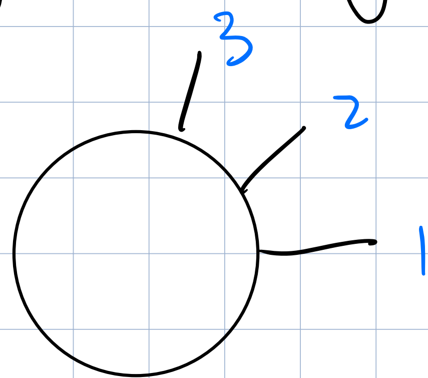
If we don't allow sep edges we get  $= \mathcal{MG}^{1PI}_{1,2}$

Prop $\mathcal{MG}_{1,2} \simeq \mathcal{MG}^{1PI}_{1,2}$. (shrink sep edges linearly)

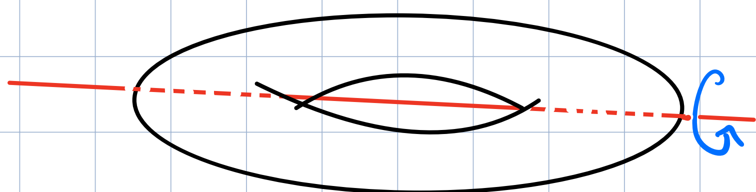
So $\mathcal{MG}_{1,2}$ =  (think of $S^1 \subset \mathbb{C}$ $z \sim \bar{z}$.)

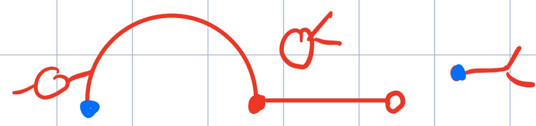
If ==we don't allow sep edges== we get  $= \mathcal{MG}_{1,2}^{1PI}$

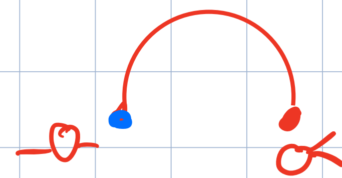
<u>Prop</u> $\mathcal{MG}_{1,2} \simeq \mathcal{MG}_{1,2}^{1PI}$. (shrink sep edges linearly)

$\boxed{\mathcal{MG}_{1,3}^{1PI}}$ 

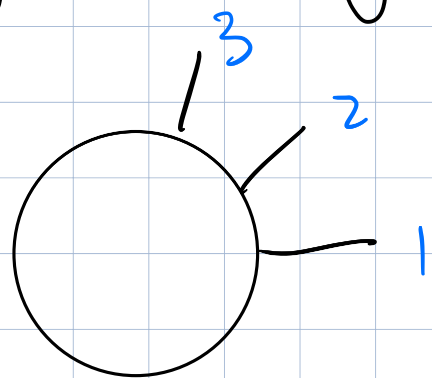
$\simeq S^1 \times S^1 \big/ (z_1, z_2) \sim (\bar{z}_1, \bar{z}_2)$

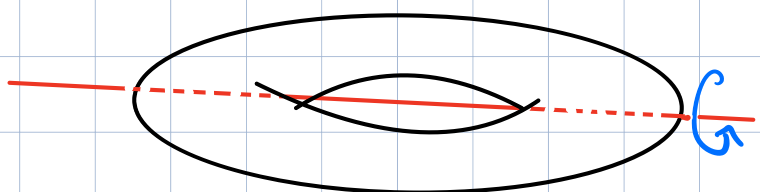
 $\approx S^2 =$ Torus / elliptic involution

So $\mathcal{M}g_{1,2}$ =  (think of $S^1 \subset \mathbb{C}$, $z \sim \bar{z}$.)

If we don't allow sep edges we get  $= \mathcal{M}g_{1,2}^{1PI}$

<u>Prop</u> $\mathcal{M}g_{1,2} \simeq \mathcal{M}g_{1,2}^{1PI}$. (shrink sep edges linearly)

$\boxed{\mathcal{M}g_{1,3}^{1PI}}$  $\simeq S^1 \times S^1 \big/ (z_1, z_2) \sim (\bar{z}_1, \bar{z}_2)$

 $\approx S^2 = \text{Torus} \big/ \text{elliptic involution}$

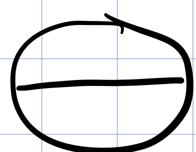
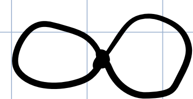
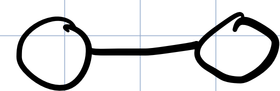
$\boxed{\mathcal{M}g_{1,s}^{1PI}} = T^{s-1} \big/ \mathbb{Z}/2\mathbb{Z}$

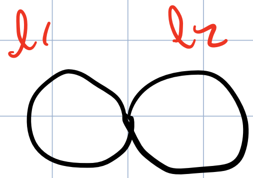
In general, $\mathcal{MG}_{n,s} \simeq \mathcal{MG}^{1PI}_{n,s}$

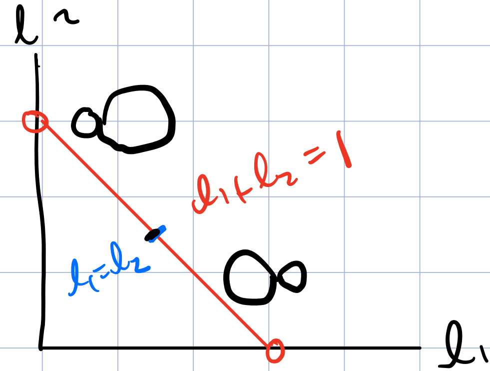
$n=1$: $\mathcal{MG}^{1PI}_{1,s}$ is compact. $= T^{s-1}/\mathbb{Z}/2\mathbb{Z}$

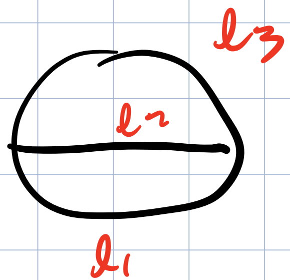
$n \geq 2$: $\mathcal{MG}^{1PI}_{2,s}$ is not compact, structure is more subtle

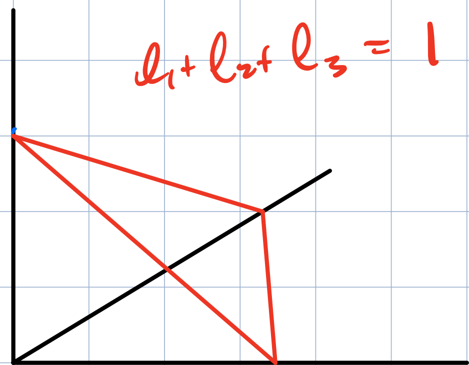
$\boxed{\mathcal{MG}_{2,0}}$ Combinatorial types: no univalent or bivalent vertices, $\pi_1 \cong F_2 \Rightarrow$

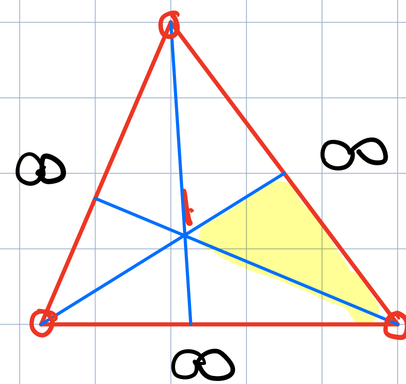
$G =$  ,  (or )

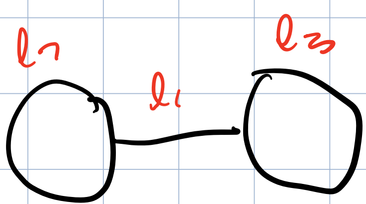

ℓ_1
ℓ_2

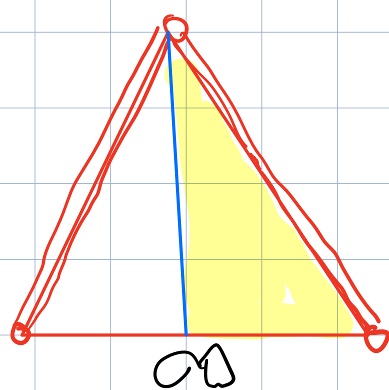

ℓ_2
$\ell_1 + \ell_2 = 1$
$\ell_1 = \ell_2$
ℓ_1


ℓ_3
ℓ_2
ℓ_1

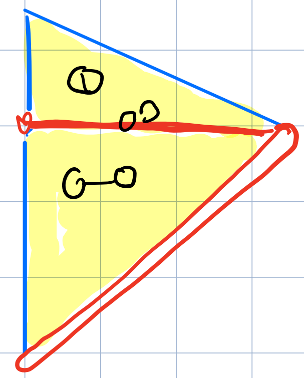

$\ell_1 + \ell_2 + \ell_3 = 1$



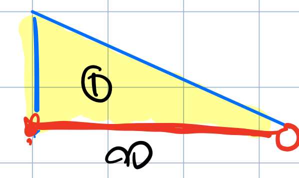

ℓ_2
ℓ_3
ℓ_1



$\mathcal{MG}_{2,0}$ =



$\mathcal{MG}_{2,0}^{1PI}$ =

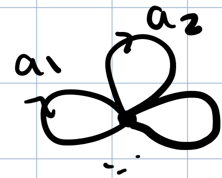


Connection with Geometric group theory

Theorem: (Culler-V 1986) $\mathcal{MG}_{n,0}$ $(n \geq 2)$ is the quotient of a contractible space by a proper action of a discrete group Γ_n.

GGT: you can use the group to understand the space, or the space to understand the group.

$\boxed{s=0}$

The group. Let $R_n =$ 

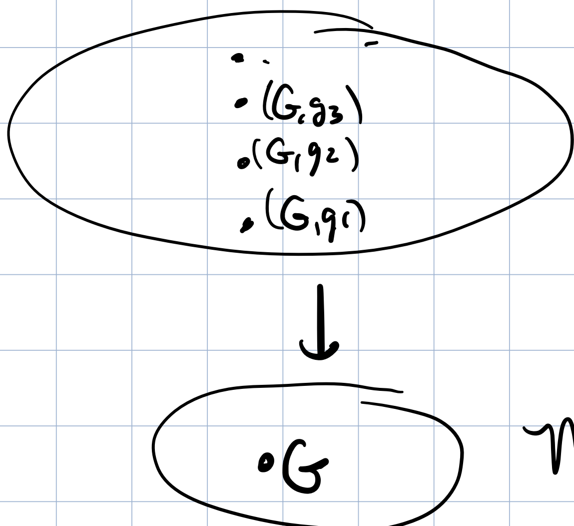
, $\pi_1(R_n, b) \equiv F_n$

A homotopy equivalence $h: R_n \to R_n$ induces an (outer) automorphism $\pi_1 R_n \longrightarrow \pi_1 R_n$

Then $\pi_0 HE(R_n) = Out(F_n)$ is the group

The contractible space CV_n

Point in CV_n = pair (G,g) $G \in \mathcal{MG}_{n,0}$, $g: R_n \xrightarrow{\simeq} G$



CV_n

$\mathcal{MG}_n$

Action of $\alpha \in \pi_0 HE(R_n)$:

$(G,g)\cdot\alpha = (G, g\circ\alpha)$

$$\left(\begin{tikzcd} R_n \arrow[r, "g"] & G \\ R_n \arrow[u, "\alpha"] \arrow[ur, dashed, "g\circ\alpha"'] & \end{tikzcd}\right).$$

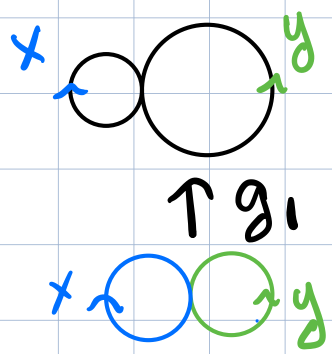
CV_n = pairs (G,g), where $G \in \mathcal{MG}_n$ and $g: R_n \xrightarrow{\simeq} G$

$(G,g_1) = (G,g_2)$ if there is an isometry

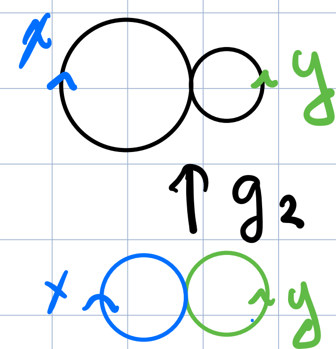
$$\begin{tikzcd} G_1 \arrow[r, "f"] & G_2 \\ & R_n \arrow[ul, "g_1"] \arrow[u, "g_2"'] \end{tikzcd}$$

such that triangle commutes
(up to homotopy)

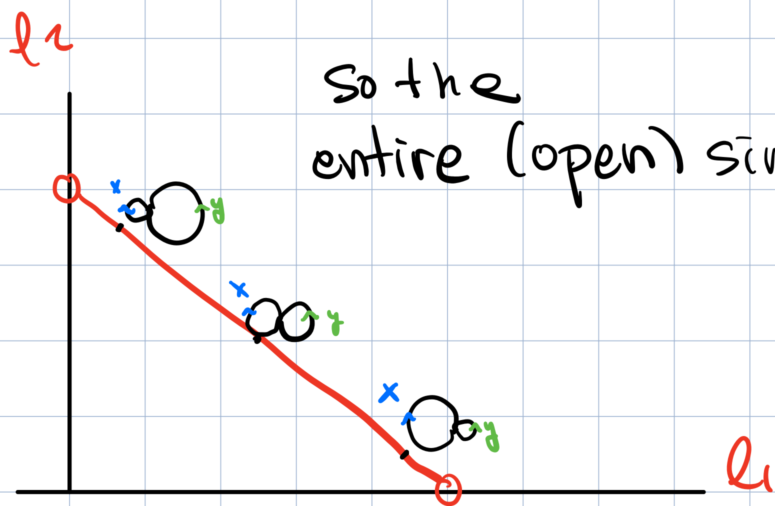
Observe:



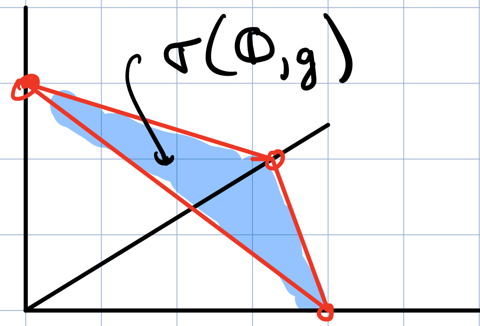
is not equal to

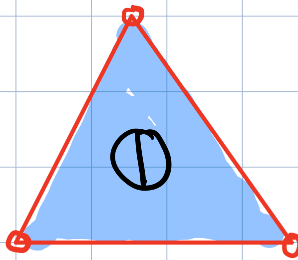


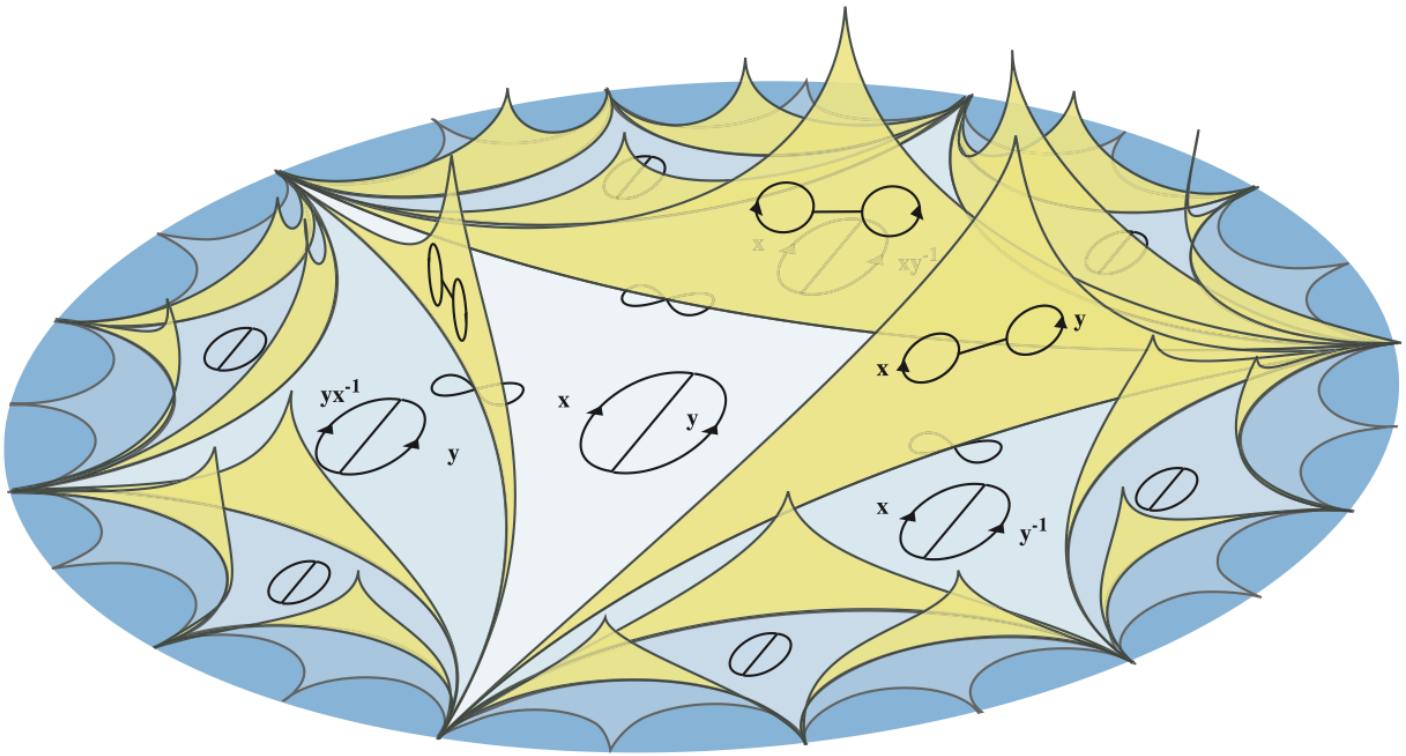
So the
entire (open) simplex $\sigma(G,g)$ is in CV_n



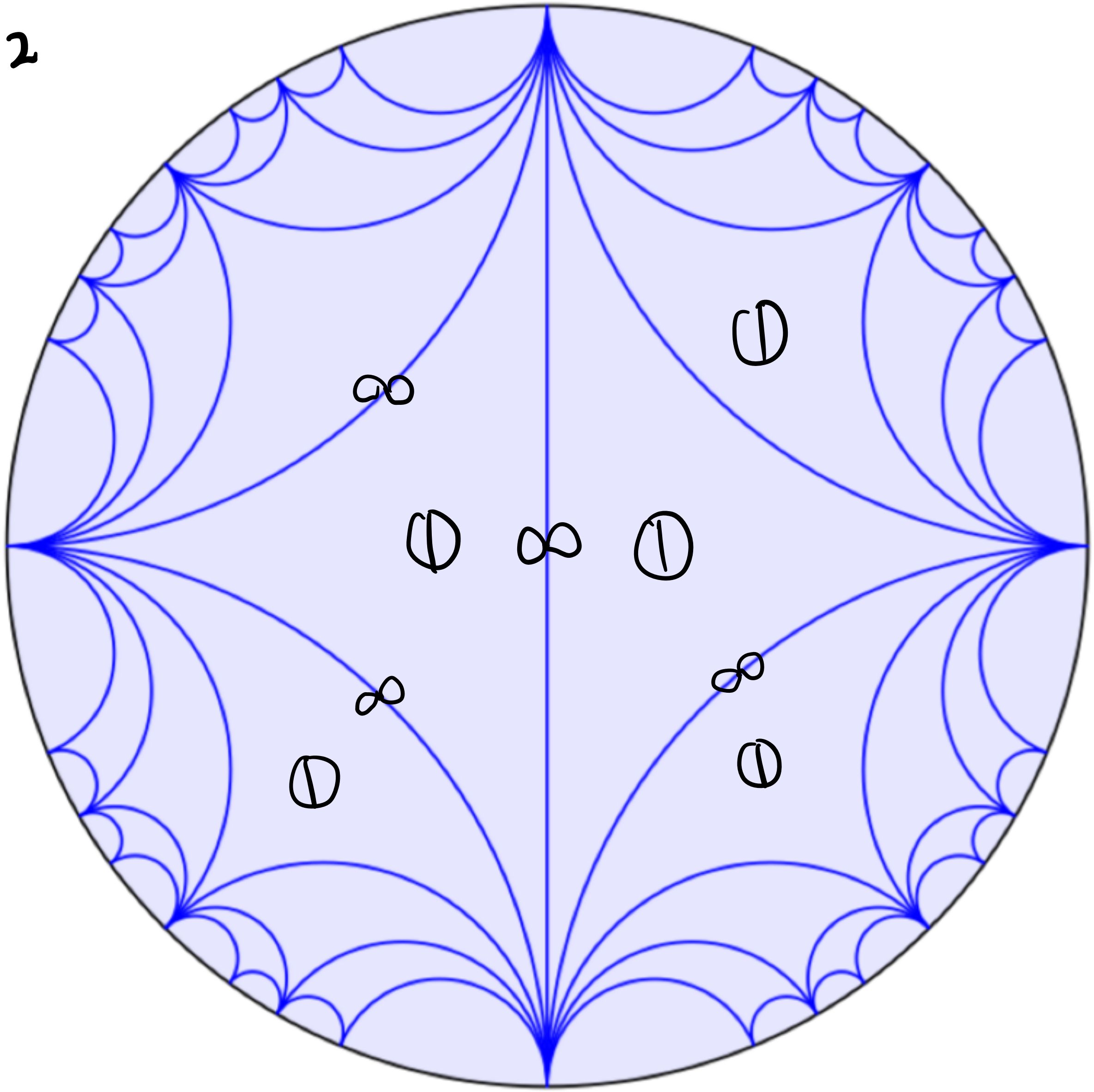
CV_2



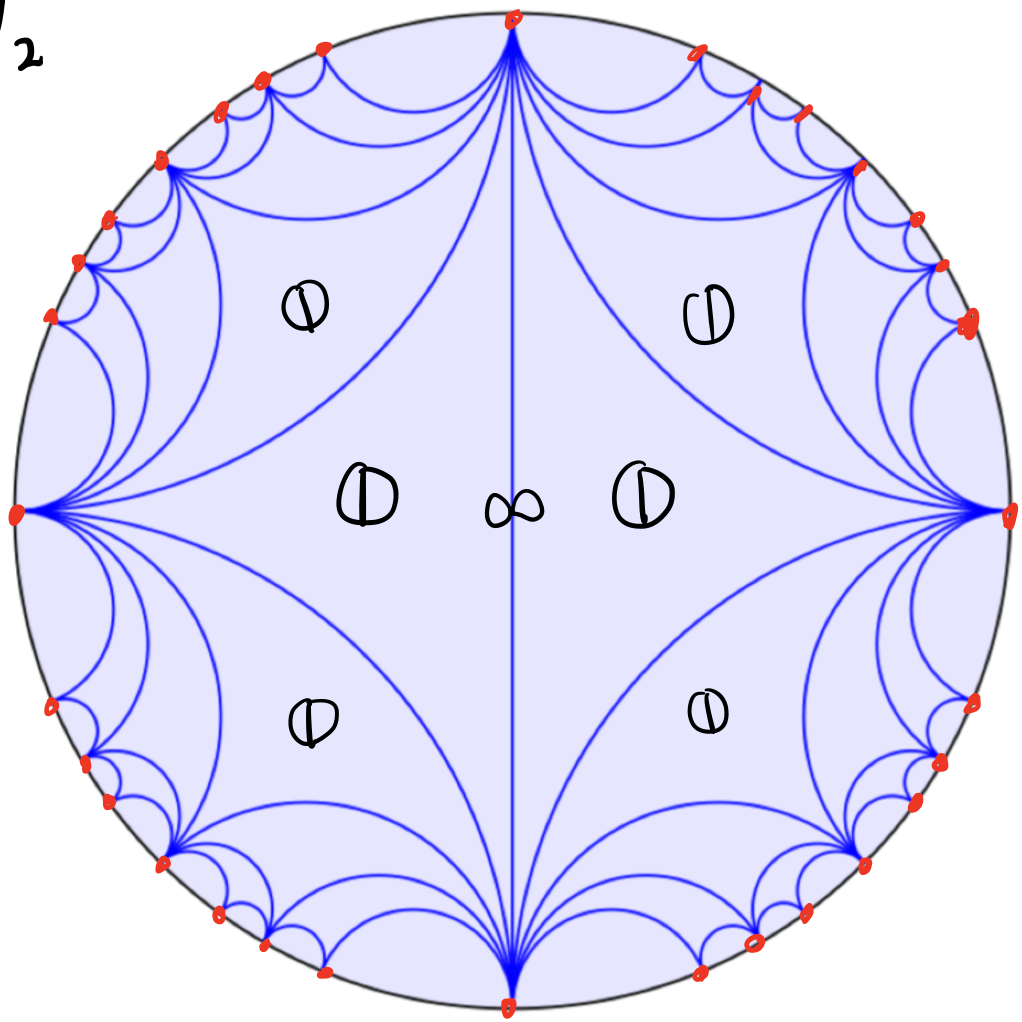




CV_2^{1PI}



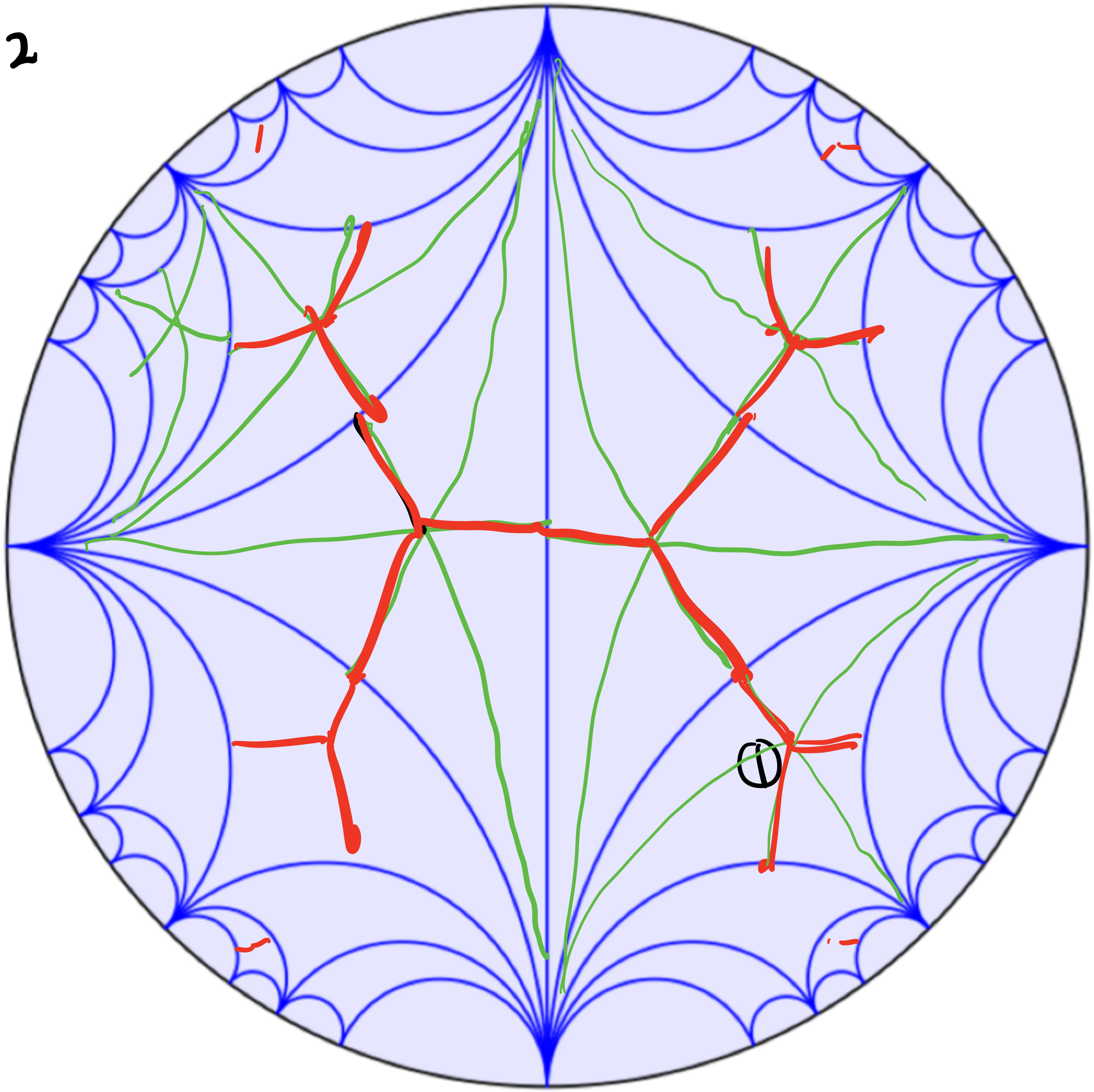
CV_2^{1PI}



Simplicial closure $CV_n^* \circlearrowleft Out(F_n)$

$CV_n^* \longrightarrow \mathcal{M}g_n^* =$ "m. space of tropical curves"

CV_2^{1P1}

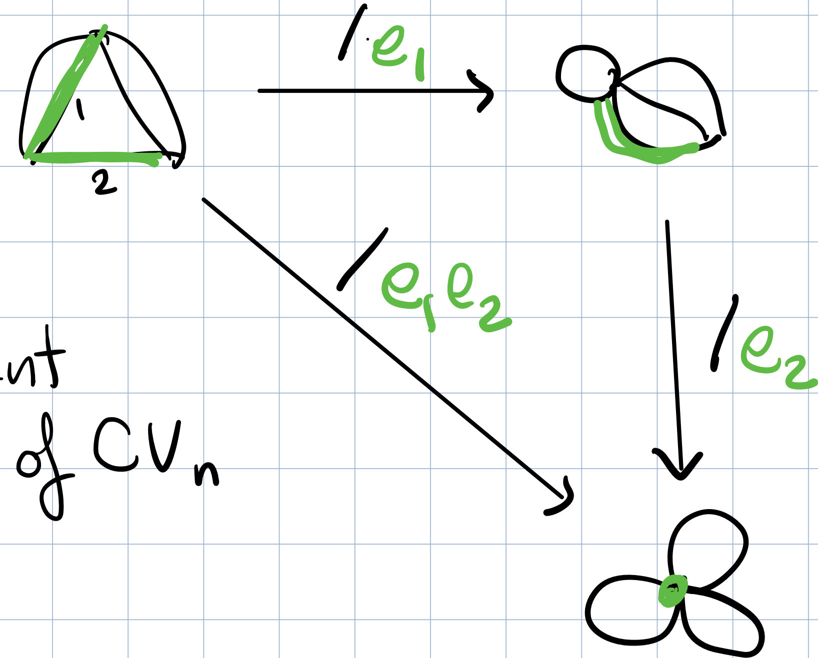


Spine K_n

K_n = simplices of barycentric subdivision $(CV_n^*)'$ of CV_n^* with no vertices at ∞

simplex $\longleftrightarrow$ chain of <u>forests</u> in G

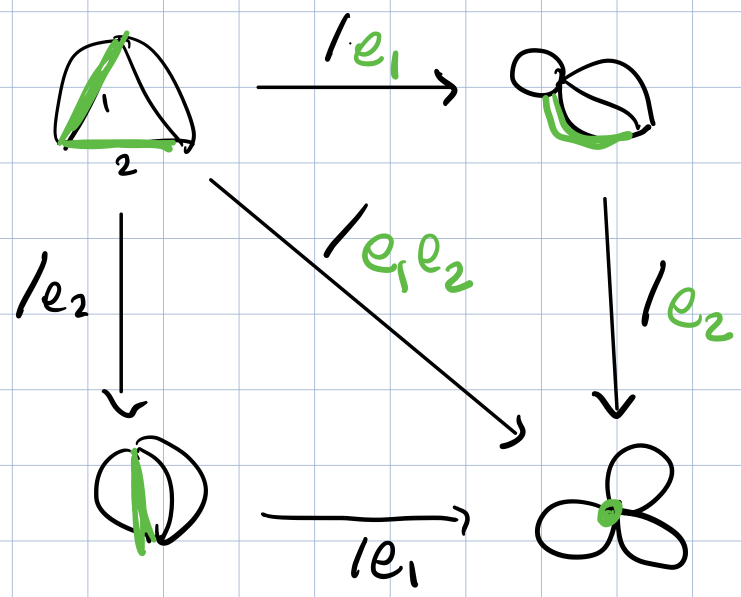
$$(G, F_1 \subset F_2 \subset \cdots \subset F_k)$$



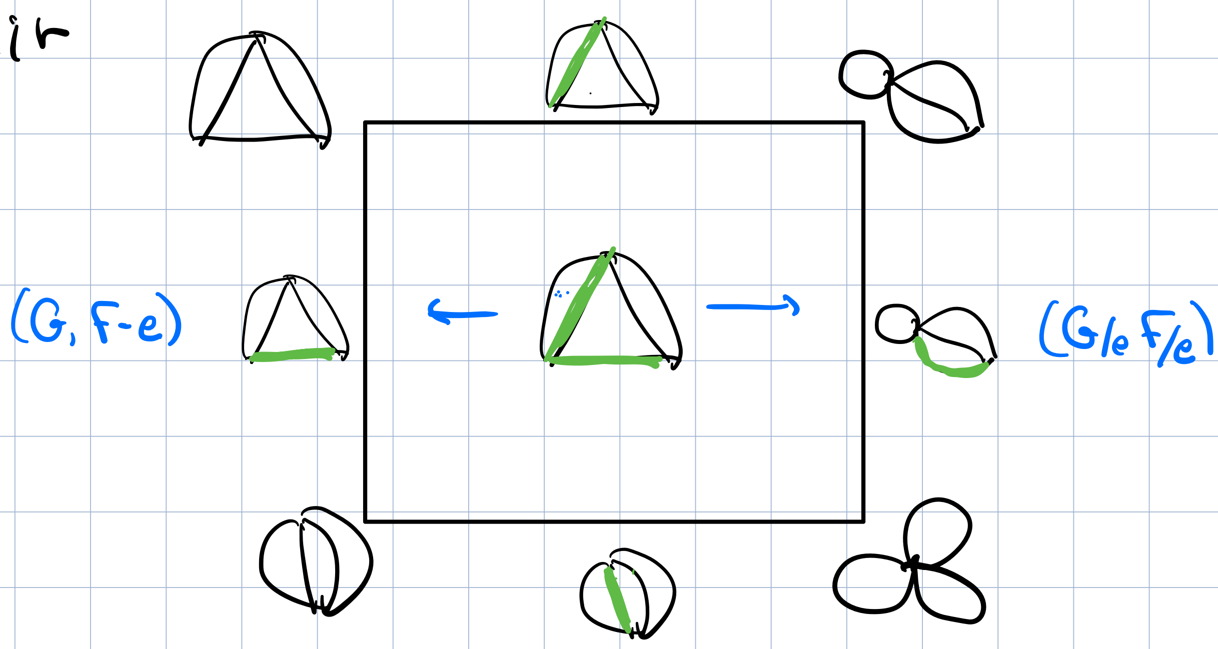
<u>Prop</u>

K_n is an equivariant deformation retract of CV_n

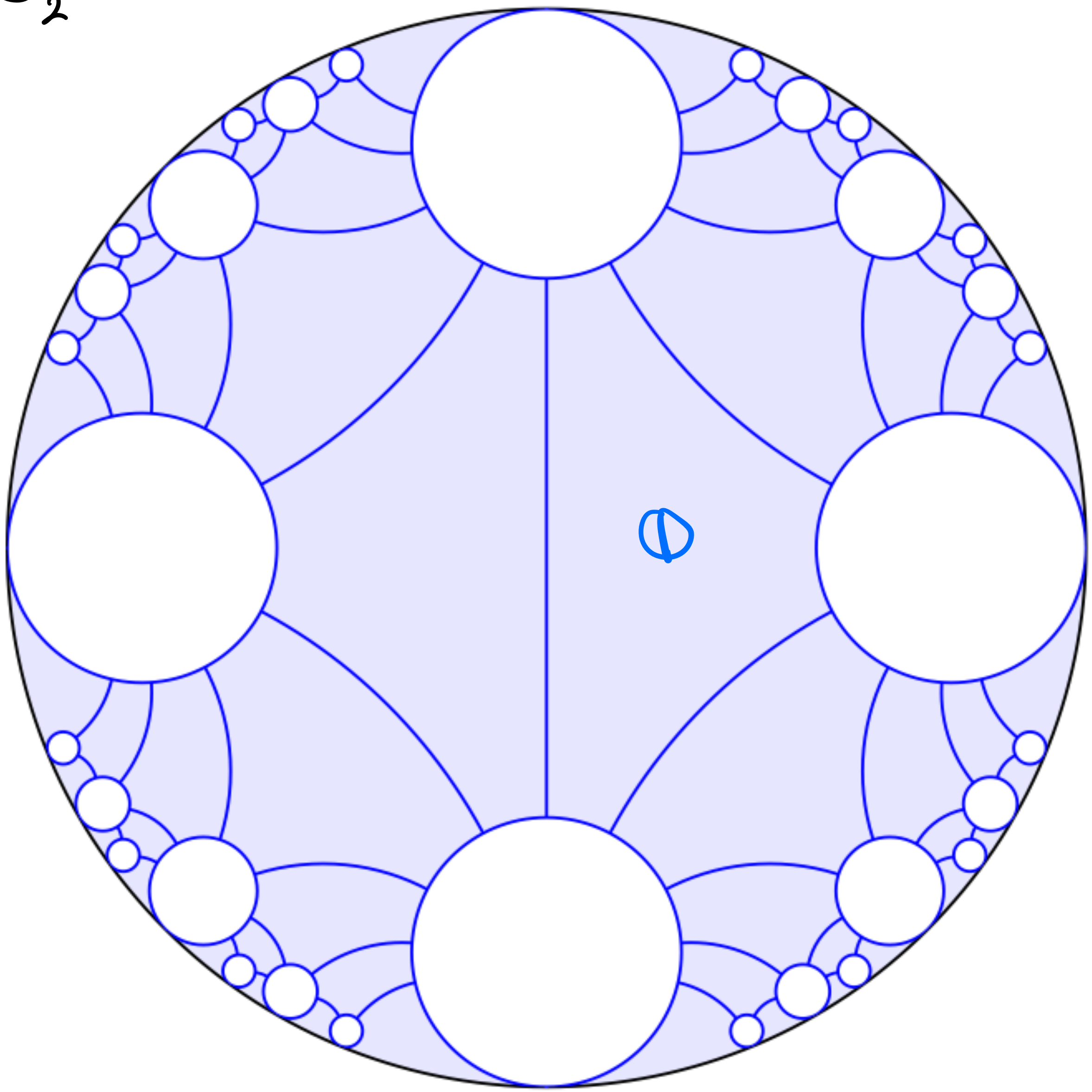
K_n = <u>cube</u> complex



k-cube $\longleftrightarrow$ pair (G, F) s.t. F has k edges

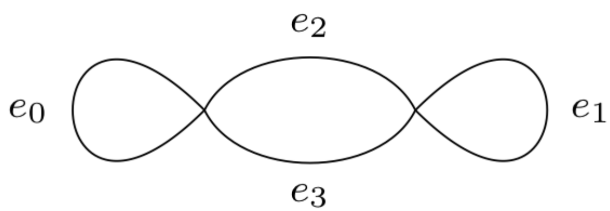


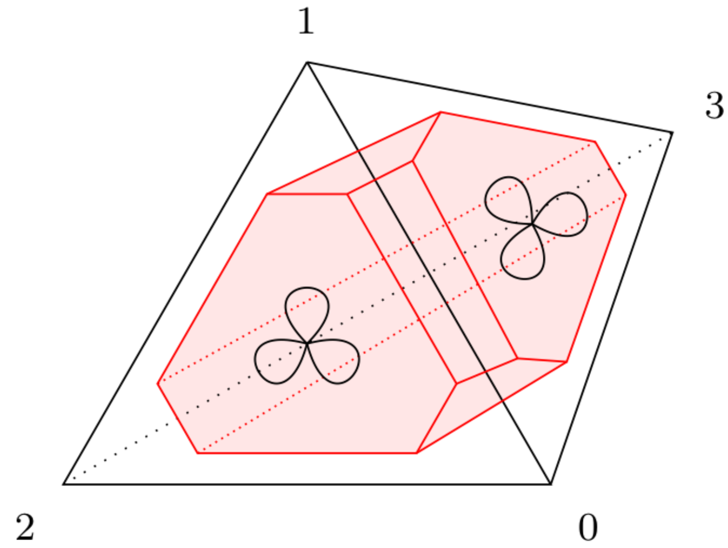
C_2^{1PI}



$$\text{Bordification} = \bigcup_{(G,g)} (\text{graph polytopes})$$

Example of a graph polytope:





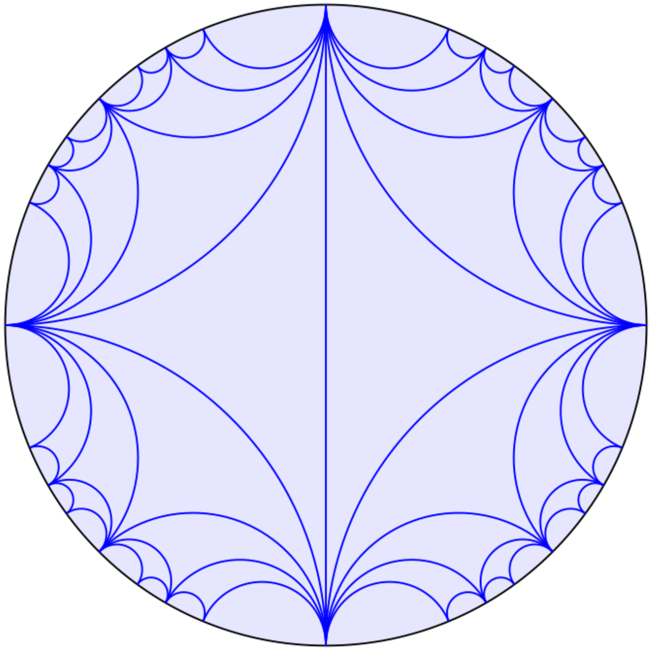
Construction: Shave faces of $\sigma(G,g)$
Opposite faces corresponding to
core subgraphs $\gamma \subset \Gamma$ (subgraphs with no bridges)

Shave deeper if core sugraph has more edges.

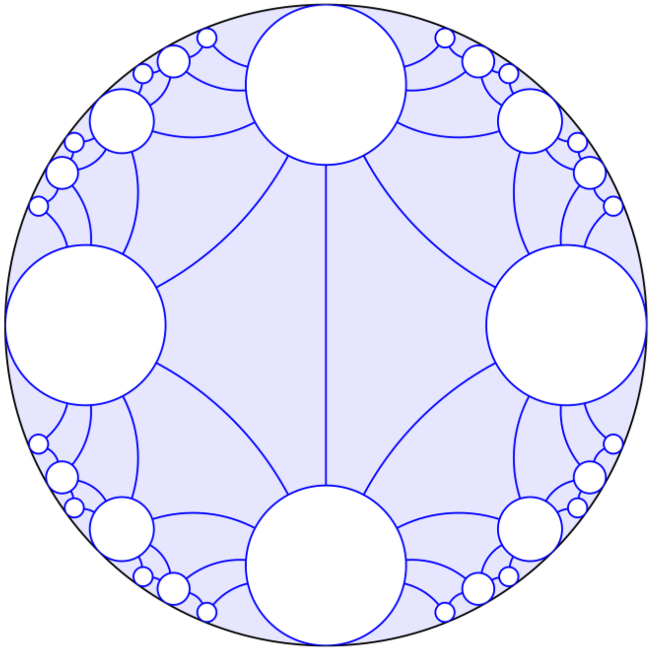
The faces at ∞ $\leftrightarrow$ core subgraphs of G

$\leftrightarrow$ zeroes of first Szymantic polynomial

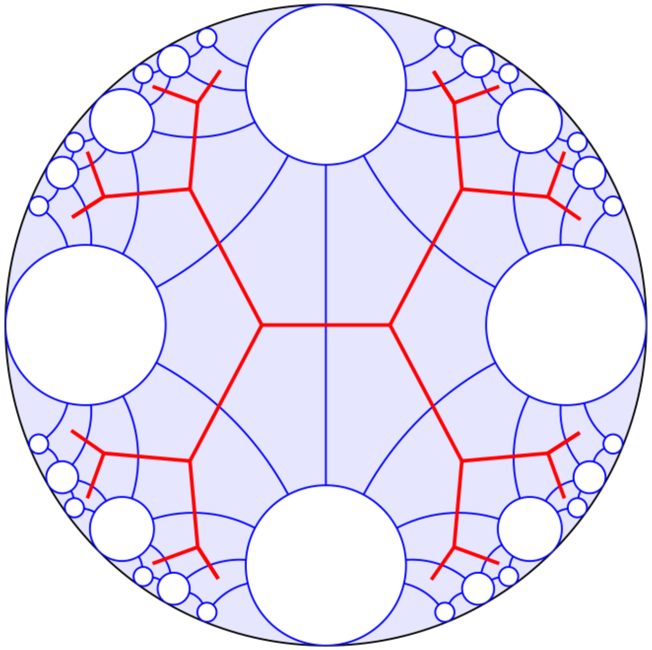
These have a recursive structure related to renormalization.



Outer space $\mathcal{O}_2$



Jeweled subspace $\mathcal{J}_2$

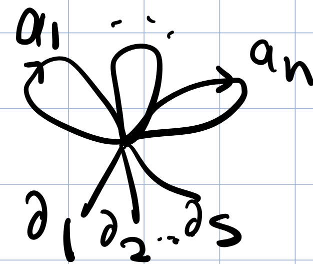


Spine $K_2 \subset \mathcal{J}_2$

The whole story is almost exactly the same for graphs with leaves

$\boxed{CV_{n,s} \text{ with } n \geq 2,\ s \geq 1}$

<u>Replace</u> R_n with $R_{n,s}$ =



$Out(F_n) = \pi_0 HE(R_n)$ with $A_{n,s} = \pi_0 HE(R_{n,s}, \partial_1, \dots, \partial_s)$

(Then $A_{n,1} \cong Aut F_n$. and $A_{n,s} = (F_n)^{s-1} \rtimes Aut F_n$)

CV_n with $CV_{n,s}$ = pairs (G, g)

with $G \in \mathcal{MG}_{n,s}$ and $g : R_{n,s} \xrightarrow{\simeq} G$

$\partial_i \longmapsto$ leaf labelled i

- Hatcher (1990) gave a new proof that CV_n is contractible, which also works for $CV_{n,s}$
- $A_{n,s}$ acts on $CV_{n,s}$ in the same way: $(G,g).\alpha = (G, g\circ\alpha)$ and the spine $K_{n,s}$ is defined in the same way, and has a cubical structure.

Note: The action of $A_{n,s}$ on $CV_{n,s}$ is not cocompact, (ie the quotient $\mathcal{M}\mathcal{G}_{n,s}$ is not compact) but the action on $K_{n,s}$ is. This makes $K_{n,s}$ useful for doing geometric group theory, since it implies that $K_{n,s}$ is quasi-isometric to $A_{n,s}$.