A hypergeometric view on Landau variety

Holonomic Techniques for Feynman Integrals

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Overview

Hypergeometric function:

- integral representation
- hypergeometric differential equation (holonomic system)
- "canonical" way of defining the singular locus

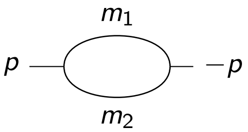
QFT:

$$G : \text{graph} \rightsquigarrow A_G = \int_{\mathbb{R}_+^n} f_G(x; \text{kinematic variables})^{-s} x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

- A_G is a special integral
- differential equation
- Singularity of each A_G matters (Landau singularity)

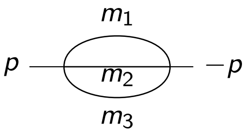
Proposal: hypergeometric study of Landau singularity

Examples



$$\Rightarrow \quad \{M m_1 m_2 \lambda(M, m_1, m_2) = 0\}$$

$$\lambda(M, m_1, m_2) = M^2 + m_1^2 + m_2^2 - 2Mm_1 - 2Mm_2 - 2m_1 m_2$$

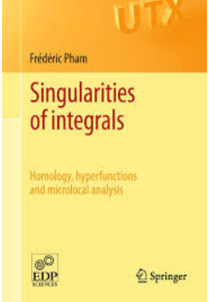


$$\Rightarrow \quad \{s m_1 m_2 m_3 \Delta(s, m_1, m_2, m_3) = 0\}$$

$$\begin{aligned}
&\Delta(s, m_1, m_2, m_3) \\
=&m_1^4 - 4\, m_1^3 m_2 + 6\, m_1^2 m_2^2 - 4\, m_1 m_2^3 + m_2^4 - 4\, m_1^3 m_3 + 4\, m_1^2 m_2 m_3 \\
&+ 4\, m_1 m_2^2 m_3 - 4\, m_2^3 m_3 + 6\, m_1^2 m_3^2 + 4\, m_1 m_2 m_3^2 + 6\, m_2^2 m_3^2 - 4\, m_1 m_3^3 \\
&- 4\, m_2 m_3^3 + m_3^4 - 4\, m_1^3 s + 4\, m_1^2 m_2 s + 4\, m_1 m_2^2 s - 4\, m_2^3 s + 4\, m_1^2 m_3 s \\
&- 40\, m_1 m_2 m_3 s + 4\, m_2^2 m_3 s + 4\, m_1 m_3^2 s + 4\, m_2 m_3^2 s - 4\, m_3^3 s + 6\, m_1^2 s^2 \\
&+ 4\, m_1 m_2 s^2 + 6\, m_2^2 s^2 + 4\, m_1 m_3 s^2 + 4\, m_2 m_3 s^2 + 6\, m_3^2 s^2 - 4\, m_1 s^3 \\
&- 4\, m_2 s^3 - 4\, m_3 s^3 + s^4
\end{aligned}$$

1. What is the singularity of an integral?

Singularities of integrals



$\pi : X \to Z$: a family of varieties.

$f_1, \dots, f_\ell : X \to \mathbb{C}^*$:invertible regular functions

$s_1, \dots, s_\ell \in \mathbb{C}$: parameters

Euler integral:

$$\int_\pi f_1^{s_1} \cdots f_\ell^{s_\ell} \xi, \qquad \xi\text{: relative differential form.}$$

Landau singularity is the locus where π fails to be a fiber bundle.

F.Pham('60-70's), Helmer-Papathanasiou-Tellander ('24)

$\to$ Whitney Stratification

This is JUST an upper bound of singularity.

Euler discriminant

Let $\pi : X \to Z$ be a (good) family of very affine varieties.

$\chi : Z \ni z \mapsto |\chi(X_z)| \in \mathbb{Z}_{\geq 0}$ locally constant function.

Euler discriminant is defined by

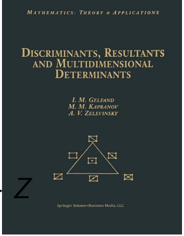
$$\nabla_\chi(Z) := \{z \in Z \mid \chi_z < \max \chi_z\}.$$

- Euler discriminant is a generalization of the principal A-determinant (Esterov '13).
- Euler discriminant will be the correct notion of singularity of an Euler integral (Fevola-Mizera-Telen '24).

Landau singularity:=Euler discriminant.

2. What does Euler discriminant mean?

Gelfand-Kapranov-Zelevinsky case



$$A \subset \mathbb{Z}^d \Rightarrow f(x;z) = \sum_{a \in A} z_a x^a \ (x = (x_1, \ldots, x_d))$$

$$Z = \mathbb{C}^A,\ X = \{(x,z) \in (\mathbb{C}^*)^d \times Z \mid f(x;z) \neq 0\} \to Z$$

$$\nabla_\chi(Z) = \{z \in Z \mid E_A(z) = 0\}$$ (Esterov'13, Amendola et al'18)

$$E_A(z) = \prod_{\substack{Q < \mathrm{New}(A) \\ Q\text{:non-defective}}} \Delta_Q(z)^{m_Q}$$: principal A-determinant (GKZ '90)

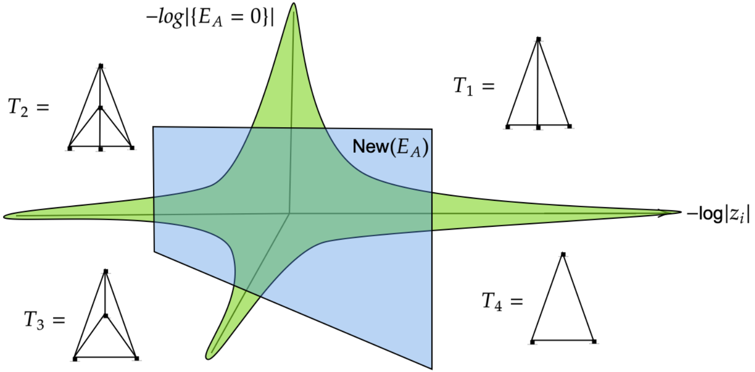
Δ_Q : Q-discriminant

$$m_Q = \max_{z \in Z} \chi_z - \max_{z \in \{\Delta_Q = 0\}} \chi_z$$

An example

$$f(x;z) = z_1x_2 + z_2x_1x_2 + z_3x_1^2x_2 + z_4x_2^2 + z_5$$

$$E_A(z) = z_3^2z_4^2z_5^2(z_1^2 - 4z_4z_5)(16z_1^2z_3^2 - 8z_1z_2^2z_3 + z_2^4 - 64z_3^2z_4z_5)$$



$$f(x;z) = \sum_{i,j,k,\ell=0}^{1} z_{ijk\ell}x_1^ix_2^jx_3^kx_4^\ell$$

$\to \deg(\Delta_A) = 24$, $\#(\text{terms of } \Delta_A(z)) = 2,894,276.$

(Huggins-Sturmfels-Yu-Yuster '08)

3. Hypergeometric Landau variety

Euler integral and the associated D-module

$Z \subset \mathbb{C}^A$: smooth subvariety, $f : (\mathbb{C}^*)^n \times Z \to \mathbb{C}$: regular function,

$\pi : X = (\mathbb{C}^*)^n \times Z \setminus V(f) \to Z$: a family of varieties.

$s, \nu_1, \ldots, \nu_n \in \mathbb{C}$: generic parameters

Euler integral: $I_\Gamma(z; s, \nu) := \int_\Gamma f(x; z)^{-s} x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1 \wedge \cdots \wedge dx_n}{x_1 \cdots x_n}$.

D_Z: ring of differential operators on Z.

$$I = \bigcap_\Gamma \mathrm{Ann}_{D_Z} I_\Gamma(z; s, \nu) \subset D_Z \text{ : left ideal}$$

$\Rightarrow M = D_Z / I$: (regular) holonomic D_Z-module

$\Rightarrow \mathrm{Char}(M) := \mathrm{Supp}(\mathrm{gr}(M)) \subset T^* Z$

$\Rightarrow \mathrm{Sing}(M) := \mathrm{proj}(\mathrm{Char}(M) \setminus T^*_Z Z) \subset Z, \quad \mathrm{proj} : T^* Z \to Z$.

General structure of Euler discriminant

Theorem

$$\mathrm{Sing}(M) = \{z \mid \exists \Gamma, I_\Gamma(z; s, \nu) \textit{ is singular at } z\}.$$

Theorem

$$\mathrm{Sing}(M) = \nabla_\chi(Z).$$

$\Rightarrow \mathrm{Sing}(M)$ is independent of generic $s, \nu_1, \ldots, \nu_n$.

Theorem

$\nabla_\chi(Z)$ *is purely one-codimensional in* Z.

Hypergeometric system

$M = D_Z/I$: holonomic D_Z-module

How to describe I?

Theorem (M.H.-Telen '23, M.H.)

$I_\Gamma(z; s, \nu)$ is annihilated by the following operators:

$$1 - \sigma_s f(\sigma_\nu; z), \nu_i - s\sigma_s\sigma_{\nu_i}\frac{\partial f}{\partial x_i}(\sigma_\nu; z), \partial_{z_j} + s\sigma_s\frac{\partial f}{\partial z_j}(\sigma_\nu; z),$$

where $\sigma_s : s \mapsto s+1$ and $\sigma_{\nu_i} : \nu_i \mapsto \nu_i + 1$. They generate the left annihilater ideal J of $I_\Gamma(z; s, \nu)$ in the difference-differential ring.

$I = J \cap D_Z$: non-commutative elimination

$$\#\left\{(y, x) \in \mathbb{C}^{n+1} \mid 1 - yf(x; z) = \nu_i - syx_i\frac{\partial f}{\partial x_i}(x) = 0\right\} = \chi_z \text{ (Huh '13).}$$

General structure of the characteristic cycle

T^*Z: cotangent bundle, $\mathcal{O}_{T^*Z} = \mathbb{C}[z_1, z_2, \ldots, \xi_1, \xi_2, \ldots]$

Consider a family of likelihood equations:

$$J_0 := \langle 1 - yf(x;z), \nu_i - syx_i \frac{\partial f}{\partial x_i}(x), \xi_j + sy \frac{\partial f}{\partial z_j}(x;z) \rangle \subset \mathbb{C}[y,x] \otimes_{\mathbb{C}} \mathcal{O}_{T^*Z}$$

$\sigma_s \to y, \sigma_{\nu_i} \to x_i, \partial_j \to \xi_j$: dequantization

$I_0 := J_0 \cap \mathcal{O}_{T^*Z} \Rightarrow M_0 := \mathcal{O}_{T^*Z}/I_0 \Rightarrow \mathrm{gr}(M_0)$

Theorem

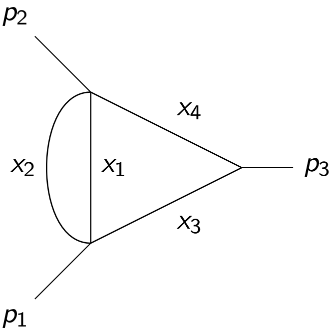
The characteristic cycle $CC(M)$ is given by

$$CC(M) = \sum_{\mathfrak{p} \in \mathrm{Ass}_{\mathcal{O}_{T^*Z}}(M_0)} m_{\mathfrak{p}} V(\mathfrak{p}), \quad m_{\mathfrak{p}} = \mathit{length}_{\mathcal{O}_{T^*Z,\mathfrak{p}}}(M_{\mathfrak{p}}).$$

In particular, $\mathrm{Char}(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_{\mathcal{O}_{T^*Z}}(M_0)} V(\mathfrak{p})$

$\mathrm{codim} \pi(V(\mathfrak{p})) = 1 \Rightarrow m_{\mathfrak{p}} = \chi^* - \chi_{\mathfrak{p}}$ (Kashiwara's index theorem)

Example:parachute



$$f(x;z) = (1 - \sum_{i=1}^{4} m_i x_i)((x_1 + x_2)(x_3 + x_4) + x_1 x_2$$
$$+ x_1 x_2 (M_1 x_3 + M_2 x_4) + M_3 x_3 x_4 (x_1 + x_2))$$

Simplification: $m_2 = m_3 = M_2 = 0$

List of $(\pi(V(\mathfrak{p})), m_\mathfrak{p})$

codim 0: $\{\{0\},8\} \Rightarrow$ Conjecture: $\mathrm{rank}(M) = \chi^*$ if f is "good".

codim 1:

$\{\{m_4 M_1 - m_1 M_3\},1\},\{\{m_1 - m_4 - M_1 + M_3\},1\},\{\{m_1 - m_4\},1\},$
$\{\{m_1 - M_1\},1\},\{\{m_1\},4\},\{\{m_4 + M_1 - M_3\},1\},\{\{m_4 - M_3\},2\},$
$\{\{m_4\},2\},\{\{M_1 - M_3\},2\},\{\{M_1\},2\},\{\{M_3\},2\}$

codim 2:

$\{\{m_4 + M_1 - M_3, m_1\},1\},\{\{m_4 - M_3, m_1 - M_1\},1\},\{\{m_4 - M_3, m_1\}$
$\{\{m_1, m_4\},2\},\{\{M_1 - M_3, m_1 - m_4\},1\},\{\{M_1 - M_3, m_1\},1\},$
$\{\{M_1 - M_3, m_4\},1\},\{\{m_1, M_1\},2\},\{\{M_1, m_4 - M_3\},1\},$
$\{\{m_1, M_3\},1\},\{\{m_4, M_3\},2\},\{\{M_1, M_3\},2\}$

codim 3:

$\{\{M_1 - M_3, m_4, m_1\},1\},\{\{M_1, m_4 - M_3, m_1\},1\},\{\{m_1, m_4, M_3\},1\},$
$\{\{m_1, M_1, M_3\},1\},\{\{m_4, M_1, M_3\},1\}$

codim 4: $\{\{m_1, m_4, M_1, M_3\},1\}$

Dequantization

Scaling:

$$I_\Gamma(z;s,\nu) \rightsquigarrow I(z;s/\hbar,\nu/\hbar) = \int_\Gamma e^{\phi_{s,\nu}/\hbar} \frac{dx_1 \wedge \cdots \wedge dx_n}{x_1 \cdots x_n}$$

If $\exists! \hat{x} \in \Gamma \cap \{d\phi_{s,\nu}(x) = 0\}$, then

$$I_\Gamma(z;s/\hbar,\nu/\hbar) \sim (2\pi)^{\frac{n}{2}} \hbar^{-\frac{n}{2}} e^{\phi_{s,\nu}(\hat{x})/\hbar} \frac{1}{\sqrt{-\mathrm{Hess}_{\mathrm{toric}}\phi_{s,\nu}(\hat{x})}}(1+o(\hbar))$$

$\mathrm{Hess}_{\mathrm{toric}}\phi_{s,\nu}(\hat{x}) = \det\left(x_i\partial_{x_i}(x_j\partial_{x_j}\phi_{s,\nu})\right)_{i,j}(\hat{x})$: toric Hessian

On the level of difference-differential rings:

$$[\partial_{z_i}, z_j] = \hbar\delta_{ij}, \quad [\sigma_s, s] = \hbar\sigma_s, \quad [\sigma_{\nu_i}, \nu_j] = \hbar\delta_{ij}\sigma_{\nu_i}$$

Summary

- Singularity of integrals is of algebro-topological origin.
- Euler discriminant is a generalization of the principal A-determinant.
- Singular locus of a D-module is Euler discriminant.
- Hypergeometric system can be dequantized.

Thank you for your attention!