Euler Discriminant of Complements of Hyperplanes

Ongoing with Saiei-Jaeyeong Matsubara-Heo

Claudia Fevola

Inria

FONDATION MATHÉMATIQUE JACQUES HADAMARD

Holonomic Techniques for Feynman Integrals

Max Planck Institute for Physics - October 14-18, 2024

Generalised Euler Integrals of Linear Forms

$$I_\Gamma(z) = \int_\Gamma h_{k+1}(\alpha; z)^{\mu_1} \cdots h_n(\alpha; z)^{\mu_n} \alpha_1^{\nu_1} \cdots \alpha_k^{\nu_k} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\alpha_k}{\alpha_k}$$

- $h_j(\alpha; z) = z_{0j} + z_{1j}\alpha_1 + \ldots + z_{kj}\alpha_k$
- $\mu_j, \nu_i \in \mathbb{C}$
- $\mathscr{A}_z := \bigcup_{j=0}^{n} \{\alpha \in (\mathbb{C}^*)^k \, : \, h_j(\alpha; z) = 0\}$
- Γ is a twisted cycle on $X_z := (\mathbb{C}^*)^k \setminus \mathscr{A}_z$

Mellin Integrals of Linear Forms

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A-hypergeometric Integrals of Linear Forms

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Aomoto-Gelfand Integrals of Linear Forms

$$I_\Gamma(z) \;=\; \int_\Gamma h_{k+1}(\alpha;z)^{\mu_1} \cdots h_n(\alpha;z)^{\mu_n}\,\alpha_1^{\nu_1} \cdots \alpha_k^{\nu_k}\,\frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\alpha_k}{\alpha_k}$$

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Two matrices:

$$I_\Gamma(z) = \int_\Gamma h_{k+1}(\alpha; z)^{\mu_1} \cdots h_n(\alpha; z)^{\mu_n} \alpha_1^{\nu_1} \cdots \alpha_k^{\nu_k} \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_k}{\alpha_k}$$

Monomial support:

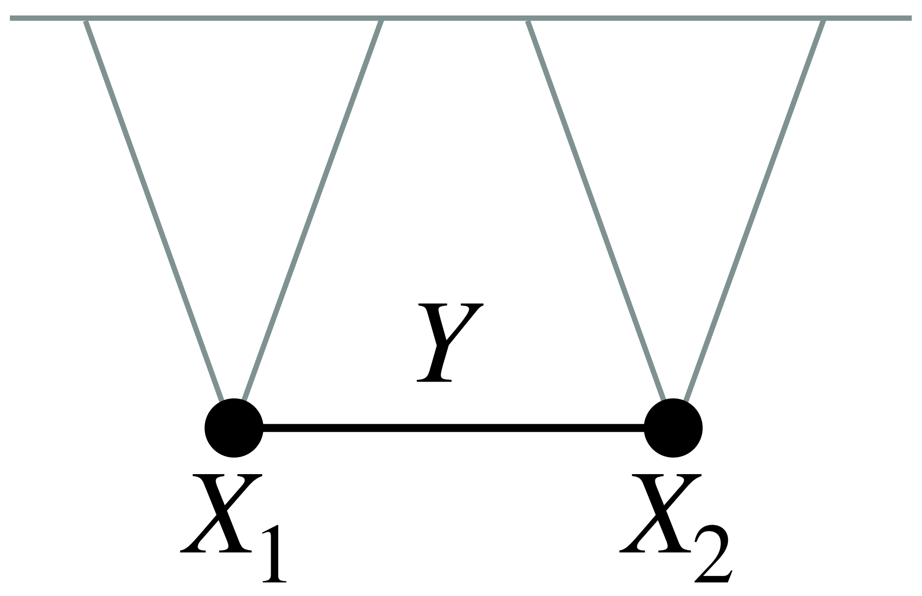
$$A = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ & \vdots & & & \vdots & & \cdots & & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ \hline & A_{k+1} & & & A_{k+2} & & & & A_n & \end{array}\right)$$

Coefficients:

$$z = \begin{bmatrix} z_{0,k+1} & z_{0,k+2} & \cdots & z_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k,k+1} & z_{k,k+2} & \cdots & z_{k,n} \end{bmatrix}$$

$$\quad h_{k+1} \qquad h_{k+2} \qquad\qquad h_n$$

Motivating example: **Cosmological Integrals**



$$\psi_2(X_1, X_2, Y, \varepsilon) = \int_\Gamma \frac{2 \cdot Y \cdot \alpha_1^\varepsilon \alpha_2^\varepsilon}{(X_1 + X_2 + \alpha_1 + \alpha_2)(X_1 + Y + \alpha_1)(X_2 + Y + \alpha_2)} \mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_2$$

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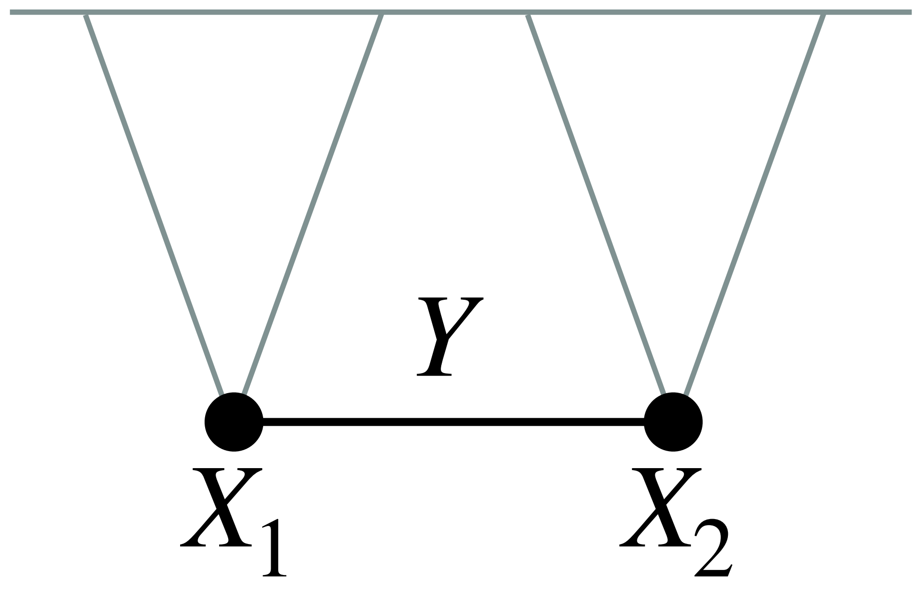

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Canonical form of a cosmological polytope

Thu @9:30: Juhnke-Kubitzke

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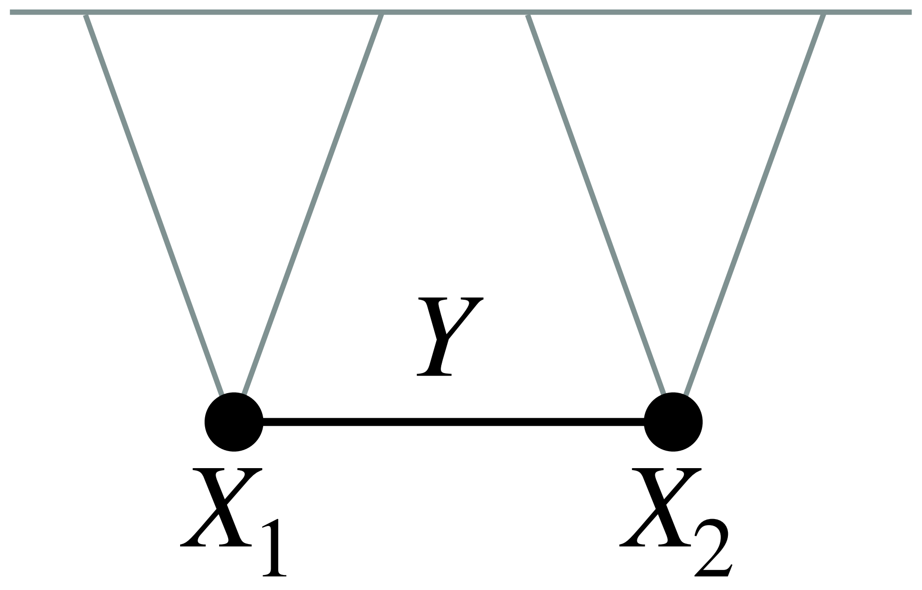

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Canonical form of a cosmological polytope

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$$A = \left(\begin{array}{ccc|cc|cc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right) \qquad z_2 = \begin{bmatrix} z_{03} & z_{04} & z_{05} \\ z_{13} & z_{14} & 0 \\ z_{23} & 0 & z_{25} \end{bmatrix}$$

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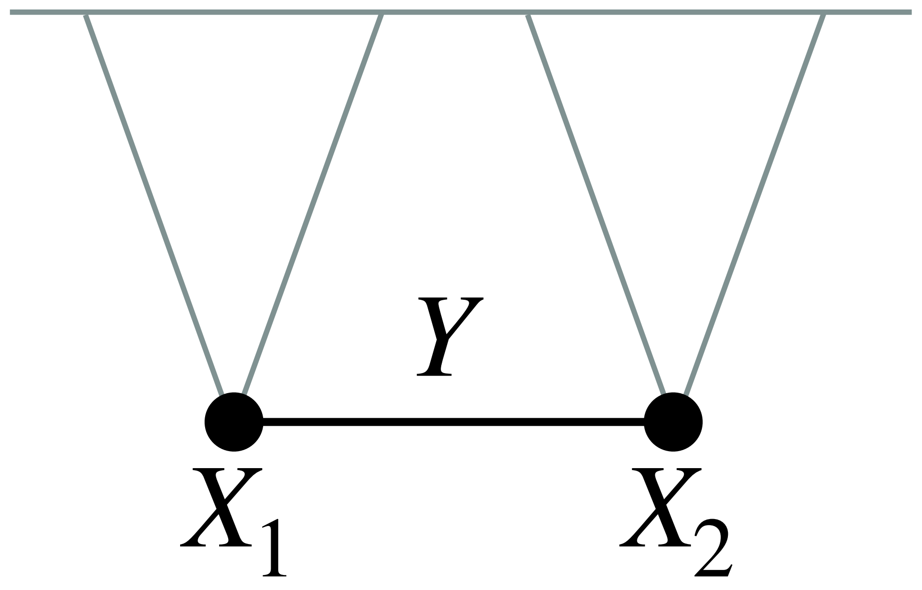

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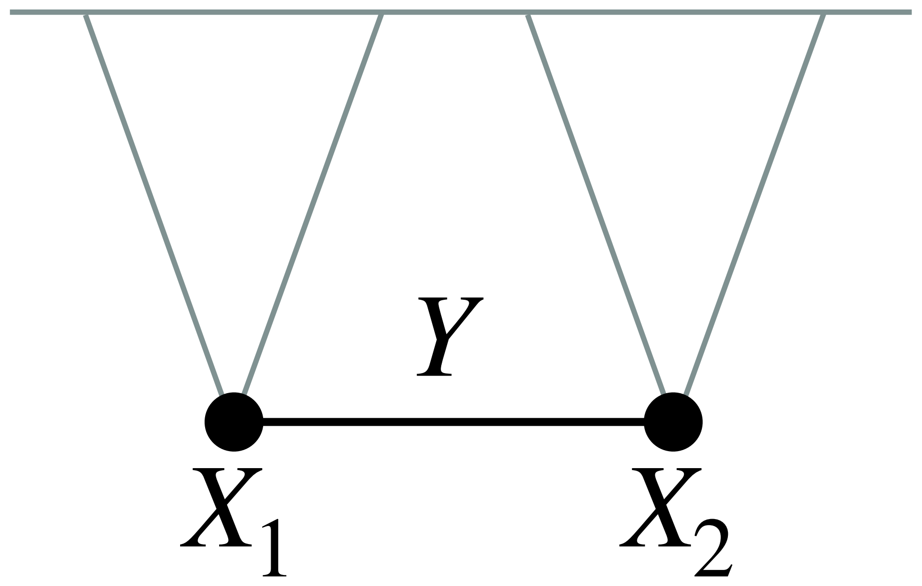
Motivating example: **Cosmological Integrals**



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Canonical form of a cosmological polytope

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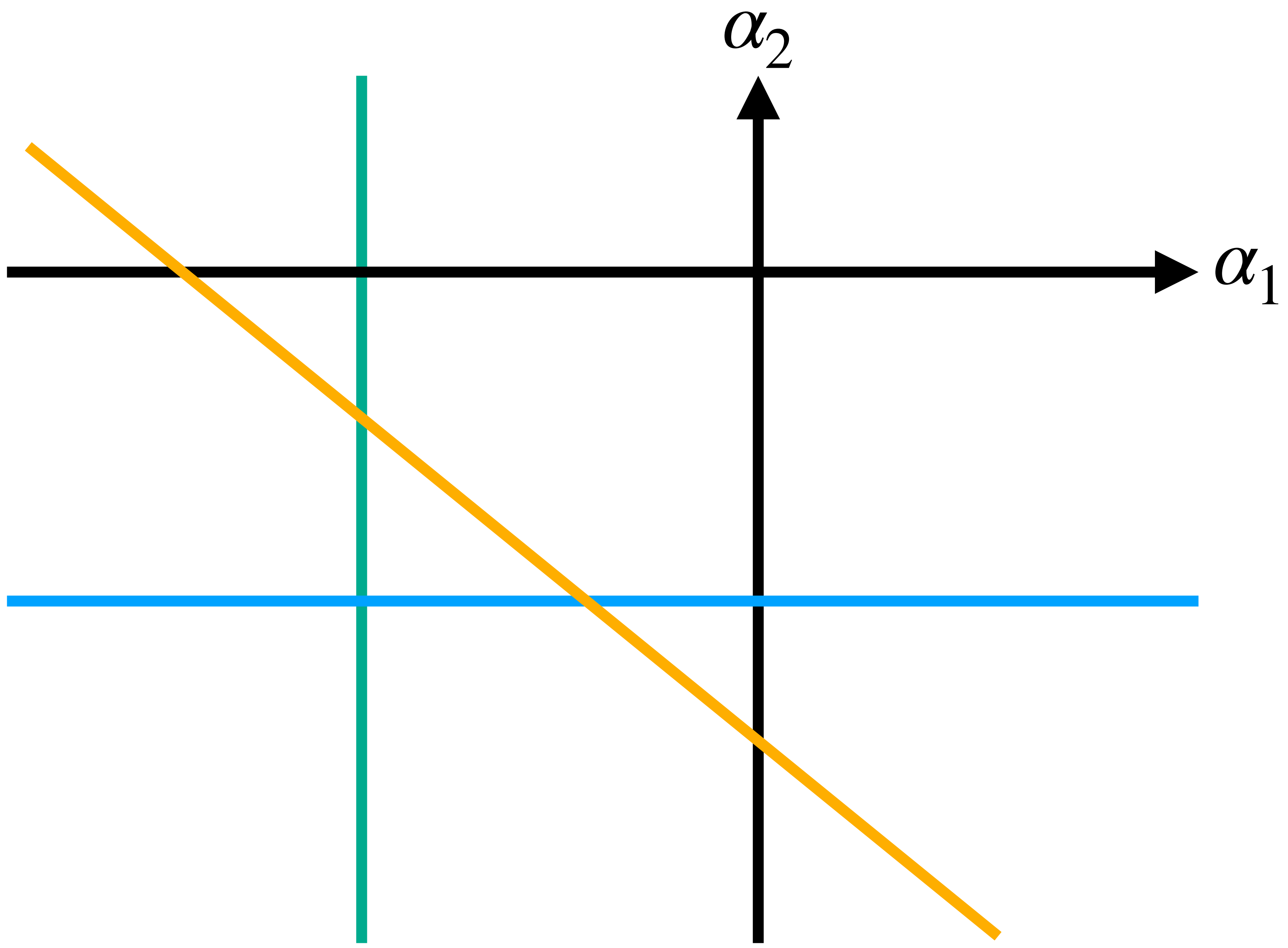
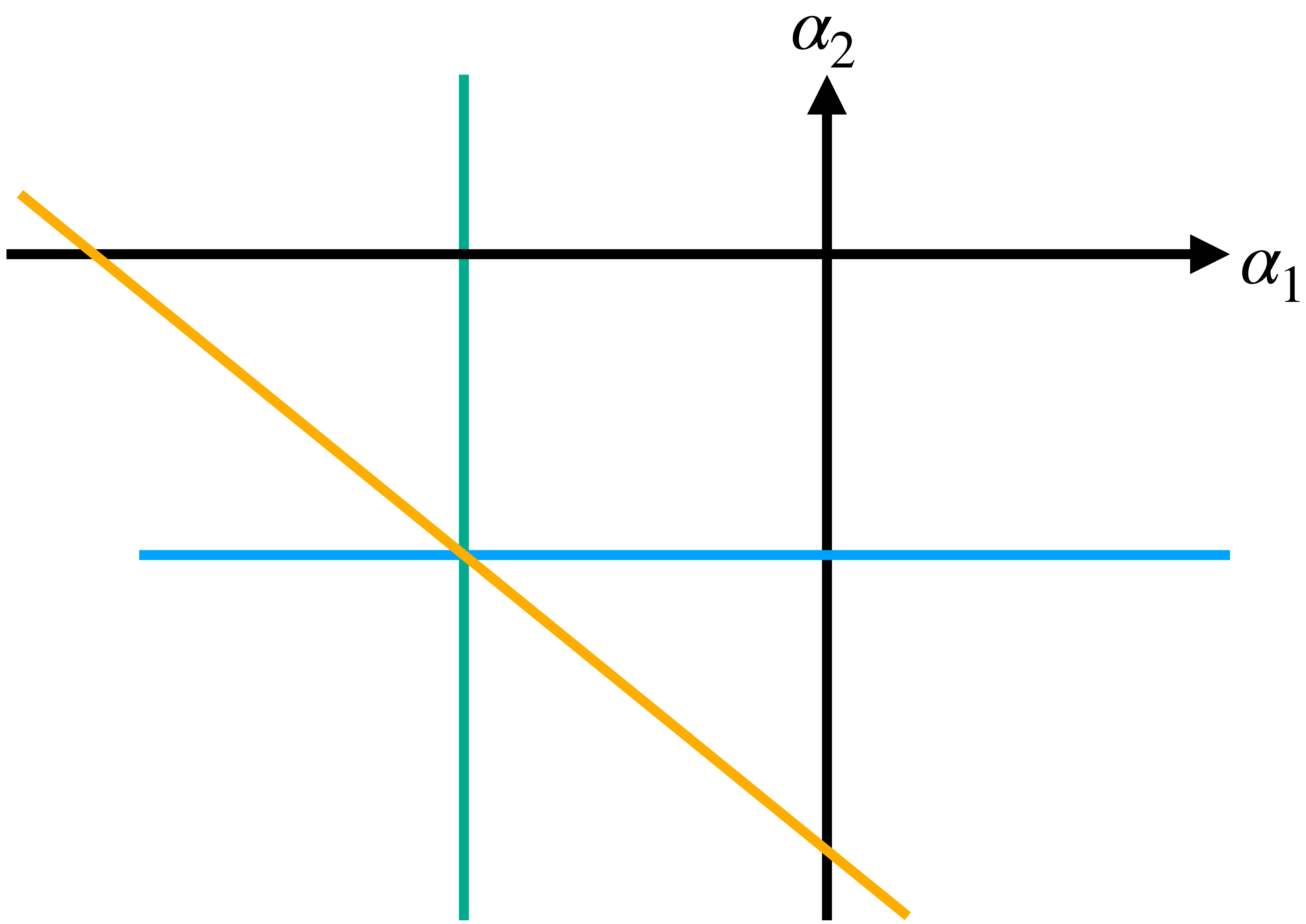


Question:

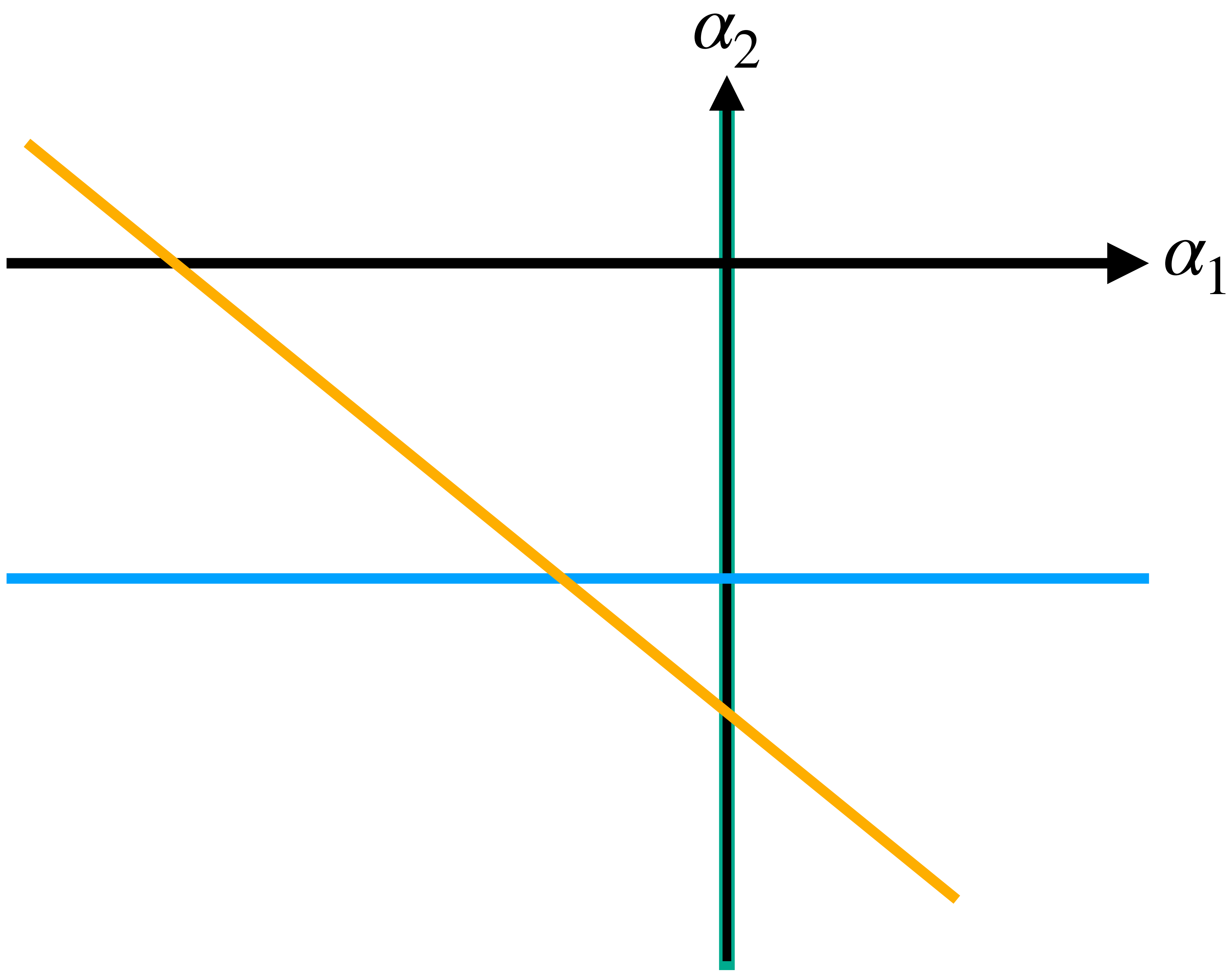
For which values of X_1, X_2, Y
The number of bounded regions is smaller than 4?

…for example

$Y = 0$



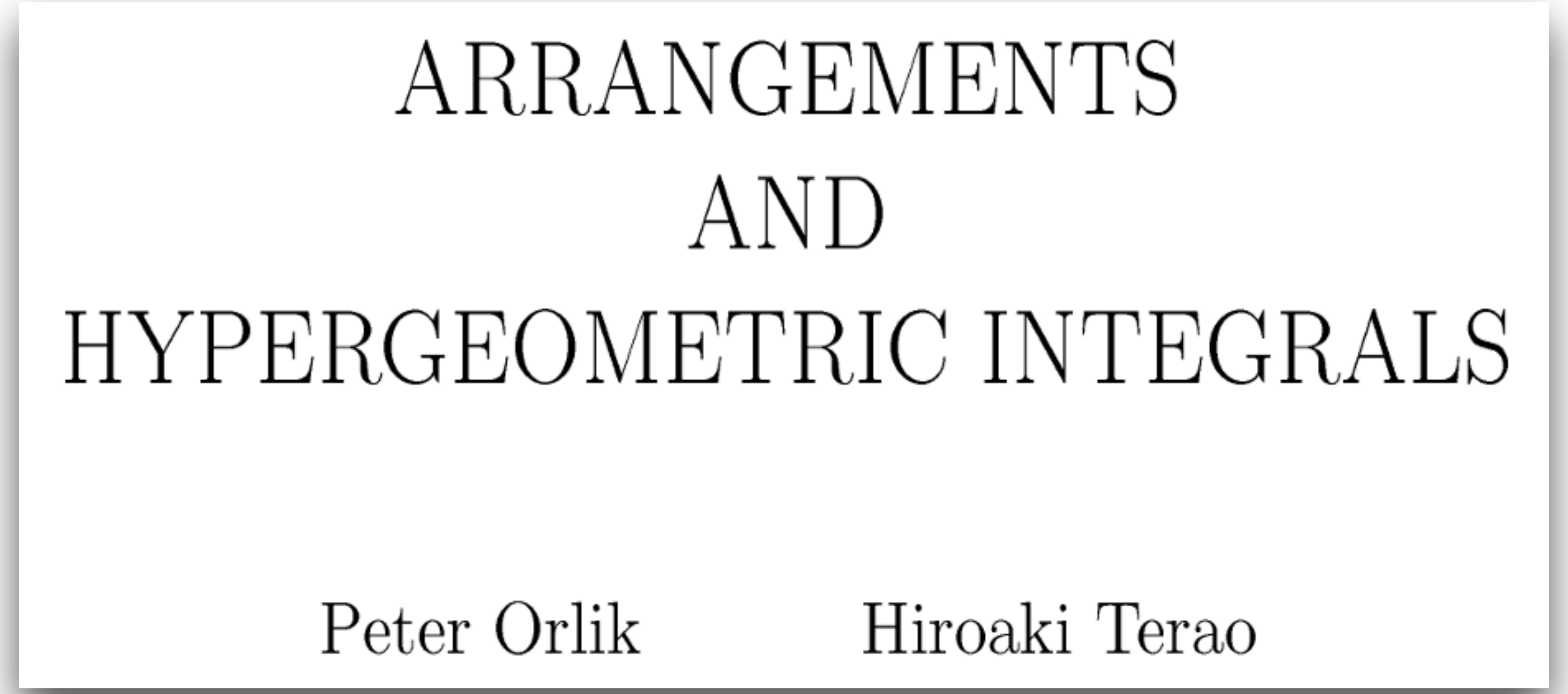
$X_2 + Y = 0$



Why?

Real arrangements:

Number of bounded chambers = $(-1)^k \cdot \chi(X_z)$


ARRANGEMENTS
AND
HYPERGEOMETRIC INTEGRALS

Peter Orlik   Hiroaki Terao

Why?

Real arrangements:

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The decrease in the Euler characteristic characterises:

@~45 mins ago: Matsubara-Heo

- The singularities of the integrals
- The singular locus of a $D-$module

Why?

Real arrangements:

Number of bounded chambers = $(-1)^k \cdot \chi(X_z)$

The decrease in the Euler characteristic characterises:

@~45 mins ago: Matsubara-Heo

- The singularities of the integrals
- The singular locus of a $D-$module

Theorem (Amendola, Bliss, Burke, Gibbons, Helmer, Hoşten, Nash, Rodriguez, Smolkin, 2012):

$$|\chi(X_z)| = \mathrm{vol}(A) \quad \Longleftrightarrow \quad z^* \in \mathbb{C}^A \setminus \{E_A(z) = 0\}$$

Principal A-determinant

Moreover, when $E_A(z) = 0$, we have $|\chi(X_z)| < \mathrm{vol}(A)$.

Thu @15:00: Dlapa

How?

$Z \subset \mathbb{C}^A$ smooth subvariety

$$\nabla_\chi(Z) \;=\; \{z \in Z \;:\; |\chi(X_z)| < |\chi^*|\}$$

Euler discriminant

Generic signed Euler characteristic

Advances in Mathematics

Volume 245, 1 October 2013, Pages 534-572

The discriminant of a system of equations

Alexander Esterov

Computer Programs in Physics

Principal Landau determinants

Claudia Fevola [a], Sebastian Mizera [b], Simon Telen [c]

How?

$Z \subset \mathbb{C}^A$ smooth subvariety

Generic signed Euler characteristic

$$\nabla_\chi(Z) \; = \; \{ z \in Z \; : \; |\chi(X_z)| < |\chi^*| \}$$

Euler discriminant

- $\nabla_\chi(Z)$ is a closed subvariety of Z
- If $Z = \mathbb{C}^A$, $\chi^* = \mathrm{vol}(A)$, then

$$\nabla_\chi(Z) = \{E_A = 0\}$$

Principal A-determinant

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ELSEVIER

The discriminant of a system of equations

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Principal Landau determinants

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Principal A-determinants [GKZ]

$$f_A(\alpha; z) \;=\; z_1\alpha^{m_1} + z_2\alpha^{m_2} + \ldots + z_s\alpha^{m_s}$$

$$E_A(z_1, \ldots, z_s) \;=\; \prod_{Q \in F(A)} \Delta_{A\cap Q}^{m_Q}$$

$m_Q \in \mathbb{N}$

$$A \cap Q = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ m_1 & m_2 & \cdots & m_s \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

$m_i \in Q$

$F(A)$: Set of faces of Conv(A)

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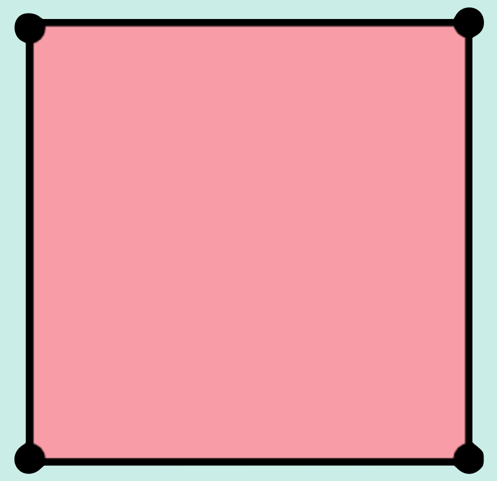
Set of faces of Conv(A)

Example $\quad f_A(\alpha, z) \;=\; z_1 + z_2\,\alpha_1 + z_3\,\alpha_2 + z_4\,\alpha_1\alpha_2$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

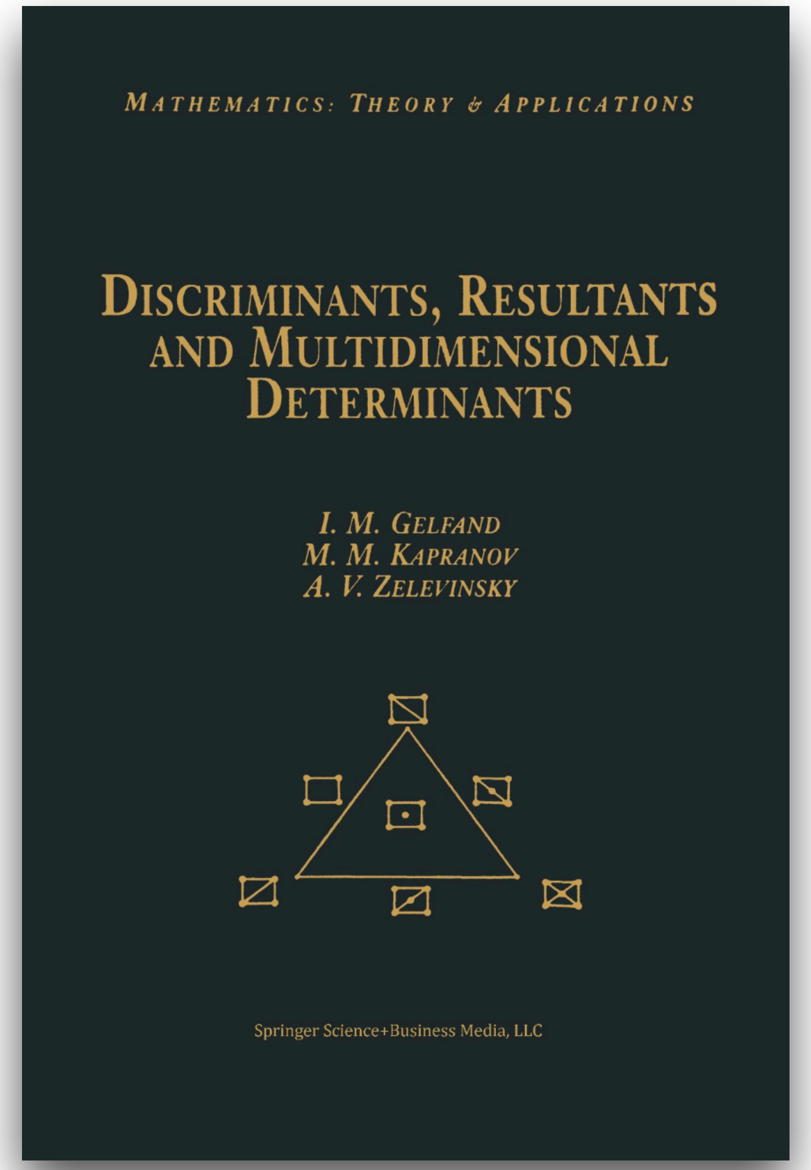
$$E_A \;=\; z_1 \cdot z_2 \cdot z_3 \cdot z_4 \cdot (z_1 z_4 - z_2 z_3)$$



$$Z = \mathbb{C}^{(k+1)(n-k)}$$

$$z := \begin{bmatrix} z_{0,k+1} & z_{0,k+2} & \cdots & z_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k,k+1} & z_{k,k+2} & \cdots & z_{k,n} \end{bmatrix}$$

$$E_A(z) = \prod_{I,J:|I|=|J|} \det(z_{I,J})$$



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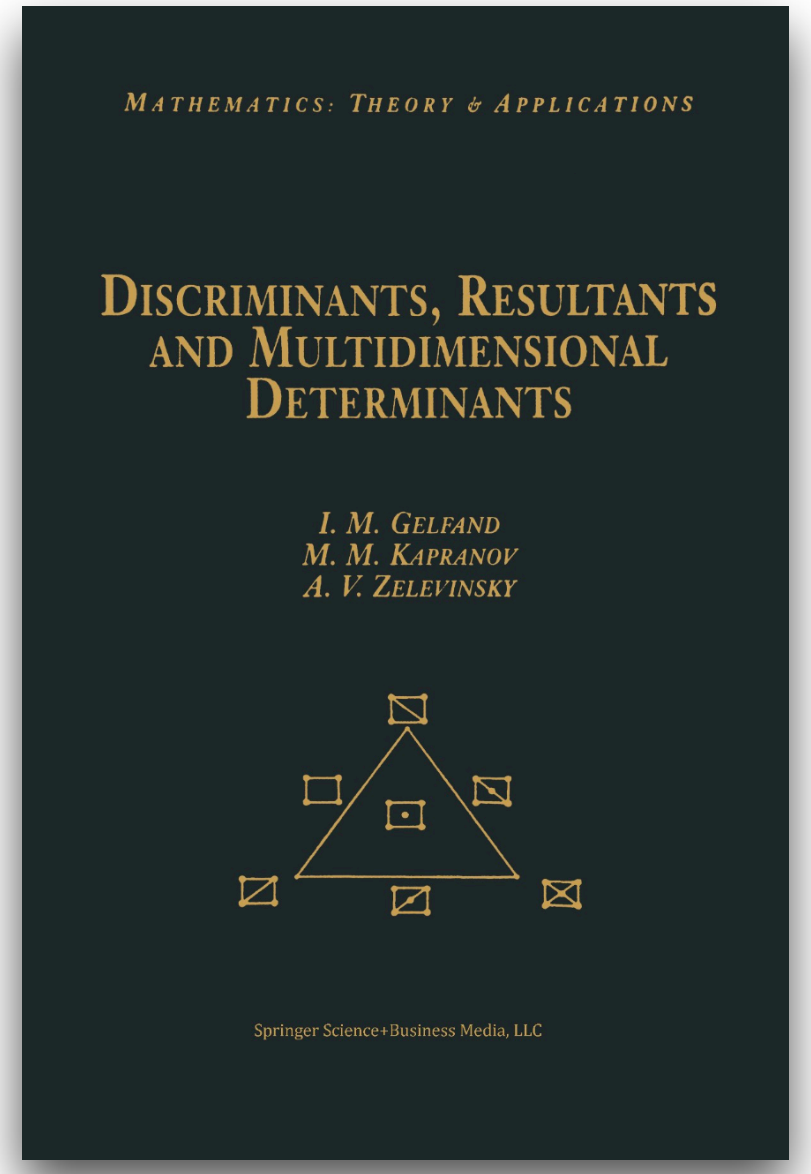

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$$k = 2, n = 6$$

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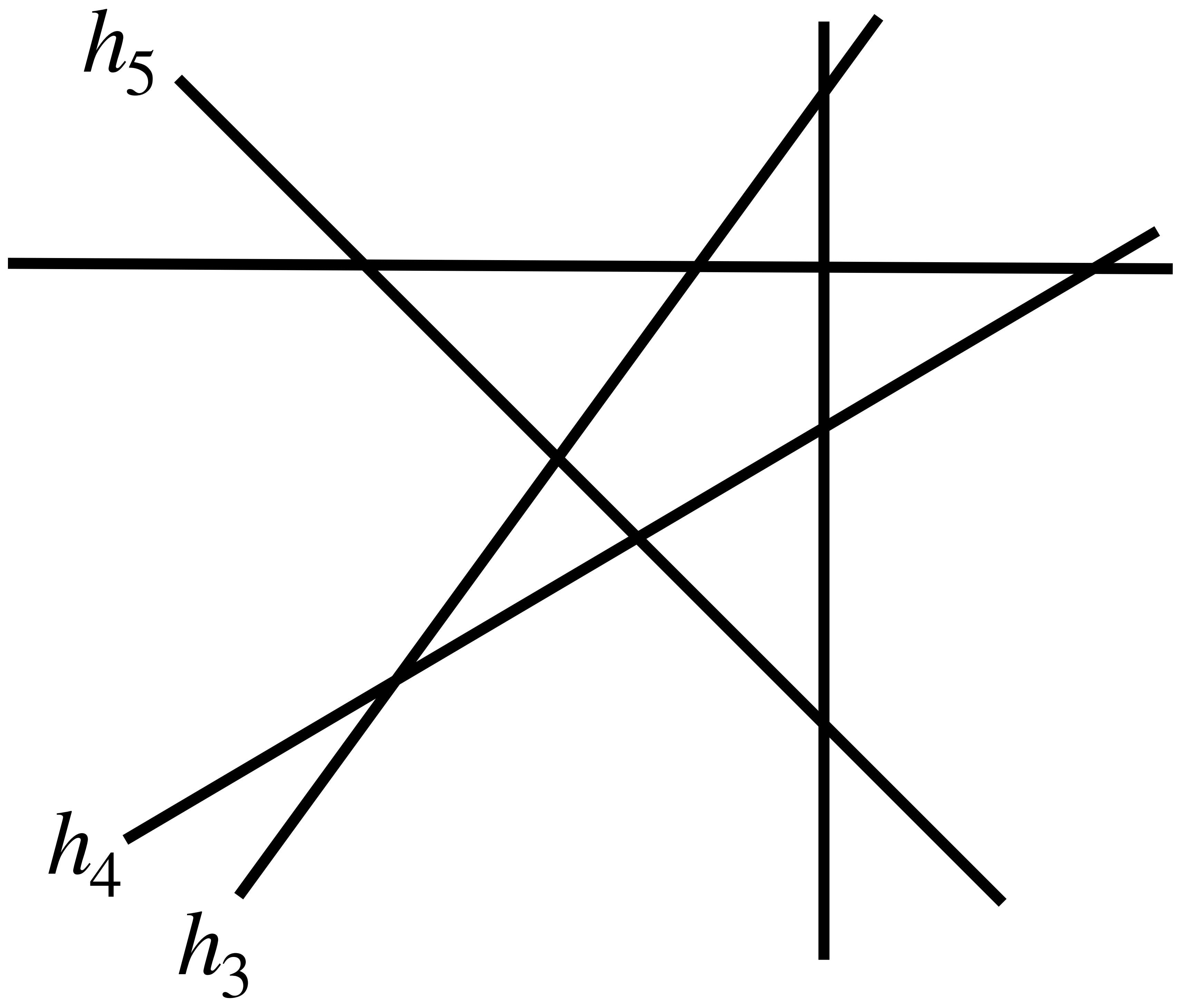

Sparse Arrangements

$$z := \begin{bmatrix} z_{0,k+1} & z_{0,k+2} & \cdots & z_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k,k+1} & z_{k,k+2} & \cdots & z_{k,n} \end{bmatrix}$$

Some of the $z_{i,j}$ are zero!

The Euler discriminant still coincides with the zero-locus of a principal A-determinant

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Theorem (F.-Matsubara-Heo):

$$E_A(z) = \prod_{\substack{I,J:|I|=|J|,\\ \det(z_{I,J})\neq 0}} \det(z_{I,J})^{m_{IJ}}$$

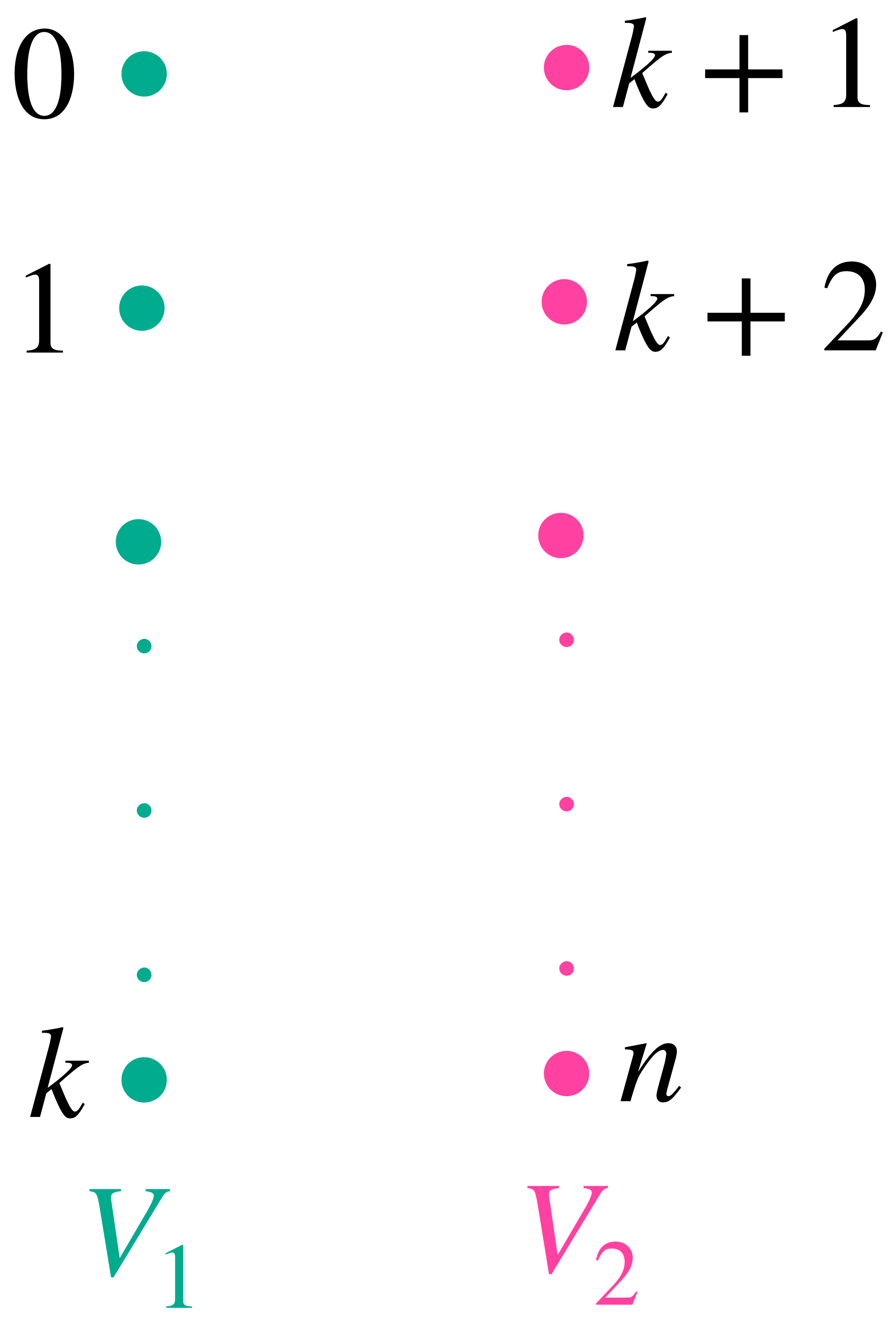
non-defective

Edge Polytopes

$$z := \begin{bmatrix} z_{0,k+1} & z_{0,k+2} & \cdots & z_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k,k+1} & z_{k,k+2} & \cdots & z_{k,n} \end{bmatrix}$$

$G = (V, E)$ bipartite graph

$V = \{0, \ldots, n\}$



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Edmonds matrix

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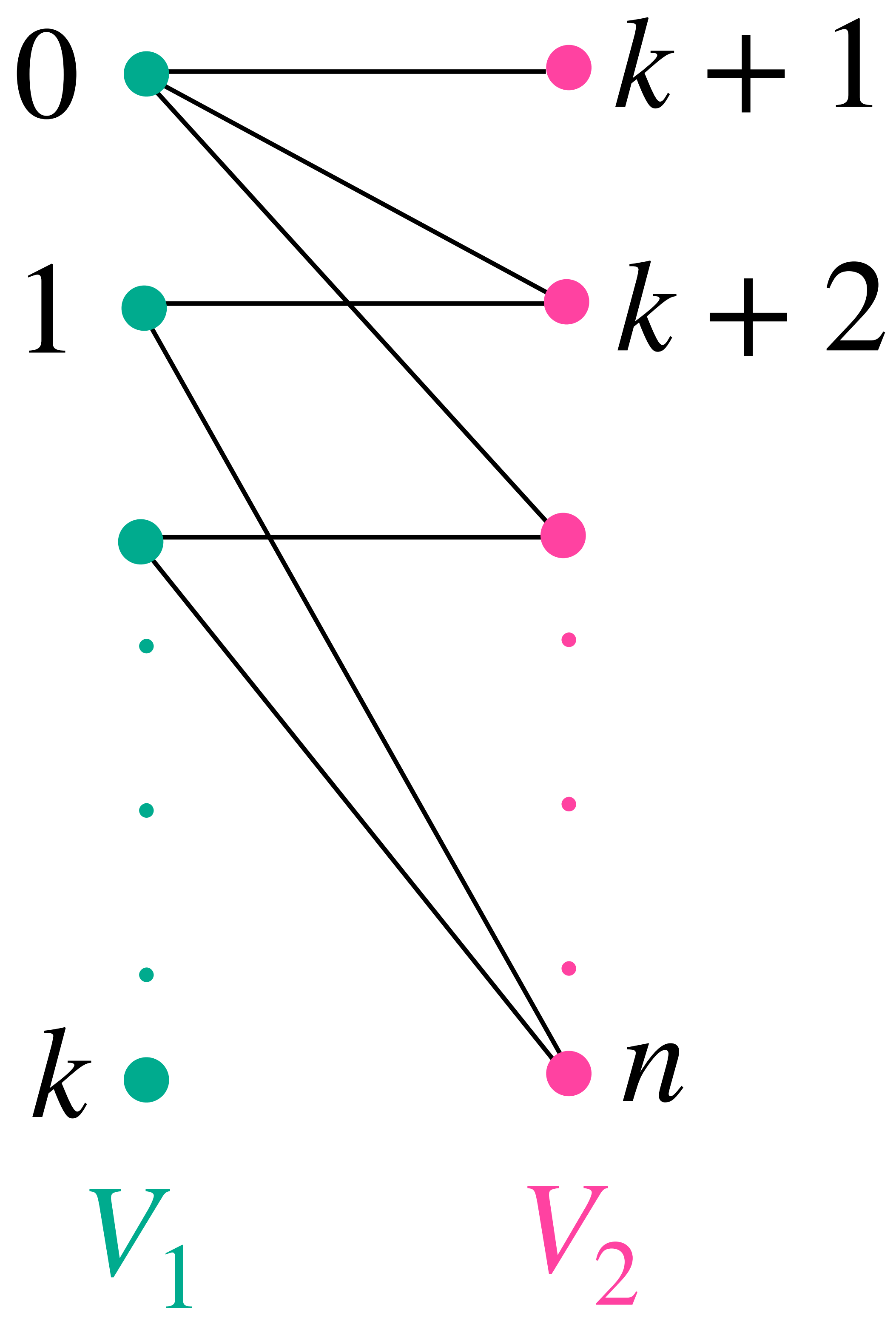

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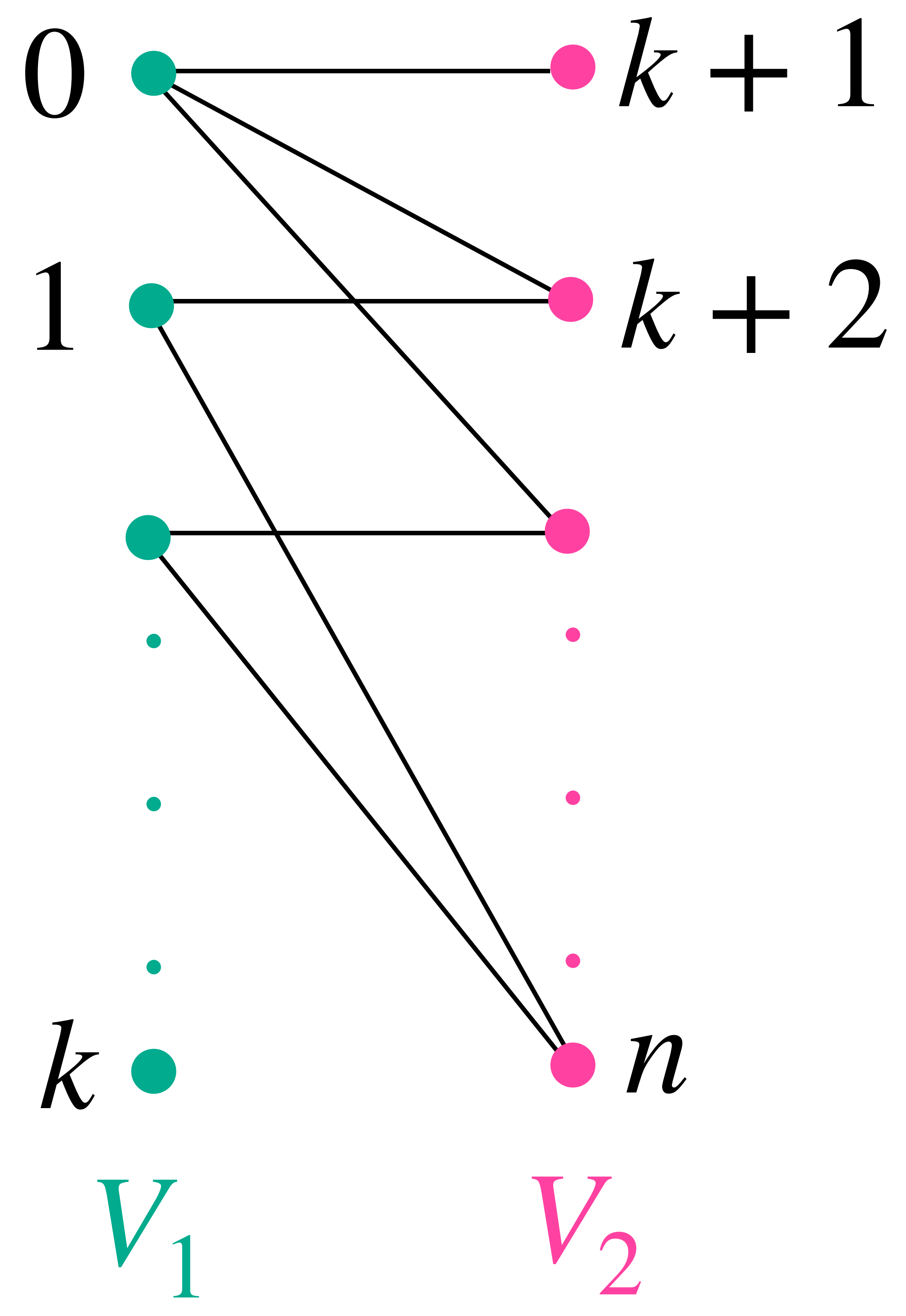

$A_G := \{a_{ij} := e_i + e_j \mid ij \in E(G)\}$

$P_G = \mathrm{conv}(A_G)$ edge polytope of G

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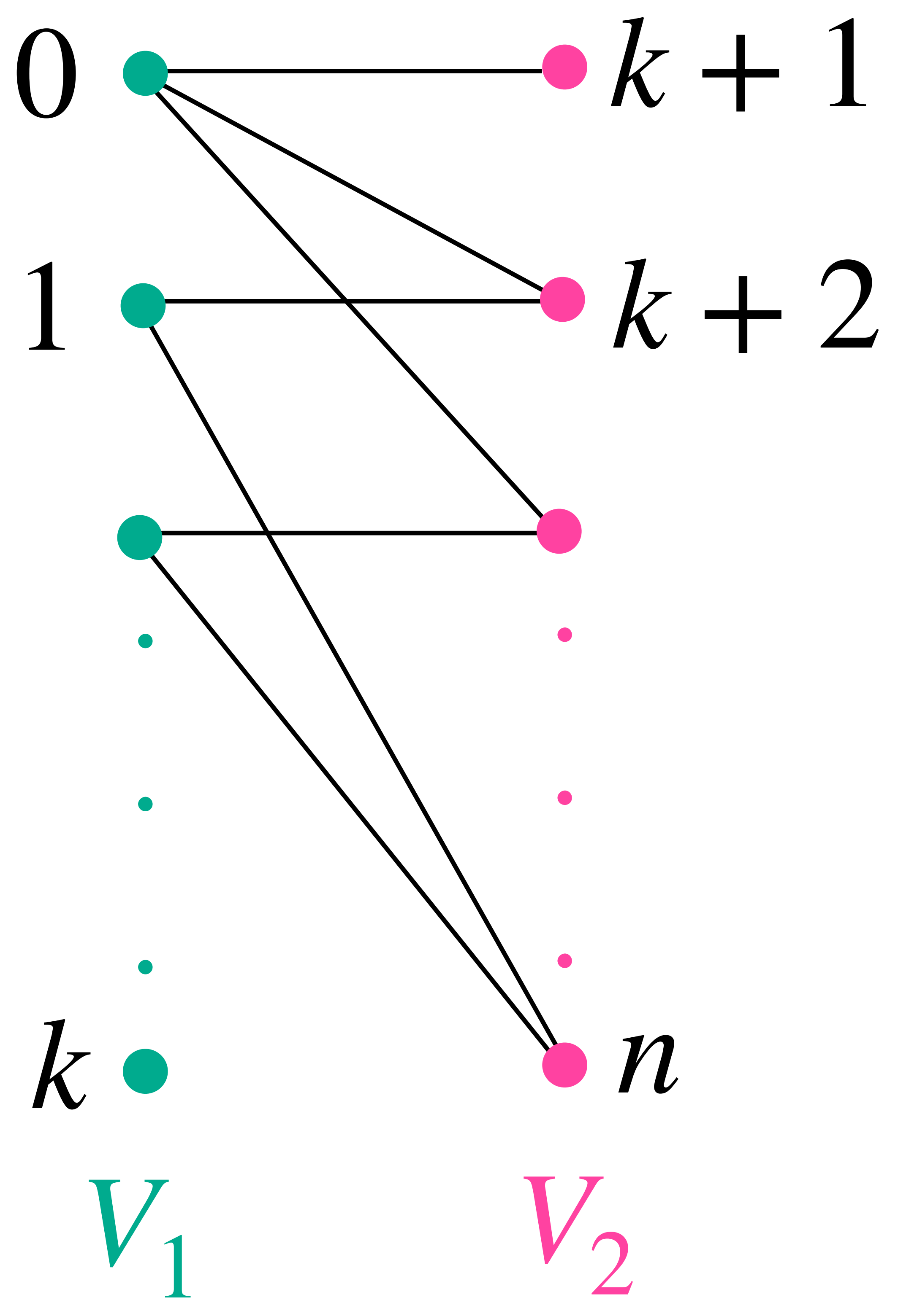
$$A = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ & \vdots & & & \vdots & & \cdots & & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ & A_{k+1} & & & A_{k+2} & & & & A_n & \end{pmatrix}$$



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Principal A-Determinant of a sparse arrangement

Lemma (F.-Matsubara-Heo):

Let $\emptyset \neq I \subset V_1$ and $\emptyset \neq J \subset V_2$. Then, $Q_{I,J}$ is a face of the polytope P_G.


ELSEVIER

Journal of Algebra

Volume 207, Issue 2, 15 September 1998, Pages 409-426

Regular Article

Normal Polytopes Arising from Finite Graphs ☆

Hidefumi Ohsugi *, Takayuki Hibi †

Principal A-Determinant of a sparse arrangement

Theorem (F.-Matsubara-Heo):

Let G be a connected bipartite graph. Then, one has the formula

$$E_{A_G}(z) = \prod_{\substack{I,J:|I| = |J|, \\ G_{I\cup J} \text{ is connected} \\ \text{and } (*)}} \det(z_{I,J})^{m_{IJ}}$$

Principal A-Determinant of a sparse arrangement

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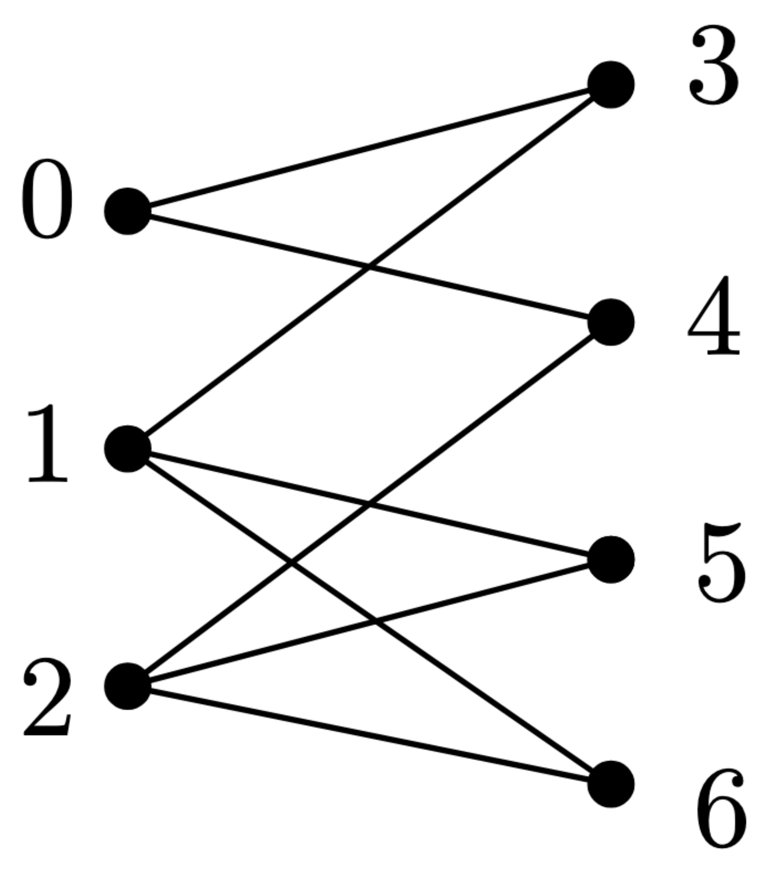
Non-defective $\iff$ Condition on subgraph

Change of matroid $=$ Change of χ

Example (k=2,n=6)

$G =$



$$z_G := \begin{bmatrix} z_{0,3} & z_{0,4} & 0 & 0 \\ z_{1,3} & 0 & z_{1,5} & z_{1,6} \\ 0 & z_{2,4} & z_{2,5} & z_{2,6} \end{bmatrix}$$

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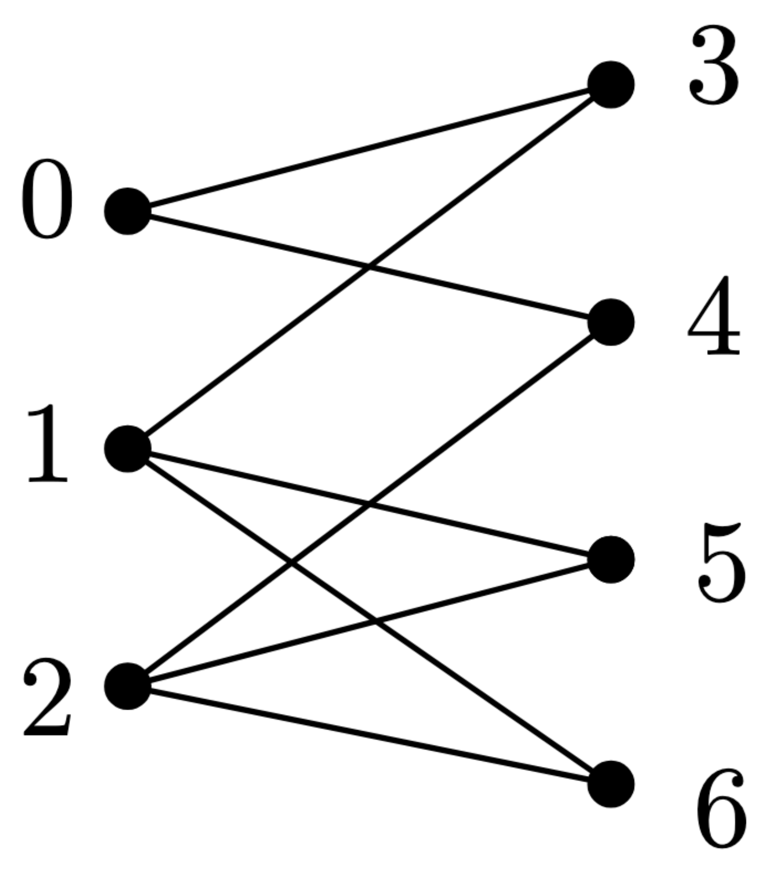
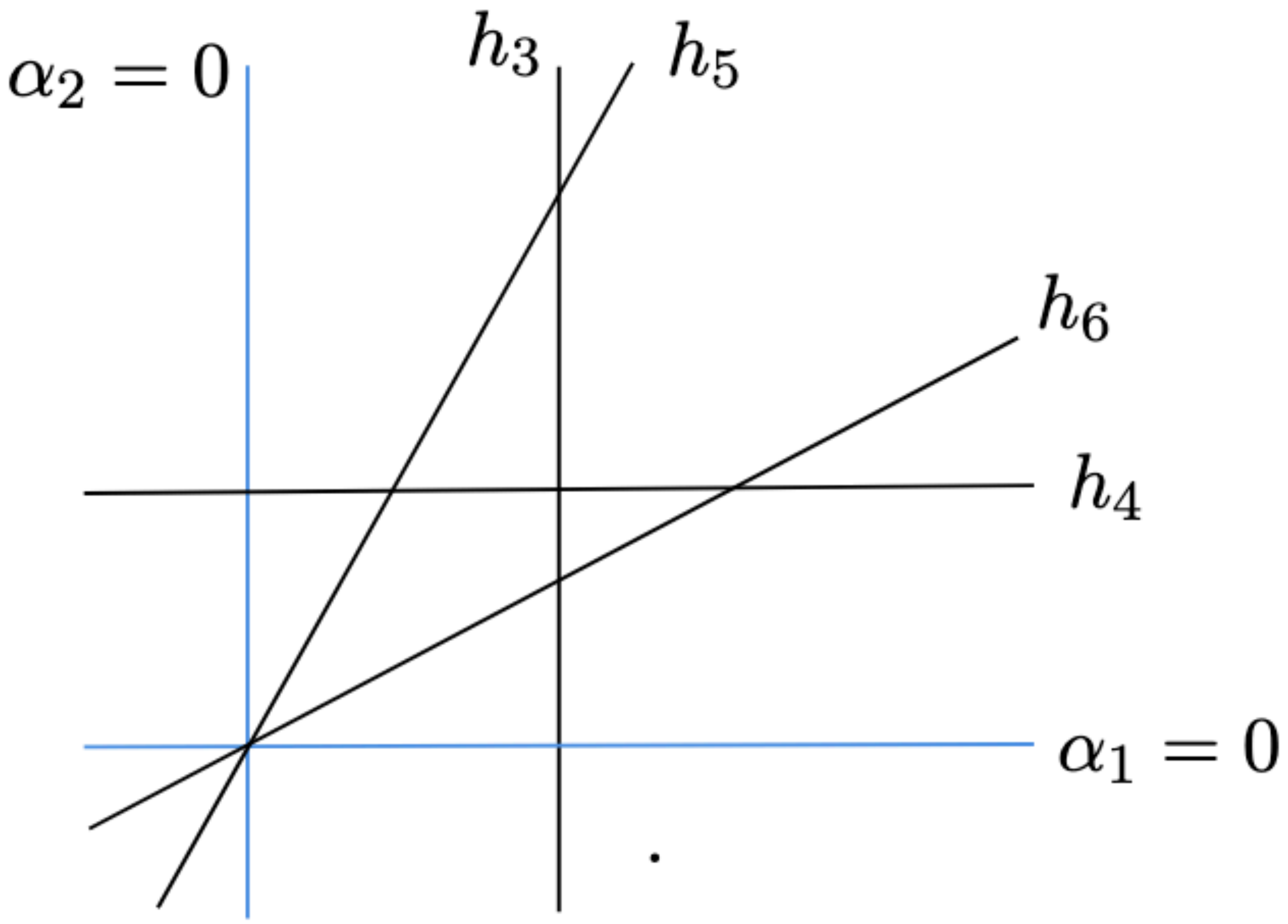
$G =$ 



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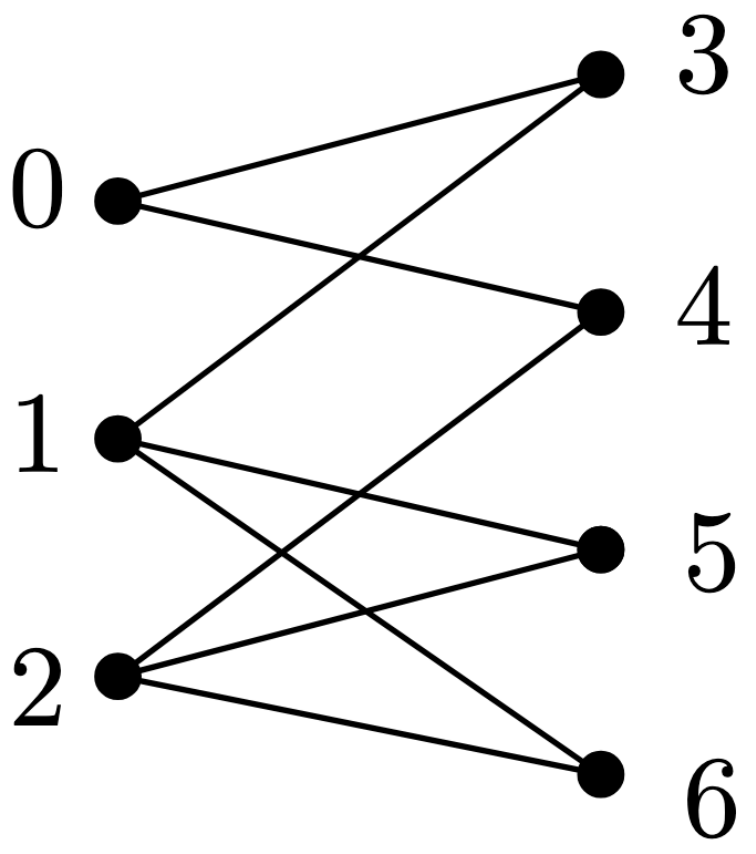
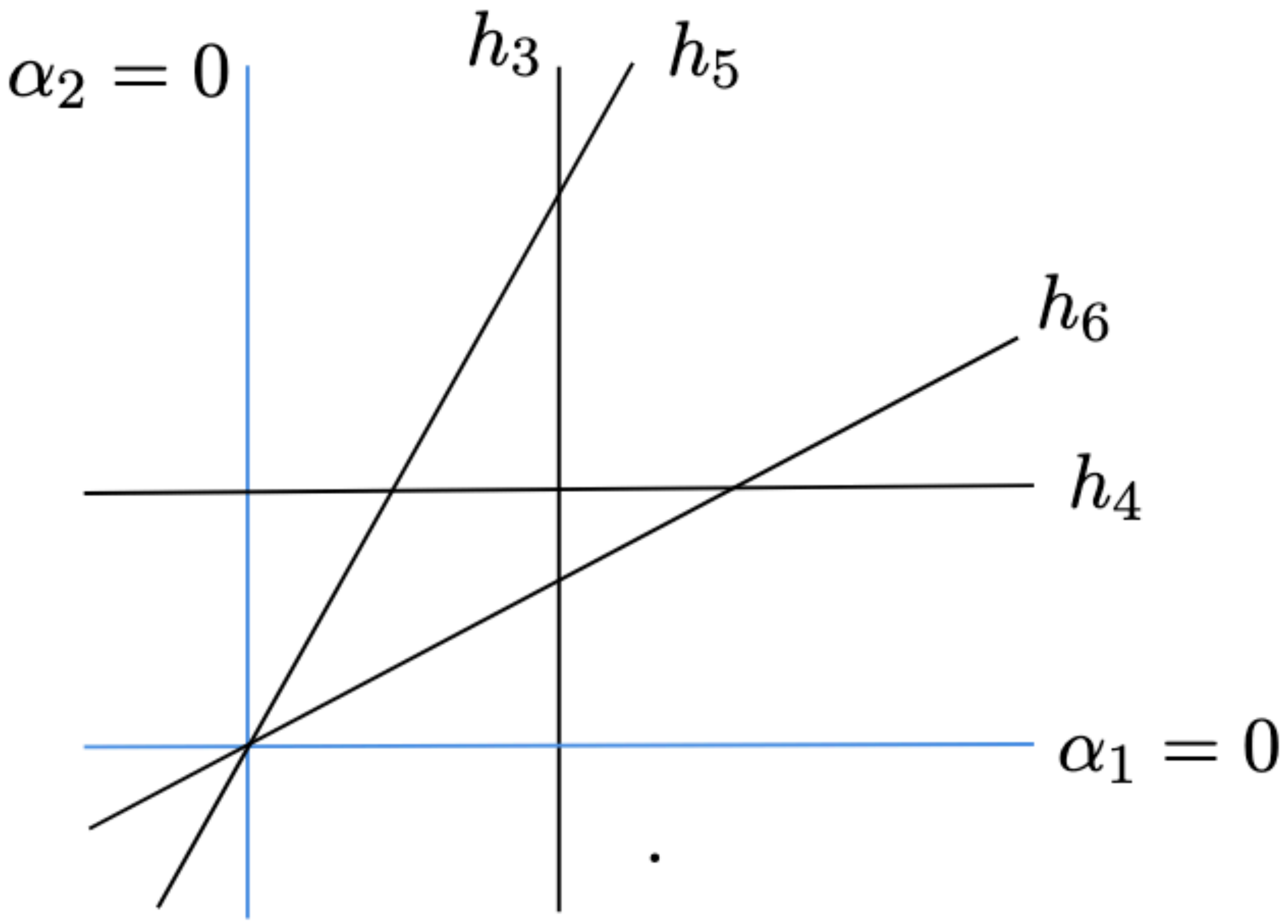
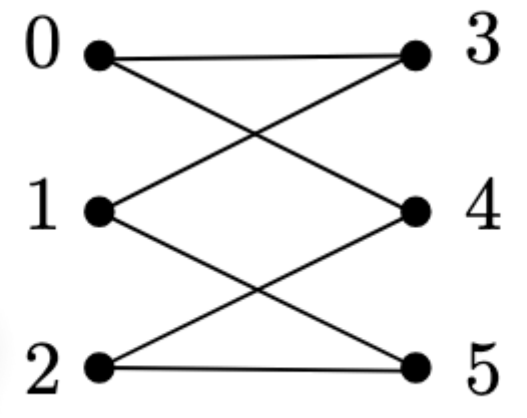
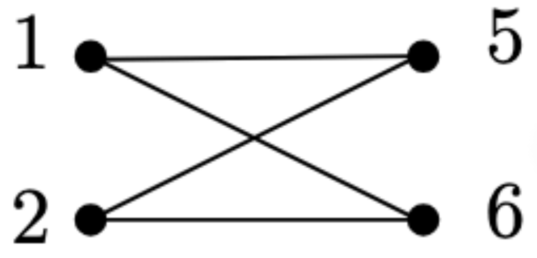


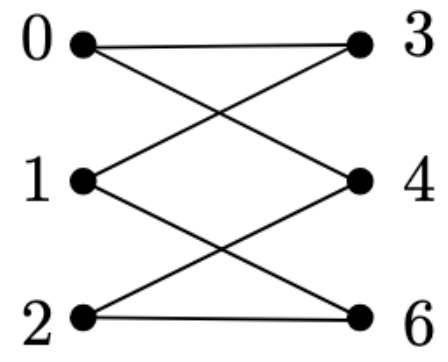


$$z_G := \begin{bmatrix} z_{0,3} & z_{0,4} & 0 & 0 \\ z_{1,3} & 0 & z_{1,5} & z_{1,6} \\ 0 & z_{2,4} & z_{2,5} & z_{2,6} \end{bmatrix}$$



$$E_{A_G}(z_G) = z_{03}^3 z_{13}^3 z_{04}^3 z_{24}^3 z_{15}^2 z_{25}^2 z_{16}^2 z_{26}^2 (z_{15}z_{26} - z_{16}z_{25})^2 (z_{03}z_{24}z_{15} + z_{13}z_{04}z_{25}) (z_{03}z_{24}z_{16} + z_{13}z_{04}z_{26})$$





Euler Discriminant

$$z := \begin{bmatrix} z_{0,k+1} & z_{0,k+2} & \cdots & z_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k,k+1} & z_{k,k+2} & \cdots & z_{k,n} \end{bmatrix} \in Z \subset \mathbb{C}^A$$

$$E_\chi(z) := \prod_{\substack{(I,J),\,|I|=|J| \\ \det(z_{I,J}) \neq 0}} \det(z_{I,J})$$

Euler Discriminant

$$z := \begin{bmatrix} z_{0,k+1} & z_{0,k+2} & \cdots & z_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k,k+1} & z_{k,k+2} & \cdots & z_{k,n} \end{bmatrix} \in Z \subset \mathbb{C}^A$$

$$E_\chi(z) := \prod_{\substack{(I,J),\, |I| = |J| \\ \det(z_{I,J}) \neq 0}} \det(z_{I,J})$$

Theorem (F.-Matsubara-Heo):

Let $\chi(X_z) > 0$ for some $z \in Z$, then

$$\nabla_\chi(Z) = \{z \in Z \mid E_\chi(z) = 0\}$$

Example (k=2,n=5)

$$\psi_2(X_1, X_2, Y, \varepsilon) = \int_\Gamma \frac{2 \cdot Y \cdot \alpha_1^\varepsilon \alpha_2^\varepsilon}{(X_1 + X_2 + \alpha_1 + \alpha_2)(X_1 + Y + \alpha_1)(X_2 + Y + \alpha_2)} d\alpha_1 \wedge d\alpha_2$$

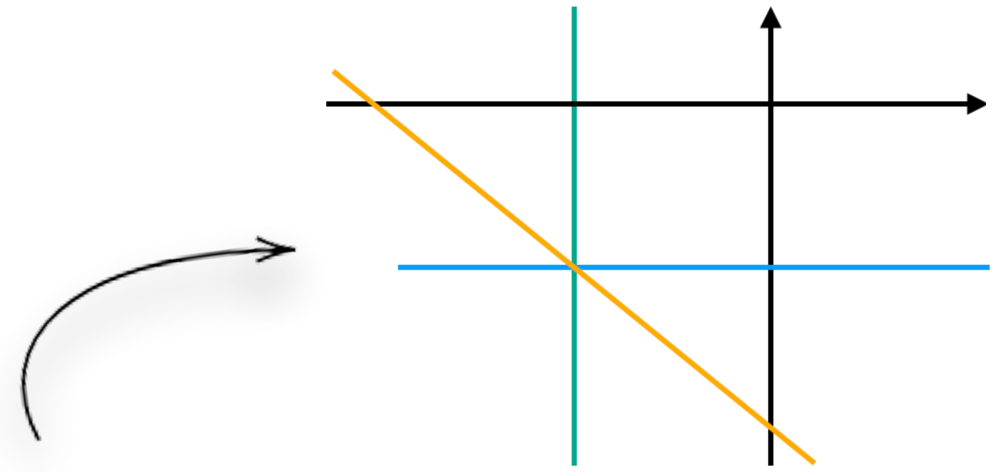
$$z_2(X, Y) \;=\; \begin{bmatrix} X_1 + X_2 & X_1 + Y_{12} & X_2 + Y_{12} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_\chi(z) = (X_1 + X_2)(X_1 + Y)^2(X_2 + Y)^2(X_1 - Y)(X_2 - Y)Y$$

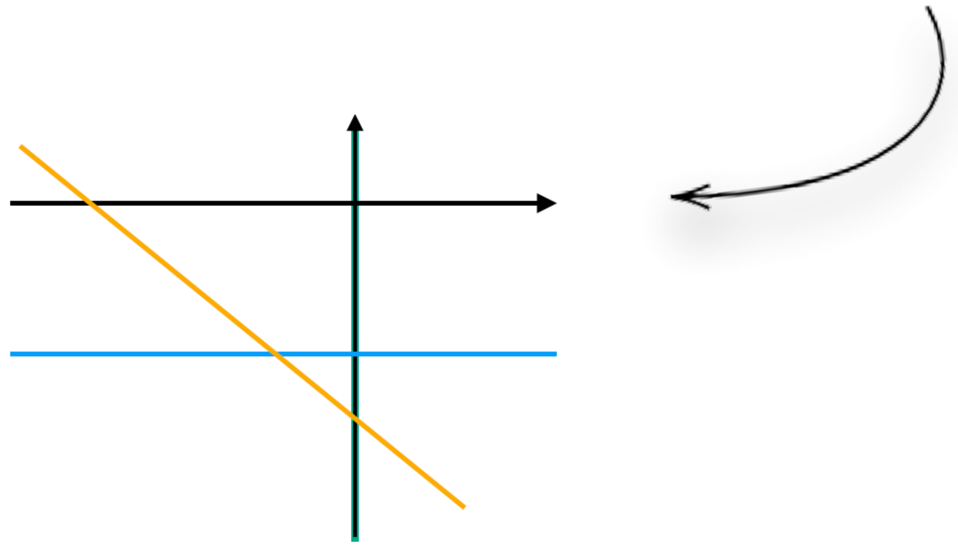
Example (k=2,n=5)

$$\psi_2(X_1, X_2, Y, \varepsilon) = \int_\Gamma \frac{2 \cdot Y \cdot \alpha_1^\varepsilon \alpha_2^\varepsilon}{(X_1 + X_2 + \alpha_1 + \alpha_2)(X_1 + Y + \alpha_1)(X_2 + Y + \alpha_2)} \mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_2$$

$$z_2(X, Y) = \begin{bmatrix} X_1 + X_2 & X_1 + Y_{12} & X_2 + Y_{12} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$



$$E_\chi(z) = (X_1 + X_2)(X_1 + Y)^2(X_2 + Y)^2(X_1 - Y)(X_2 - Y)Y$$



Multiplicities

WORK in PROGRESS

Theorem (Esterov): For a face Q of $\mathrm{Conv}(A)$

$$m_Q = \chi^* - \max_{z \in \Delta_{A \cap Q}} \chi_z$$

```
Generic |Euler characteristic|, χ* = 5
candidates = Any[z03, z03*z15*z24 + z04*z13*z25, z03*z16*z24 + z04*z13*z26, z04, z13, z15, z15*z26 - z16*z25, z1
6, z24, z25, z26]
Subspace z03 has χ = 2 < χ*
Subspace z03*z15*z24 + z04*z13*z25 has χ = 4 < χ*
Subspace z03*z16*z24 + z04*z13*z26 has χ = 4 < χ*
Subspace z04 has χ = 2 < χ*
Subspace z13 has χ = 2 < χ*
Subspace z15 has χ = 3 < χ*
Subspace z15*z26 - z16*z25 has χ = 3 < χ*
Subspace z16 has χ = 3 < χ*
Subspace z24 has χ = 2 < χ*
Subspace z25 has χ = 3 < χ*
Subspace z26 has χ = 3 < χ*
```

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Multiplicities

WORK in PROGRESS

Subdiagram Volume

$$m_Q = i(Q, A) \cdot u(S(A)/Q) = u(S(A)/Q) = \mathrm{mult}_0(Y)$$

$i(Q, A) \to 1$

Toric variety constructed from $S(A)/Q$

Multiplicities

WORK in PROGRESS

Subdiagram Volume

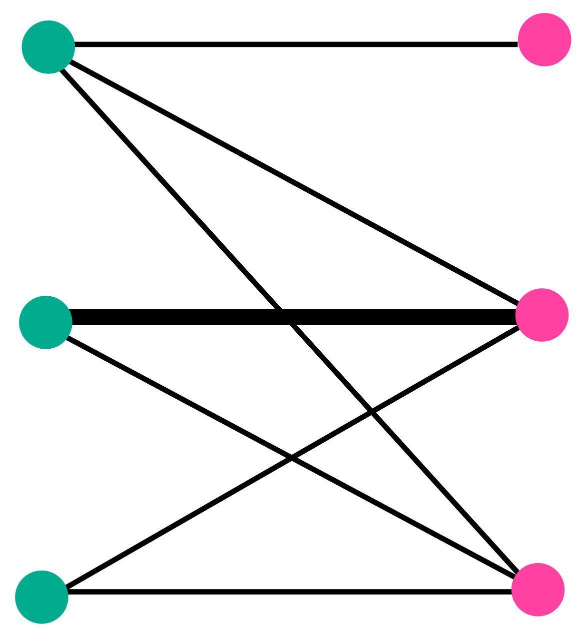
$$m_Q = i(Q,A) \cdot u(S(A)/Q) = u(S(A)/Q) = \text{mult}_0(Y)$$

$i(Q,A) = 1$

Toric variety constructed from $S(A)/Q$

Question: Can we derive it from the graph?

- Different vertices can have different multiplicities: it is about the graph G/Q



Strategy: Construct a toric ideal associated to G/Q and compute $\text{mult}_0(Y_{G/Q})$

On the note of:
Advancing diversity in math and physics

Rita Teixeira da Costa
University of Cambridge

Chiara Meroni
ETH ITS, Zürich

She^{+}Maths

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Thank you!

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A-discriminants

Question: Where do solutions $\mathscr{I}_\Gamma(z)$ to the GKZ system develop singularities?

$$\nabla_A^\circ = \left\{ z \in \mathbb{C}^s \,:\, \exists \alpha \in (\mathbb{C}^*)^n \text{ s.t. } f_A(\alpha; z) = \partial_\alpha f_A(\alpha; z) = 0 \right\}$$

The A-discriminant variety $\nabla_A = \overline{\nabla_A^\circ}$ records values of z for which $V_{A,z}$ is singular.

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad f_A(\alpha, z) = z_1 + z_2\,\alpha_1 + z_3\,\alpha_2 + z_4\,\alpha_1\alpha_2$$

$$\Delta_A = \det \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = z_1 z_4 - z_2 z_3$$