Canonical Differential Equations for Maximal Cuts of *Hyperelliptic* Feynman Integrals

UNIVERSITÄT BONN

Funded by the European Union

European Research Council
Established by the European Commission

bctp Bethe Center for Theoretical Physics

Franziska Porkert

with Claude Duhr, Cathrin Semper & Sven Stawinski

work in progress (arXiv 24XX.XXXX)

arXiv: 2408.04904

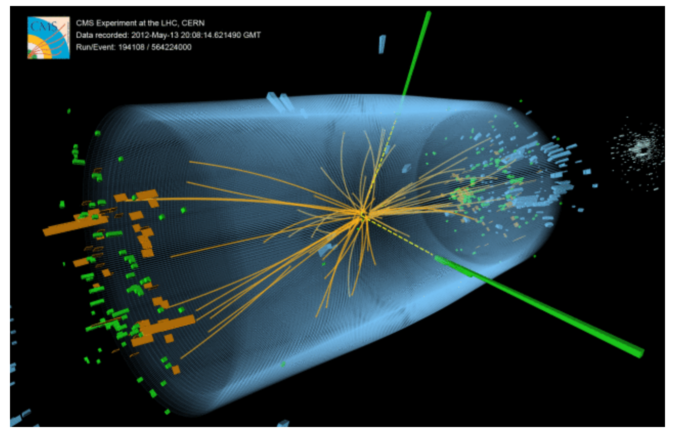
arXiv: 2407.17175

Holonomic Techniques for Feynman Integrals, 15.10.2024

MOTIVATION

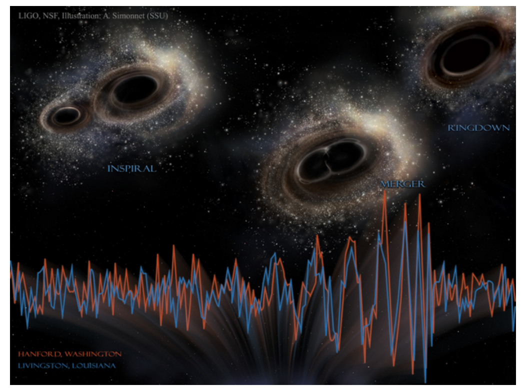
The (canonical) differential equation

The function space

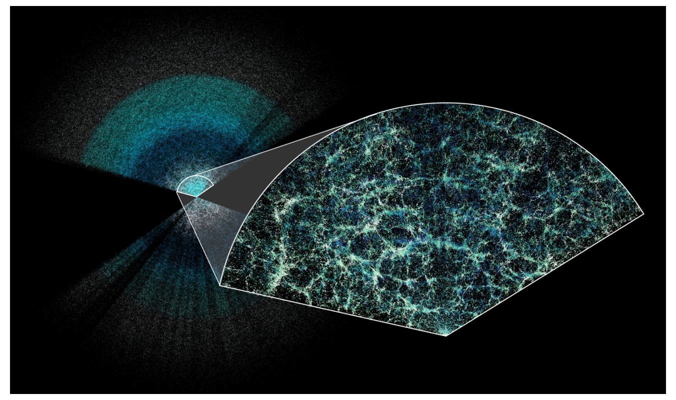


CERN

FEYNMAN INTEGRALS



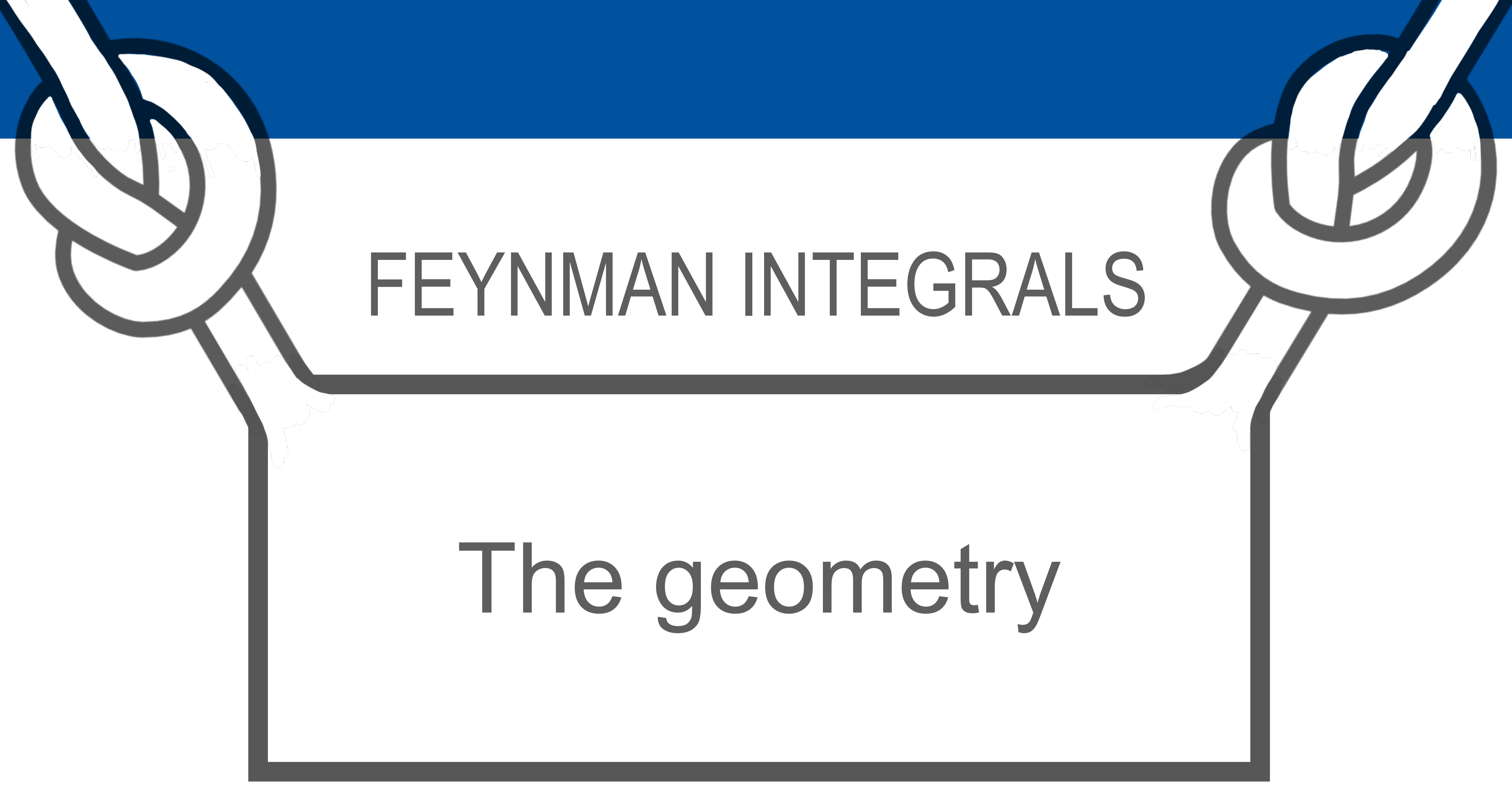
Aurore Simones (Sonoma State U.)



Claire Lamman/DESI

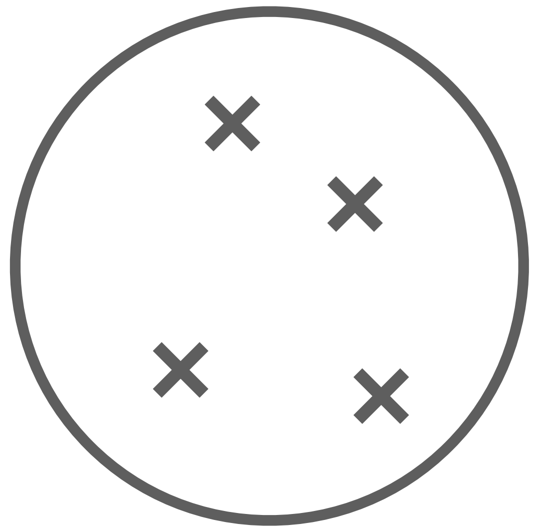
The geometry

MOTIVATION

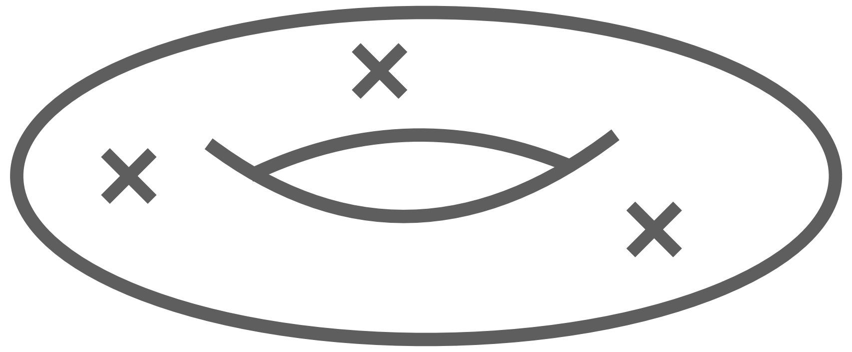


FEYNMAN INTEGRALS

The geometry



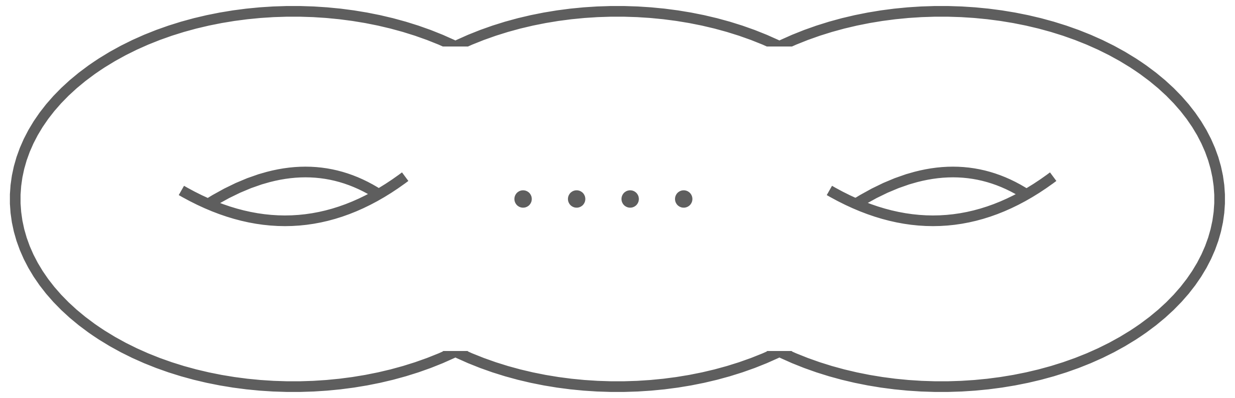
Riemann sphere



Torus

Calabi-Yau

Feynman integrals related to these geometries are well studied.
Still many open questions/problems!



Riemann surface of genus $g > 1$

THIS TALK!

In String Theory: See Carlos' & Konstantin's posters!

CONTENT

◆ Motivation

I. Feynman Integrals

- Geometry
- Canonical differential equations
- Maximal cuts

II. Maximal Cuts of (Hyper-)Elliptic Feynman Integrals — a genus two example

◆ Summary + Outlook

FEYNMAN

INTEGRALS

FEYNMAN INTEGRALS AND *THEIR* GEOMETRIES

How do we associate one (or multiple) geometries to a Feynman integral (family)?

$$I_{\boldsymbol{\nu}} \sim \int \frac{\prod_{j=1}^{L} \mathrm{d}\ell_j^D}{\prod_{i=1}^{N} D_i^{\nu_i}}$$

$D = d - 2\varepsilon$

Symanzik polynomials

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_k \alpha_k^{\nu_k - 1} \mathrm{d}\alpha_k \right) \frac{\mathcal{U}(\boldsymbol{\alpha})^{\nu-(l+1)D/2}}{\mathcal{F}(\boldsymbol{\alpha})^{\nu - lD/2}}$$

Find variety

Maximal cut $(I_{\boldsymbol{\nu}}) \sim \int \frac{\prod_{j=1}^{L} \mathrm{d}\ell_j^D}{\prod_{i=1}^{N} D_i^{\nu_i}} \bigg| \frac{1}{D_i} \to \delta(D_i)$

Picard Fuchs operator

Polynomial equation
(via Baikov)

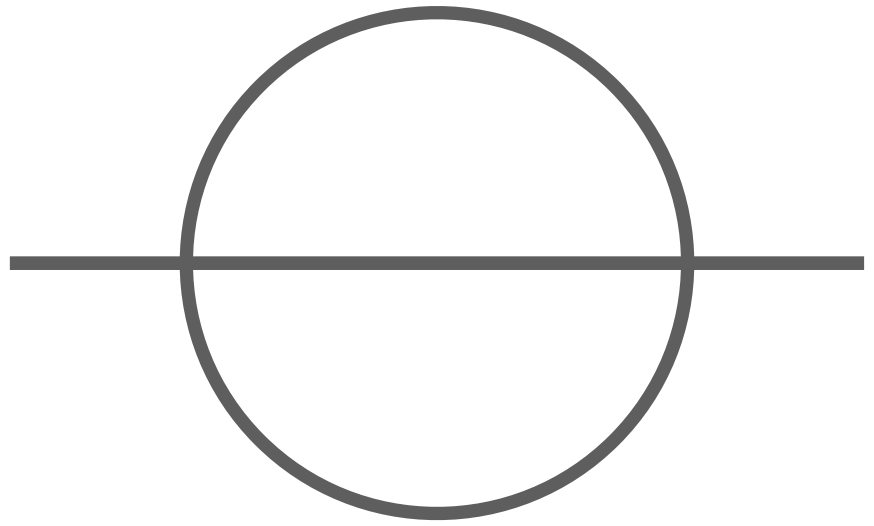
Not necessarily unique!

[Marzucca, McLeod, Page, Pögel, Weinzierl | Jockers, Kotlewski, Kuusela, McLeod, Pögel, Sarve, Wang, Weinzierl]

EXAMPLES FOR (HYPER-)ELLIPTIC FEYNMAN INTEGRALS

ELLIPTIC EXAMPLE: SUNRISE

$D = 2 - 2\varepsilon$



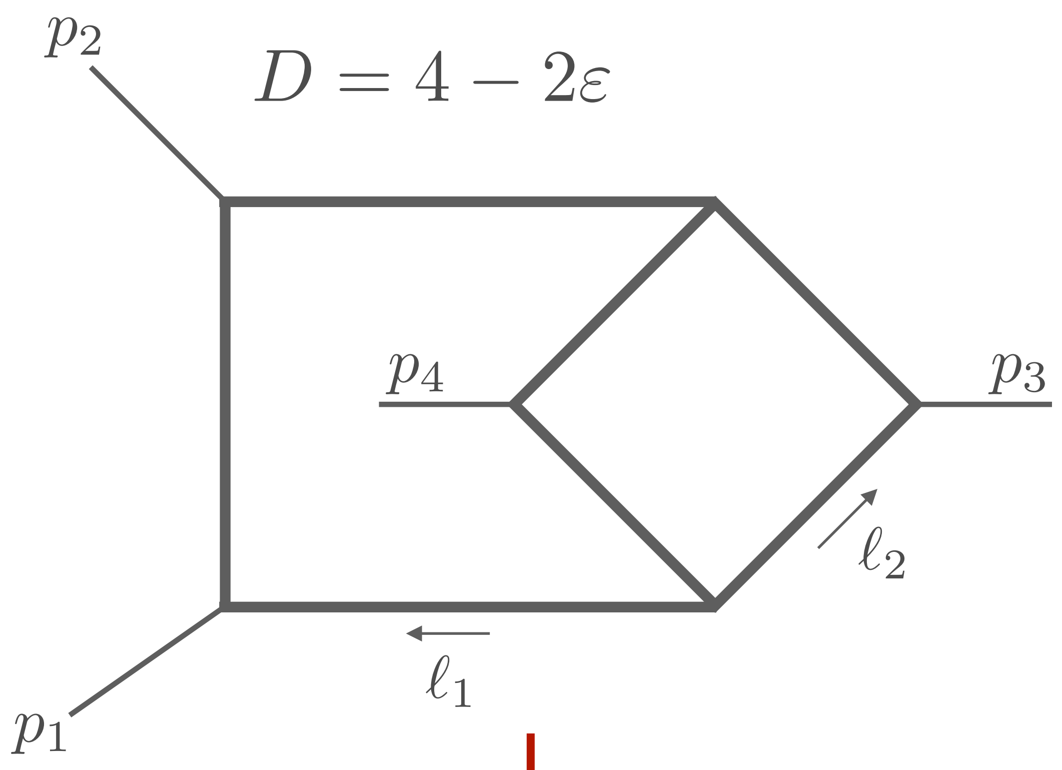
maximal cut
loop - by - loop

$$\int_\Gamma \mathrm{d}x\, x^\varepsilon \left[\boxed{(x-\mu_1)(x-\mu_2)(x-\mu_3)(x-\mu_4)}\right]^{-\frac{1}{2}-\varepsilon}$$

even **elliptic** curve of **genus 1**:

$$y^2 = \boxed{(x-\mu_1)(x-\mu_2)(x-\mu_3)(x-\mu_4)}$$

HYPERELLIPTIC EXAMPLE: NON-PLANAR CROSSED BOX



maximal cut
loop - by - loop

$$\int_\Gamma \mathrm{d}x \boxed{\left[(x-\lambda_1)(x-\lambda_2)\right]^{-\frac{1}{2}}\left[(x-\lambda_3)(x-\lambda_4)(x-\lambda_5)(x-\lambda_6)\right]}^{-\frac{1}{2}-\varepsilon}$$

[Huang, Zhang | Georgoudis, Zhang | Marzucca, McLeod, Page, Pögel, Weinzierl]

even **hyperelliptic** curve of **genus 2**:

$$y^2 = \boxed{(x-\lambda_1)\,(x-\lambda_2)\,(x-\lambda_3)\,(x-\lambda_4)\,(x-\lambda_5)\,(x-\lambda_6)}$$

FEYNMAN INTEGRALS FROM DIFFERENTIAL EQUATIONS

We want to compute a **Feynman integral family** analytically with ***differential equations***.

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{D_i^{\nu_i}}$$

- Use IBPs to find a **basis of master integrals** for the integral family

  See talks by G. Fontana, T.Huber.

- Set up a **differential equation** w.r.t the external (kinematic) parameters

  $\mathrm{d}\mathbf{I}(\mathbf{X}) = A(\mathbf{X}, \varepsilon)\mathbf{I}(\mathbf{X})$ with $\mathrm{d} = \sum \mathrm{d}X_i \partial_{X_i}$ where X_i are kinematic variables

- Find a **canonical differential equation** & solve in terms of **iterated integrals.**

  [Henn]

$$\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X}) \text{ with } \mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X})$$

$$\text{and } \varepsilon B(\boldsymbol{X}) = (\mathrm{d}\boldsymbol{U}) \cdot \boldsymbol{U}^{-1} + \boldsymbol{U} \cdot \boldsymbol{A}(\boldsymbol{X}, \varepsilon) \cdot \boldsymbol{U}^{-1}$$

$$\mathbf{J}(\mathbf{X}) = \mathbb{P}\mathrm{exp}\left(\varepsilon \int_\gamma B \right) \cdot \mathbf{J}\left(\text{some point } \mathbf{X}^0\right) = \left(1 + \varepsilon \int_\gamma B + \varepsilon^2 \int_\gamma B \int_\gamma B + \ldots \right) \cdot \mathbf{J}(\mathbf{X}^0)$$

FEYNMAN INTEGRALS FROM DIFFERENTIAL EQUATIONS

We want to compute a **Feynman integral family** analytically with ***differential equations***.

hyperelliptic

- Find a **canonical differential equation** & solve in terms of **iterated integrals.**
  [Henn]

$\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X})$ ⟶ **How do we find this (systematically)? This talk!**

$$\mathbf{J}(\mathbf{X}) = \left(1 + \varepsilon \int_\gamma B + \varepsilon^2 \int_\gamma B \int_\gamma B + \dots \right) \cdot \mathbf{J}(\mathbf{X}^0)$$ ⟶ **What are these? Ongoing work!**

See also Konstantin's poster!

- **Genus 0:** $\mathcal{E}$- form + dLog forms
- **Genus 1:** $\mathcal{E}$- form + simple poles (+ e.g. quasi modular forms)
- **Genus > 1:** $\mathcal{E}$ - form + simple poles + **???** ← generalization of Siegel modular forms (?)

FEYNMAN INTEGRALS IN C-FORM

Consider a (Feynman integral -) differential equation of the form

$$\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X}) \quad \text{with} \quad B(\boldsymbol{X})_{ij} = \sum_{k=1}^{n} \mathrm{d}X_k f_{ijk}$$

Define:

$\mathcal{A}$ $=$ $\mathbb{K}$ - algebra of functions that contains all f_{ijk} and:

- Differentially closed ($f \in \mathcal{A} \Rightarrow \partial_{X_i} f \in \mathcal{A} \, \forall i$)
- Constants = $\mathbb{K}$ ($\partial_{X_i} f = 0 \, \forall i \Rightarrow f \in \mathbb{K}$)

$\mathbb{A}$ $=$ $\mathbb{K}$ - vector space of closed differential forms generated by the forms appearing in $B(\boldsymbol{X})$

$\mathcal{F}_{\mathbb{C}}$ $=$ $\mathrm{Frac}\,(\mathbb{C} \otimes_{\mathbb{K}} \mathcal{A})$

An ε- factorised differential equation is in C-form, if $\mathbb{A} \cap \mathrm{d}\mathcal{F}_{\mathbb{C}} = \{0\}$.

[Duhr, Semper, Stawiński, FP]

All known (to us) canonical DEQS for Feynman integrals are also in C-fom!

EXAMPLE:

$B(\boldsymbol{X})$ in dLog-form, i.e.:

$$f_{ij} = \sum_r \frac{1}{a_{ijr} - X}$$

- $\mathcal{A}_{\mathrm{dLog}} =$
  Rational functions in X with singularities at the a_{ijr}
- $\mathbb{A}_{\mathrm{dLog}} = \left\langle \frac{\mathrm{d}X}{a_{ijr} - X} \,\middle|\, \text{all } i, j, r \right\rangle$

Elements of $\mathrm{d}\mathcal{F}_{\mathbb{C}}$:

- no pole/ pole of order > 1

or

- 0

$$\Rightarrow \mathbb{A}_{\mathrm{dLog}} \cap \mathrm{d}\mathcal{F}_{\mathbb{C}} = \{0\}$$

MAXIMAL CUTS OF FEYNMAN INTEGRALS

How to understand maximal cuts for the rest of this talk:

The **period matrix** of a twisted cohomology group defined by the Feynman integrand after taking residues.

The **fundamental solution** for the homogenous differential equation of the top sector.

$$\int \frac{\prod_{j=1}^{L} \mathrm{d}\ell_j^D}{\prod_{i=1}^{N} D_i^{\nu_i}}\Bigg|\frac{1}{D_i} \to \delta(D_i)$$

$$\sim \int_\Gamma \mathrm{d}x\, \underbrace{x^\varepsilon \left[(x-\mu_1)(x-\mu_2)(x-\mu_3)(x-\mu_4)\right]^{-\frac{1}{2}-\varepsilon}}_{}$$

Φ = twist (generally ε-dependent)

$$\boldsymbol{P} = \left(\int_{\gamma_i} \Phi\, \varpi_j\right)_{ij}$$

Bases of twisted (co-)homology groups

$$\mathrm{d}\begin{pmatrix} I_{110} \\ I_{101} \\ I_{011} \\ I_{111} \\ I_{211} \\ I_{121} \\ I_{112} \end{pmatrix} = \begin{bmatrix} \bullet & - & - & - & - & - & - \\ - & \bullet & - & - & - & - & - \\ - & - & \bullet & - & - & - & - \\ - & - & - & \bullet & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{pmatrix} I_{110} \\ I_{101} \\ I_{011} \\ I_{111} \\ I_{211} \\ I_{121} \\ I_{112} \end{pmatrix}$$

$$\mathrm{d}\boldsymbol{P} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \boldsymbol{P}$$

MAXIMAL CUTS OF FEYNMAN INTEGRALS

The maximal cut:

The **period matrix** of a twisted cohomology group defined by the Feynman integrand after taking residues.

$$\boldsymbol{P} = \left(\int_{\underset{\downarrow}{\gamma_i}} \Phi \, \underset{\downarrow}{\varpi_j} \right)_{ij}$$

Bases of twisted (co-)homology groups

In this framework: We also have a dual twisted cohomology and homology group with bases $\{\check{\varpi}_i\}, \{\check{\gamma}_i\}$ and:

$$\check{\boldsymbol{P}} = \left(\int_{\check{\gamma}_i} \Phi^{-1} \check{\varpi}_j \right)_{ij}$$

dual period matrix

$$\boldsymbol{C} = \frac{1}{(2\pi i)^n} \left(\int \varpi_i \wedge \check{\varpi}_j \right)_{ij}$$

cohomology intersection matrix

$$\boldsymbol{H} = \left(\text{weighted topological intersections of } \gamma_i, \check{\gamma}_j \right)_{ij}$$

homology intersection matrix

For maximal cuts, we can choose the bases such that $\check{\boldsymbol{P}}(\varepsilon) = \boldsymbol{P}(-\varepsilon)$ and we obtain bilinear relations between the maximal cut entries (from twisted Riemann bilinear). Connected to self-duality; See talk by S.Weinzierl!

[Duhr, Semper, Stawiński, FP]

MAXIMAL CUTS IN C-FORM

The maximal cut:

The **period matrix** of a twisted cohomology group defined by the Feynman integrand after taking residues.

$$\boldsymbol{P} = \left(\int_{\gamma_i} \Phi \, \varpi_j \right)_{ij} \qquad \text{with} \qquad \underbrace{\mathrm{d}\boldsymbol{P} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \boldsymbol{P}}_{\text{Goal: } \varepsilon\text{-form and C-form}}$$

Bases of twisted (co-)homology groups

$\Rightarrow$ Goal: Good basis of master integrals $\Leftrightarrow$ Good basis of the twisted cohomology group

MAXIMAL CUTS IN C-FORM

The maximal cut:

The **period matrix** of a twisted cohomology group defined by the Feynman integrand after taking residues.

$$\boldsymbol{P} = \left(\int_{\gamma_i} \Phi \, \varpi_j \right)_{ij} \qquad \text{with} \qquad \underbrace{\mathrm{d}\boldsymbol{P} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \boldsymbol{P}}_{\text{Goal: } \mathcal{E}\text{-form and C-form}}$$

Bases of twisted (co-)homology groups

⇒ Goal: Good basis of master integrals ⇔ Good basis of the twisted cohomology group

Slogan:

Basis and dual basis are in $\mathcal{E}$**-form** and **C-form** (with $\check{\boldsymbol{P}}(\varepsilon) = \boldsymbol{P}(-\varepsilon)$) ⇒ The intersection matrix is **constant** in the external variables, $\mathrm{d}\boldsymbol{C} = 0$.

[Duhr, Semper, Stawiński, FP]

THREE KEY-POINTS TO REMEMBER

Maximal cut: Period matrix of twisted cohomology and fundamental solution of homogenous DEQ

Good basis and dual basis of differentials $\Longrightarrow$ The intersection matrix is constant.

Maximal cut also defines a geometry and the *good* basis is connected to this geometry.

MAXIMAL CUTS OF (HYPER-)ELLIPTIC FEYNMAN INTEGRALS

FINDING CANONICAL DEQs

$\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X}) \longrightarrow$ **How do we find this (systematically)?**

Different methods for finding canonical DEQ of Feynman integrals with elliptic curve or CY geometry.
[Brösel, Duhr, Dulat, Penante, Tancredi | Pögel, Wang, Weinzierl | Görges, Nega, Tancredi, Wagner]

Short review of the algorithm by [Görges, Nega, Tancredi, Wagner] (applied to maximal cut):

FINDING CANONICAL DEQs

$\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X}) \longrightarrow$ **How do we find this (systematically)?**

Different methods for finding canonical DEQ of Feynman integrals with elliptic curve or CY geometry.

[Brösel, Duhr, Dulat, Penante, Tancredi | Pögel, Wang, Weinzierl | Görges, Nega, Tancredi, Wagner]

Short review of the algorithm by [Görges, Nega, Tancredi, Wagner] (applied to maximal cut):

1. Make a good choice for the **starting basis**
   (Inspired by simple basis of Abelian differentials; derivative basis)

FINDING CANONICAL DEQs

$\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X})$ ⟶ **How do we find this (systematically)?**

Different methods for finding canonical DEQ of Feynman integrals with elliptic curve or CY geometry.

[Brösel, Duhr, Dulat, Penante, Tancredi | Pögel, Wang, Weinzierl | Görges, Nega, Tancredi, Wagner]

Short review of the algorithm by [Görges, Nega, Tancredi, Wagner] (applied to maximal cut):

1. Make a good choice for the **starting basis**
   (Inspired by simple basis of Abelian differentials; derivative basis)

2. Compute the period matrix at $\varepsilon = 0$ and split it in semi-simple and unipotent parts.
   Rotate the initial basis with the **inverse of the semi-simple part**.
   (*Geometry inspired* step)

FINDING CANONICAL DEQs

$\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X}) \longrightarrow$ **How do we find this (systematically)?**

Different methods for finding canonical DEQ of Feynman integrals with elliptic curve or CY geometry.

[Brösel, Duhr, Dulat, Penante, Tancredi | Pögel, Wang, Weinzierl | Görges, Nega, Tancredi, Wagner]

Short review of the algorithm by [Görges, Nega, Tancredi, Wagner] (applied to maximal cut):

1. Make a good choice for the **starting basis**
   (Inspired by simple basis of Abelian differentials; derivative basis)
2. Compute the period matrix at $\varepsilon = 0$ and split it in semi-simple and unipotent parts.
   Rotate the initial basis with the **inverse of the semi-simple part**.
   (*Geometry inspired* step)
3. Make further simple rotations (exchanges of basis elements + powers of ε)
   to make the remaining **non-canonical part lower-triangular**.
   (*Adjustment* step)

FINDING CANONICAL DEQs

$\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X}) \longrightarrow$ **How do we find this (systematically)?**

Different methods for finding canonical DEQ of Feynman integrals with elliptic curve or CY geometry.

[Brösel, Duhr, Dulat, Penante, Tancredi | Pögel, Wang, Weinzierl | Görges, Nega, Tancredi, Wagner]

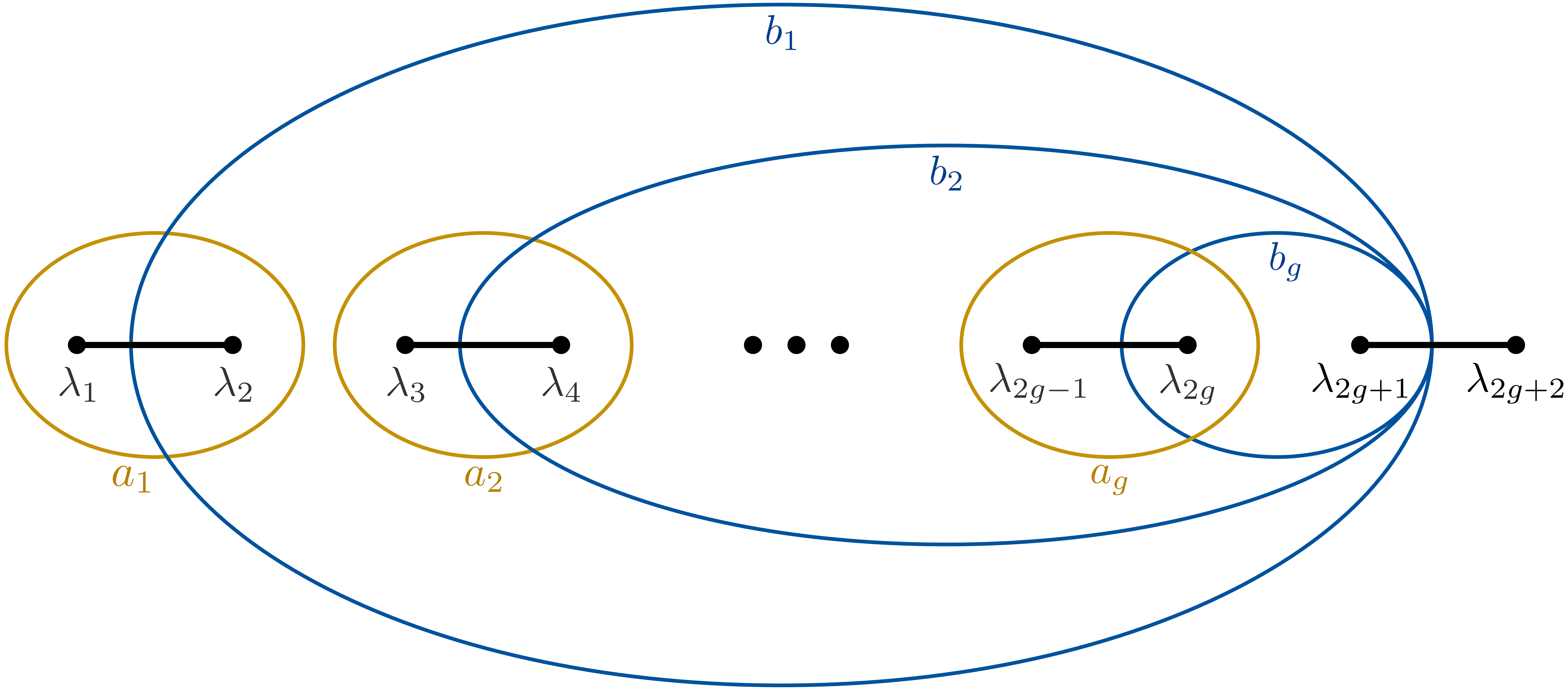
Short review of the algorithm by [Görges, Nega, Tancredi, Wagner] (applied to maximal cut):

1. Make a good choice for the **starting basis**
   (Inspired by simple basis of Abelian differentials; derivative basis)
2. Compute the period matrix at $\varepsilon = 0$ and split it in semi-simple and unipotent parts.
   Rotate the initial basis with the **inverse of the semi-simple part**.
   (*Geometry inspired* step)
3. Make further simple rotations (exchanges of basis elements + powers of ε)
   to make the remaining **non-canonical part lower-triangular**.
   (*Adjustment* step)
4. Make **ansatz** to remove these remaining **non-canonical** entries and
   solve the resulting differential equations.
   (*New objects* step)

CYCLES ON HYPERELLIPTIC CURVES

Even hyperelliptic curve: $y^2 = \prod_{i=1}^{2g+2} (x - \lambda_i)$.

We have the following a- and b-cycles:



DIFFERENTIALS ON HYPERELLIPTIC CURVES

ANY GENUS

ABELIAN DIFFERENTIALS

… OF THE FIRST KIND:

Holomorphic

$$\frac{\mathrm{d}x}{y} \dots \frac{x^{g-1}\mathrm{d}x}{y}$$

$$g = 1 \quad \frac{\mathrm{d}x}{y}$$

$$g = 2 \quad \frac{\mathrm{d}x}{y}, \frac{x\mathrm{d}x}{y}$$

… OF THE SECOND KIND:

Meromorphic
vanishing residue

$$\frac{\Phi_1(x)\mathrm{d}x}{y} \dots \frac{\Phi_g(x)\mathrm{d}x}{y}$$

$$\Phi_1(x) = x^{g+1} + \dots$$

$$\Phi_g(x) = x^{2g} + \dots$$

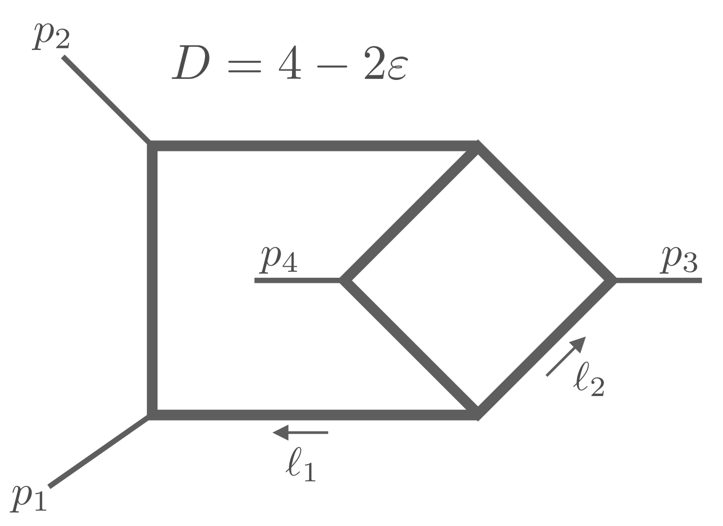
… OF THE THIRD KIND:

Meromorphic
non-zero residue

$$\frac{x^g\mathrm{d}x}{y}, \frac{\mathrm{d}x}{y(x-c)}, \frac{\mathrm{d}x}{x-c}$$

EXAMPLE: MAXIMAL CUT FOR THE NON-PLANAR CROSSED BOX

GENUS 2

MC $\Bigg($  $\Bigg)$

$$L(\boldsymbol{\lambda}, \boldsymbol{a}) = \int_{\lambda_1}^{\lambda_2} \mathrm{d}x \, (1 - \lambda_1^{-1} x)^{-\frac{1}{2} + a_1 \varepsilon} \ldots (1 - \lambda_6^{-1} x)^{-\frac{1}{2} + a_6 \varepsilon}$$

Twist $= \dfrac{\Phi}{y}$ with $\Phi = \prod_{i=1}^{6} (1 - \lambda_i^{-1} x)^{a_i \varepsilon}$ & $y = \prod_{i=1}^{6} \sqrt{(1 - \lambda_i^{-1} x)}$

BASIS OF DIFFERENTIALS:

$\varphi_1^{(0)} = \dfrac{\mathrm{d}x}{y} \Phi \,, \quad \varphi_2^{(0)} = \dfrac{x \mathrm{d}x}{y} \Phi$, $\varphi_3^{(0)} = \dfrac{\Phi_1(x) \mathrm{d}x}{y} \Phi \,, \quad \varphi_4^{(0)} = \dfrac{\Phi_2(x) \mathrm{d}x}{y} \Phi$ and $\varphi_5^{(0)} = \dfrac{x^2 \mathrm{d}x}{y} \Phi$

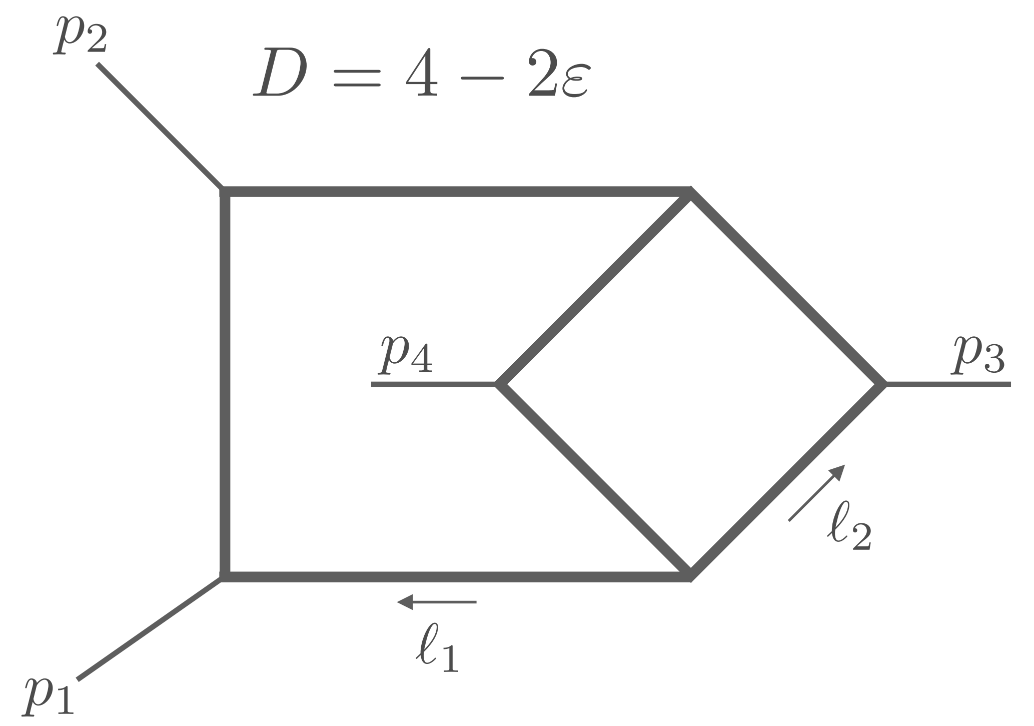
„first kind“ „second kind“ „third kind“

BASIS OF CYCLES:

$[\lambda_1, \lambda_2] \,, [\lambda_3, \lambda_4]$ $\quad$ $[\lambda_2, \lambda_3] + [\lambda_4, \lambda_5] \;, \; [\lambda_4, \lambda_5]$ $\quad$ $[\lambda_1, \lambda_2] + [\lambda_3, \lambda_4] + [\lambda_5, \lambda_6]$

$a_1 \,, a_2$ $\qquad$ $b_1 \,, b_2$

EXAMPLE: MAXIMAL CUT FOR THE NON-PLANAR CROSSED BOX



$$\mathrm{MC}\left(\quad \right)_{\varepsilon \to 0} = \mathcal{P}\Big|_{\epsilon\to 0} = \begin{array}{cccc} \text{a-cycles} & \text{b-cycles} & & \\ \mathcal{A} & \mathcal{B} & 0 & \text{first kind} \\ \tilde{\mathcal{A}} & \tilde{\mathcal{B}} & 0 & \text{second kind} \\ \star & \star & \star & \end{array}$$

$\downarrow$

„normalized“ period

$$\Omega = \mathcal{A}^{-1} \cdot \mathcal{B}$$

Genus 1:

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix}$$

$$\tau = \omega_2 \cdot \omega_1^{-1}$$

STEP 1: DERIVATIVE BASIS

$$\varphi_1^{(0)} = \frac{\mathrm{d}x}{y}\Phi\,,\ \varphi_2^{(0)} = \frac{x\mathrm{d}x}{y}\Phi\,,\quad \varphi_3^{(0)} = \frac{\Phi_1(x)\mathrm{d}x}{y}\Phi\,,\ \varphi_4^{(0)} = \frac{\Phi_2(x)\mathrm{d}x}{y}\Phi$$

and

$$\varphi_5^{(0)} = \frac{x^2\mathrm{d}x}{y}\Phi$$

$$\mathrm{d}\boldsymbol{\varphi}^{(0)} = \left[\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}\varepsilon\right]\boldsymbol{\varphi}^{(0)}$$

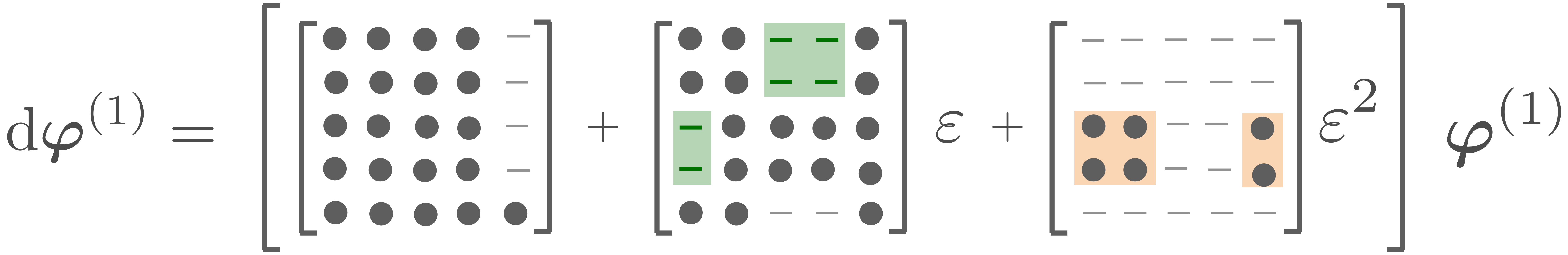
$$\downarrow$$

$$\boldsymbol{\varphi}^{(1)} = U_6^{(1)}\boldsymbol{\varphi}^{(0)}$$

$$\varphi_1^{(1)} = \varphi_1^{(0)},\quad \varphi_2^{(1)} = \varphi_2^{(0)}\qquad \varphi_3^{(1)} = \frac{\partial}{\partial\lambda_1}\varphi_1^{(0)}\quad,\quad \varphi_4^{(1)} = \frac{\partial}{\partial\lambda_2}\varphi_2^{(0)}$$

and

$$\varphi_5^{(1)} = \varphi_5^{(0)}$$

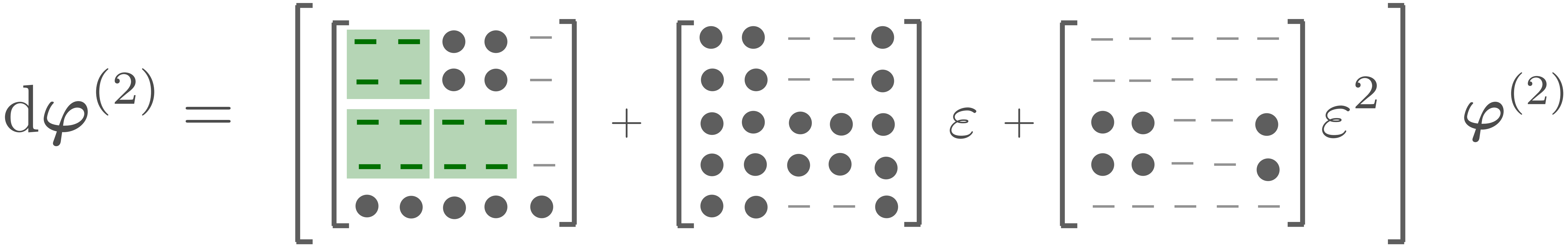


$$\mathrm{d}\boldsymbol{\varphi}^{(1)} = \left[\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & - & - & \bullet \\ \bullet & \bullet & - & - & \bullet \\ - & \bullet & \bullet & \bullet & \bullet \\ - & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & - & - & \bullet \end{bmatrix}\varepsilon + \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \bullet & \bullet & - & - & \bullet \\ \bullet & \bullet & - & - & \bullet \\ - & - & - & - & - \end{bmatrix}\varepsilon^2\right]\boldsymbol{\varphi}^{(1)}$$

STEP 2: SEMI-SIMPLE ROTATION

$$\lim_{\varepsilon \to 0} \mathcal{P}_{(1)} = \lim_{\varepsilon \to 0} U_6^{(1)} \mathcal{P}_{(0)} = \lim_{\varepsilon \to 0} U_6^{(1)} \begin{pmatrix} \mathcal{A} & \mathcal{B} & 0 \\ \tilde{\mathcal{A}} & \tilde{\mathcal{B}} & 0 \\ \star & \star & \star \end{pmatrix} = \underbrace{\lim_{\varepsilon \to 0} U_6^{(1)} \begin{pmatrix} \mathcal{A} & 0 & 0 \\ \tilde{\mathcal{A}} & 2\pi i \cdot \mathbf{1} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \begin{pmatrix} \mathbf{1} & \Omega & 0 \\ \mathbf{0} & \mathbf{1} & 0 \\ \star & \star & \star \end{pmatrix}$$

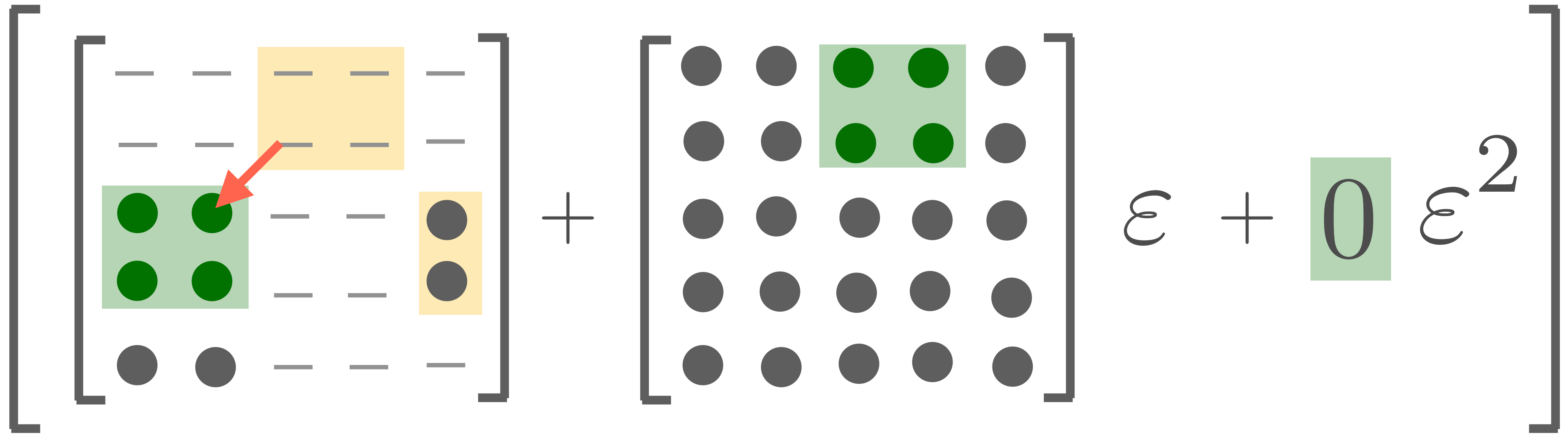
$$\varphi^{(2)} = \boldsymbol{S}^{-1} \varphi^{(1)}$$



EXAMPLE STEP 3: *ADJUSTEMENTS*

Remove ε^2 - terms:

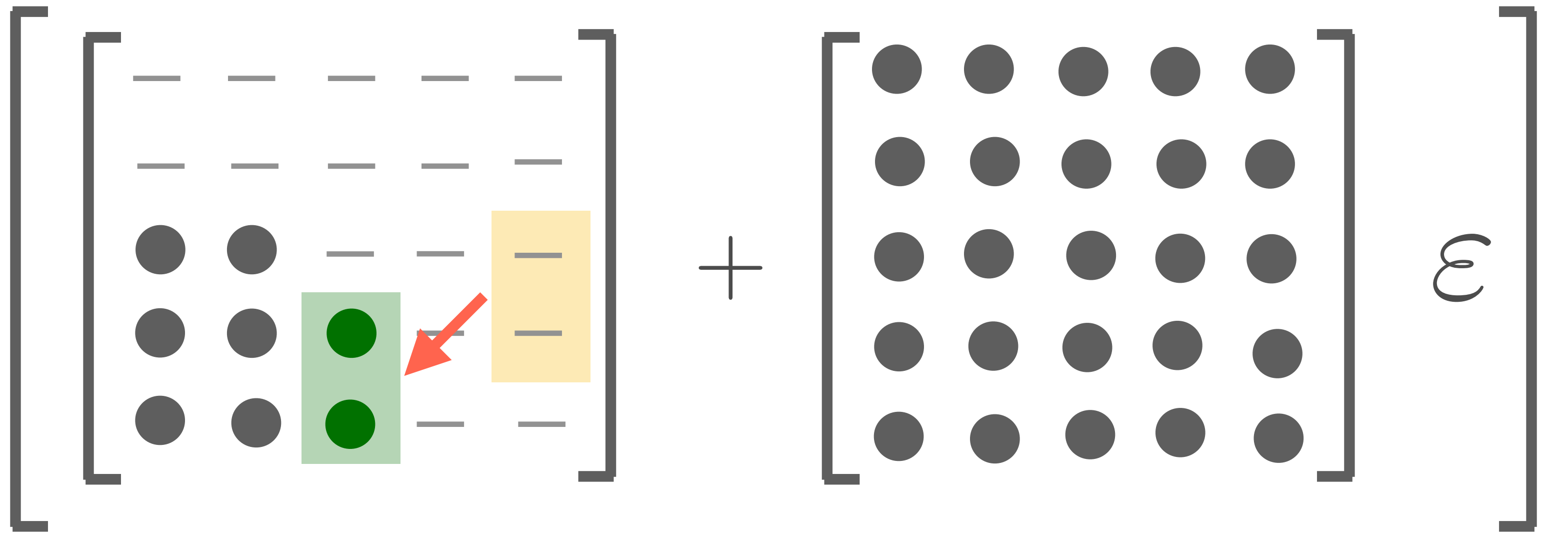
$$\varphi^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \varphi^{(2)} \longrightarrow \mathrm{d}\varphi^{(3)} =$$



$$\varphi^{(3)}$$

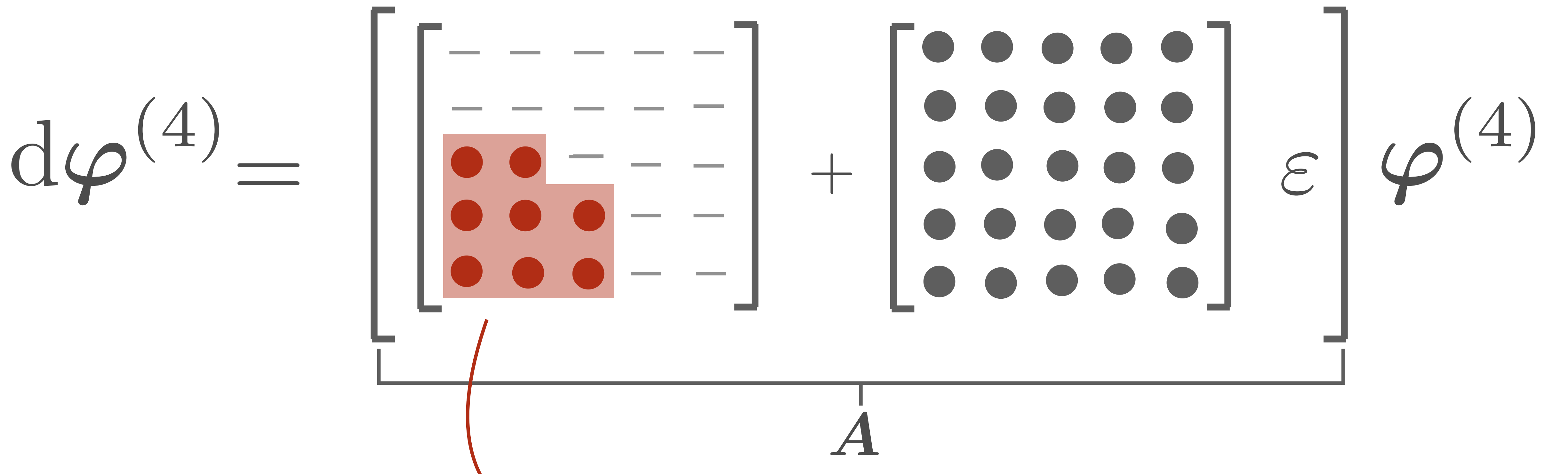
Lower triangular ε^0 - terms:

$$\varphi^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \varphi^{(3)} \longrightarrow \mathrm{d}\varphi^{(4)} =$$



$$\varphi^{(4)}$$

EXAMPLE STEP 4: ANSATZ



$$d\boldsymbol{\varphi}^{(4)} = \underbrace{\left[\begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \bullet & \bullet & - & - & - \\ \bullet & \bullet & \bullet & - & - \\ \bullet & \bullet & \bullet & - & - \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \varepsilon\right]}_{\boldsymbol{A}} \boldsymbol{\varphi}^{(4)}$$

We want to remove these entries!

Find final transformation:

1. Make an ansatz:

$$d\varphi^{(5)} = U_6^{(5)} \varphi^{(4)} \quad \text{with} \quad U_6^{(5)} = \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \star & \star & - & - & - \\ \star & \star & \star & - & - \\ \star & \star & \star & - & - \end{bmatrix}$$

unknowns

2. Transform the differential equation:

$$d\boldsymbol{\varphi}^{(5)} = \left[\left(dU_6^{(5)}\right)\left(U_6^{(5)}\right)^{-1} + U_6^{(5)} \boldsymbol{A} \left(U_6^{(5)}\right)^{-1}\right] \boldsymbol{\varphi}^{(5)}$$

3. Require that the ε^0- entries vanish

8 coupled differential equations of 8 **unknowns** ★

EXAMPLE STEP 4: ANSATZ

3. Require that the ε^0- entries vanish

8 coupled differential equations of 8 **unknowns** ★

- Non-trivial to solve !
- Undetermined (at most 8) number of *new* functions !
  (not expressible just in periods and branch points)

↓

Can we simplify this?

EXAMPLE STEP 4: ANSATZ

3. Require that the ε^0- entries vanish

8 coupled differential equations of 8 **unknowns** ★

↓

We can simplify this, using the **intersection matrix**!

Slogan:

Basis and dual basis are in $\mathcal{E}$**-form** and **C-form** (with $\check{\boldsymbol{P}}(\varepsilon) = \boldsymbol{P}(-\varepsilon)$) $\Rightarrow$ The intersection matrix is **constant** in the external variables, $\mathrm{d}\boldsymbol{C} = 0$.

[Duhr, Semper, Stawiński, FP]

Use this condition constructively:

1. Choose basis $\varphi^{(5)}$ & dual basis $\check{\varphi}^{(5)}$, so that $\check{\boldsymbol{P}}(\varepsilon) = \boldsymbol{P}(-\varepsilon)$.
2. Compute intersection matrix C [Contains the 8 unknowns ★ of $U_6^{(5)}$]
3. Require all entries of C to be constant in parameters λ_i and solve for (some) ★.

EXAMPLE STEP 4: ANSATZ

GENUS 2

3. Require that the ε^0- entries vanish

8 coupled differential equations of 8 **unknowns** ★

Slogan:

Basis and dual basis are in $\mathcal{E}$**-form** and **C-form** (with $\check{\boldsymbol{P}}(\varepsilon) = \boldsymbol{P}(-\varepsilon)$) $\Rightarrow$ The intersection matrix is **constant** in the external variables, $\mathrm{d}\boldsymbol{C} = 0$.

[Duhr, Semper, Stawiński, FP]

Use this condition constructively:

$$U_6^{(5)} = \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \bullet & \bullet & - & - & - \\ \star & \bullet & \star & - & - \\ \bullet & \bullet & \star & - & - \end{bmatrix} \longrightarrow$$

All but three entries of the final transformation (expressed in periods, branch points & the three remaining new functions)

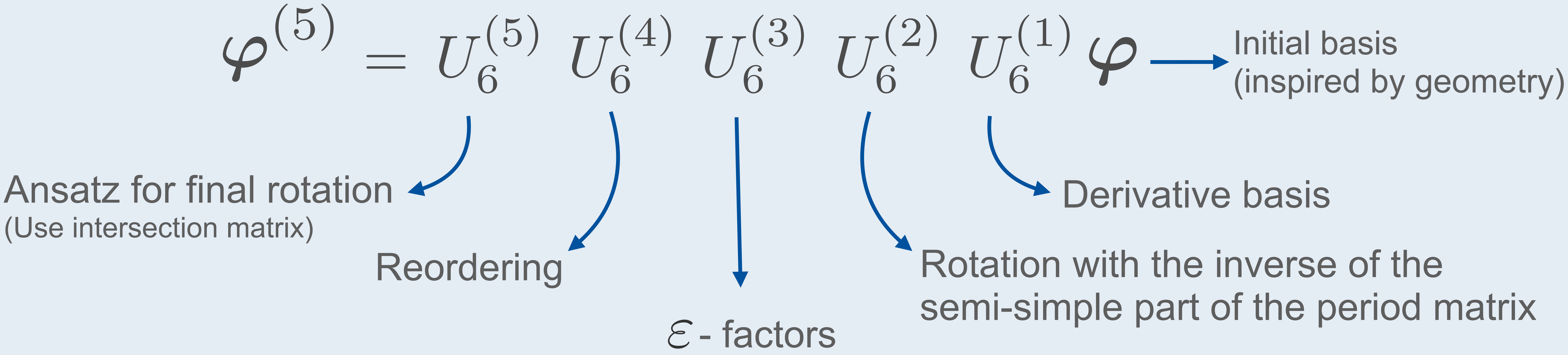
Find from (now) simpler differential equations

$$\boldsymbol{C} = \begin{bmatrix} - & - & - & - & \bullet \\ - & - & - & \bullet & - \\ - & - & \bullet & - & - \\ - & \bullet & - & - & - \\ \bullet & - & - & - & - \end{bmatrix} \longrightarrow$$

A constant skew-diagonal intersection

The requirement, that the **intersection matrix** is constant, can be used ***constructively***!

RESULT



$$\boldsymbol{\varphi}^{(5)} = U_6^{(5)}\ U_6^{(4)}\ U_6^{(3)}\ U_6^{(2)}\ U_6^{(1)}\ \boldsymbol{\varphi}$$

Initial basis
(inspired by geometry)

Ansatz for final rotation
(Use intersection matrix)

Reordering

$\mathcal{E}$ - factors

Rotation with the inverse of the
semi-simple part of the period matrix

Derivative basis

$$\mathrm{d}\varphi^{(5)} = \varepsilon \boldsymbol{B}(\underline{\lambda})\ \ \varphi^{(5)}$$ in $\mathcal{E}$-form and C-form

forms **to be** classified (Siegel (quasi-)modular?)

Preliminary results:
For the Lauricella function with 5 branch points (odd hyperelliptic curve of genus 2), we obtain Siegel modular forms.

SUMMARY: THREE TAKEAWAYS

The algorithm by [Görges,Nega,Tancredi,Wagner] also works for hyperelliptic maximal cuts!

Differential equation for maximal cut in ε- form and C-form ⟹ constant intersection matrix!
Can be used constructively!

Preliminary evidence for Siegel modular forms (+ generalisations) in Feynman integrals

OUTLOOK

- Better understanding of the appearing (Siegel modular?) forms
- Numerical evaluation of hyperelliptic Feynman integrals
- Compute the full non-planar double box (beyond the maximal cut)
- Better understanding of the role of the C-form (more generally)

THANK YOU!

Quasi - modular forms

Under a modular transformation, the periods and punctures transform in the following way:

$$z \mapsto \frac{z}{c\tau + d} \quad , \quad \tau \mapsto \frac{a\tau + b}{c\tau + d}$$

$$\psi_1 \mapsto (c\tau + d)\psi_1 \quad , \quad \psi_2 \mapsto (a\tau + b)\psi_2$$

$$\partial_0\psi_1 \mapsto (c\tau + d)\partial_t\psi_1 + c\psi_1\partial_0\tau$$

A quasi-modular form of weight k and depth p transforms in the following way:

$$f(z,\tau) \mapsto \sum_{i=0}^{p} (c\tau + d)^{k+2} \left(\frac{cz}{cz + d}\right)^i f_i(z,\tau)$$

Siegel modular forms

Symplectic group of level p:

$$\Gamma^{\text{symp}}(p) = \left\{ \gamma \in Gl_{2g}(\mathbb{Z}) \middle| \gamma^T \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \gamma = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \text{ and } \gamma = \mathbf{1} \text{ mod } p \right\}$$

The normalized period matrix transforms as:

$$\gamma \circ \Omega = (\boldsymbol{A} \cdot \Omega + \boldsymbol{B}) (\boldsymbol{C} \cdot \Omega + \boldsymbol{D})^{-1} \text{ for } \gamma = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}$$

$$g(\gamma \circ \Omega) = \det (\boldsymbol{C} \cdot \Omega + \boldsymbol{D})^k g(\Omega) \text{ for all } \gamma = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \in \Gamma^{\text{symp}}(p)$$

Independent differentials on hyperelliptic curve

BACKUP

We are interested in integrals over rational functions $\mathrm{R}(x,y)$ with $y^2 = \prod_{i=1}^{2g+2} (x-\lambda_i)$.

$$\int \mathrm{d}x\, \mathrm{R}(x,y) = \int \frac{\mathrm{d}x}{y} R_1(x) + \int \mathrm{d}x\, R_2(x)$$

Partial fractioning

$$\int \mathrm{d}x \frac{x^k}{y} \quad \text{and} \quad \int \mathrm{d}x \frac{1}{y(x-c)^k}$$

MPLs: $\int \mathrm{d}x \frac{1}{x-c}$

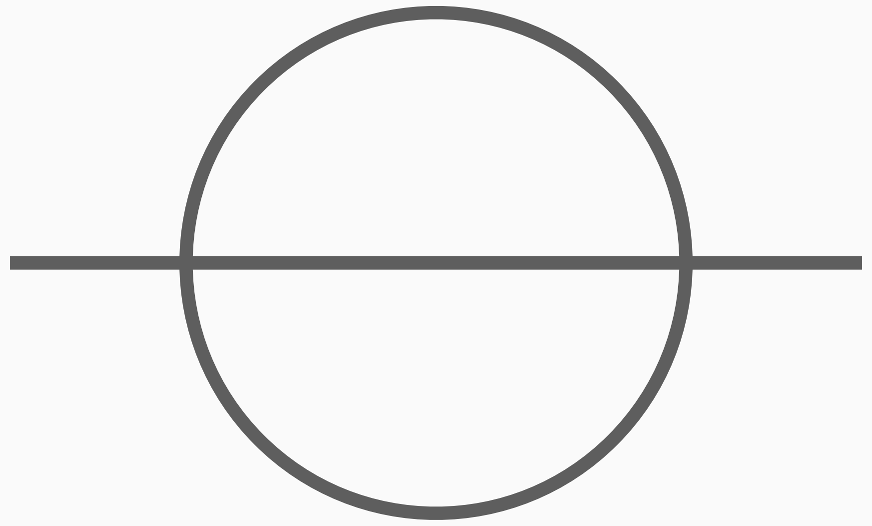
Integration by parts

$$\int \mathrm{d}x \frac{x^k}{y} \ \text{with} \ k = 0, \ldots, 2g \ \text{and} \ \int \mathrm{d}x \frac{1}{y(x-c)} \quad \text{for} \quad c \neq \lambda_i$$

FEYNMAN INTEGRAL WITH AN ELLIPTIC CURVE

ELLIPTIC EXAMPLE: SUNRISE

$$D = 2 - 2\varepsilon$$



$\downarrow$ maximal cut
loop - by - loop

$$\int_\Gamma \mathrm{d}x\, x^\varepsilon \left[\boxed{(x-\mu_1)(x-\mu_2)(x-\mu_3)(x-\mu_4)}\right]^{-\frac{1}{2}-\varepsilon}$$

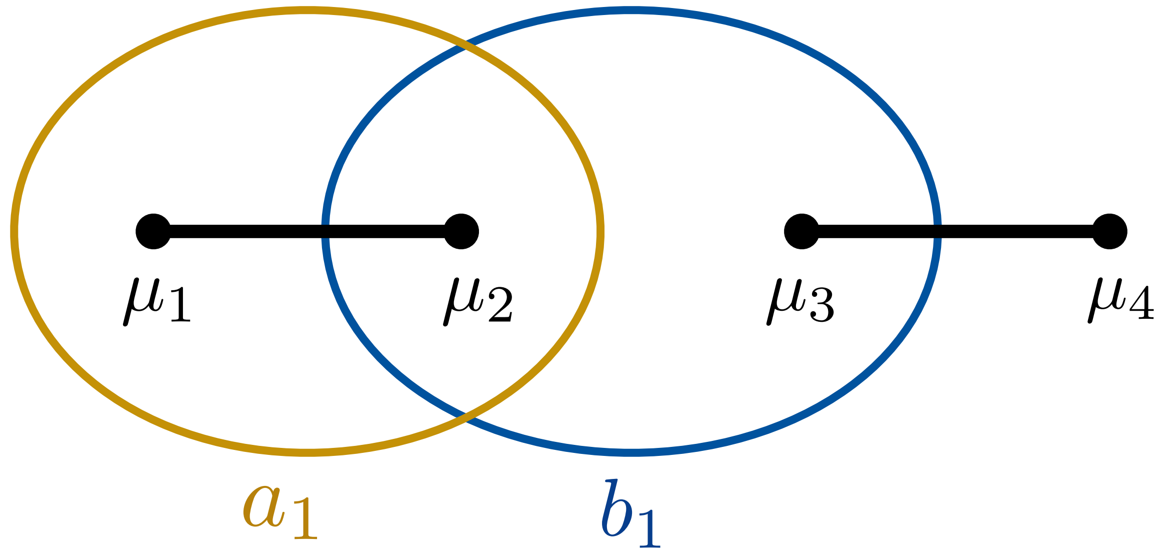
$\downarrow$

even elliptic curve of **genus 1**:

$$y^2 = \boxed{(x-\mu_1)(x-\mu_2)(x-\mu_3)(x-\mu_4)}$$

$$\boxed{y^2 = (x-\mu_1)(x-\mu_2)(x-\mu_3)(x-\mu_4)}$$

Homology basis:



Basis of differentials: $\left\{ \frac{\mathrm{d}x}{y}, \frac{x\mathrm{d}x}{y} \right\}$

Periods and quasi-periods:

$$\omega_1 = 2\int_{a_1} \frac{\mathrm{d}x}{y} \qquad \eta_1 = 2\int_{a_1} \frac{x\,\mathrm{d}x}{y}$$

$$\omega_2 = 2\int_{b_1} \frac{\mathrm{d}x}{y} \qquad \eta_2 = 2\int_{b_1} \frac{x\,\mathrm{d}x}{y}$$

$$\longrightarrow \quad \tau = \frac{\omega_2}{\omega_1}$$

EXAMPLE: ELLIPTIC LAURICELLA

A toy model for Feynman integral with even elliptic curve of **genus 1**:

$$\mathcal{L}_4(\boldsymbol{\mu}, \boldsymbol{a}) = \int_{\mu_1}^{\mu_2} \underbrace{(1-\mu_1^{-1}x)^{-\frac{1}{2}+a_1\varepsilon}(1-\mu_2^{-1}x)^{-\frac{1}{2}+a_2\varepsilon}(1-\mu_3^{-1}x)^{-\frac{1}{2}+a_3\varepsilon}(1-\mu_4^{-1}x)^{-\frac{1}{2}+a_4\varepsilon}}_{\text{Twist: } \Phi_4 = \frac{\Psi_4}{y} \text{ with } y^2 = (1-\mu_1^{-1}x)(1-\mu_2^{-1}x)(1-\mu_3^{-1}x)(1-\mu_4^{-1}x)} \mathrm{d}x$$

We use the algorithm by [Görges, Nega, Tancredi, Wagner]:

1. Make a good choice for the starting basis

- $I_1^{(0)} = \int \Psi_4 \frac{\mathrm{d}x}{y}$
- $I_2^{(0)} = \int \frac{\partial}{\partial \mu_1}\left(\Psi_4 \frac{\mathrm{d}x}{y}\right)$
- $I_3^{(0)} = \int \Psi_4 \frac{x\,\mathrm{d}x}{y}$

$$\longrightarrow \quad \mathrm{d}\begin{pmatrix} I_1^{(0)} \\ I_2^{(0)} \\ I_3^{(0)} \end{pmatrix} = \left(\begin{bmatrix} \bullet & \bullet & - \\ \bullet & \bullet & - \\ \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & - & \bullet \\ - & \bullet & \bullet \\ \bullet & - & \bullet \end{bmatrix} \varepsilon + \begin{bmatrix} - & - & - \\ \bullet & - & \bullet \\ - & - & - \end{bmatrix} \varepsilon^2 \right) \begin{pmatrix} I_1^{(0)} \\ I_2^{(0)} \\ I_3^{(0)} \end{pmatrix}$$

EXAMPLE: ELLIPTIC LAURICELLA

2. Rotation with semi-simple part of the period matrix at $\varepsilon = 0$:

$$\mathcal{P}_0 = \begin{pmatrix}\omega_1 & \omega_2\\ \eta_1 & \eta_2\end{pmatrix} = \frac{1}{\sqrt{\lambda}} \underbrace{\begin{pmatrix}\omega_1 & 0 \\ \eta_1 & -\frac{2\pi i}{\omega_1}\end{pmatrix}}_{S} \begin{pmatrix}1&\tau\\0&1\end{pmatrix} \longrightarrow \begin{pmatrix}I_1^{(1)}\\I_2^{(1)}\\I_3^{(1)}\end{pmatrix} = \left[\begin{array}{c|c} S^{-1} & \begin{matrix}0\\0\end{matrix}\\ \hline \begin{matrix}0&0\end{matrix} & 1\end{array}\right]\begin{pmatrix}I_1^{(0)}\\I_2^{(0)}\\I_3^{(0)}\end{pmatrix}$$

$$\longrightarrow \mathrm{d}\begin{pmatrix}I_1^{(1)}\\I_2^{(1)}\\I_3^{(1)}\end{pmatrix} = \left(\begin{bmatrix}- & \bullet & -\\ \bullet & - & -\\ \bullet & \bullet & \bullet\end{bmatrix} + \begin{bmatrix}\bullet & - & \bullet\\ \bullet & \bullet & \bullet\\ \bullet & - & \bullet\end{bmatrix}\varepsilon + \begin{bmatrix}- & - & -\\ \bullet & - & \bullet\\ - & - & -\end{bmatrix}\varepsilon^2\right)\begin{pmatrix}I_1^{(1)}\\I_2^{(1)}\\I_3^{(1)}\end{pmatrix}$$

3. Reordering and ε Rotation:

$$\begin{pmatrix}I_1^{(2)}\\I_2^{(2)}\\I_3^{(2)}\end{pmatrix} = \begin{pmatrix}\varepsilon & 0 & 0\\ 0 & 0 & 1\\ 0 & \varepsilon & 0\end{pmatrix}\begin{pmatrix}I_1^{(1)}\\I_2^{(1)}\\I_3^{(1)}\end{pmatrix} \longrightarrow \mathrm{d}\begin{pmatrix}I_1^{(2)}\\I_2^{(2)}\\I_3^{(2)}\end{pmatrix} = \left(\begin{bmatrix}- & - & -\\ \bullet & \bullet & -\\ \bullet & \bullet & \bullet\end{bmatrix} + \begin{bmatrix}\bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet\end{bmatrix}\varepsilon\right)\begin{pmatrix}I_1^{(2)}\\I_2^{(2)}\\I_3^{(2)}\end{pmatrix}$$

4. Integrate out remaining entries

= Make an Ansatz for the final transformation, require ε-form and solve the resulting differential equations.

$\longrightarrow$ One *new* object (not rational function of periods & branch points).