Linear PDE with Constant Coefficients

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Primary Ideals and their Differential Equations
(with Yairon Cid-Ruiz and Roser Homs)

Primary Decomposition with Differential Operators
(with Yairon Cid-Ruiz)

Noetherian Operators and Primary Decomposition
(by Justin Chen, Marc Härkönen, Robert Krone, Anton Leykin)

History

In his 1938 article on foundations of algebraic geometry, Gröbner introduced differential operators to characterize **membership in a polynomial ideal**. He derived this for zero-dimensional ideals (Macaulay's inverse systems), and he envisioned it for all ideals. Gröbner wanted algorithmic solutions. *We provide them.*

Wolfgang Gröbner: Über die algebraischen Eigenschaften der Integrale von linearen Differentialgleichungen mit konstanten Koeffizienten, *Monatshefte für Mathematik und Physik* (1939)

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Analysts made substantial contributions to this subject.

In the 1960s, Ehrenpreis and Palamodov studied solutions to linear partial differential equations (PDE) with constant coefficients. A main step was the characterization of membership in a primary ideal by *Noetherian operators*.

Their celebrated **Fundamental Principle** appears in the books

Leon Ehrenpreis: Fourier Analysis in Several Complex Variables, 1970

Victor Palamodov: Linear Differential Operators w Constant Coeffs, 1970

Four Exercises

Question 1: Solve the system of polynomial equations

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$$\frac{\partial^2 \phi}{\partial x^2} \;=\; \frac{\partial^2 \phi}{\partial y^2} \;=\; \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^3 \phi}{\partial y \partial z^2} \;=\; 0.$$

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Question 4: We presented a subscheme of affine 3-space. Describe it.

Four Solutions

Answer 1: Our equations $x^2 = y^2 = xz - yz^2 = 0$ define the *z-axis*:

$$x \;=\; y \;=\; 0.$$

Answer 2: A sufficiently differentiable function ϕ satisfies

$$\frac{\partial^2\phi}{\partial x^2} \;=\; \frac{\partial^2\phi}{\partial y^2} \;=\; \frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^3\phi}{\partial y\partial z^2} \;=\; 0$$

if and only if it decomposes into four summands as follows:

$$\phi(x,y,z) \quad = \quad \xi(z) \;+\; \big(\, y\psi(z) + x\psi'(z) \,\big) \;+\; \alpha xy \;+\; \beta x.$$

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Answer 3: A polynomial f lies in $I = \langle x^2, y^2, x - yz \rangle \cap \langle x^2, y^2, z \rangle$ if and only if the following four conditions hold: Both f and $\frac{\partial f}{\partial y} + z \frac{\partial f}{\partial x}$ vanish on the z-axis, and both $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial f}{\partial x}$ vanish at the origin.

Answer 4: The scheme is a double z-axis with an embedded point of length two at the origin. The arithmetic multiplicity of I is four.

Prime Ideals

Let P be a prime ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and $V(P)$ its variety in $\mathbb{C}^n$.
A polynomial f is in the ideal P if and only if f vanishes on $V(P)$.

Setting $x_i = \partial_{z_i}$, view P as PDE for an unknown function $\phi(z_1, \ldots, z_n)$.

Remark

For $y \in \mathbb{C}^n$, the exponential function

$$z \ \mapsto \ \exp(\mathrm{y}^t \mathrm{z}) \ = \ \exp(y_1 z_1 + \cdots + y_n z_n)$$

satisfies the PDE given by P if and only if $\mathrm{y} \in V(P)$.

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Proposition

Each solution to P admits an integral representation

$$\phi(\mathrm{z}) \quad = \quad \int_{V(P)} \exp\left(\mathrm{y}^t\, \mathrm{z}\right) d\mu(\mathrm{y}),$$

where μ is a measure on the irreducible variety $V(P)$.

Primary Ideals

$$m \;=\; \mathrm{length}\big(R_P/QR_P\big) \;=\; \frac{\texttt{degree}(Q)}{\texttt{degree}(P)}.$$

Fix a prime P of codimension c in $R = \mathbb{C}[x_1, \ldots, x_n]$, in Noether position. Write $\mathbb{F} = \mathbb{C}(u_1, \ldots, u_n)$ for the field of fractions of R/P.

Theorem

The following sets are in bijective correspondences:

(a) *P-primary ideals Q in R of multiplicity m,*

(b) *points in the punctual Hilbert scheme $\mathrm{Hilb}^m(\mathbb{F}[[y_1, \ldots, y_c]])$,*

(c) *m-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}[z_1, \ldots, z_c]$ that are closed under differentiation,* *Inverse systems*

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(c) *m-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}[z_1, \ldots, z_c]$ that are closed under differentiation,* *Inverse systems*

(d) *m-dimensional $\mathbb{F}$-subspaces of the Weyl-Noether module $\mathbb{F} \otimes_R D_{n,c}$ that are R-bi-modules, where $D_{n,c} = R\langle \partial_{x_1}, \ldots, \partial_{x_c} \rangle$.*

Any basis of the $\mathbb{F}$-subspace in (d) lifts to Noetherian operators $A_1, \ldots, A_m \in D_{n,c}$. These characterize **ideal membership** *in Q.*

Ehrenpreis-Palamodov

Each A_l in $D_{n,c}$ is written uniquely as $\sum_{\alpha,\beta} c_{\alpha,\beta} \mathrm{x}^\alpha \partial_{\mathrm{x}}^\beta$.
Replace ∂_{x} by z to get polynomials

$$B_l(\mathrm{x}, \mathrm{z}) \;:=\; A_l(\mathrm{x}, \partial_{\mathrm{x}})|_{\partial_{x_1} \mapsto z_1, \ldots, \partial_{x_c} \mapsto z_c} \quad \text{for} \quad l = 1, \ldots, m.$$

The *Noetherian multipliers* $B_1, \ldots, B_m$ span the inverse system (c).

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Theorem (Ehrenpreis-Palamodov Fundamental Principle)

Consider the PDE given by a P-primary ideal Q.

Any *sufficiently nice* *solution ψ has an integral representation*

$$\psi(\mathrm{z}) \ = \ \sum_{l=1}^{m} \int_{V(P)} B_l\,(\mathrm{x},\mathrm{z}) \exp\left(\mathrm{x}^t\,\mathrm{z}\right)\, d\mu_l(\mathrm{x})$$

for suitable measures μ_l supported in the variety $V(P)$.

Conversely, all such functions are solutions.

From (a) to (d)

Algorithm (From ideal generators to Noetherian operators)

Input: *Generators of a P-primary ideal Q in $R = \mathbb{C}[x_1, \ldots, x_n]$.*

Output: *Operators $A_1, \ldots, A_m$ in the relative Weyl algebra $D_{n,c}$ with* $Q = \{ f \in \mathbb{C}[x_1, \ldots, x_n] : A_i \bullet f \in P \text{ for all } i \}$.

$$\text{Set} \quad \gamma : R \hookrightarrow \mathbb{F}[y_1, \ldots, y_c]\,, \quad \begin{array}{ccc} x_i & \mapsto & y_i + u_i \\ x_j & \mapsto & u_j \end{array} \qquad \begin{array}{l} \text{for } 1 \le i \le c, \\ \text{for } c+1 \le j \le n. \end{array}$$

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Set $\gamma : R \hookrightarrow \mathbb{F}[y_1, \ldots, y_c]$, $\quad x_i \mapsto y_i + u_i \quad$ for $1 \le i \le c$,
$\quad x_j \mapsto u_j \quad$ for $c+1 \le j \le n$.

1. *Find generators of the 0-dim'l ideal $I = \langle y_1, \ldots, y_c \rangle^m + \gamma(Q)$.*
2. *Using linear algebra over $\mathbb{F} = \mathbb{C}(u_1, \ldots, u_n)$, compute a basis $\{B_1, \ldots, B_m\}$ for the inverse system $I^\perp$ in $\mathbb{F}[z_1, \ldots, z_c]$.*
3. *Lift $B_i(\mathrm{u}, \mathrm{z})$ to obtain the Noetherian multipliers $B_i(\mathrm{x}, \mathrm{z})$.*
4. *Replace z by ∂_{x} to get the Noetherian operators $A_i(\mathrm{x}, \partial_{\mathrm{x}})$.*

Available in `Macaulay2`, as part of J. Chen, Y. Cid-Ruiz, M. Härkönen, R. Krone, A. Leykin: *Noetherian operators in Macaulay2*, January 2021.

Operators versus Multipliers

Input: Primary ideal $Q = \langle x_1^2, x_2^2, x_1 - x_2 x_3 \rangle$.

Here $n = 3$, $c = m = 2$ and $P = \langle x_1, x_2 \rangle$.

Output in Step 4: The Noetherian operators

$$A_1(x, \partial_x) = 1 \quad \text{and} \quad A_2(x, \partial_x) = x_3 \partial_{x_1} + \partial_{x_2}.$$

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Ehrenpreis-Palamodov: Solutions to $\phi_{z_1z_1} = \phi_{z_2z_2} = \phi_{z_1} - \phi_{z_2z_3} = 0$:

$$\phi_1(z) \;=\; \int 1 \cdot \exp(0z_1 + 0z_2 + x_3z_3)\, d\mu_x \;=\; \xi(z_3) \qquad \text{and}$$

$$\begin{aligned} \phi_2(z) &= \int (z_2 + z_1x_3) \cdot \exp(0z_1 + 0z_2 + x_3z_3)\, d\mu_x \\ &= z_2 \int exp(0z_1 + 0z_2 + x_3z_3) d\mu_x + z_1 \int x_3\, exp(0z_1 + 0z_2 + x_3z_3) d\mu_x \\ &= z_2\, \psi(z_3) + z_1\, \psi'(z_3). \end{aligned}$$

Gröbner's Dream

Consider **any** ideal $I \subset R$ with associated primes $P_1, \ldots, P_k$. Its *arithmetic multiplicity* is $\mathrm{amult}(I) = \sum_{j=1}^{k} \mathrm{mult}_I(P_j)$, where

$$\mathrm{mult}_I(P) = \frac{\texttt{degree(saturate(I,P)/I)}}{\texttt{degree(P))}}$$

is the length of the largest ideal of finite length in R_P/IR_P.

A *differential primary decomposition* of I is a list $(P_1, \mathcal{A}_1), \ldots, (P_k, \mathcal{A}_k)$ where $\mathcal{A}_i$ is a finite subset of $D_{n,n}$ with

$$I = \{ f \in R \mid \delta \bullet f \in P_i \text{ for all } \delta \in \mathcal{A}_i \text{ and } i = 1, \ldots, k \}.$$

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$$I = \left\{ f \in R \mid \delta \bullet f \in P_i \text{ for all } \delta \in \mathcal{A}_i \text{ and } i = 1, \ldots, k \right\}.$$

Theorem

The size of a differential primary decomposition is at least $\mathrm{amult}(I)$, *and this lower bound is tight. More precisely:*

(i) *The ideal I has a differential primary decomposition $(P_1, \mathcal{A}_1), \ldots, (P_k, \mathcal{A}_k)$ such that $|\mathcal{A}_i| = \mathrm{mult}_I(P_i)$.*

(ii) *If $(P_1, \mathcal{A}_1), \ldots, (P_k, \mathcal{A}_k)$ is any differential primary decomposition for I, then $|\mathcal{A}_i| \geq \mathrm{mult}_I(P_i)$.*

Macaulay 2

Computing a minimal differential primary decomposition:

```
i1 : load "modulesNoetherianOperators.m2"

i2 : R = QQ[x,y,z]

i3 : I = ideal(x^2,y^2,x*z-y*z^2);

i4 : amult(I)

o4 = 4

i5 : netList solvePDE(I)

     +---------------+-------------------+
o5 = |ideal (y, x)   |{| 1 |, | dxz+dy |}|
     +---------------+-------------------+
     |ideal (z, y, x)|{| dx |, | dxdy |} |
     +---------------+-------------------+
```

This is Answer 2 & 3 for our double line:

$P_1 = \langle x, y \rangle\,, \;\; \mathcal{A}_1 = \{1, z\partial_x + \partial_y\}$

$P_2 = \langle x, y, z \rangle\,, \; \mathcal{A}_2 = \{\partial_x,\, \partial_x\partial_y\}$

Modules

The treatment of Ehrenpreis-Palamodov in books on analysis emphasizes PDE for vector-valued functions $\psi : \mathbb{C}^n \to \mathbb{C}^k$.

[J.-E. Björk: Rings of Differential Operators], [L. Hörmander: An Introduction to Complex Analysis in Several Variables]

In calculus we learn how to rewrite one higher-order ODE as a system of first order ODE, and in algebraic geometry we learn how to appreciate matrix representations of geometric objects:

Ideals	$\longrightarrow$	Schemes
Modules	$\longrightarrow$	Coherent Sheaves

A system of ℓ linear PDE for ψ is represented by a $k \times \ell$ matrix with entries in $R = \mathbb{C}[x_1, \ldots, x_n]$. The image of this matrix is a submodule M of R^k. Primary decomposition makes sense here:

$$M \;=\; M_1 \cap \cdots \cap M_k.$$

... and so does differential primary decomposition

Coherent Sheaves

Let $M \subset R^2$ be the module spanned by the columns of

$$\begin{bmatrix} \partial_1\partial_3 & \partial_1\partial_2 & \partial_1^2\partial_2 \\ \partial_1^2 & \partial_2^2 & \partial_1^2\partial_4 \end{bmatrix}.$$

This represents PDE for functions $\psi : \mathbb{C}^4 \to \mathbb{C}^2$. We seek $\psi(z) = \big(\psi_1(z_1, z_2, z_3, z_4), \psi_2(z_1, z_2, z_3, z_4)\big)$ such that

$$\frac{\partial^2\psi_1}{\partial z_1 \partial z_3} + \frac{\partial^2\psi_2}{\partial z_1^2} = \frac{\partial^2\psi_1}{\partial z_1 \partial z_2} + \frac{\partial^2\psi_2}{\partial z_2^2} = \frac{\partial^3\psi_1}{\partial z_1^2 \partial z_2} + \frac{\partial^3\psi_2}{\partial z_1^2 \partial z_4} = 0.$$

The module M has six associated primes, namely $P_1 = \langle \partial_1 \rangle$, $P_2 = \langle \partial_2, \partial_4 \rangle$, $P_3 = \langle \partial_2, \partial_3 \rangle$, $P_4 = \langle \partial_1, \partial_3 \rangle$, $P_5 = \langle \partial_1, \partial_2 \rangle$, $P_6 = \langle \partial_1^2 - \partial_2\partial_3, \partial_1\partial_2 - \partial_3\partial_4, \partial_2^2 - \partial_1\partial_4 \rangle$. Primes P_4, P_5 are embedded. Arithmetic multiplicity: $1{+}1{+}1{+}1{+}4{+}1 = 9 = \mathrm{amult}(M)$.

To solve the PDE, we compute a differential primary decomposition.

Macaulay 2

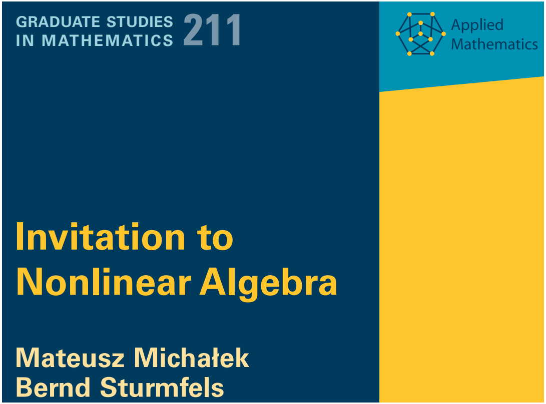
$$\frac{\partial^2\psi_1}{\partial z_1\partial z_3}+\frac{\partial^2\psi_2}{\partial z_1^2}=\frac{\partial^2\psi_1}{\partial z_1\partial z_2}+\frac{\partial^2\psi_2}{\partial z_2^2}=\frac{\partial^3\psi_1}{\partial z_1^2\partial z_2}+\frac{\partial^3\psi_2}{\partial z_1^2\partial z_4}=0.$$

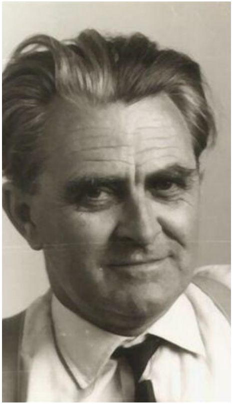
```
i1 : load "modulesNoetherianOperators.m2"

i2 : R = QQ[x1,x2,x3,x4]

i3 : M = image matrix{
         {x1*x3, x1*x2, x1^2*x2 },
         { x1^2, x2^2, x1^2*x4} };

i4 : amult(M)

o4 = 9

i5 : S =  solvePDE(M)

o5 = {ideal x1, {| 1 |}}
                 | 0 |

               2                            2
     {ideal (x2  - x1*x4, x1*x2 - x3*x4, x1  - x2*x3), {| -x4 |}}
                                                        | x2  |

     {ideal (x4, x2), {| -x1 |}}
                       | x3  |

     {ideal (x2, x1), {| 0 |, | 0   |, | 0   |, | 0       |}}
                       | 1 |  | dx1 |  | dx2 |  | dx1dx2 |

     {ideal (x3, x2), {| 1 |}}
                       | 0 |

     {ideal (x3, x1), {| -dx1x2 |}}
                       | 1      |
```

Solutions (ψ_1,ψ_2)?

For Students

Wolfgang Gröbner





Theorem 3.27. *Let I be a zero-dimensional ideal in $\mathbb{C}[x_1, \ldots, x_n]$, here interpreted as a system of linear PDEs. The space of holomorphic solutions has dimension equal to the degree of I. There exist nonzero polynomial solutions if and only if the maximal ideal $M = \langle x_1, \ldots, x_n \rangle$ is an associated prime of I. In that case, the polynomial solutions are precisely the solutions to the system of PDEs given by the M-primary component $(I : (I : M^\infty))$.*

Calculus Homework

Given three distinct integers $a, b, c > 0$, describe the space of all functions $\phi = \phi(x, y, z)$ that satisfy the three PDE

$$\frac{\partial^a \phi}{\partial x^a} + \frac{\partial^a \phi}{\partial y^a} + \frac{\partial^a \phi}{\partial z^a} = \frac{\partial^b \phi}{\partial x^b} + \frac{\partial^b \phi}{\partial y^b} + \frac{\partial^b \phi}{\partial z^b} = \frac{\partial^c \phi}{\partial x^c} + \frac{\partial^c \phi}{\partial y^c} + \frac{\partial^c \phi}{\partial z^c} = 0.$$

For $(a, b, c) = (1, 2, 3)$ get $\phi = (x-y)(x-z)(y-z)$ and its derivatives.

To gain insight, start with $(a, b, c) = (2, 5, 8)$.

Due Date: **Tomorrow**

Submit your solution to: `bernd@mis.mpg.de`

No late homework, please

Many thanks for your attention!