Relations in the differential equation for Feynman integrals

Stefan Weinzierl

in collaboration with Sebastian Pögel, Xing Wang, Konglong Wu, Xiaofeng Xu

October 15, 2024

Content

I. Self-duality of Feynman integrals

II. Galois symmetries

Review: ε-factorised differential equation

Notation:

$I = (I_1, \ldots, I_{N_F})$, set of master integrals,

$x = (x_1, \ldots, x_{N_B})$, set of kinematic variables the master integrals depend on.

ε-factorised differential equation: *(Henn '13)*

$$dI(\varepsilon, x) = \varepsilon A(x)\, I(\varepsilon, x)$$

- **Conjecture**: A change of the basis of master integrals to an ε-factorised differential equation always exists.
- The ε-factorised form **is preserved** under constant (i.e. x-independent) $\mathrm{GL}(N_F, \mathbb{C})$-rotations.

Review: Transformation to an ε-factorised form

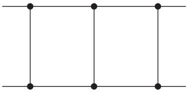
The transformation from a pre-canonical form to an ε-factorised form may involve

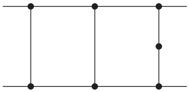
- rational functions
- algebraic functions (square roots)
- periods of elliptic curves
- periods of Calabi-Yau manifolds
- ...

Beyond rational functions, there is typically a choice involved (the sign of a square root, basis vector in a lattice, etc.).

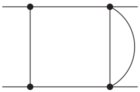
Review: Sectors with more than one master integral

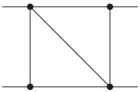
- Starting from two-loops there can be sectors with more than one master integral.





- The differential equation relates in general a sector to itself and to sub-sectors, obtained by pinching.





Review: Block-triangular structure of the matrix A

Order the set of master integrals $\vec{I} = (I_1, \dots, I_{N_F})^T$ such that I_1 is the simplest integral and I_{N_F} the most complicated integral.

The matrix A has a lower block-triangular structure:

$$A = \begin{pmatrix} D_1 & 0 & 0 & 0 \\ N_{21} & \multicolumn{2}{c}{D_2} & 0 \\ & & & 0 \\ N_{31} & \multicolumn{2}{c}{N_{32}} & D_3 \end{pmatrix}$$

Diagonal blocks: D_1, D_2, D_3

Non-diagonal blocks: N_{21}, N_{31}, N_{32}

Outline

- Question: Given an ε-factorised differential equation, is there a constant rotation, preserving the block-triangular structure, such that (some) entries of A are related by a symmetry?
- Answer: There is evidence, that sectors with two or more master integrals have extra symmetries:
  - Self-duality
  - Galois symmetries
- In practice: Assuming these additional symmetries is very helpful in finding an ε-factorised differential equation.

Pögel, Wang, S.W., Wu, Xu, '24

Section 1

Self-duality

Self-duality

Let us consider a diagonal block (i.e. a maximal cut)

$$D = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1(n-1)} & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2(n-1)} & d_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ d_{(n-1)1} & d_{(n-1)2} & \dots & d_{(n-1)(n-1)} & d_{(n-1)n} \\ d_{n1} & d_{n2} & \dots & d_{n(n-1)} & d_{nn} \end{pmatrix}.$$

Self-duality is the statement that there is a basis such that

$$d_{ij} = d_{(n+1-j)(n+1-i)},$$

i.e. D is symmetric with respect to the anti-diagonal.

Self-duality

- Self-duality first observed on the maximal cut of the equal-mass l-loop banana integrals
- Provides algebraic equations (as opposed to differential equations) to construct an ε-factorised form.
- Evidence that self-duality is not restricted to Calabi-Yau Feynman integrals, but holds more generally.
- Self-duality is a property of the maximal cut.

Essentially self-adjoint operators

- Consider a differential operator L in one variable y.
- The adjoint operator L^* of an operator L is defined to be

$$L = \sum_{j=0}^{l} r_j(y) \frac{d^j}{dy^j} \quad \Rightarrow \quad L^* = \sum_{j=0}^{l} (-1)^{l-j} \frac{d^j}{dy^j} r_j(y)$$

- An operator L is called self-adjoint, if $L^* = L$.
- An operator L is called **essentially self-adjoint** or **self-dual**, if there exists a function $\alpha(y)$ such that

$$\alpha L^* = L\alpha.$$

Fact

The Picard-Fuchs operator for the l-loop equal-mass banana integral in $D = 2$ space-time dimensions is self-dual.

Self-duality and twisted cohomology

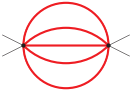
- Feynman integrals can be viewed as a pairing between twisted cocyles (the integrand) and cycles (the integration domain).
- For a sector with n master integrals: There are n independent cycles.
- We may define a $n \times n$ period matrix.
- To any twisted cocyle we may define its dual, similar for the cycles.
- This defines the dual period matrix.
- Self-duality is a relation between the period matrix and the dual period matrix.
- If n is even and $n \geq 4$ it is not excluded that

$$D = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} D^T \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$$

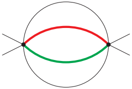
Duhr, Porkert, Semper, Stawinski, '24

Examples of self-duality with sectors of four master integrals

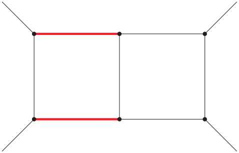
- Equal-mass four-loop banana



- Higgs self-enery: Three-loop banana with mass configuration $(0, 0, m_1, m_2)$.



- Drell-Yan double-box integral



Section 2

Galois symmetries

Galois theory

Definition

Given a non-constant polynomial $p(x)$ with coefficients from a field F, the roots of $p(x)$ may not lie in F. The splitting field L/F is the smallest field extension that contains all the roots of $p(x)$. The Galois group

$$G(L/F) \quad = \quad \{ \sigma \in \mathrm{Aut}(L) \mid \sigma|_F = \mathrm{id} \}$$

is the subgroup of the automorphism group of L, which keeps F fixed.

Example

The roots of $x^2 - 3 \in \mathbb{Q}[x]$ lie in $\mathbb{Q}[\sqrt{3}]$ and the Galois group is

$$G\left(\mathbb{Q}[\sqrt{3}]/\mathbb{Q}\right) \quad = \quad \mathbb{Z}_2,$$

generated by

$$\begin{aligned} \sigma \quad : \quad & \mathbb{Q}[\sqrt{3}] \rightarrow \mathbb{Q}[\sqrt{3}], \\ & \sigma\left(\sqrt{3}\right) \; = \; -\sqrt{3}. \end{aligned}$$

Galois theory

- In the application towards Feynman integrals we often encounter roots r of quadratic equations, where the Galois group acts as $r \to -r$. A typical example is the square root

$$r = \sqrt{-s\left(4m^2-s\right)}.$$

- Typical Galois groups are products of $\mathbb{Z}_2$.
- Nested roots: Two-loop calculation for $pp \to t\bar{t}H$.

  Febres Cordero, Figueiredo, Kraus, Page, Reina, ’23

Galois theory

- We are interested in Galois symmetries in addition to self-duality.
- If σ is an element of the Galois group, we ask if in addition to self duality we may choose master integrals such that for example

$$J_2 = \sigma(J_1),$$

- Galois symmetries provide relations beyond the maximal cut.

A simple example

For a sector with two master integrals it is often possible to find a basis $J = (J_1, J_2)^T$ such that

$$dJ = \varepsilon A J, \quad J_2 = \sigma(J_1),$$

and A has the structure

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

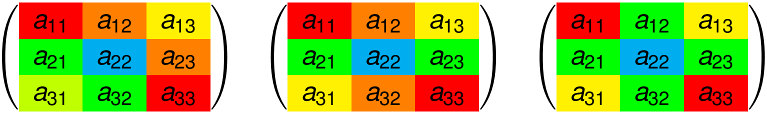
where entries with the same background colour are related by a symmetry. We have the relations

$$a_{11} = a_{22}, \quad a_{11} = \sigma(a_{11}), \quad a_{12} = \sigma(a_{21}).$$

A sector with three master integrals

A sector with three master integrals $I = (I_1, I_2, I_3)^T$ and a Galois symmetry, which relates I_1 and I_3

$$I_3 = \sigma(I_1)$$



self-duality Galois both

An example with subsectors

A system with two sectors with two master integrals each and Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$I_2 = \sigma(I_1) \quad \text{and} \quad I_4 = \sigma'(I_3).$$

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

A realistic example

A system with 16 master integrals (related to Drell-Yan):

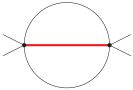
$$A = \begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{51} & a_{52} & a_{53} & 0 & a_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{61} & 0 & 0 & a_{64} & 0 & a_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{71} & 0 & 0 & 0 & 0 & 0 & a_{77} & a_{78} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{81} & 0 & 0 & 0 & 0 & 0 & a_{87} & a_{88} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{95} & 0 & 0 & 0 & a_{99} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{A6} & 0 & 0 & 0 & a_{AA} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{B1} & a_{B2} & a_{B3} & 0 & a_{B5} & 0 & a_{B7} & a_{B8} & 0 & 0 & a_{BB} & 0 & 0 & 0 & 0 & 0 \\
a_{C1} & 0 & a_{C3} & a_{C4} & 0 & a_{C6} & 0 & 0 & 0 & 0 & 0 & a_{CC} & a_{CD} & 0 & 0 & 0 \\
a_{D1} & a_{D2} & 0 & a_{D4} & 0 & a_{D6} & 0 & 0 & 0 & 0 & 0 & a_{DC} & a_{DD} & 0 & 0 & 0 \\
a_{E1} & 0 & 0 & a_{E4} & 0 & a_{E6} & a_{E7} & a_{E8} & 0 & 0 & 0 & 0 & 0 & a_{EE} & a_{EF} & 0 \\
a_{F1} & 0 & 0 & a_{F4} & 0 & a_{F6} & a_{F7} & a_{F8} & 0 & 0 & 0 & 0 & 0 & a_{FE} & a_{FF} & 0 \\
a_{01} & a_{02} & a_{03} & 0 & a_{05} & a_{06} & a_{07} & a_{08} & a_{09} & a_{0A} & a_{0B} & a_{0C} & a_{0D} & a_{0E} & a_{0F} & a_{00}
\end{pmatrix}$$

Section 3

Details

Constant square roots

- The requirement of self-duality may introduce constant square roots like $\sqrt{3}$.
- This in turn may lead to a Galois symmetry $\sqrt{3} \rightarrow -\sqrt{3}$.
- Two-loop sunrise integral with mass configuration $(0, 0, m)$: Two master integrals, no sub-sectors, no kinematic square root.



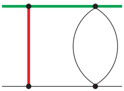
Non-uniqueness

- An example with a sector with two master integrals and two kinematic square roots:

$$r_1 = \sqrt{-t\left(4m_1^2 - t\right)}, \qquad r_3 = \sqrt{-m_2^2\left(4m_1^2 - m_2^2\right)}.$$

- Standard integrals for an ε-factorised form are

$$\begin{aligned} I_5 &= \varepsilon^3 r_1 \, I_{011012000}, \\ I_6 &= \varepsilon^2 r_3 \, \mathbf{D}^- \, I_{011(-1)11000}. \end{aligned}$$



- For self-duality and Galois symmetry we may either choose

$$J_5 = I_5 + \frac{i}{6}\sqrt{3} I_6, \qquad J_6 = I_5 - \frac{i}{6}\sqrt{3} I_6,$$

or

$$J_5' = I_6 - 2i\sqrt{3} I_5, \qquad J_6' = I_6 + 2i\sqrt{3} I_5.$$

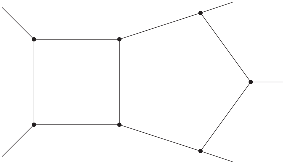
Parameterised Galois symmetries

- It might occur that the transformation

$$
\begin{aligned}
J_1 &= I_1 + r I_2, \qquad r = \sqrt{\lambda}, \\
J_2 &= I_1 - r I_2,
\end{aligned}
$$

realises self-duality and Galois symmetry for any value $\lambda \in \mathbb{Q}$ that is not a perfect square.

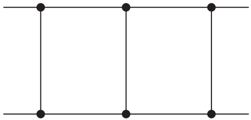
- Example: Pentabox



Limit Galois symmetries

- We first divide the rational numbers $\mathbb{Q}$ into perfects squares $\mathbb{PS}$ and not perfect squares $\mathbb{NPS}$.
- Consider sequences $(\lambda_n) \in \mathbb{NPS}$ with

$$\lim_{n\to\infty} \lambda_n \quad = \quad \lambda \in \mathbb{PS}.$$



- For each such sequence redefine the master integrals for example as as

$$J_1^{(n)} = I_1 + \sqrt{\lambda_n} I_2, \qquad J_2^{(n)} = I_1 - \sqrt{\lambda_n} I_2.$$

- Set

$$J_1 = \lim_{n\to\infty} J_1^{(n)} = I_1 + \sqrt{\lambda} I_2, \qquad J_2 = \lim_{n\to\infty} J_2^{(n)} = I_1 - \sqrt{\lambda} I_2.$$

Section 4

Remarks

Always walk on the physics side of life

- No rigorous proof in this talk.
- Recall: Assuming self-dualtiy and Galois symmetries is very helpful in finding an ε-factorised differential equation, i.e. a change of the basis of master integrals $I' = UI$.
- Suppose we have an **educated guess** for $U(\varepsilon, x)$. It is easy to check, if this transformation factors out ε: Simply compute

$$A' = UAU^{-1} + UdU^{-1}.$$

- Compare to the following situation: Suppose N is the product of two prime numbers. It is simple to check if p is a factor of N, this requires only one division.
- No mathematical rigour required for our educated guess (... still it would be nice to have a proof...).

Conclusions

- We certainly would like to choose our master integrals such that they satisfy an ε-factorised differential equation.
- I presented evidence, that in addition we may choose the master integrals such that we realise self-duality and Galois symmetries.
- Assuming these additional symmetries is very helpful in finding an ε-factorised differential equation.