IBP Reduction using Gröbner bases

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Collaborative Research Center TRR 257
Particle Physics Phenomenology after the Higgs Discovery

based on	M. Barakat, R. Brüser, J. Piclum, TH	2207.09275 (PoS),
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	M. Barakat, R. Brüser, J. Müller, TH	in progress

Holonomic Techniques for Feynman Integrals, Munich, October 16th, 2024

Outline

- Introduction / motivation
- Mathematical framework
- Gröbner basis of the ideal of IBP relations in the double-shift algebra
- Special IBP relations
- Linear algebra ansatz
- Conclusion and outlook

Motivation

- Reveal the mathematical structure of Feynman integrals
- The standard model of particle physics is extremely successful!
- Contemporary paradigm:

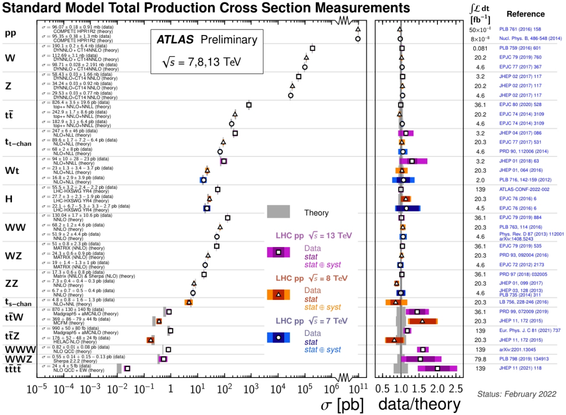
Discovery through precision

exptl. measurements and theo. predictions

$\Downarrow$

search for deviations from the SM

- LHC and other accelerators (Belle II, . . .) are precision machines



Setting the stage

- Integral reduction is an indispensable tool for higher-order calculations
- Based on integration-by-parts (IBP) relations
  [Tkachov'81; Chetyrkin,Tkachov'81]
- Many sophisticated public and private tools exist
  - AIR [Anastasiou,Lazopoulos'04]
  - Reduze [Studerus'09; Manteuffel,Studerus'12]
  - FIRE [Smirnov et al.'08+]
  - LiteRed [Lee'12+]
  - Kira [Maierhöfer,Usovitsch,Uwer'17; Klappert,Lange,Maierhöfer,Usovitsch'20]
  - Blade [Guan,Liu,Ma,Wu'24]
  - Crusher . . . [Marquard,Seidel]

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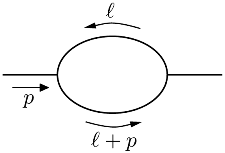
- Mostly based on Laporta's algorithm [Laporta'01]
  - Solves IBP equations for numerical values of indices with Gaussian elimination.
  - Has vastly served the community over the past $\sim$ two decades
- Several refinements exist, e.g.
  - Parallelization
  - Methods from finite fields

    [v. Manteuffel,Schabinger'14; Smirnov,Chukharev'19]
    [Peraro'16'19; Klappert,Klein,Lange'19'20]
  - Block-triangular form [Guan,Liu,Ma,Wu'24]
- Drawbacks
  - Compute many more integrals than required
  - Large storage required for results of $10^{\sim 4-6}$ integrals

Example: one-loop bubble

- Start with one-loop massless bubble

[figures courtesy by Robin Brüser]



$(p^2 \neq 0)$

$$F(a_1, a_2) = \int \frac{\mathrm{d}^d\ell}{i\pi^{d/2}} \frac{1}{D_1^{a_1}\, D_2^{a_2}}\,, \qquad a_i \in \mathbb{Z}$$

$$\begin{aligned} D_1 &= -\ell^2 \\ D_2 &= -(\ell + p)^2 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} \ell^2 &= -D_1 \\ \ell \cdot p &= \frac{1}{2}(D_1 - D_2 - p^2) \end{aligned}$$

- Symmetry: $F(a_1, a_2) = F(a_2, a_1)$
- Scaleless integrals:

$F(a_1, a_2) = 0$ if $a_1 \leq 0$ or $a_2 \leq 0$

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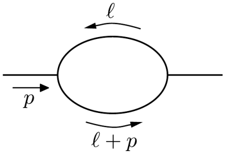

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- IBP equations

[Tkachov'81;Chetyrkin,Tkachov'81]

$$\int \frac{d^d\ell_1}{i\pi^{d/2}} \cdots \int \frac{d^d\ell_L}{i\pi^{d/2}} \frac{\partial}{\partial \ell_i^\mu} \frac{v_j^\mu}{D_1^{a_1} \cdots D_n^{a_n}} = 0$$

- Standard IBPs for one-loop bubble

$\underline{v = \ell:}$

$$\begin{aligned} 0 = {} & (d - a_2 - 2a_1)F(a_1, a_2) - a_2 p^2 F(a_1, a_2+1) \\ & - a_2 F(a_1 - 1, a_2 + 1) \end{aligned}$$

$\underline{v = p:}$

$$\begin{aligned} 0 = {} & (a_1 - a_2)F(a_1, a_2) + a_2 p^2 F(a_1, a_2 + 1) - a_1 p^2 F(a_1 + 1, a_2) \\ & + a_2 F(a_1 - 1, a_2 + 1) - a_1 F(a_1 + 1, a_2 - 1) \end{aligned}$$

- Laporta algorithm: Plug in integers for $a_{1,2}$, solve generated system of eqs., e.g. by Gaussian elimination. Obtain each integral in terms of master integral $F(1, 1)$. $F(2,3) = \cdots = \frac{(d-8)(d-5)(d-3)}{2\,p^6} F(1,1)$

Setting the stage

- New ideas from
  - syzygy equations
    [Kosower et al.’10’18; Schabinger et al.’11’20; Ita’15; Böhm et al.’17]
  - algebraic geometry and module intersection
    [Larsen,Zhang’14; Böhm et al.’18’19; Wu,Böhm,Ma,Xu,Zhang’23]
  - intersection numbers
    [Mastrolia,Mizera’18; Frellesvig et al.’19’20; Weinzierl’20]
- Our approach
  - Leave propagator powers symbolic
  - Find a Gröbner basis of the left ideal of IBP relations in the double-shift algebra
  - Derive normal form from Gröbner basis

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- Preliminary work using Gröbner bases
  - in combination with PDE and dimension shift
    [Tarasov'98'04]
  - in combination with shift algebras
    [Gerdt'05 (but chosen example is scaleless and trivial)]
  - MAPLE implementation of IBP reduction with shift algebras [Gerdt,Robertz'06]
  - introduction of sector bases (s-bases)
    [Smirnov,Smirnov'05-'08]
  - discussion of Gröbner bases and left and right ideals (for IBPs and scaleless ints) [Lee'08]

Reminder about mathematical quantities

Let R be a ring over a field $\mathbb{K}$.

- A subset $I \subseteq R$ is an **ideal** if it forms an additive group and fulfils

$$x \in R \wedge y \in I \implies xy \in I \ \wedge \ yx \in I.$$

  - Example: Set of even integers is an ideal in the ring of integers.
  - Left and right ideal analogous

- Multi-index notation

$$x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

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$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

- A **monomial order** on R is a total order $>$ s.t.
$$x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta \quad \forall \alpha, \beta, \gamma \in \mathbb{N}^n$$
  - **Lexicographic order**
$$x^\alpha >_{\text{lex}} x^\beta \quad \iff \quad \text{first nonzero entry of } \alpha - \beta > 0.$$
  - **Degree lexicographic order**
$$x^\alpha >_{\text{dlex}} x^\beta \iff \deg x^\alpha > \deg x^\beta \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and first nonzero entry of } \alpha - \beta > 0).$$
  - **Degree reverse lexicographic order**
$$x^\alpha >_{\text{drlex}} x^\beta \iff \deg x^\alpha > \deg x^\beta \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and last nonzero entry of } \alpha - \beta < 0).$$
  - Example: $x_1^2 x_2 x_3^3 >_{\text{lex}} x_1 x_2^3 x_3^2 >_{\text{lex}} x_1 x_2 x_3^5$
$$x_1 x_2 x_3^5 >_{\text{dlex}} x_1^2 x_2 x_3^3 >_{\text{dlex}} x_1 x_2^3 x_3^2$$
$$x_1 x_2 x_3^5 >_{\text{drlex}} x_1 x_2^3 x_3^2 >_{\text{drlex}} x_1^2 x_2 x_3^3$$

Reminder about mathematical quantities

- For $f \in R$, the **leading term** $\boldsymbol{L}_>(f)$ w.r.t. $>$ is the largest term in f w.r.t. $>$.
- A finite subset $G = \{g_1, \ldots, g_r\} \subset I$ is a **Gröbner basis for** $\boldsymbol{I}$ if

$$\boldsymbol{L}_>(I) = \boldsymbol{L}_>(G)\,,$$

    - i.e. the leading submodule of I is generated by the leading terms of the elements of G.
    - Hence G generates I.
- One way of computing Gröbner bases is via **Buchberger's algorithm**.
    - Also applicable to the non-commutative case (left ideal).
- The **remainder** h of

$$g = \sum_{i=1}^{r} f_i\, g_i + h$$

  is uniquely determined by g, I, and $>$.
- Call $\mathrm{NF}_{I,>}(g) = \mathrm{NF}_G(g) = h$ the **normal form** of g mod I w.r.t. $>$.

Operators and algebra

- Introduce operators with **partial right** action. For each $i = 1, \dots, n$ have

$$I(\dots, z_i, \dots) \bullet D_i = I(\dots, z_i - 1, \dots), \qquad \underbrace{I(\dots, z_i, \dots)}_{\text{not scaleless}} \bullet D_i^- = I(\dots, z_i + 1, \dots),$$

[note that $D_i \sim \boldsymbol{i}^-$ and $D_i^- \sim \boldsymbol{i}^+$]

$$I(\dots, z_i, \dots) \bullet a_i = z_i I(\dots, z_i, \dots), \qquad I(\dots, \underbrace{z_i}_{\neq 0}, \dots) \bullet a_i^{-1} = \frac{1}{z_i} I(\dots, z_i, \dots).$$

- The following computations will take place in the **non-commutative rational** double-shift algebra

$$Y := \mathbb{Q}(d, s_{ij}, m_i^2)(a_1, \dots, a_n)\langle D_j, D_j^- \mid j = 1, \dots, n\rangle / (D_i D_i^- = 1 = D_i^- D_i \mid i = 1, \dots, n)$$

in the indeterminates $a_1, \dots, a_n, D_1, \dots, D_n, D_1^-, \dots, D_n^-$ with relations

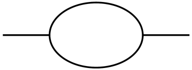
$$[a_i, D_j] = \delta_{ij} D_i\,, \qquad [a_i, D_j^-] = -\delta_{ij} D_i^-\,, \qquad D_i D_i^- = 1,$$

[no summation over repeated indices]

$$[a_i, a_j] = [D_i, D_j] = [D_i^-, D_j^-] = [D_i, D_j^-] = 0 \quad .$$

IBPs as left ideal in the double-shift algebra

- Write standard IBP relations in terms of operators, e.g. for one-loop bubble



$$0 = (d - z_2 - 2z_1)\,F(z_1, z_2) - z_2\,p^2\,F(z_1, z_2 + 1) - z_2\,F(z_1 - 1, z_2 + 1)$$

$$= F(z_1, z_2) \bullet \underbrace{\left[(d - a_2 - 2a_1) - p^2\,a_2 D_2^- - a_2\,D_1\,D_2^-\right]}_{= r_1}$$

- Similarly $r_2 = -a_1\,D_1^-\,D_2 + a_2\,D_2^-\,D_1 + p^2\,a_2\,D_2^- - p^2\,a_1\,D_1^- + a_1 - a_2$

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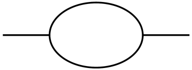

$$0 = (d - z_2 - 2z_1)\, F(z_1, z_2) - z_2\, p^2\, F(z_1, z_2 + 1) - z_2\, F(z_1 - 1, z_2 + 1)$$

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The IBP relations generate a **left** ideal in the non-commutative rational double-shift algebra Y!

$$I_{\mathrm{IBP}} := \langle r_i \mid i = 1, \ldots, L(L + E)\rangle \lhd Y$$

Formulation of the algebra and ideal was crucial for successful computation

- For the one-loop bubble $I_{\mathrm{IBP}} = \langle r_1, r_2\rangle_Y = \{u_1\, r_1 + u_2\, r_2 | u_{1,2} \in Y\}$
- By construction, have $F(z_1, z_2) \bullet r = 0$ for $r \in I_{\mathrm{IBP}}$

Goal: Compute a Gröbner basis for the left ideal I_{IBP} in Y

- Remainder (normal form) corresponds to result of reduction

Computing a Gröbner basis for $I_{\mathrm{IBP}} \lhd Y$

- The computations are done in the GAP package `LoopIntegrals`.

  https://homalg-project.github.io/pkg/LoopIntegrals

  It computes IBP relations among loop integrals.

- Dependencies
  - the computer algebra system SINGULAR for **commutative** Gröbner bases in **polynomial** rings, [Decker,Greuel,Pfister,Schönemann'19]
  - its subsystem PLURAL for **non-commutative** Gröbner bases in the double-shift algebra with **polynomial** coefficients, [Levandovskyy,Schönemann'03]
  - Chyzak's Maple package `Ore_algebra` for **noncommutative** Gröbner bases in the double-shift algebra with **rational** coefficients, [Chyzak'98]
  - the Julia package HECKE for simulating the reduction w.r.t. Gröbner bases in the rational double-shift algebra using linear algebra over the field of rational functions.
    HECKE uses finite-field methods to compute GCDs.

Gröbner basis of left ideal of IBPs in the double-shift algebra

- Rational Gröbner basis of one-loop massless bubble has 4 elements and was computed in <1s.

$$G = \Big\{(d-a_1-a_2)(d-2a_1-2a_2+2)D_2-(a_2-1)(d-2a_2)\,p^2,$$
$$(d-a_1-a_2)(d-2a_1-2a_2+2)D_1-(a_1-1)(d-2a_1)\,p^2,$$
$$a_2\,p^2\,(d-2a_2-2)D_2^- -(d-a_1-a_2-1)(d-2a_1-2a_2),$$
$$a_1\,p^2\,(d-2a_1-2)D_1^- -(d-a_1-a_2-1)(d-2a_1-2a_2)\Big\}$$

  - Leaves propagator powers parametric

- Particular interesting are IBP operators of the form ("first-order normal form IBPs")

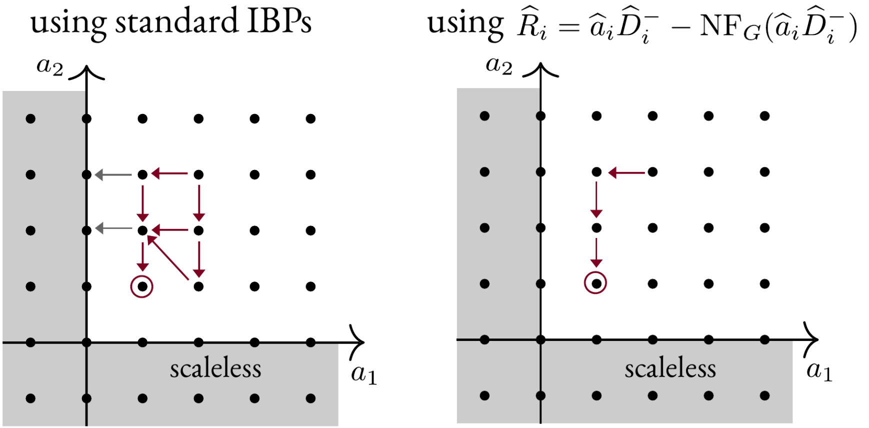
$$\boxed{R_i = a_i\,D_i^- - \mathrm{NF}_G\left(a_i D_i^-\right) \;\in\; I_{\mathrm{IBP}} \qquad \text{[no summation over } i\text{]}}$$

  - e.g. $\mathrm{NF}_G\left(a_1 D_1^-\right) = \dfrac{(d-a_1-a_2-1)(d-2a_1-2a_2)}{p^2(d-2a_1-2)}$
  - Allow for a straightforward reduction

Standard IBPs vs. Gröbner basis

- Compare reduction with standard IBPs vs. Gröbner basis for the one-loop massless bubble

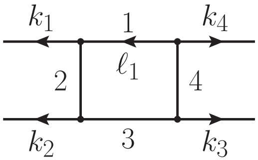
Reduction of $F(3,2)$:



using standard IBPs

using $\widehat{R}_i = \widehat{a}_i \widehat{D}_i^- - \mathrm{NF}_G(\widehat{a}_i \widehat{D}_i^-)$

One-loop massless box



$$\begin{aligned} P_1 &= -\ell_1^2, \\ P_2 &= -(\ell_1 - k_1)^2, \\ P_3 &= -(\ell_1 - k_1 - k_2)^2, \\ P_4 &= -(\ell_1 + k_4)^2. \end{aligned}$$

- Massless one-loop box, $k_i^2 = 0$
- Independent Mandelstam variables s_{12}, s_{14} (resp. s, t)
- Four standard IBPs

$$\begin{aligned} r_1 &= -a_2 D_1 D_2^- - a_3 D_1 D_3^- - a_4 D_1 D_4^- - s_{12} a_3 D_3^- + (d - 2a_1 - a_2 - a_3 - a_4), \\ r_2 &= a_1 D_1^- D_2 - a_2 D_1 D_2^- - a_3 D_1 D_3^- + a_3 D_2 D_3^- - a_4 D_1 D_4^- + a_4 D_2 D_4^- - s_{12} a_3 D_3^- + s_{14} a_4 D_4^- - a_1 + a_2, \\ r_3 &= -a_1 D_1^- D_2 + a_1 D_1^- D_3 + a_2 D_2^- D_3 - a_3 D_2 D_3^- - a_4 D_2 D_4^- + a_4 D_3 D_4^- + s_{12} a_1 D_1^- - s_{14} a_4 D_4^- - a_2 + a_3, \\ r_4 &= a_2 D_1 D_2^- + a_3 D_1 D_3^- - a_1 D_1^- D_4 - a_2 D_2^- D_4 - a_3 D_3^- D_4 + a_4 D_1 D_4^- - s_{14} a_2 D_2^- + s_{12} a_3 D_3^- + a_1 - a_4 \end{aligned}$$

One-loop massless box

- Reduced Gröbner basis over *rational* double-shift algebra has 9 elements

$$
\begin{aligned}
G = \Bigg\{ & D_4 - D_2 + \frac{(a_2 - a_4)s_{14}}{d - a_{1234}} 1, \\
& D_3 - D_1 + \frac{(a_1 - a_3)s_{12}}{d - a_{1234}} 1, \\
& \ldots , \\
& 4(a_2 - 1)(d - a_{1234})D_3 - 2(d - 2a_{134})(d - a_{1234})D_4 + (d - 2a_{14} - 2)(d - 2a_{234})s_{12} 1 \\
& - 2(d - 2a_{134})(a_2 - a_4)s_{14} 1 - \frac{(d - 2a_{14} - 2)(d - 2a_{34} - 2)a_4 s_{12} s_{14}}{d - a_{1234} - 1} D_4^- , \\
& - 2(d - 2a_{1234} + 4)(d - a_{1234} + 1)D_3^2 + (d - 2a_{123} + 2)(d - 2a_{134} + 2)s_{14} D_3 \\
& - 2(a_1 - a_3 + 1)(d - 2a_{1234} + 4)s_{12} D_3 + 4(a_1 - 1)(a_3 - 1)s_{12} D_4 \\
& - \frac{(d - 2a_{123} + 2)(a_3 - 1)(d - 2a_{34})s_{12} s_{14}}{d - a_{1234}} 1 \Bigg\},
\end{aligned}
$$

- G is rational in d, a_i, s_{ij} and polynomial in D_i, D_i^-.

One-loop massless box

- A **standard monomial** w.r.t. the Gröbner basis G of I_{IBP} is a monomial m in the D_i, D_j^- such that $\text{NF}_G(m) = m$.
- The set of standard monomials is a basis for the finite dimensional $\mathbb{K}$-vector space Y/I_{IBP}
- It corresponds to a set of master integrals with respect to e.g. the corner integral

$$\begin{aligned} \text{NF}_G(D_1) =& D_3 + \frac{(a_1 - a_3)s_{12}}{d - a_{1234}} && \rightsquigarrow D_1 \text{ is a nonstandard monomial} \\ \text{NF}_G(D_2) =& D_4 + \frac{(a_2 - a_4)s_{14}}{d - a_{1234}} && \rightsquigarrow D_2 \text{ is a nonstandard monomial} \\ \text{NF}_G(D_3) =& D_3 && \rightsquigarrow D_3 \text{ is a standard monomial} \\ \text{NF}_G(D_4) =& D_4 && \rightsquigarrow D_4 \text{ is a standard monomial} \end{aligned}$$

- The set of standard monomials with respect to G is $\{1, D_3, D_4\}$
- Corresponds to three master integrals

$$\{I(1,1,1,1), I(1,1,0,1), I(1,1,1,0)\}.$$

- Can also verify that $D_1D_2, D_1D_4, D_2D_3, D_3D_4$ are the minimal scaleless monomials w.r.t. $I(1,1,1,1)$.

One-loop massless box

- Again compute IBP relations of the form $R_i = a_i D_i^- - \mathrm{NF}_G(a_i D_i^-) \in I_{\mathrm{IBP}}$
- Normal form of operators $a_i D_i^-$ w.r.t. the Gröbner basis G of the left ideal I_{IBP}

$$\begin{aligned}\mathrm{NF}_G(a_1 D_1^-) = &-\frac{2\,(d-2a_{124})\,(d-a_{1234})\,(d-a_{1234}-1)}{(d-2a_{12}-2)\,(d-2a_{14}-2)\,s_{12}s_{14}}\,D_3\\ &+\frac{4\,(a_3-1)\,(d-a_{1234})\,(d-a_{1234}-1)}{(d-2a_{12}-2)\,(d-2a_{14}-2)\,s_{12}s_{14}}\,D_4\\ &+\frac{(d-2a_{134})\,(d-a_{1234}-1)}{(d-2a_{14}-2)\,s_{12}}\,1,\end{aligned}$$

$$\begin{aligned}\mathrm{NF}_G(a_3 D_3^-) = &-\frac{2\,(d-2a_{234})\,(d-a_{1234})\,(d-a_{1234}-1)}{(d-2a_{23}-2)\,(d-2a_{34}-2)\,s_{12}s_{14}}\,D_3\\ &+\frac{4\,(a_1-1)\,(d-a_{1234})\,(d-a_{1234}-1)}{(d-2a_{23}-2)\,(d-2a_{34}-2)\,s_{12}s_{14}}\,D_4\\ &+\frac{(d-2a_{134})\,(d-a_{1234}-1)}{(d-2a_{34}-2)\,s_{12}}\,1\\ &-\frac{2\,(a_1-a_3)\,(d-2a_{234})\,(d-a_{1234}-1)}{(d-2a_{23}-2)\,(d-2a_{34}-2)\,s_{14}}\,1,\end{aligned}$$

$$\begin{aligned}\mathrm{NF}_G(a_2 D_2^-) = &\frac{4\,(a_4-1)\,(d-a_{1234})\,(d-a_{1234}-1)}{(d-2a_{12}-2)\,(d-2a_{23}-2)\,s_{12}s_{14}}\,D_3\\ &-\frac{2\,(d-2a_{123})\,(d-a_{1234})\,(d-a_{1234}-1)}{(d-2a_{12}-2)\,(d-2a_{23}-2)\,s_{12}s_{14}}\,D_4\\ &+\frac{(d-2a_{234})\,(d-a_{1234}-1)}{(d-2a_{23}-2)\,s_{14}}\,1,\end{aligned}$$

$$\begin{aligned}\mathrm{NF}_G(a_4 D_4^-) = &\frac{4\,(a_2-1)\,(d-a_{1234})\,(d-a_{1234}-1)}{(d-2a_{14}-2)\,(d-2a_{34}-2)\,s_{12}s_{14}}\,D_3\\ &-\frac{2\,(d-2a_{134})\,(d-a_{1234})\,(d-a_{1234}-1)}{(d-2a_{14}-2)\,(d-2a_{34}-2)\,s_{12}s_{14}}\,D_4\\ &+\frac{(d-2a_{234})\,(d-a_{1234}-1)}{(d-2a_{34}-2)\,s_{14}}\,1\\ &-\frac{2\,(a_2-a_4)\,(d-2a_{134})\,(d-a_{1234}-1)}{(d-2a_{14}-2)\,(d-2a_{34}-2)\,s_{12}}\,1\end{aligned}$$

- In denominators of $\mathrm{NF}_G(a_i D_i^-)$, all a_j appear together with d.
- $\mathrm{NF}_G(a_i D_i^-)$ are $\mathbb{K}$-linear combinations of the standard monomials

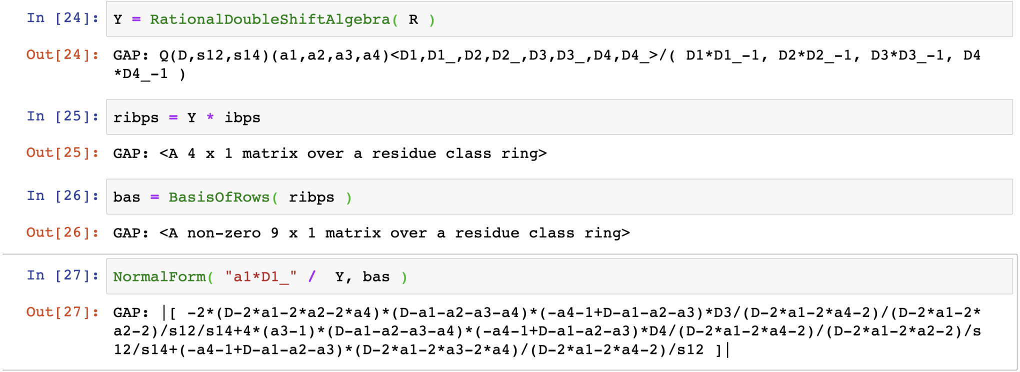
One-loop massless box

- The Jupyter notebook can be found at https://homalg-project.github.io/nb/1LoopBox/

```
In [17]: LoadPackage( "LoopIntegrals" )

In [18]: R = RingOfLoopDiagram( LD )

Out[18]: GAP: Q[D,s12,s14][D1,D2,D3,D4]

In [19]: Ypol = DoubleShiftAlgebra( R )

Out[19]: GAP: Q[D,s12,s14][a1,a2,a3,a4]<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4
         *D4_-1 )

In [20]: ibps = MatrixOfIBPRelations( LD )

Out[20]: GAP: <A 4 x 1 matrix over a residue class ring>

In [21]: r1 = ibps[1,1]

Out[21]: GAP: |[ -a2*D1*D2_-s12*a3*D3_-a3*D1*D3_-a4*D1*D4_+D-2*a1-a2-a3-a4 ]|

In [22]: bas_pol = BasisOfRows( ibps )

Out[22]: GAP: <A non-zero 28 x 1 matrix over a residue class ring>

In [23]: NormalForm( "a1*D1_" /  Ypol, bas_pol )

Out[23]: GAP: |[ a1*D1_ ]|
```

One-loop massless box

The following command needs Chyzak's Maple package `Ore_algebra` for the noncommutative Gröbner bases of the rational double-shift algebra:



```
In [24]: Y = RationalDoubleShiftAlgebra( R )

Out[24]: GAP: Q(D,s12,s14)(a1,a2,a3,a4)<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4
         *D4_-1 )

In [25]: ribps = Y * ibps

Out[25]: GAP: <A 4 x 1 matrix over a residue class ring>

In [26]: bas = BasisOfRows( ribps )

Out[26]: GAP: <A non-zero 9 x 1 matrix over a residue class ring>

In [27]: NormalForm( "a1*D1_" /  Y, bas )

Out[27]: GAP: |[ -2*(D-2*a1-2*a2-2*a4)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D3/(D-2*a1-2*a4-2)/(D-2*a1-2*
         a2-2)/s12/s14+4*(a3-1)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D4/(D-2*a1-2*a4-2)/(D-2*a1-2*a2-2)/s
         12/s14+(-a4-1+D-a1-a2-a3)*(D-2*a1-2*a3-2*a4)/(D-2*a1-2*a4-2)/s12 ]|
```

One-loop massless box

```
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )

Out[28]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )

Out[29]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )

Out[30]: GAP: |[ 0 ]|
```

One-loop massless box

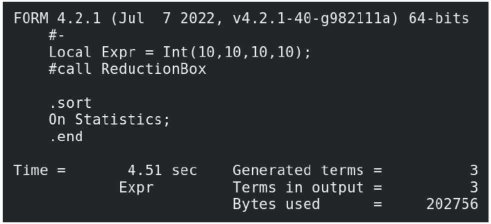
```
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )

Out[28]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )

Out[29]: GAP: |[ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ]|

In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )

Out[30]: GAP: |[ 0 ]|
```

- Virtues of the Gröbner basis / normal form
  - Contains the entire information required for reduction
  - Recognizes scaleless sectors (and in certain cases symmetries)
  - No new bottom-up reduction required for new/additional integrals
  - Ideally suited for storage, e.g. in a database

One-loop massless box

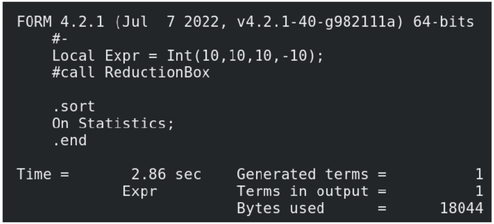
- Also easy to implement in Mathematica or FORM
  - well-suited for parallelization
  - Allows for fast reduction, also of high propagator powers

$$F(10,10,10,10)\ , \quad \sim 5 \text{ sec.}$$



```
FORM 4.2.1 (Jul  7 2022, v4.2.1-40-g982111a) 64-bits
    #-
    Local Expr = Int(10,10,10,10);
    #call ReductionBox

    .sort
    On Statistics;
    .end

Time =       4.51 sec    Generated terms =          3
            Expr         Terms in output =          3
                         Bytes used      =     202756
```

$$F(10,10,10,-10)\ , \quad \sim 3 \text{ sec.}$$



```
FORM 4.2.1 (Jul  7 2022, v4.2.1-40-g982111a) 64-bits
    #-
    Local Expr = Int(10,10,10,-10);
    #call ReductionBox

    .sort
    On Statistics;
    .end

Time =       2.86 sec    Generated terms =          1
            Expr         Terms in output =          1
                         Bytes used      =      18044
```

Dimension-shift operator

- Include the dimension-shift operators D and D^- in the algebra $Y^{(\mathrm{pol})}$ and left ideal I_{IBP}
- Action of D on integrals

$$I(d, z_1, \ldots, z_n) \bullet D = I(d-2, z_1, \ldots, z_n)$$

- Commutation relations

$$[D, D_i] = [D, a_i] = [D, D_i^-] = 0$$

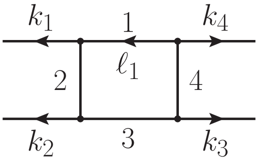
- Additional IBP relations

[Tarasov'96'97]

$$I(d-2, z_1, \ldots, z_n) = I(d, z_1, \ldots, z_n) \bullet \mathcal{U}(D_1^-, \ldots, D_n^-)$$

- Similar for $I(d, z_1, \ldots, z_n) \bullet D^- = I(d+2, z_1, \ldots, z_n)$
- Pitfall: `Ore_algebra` only allows for shifts by one unit
  - Solution: implement relations for D^2 and $(D^-)^2$

One-loop box with dimensional shift



$$\begin{aligned} P_1 &= -\ell_1^2, \\ P_2 &= -(\ell_1 - k_1)^2, \\ P_3 &= -(\ell_1 - k_1 - k_2)^2, \\ P_4 &= -(\ell_1 + k_4)^2. \end{aligned}$$

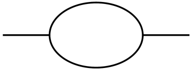
- We managed to compute the Gröbner basis of the one-loop massless box including the dimensional shift-operator, both in Y and Y^{pol}
- Comparison to calculation without dim.-shift,
  - calculations take slightly longer with dim.-shift relations included
  - Gröbner basis in Y (Y^{pol}) has 14 (56) elements, compared to 9 (28) before
  - obtain same standard monomials as before. In particular, D^2 and $(D^-)^2$ get reduced

$$D^2 = \frac{s_{14}}{(s_{12}+s_{14})(-a_1-a_2-a_3-a_4+d+1)}\,D_3 + \frac{s_{12}}{(s_{12}+s_{14})(-a_1-a_2-a_3-a_4+d+1)}\,D_4 - \frac{s_{12}s_{14}(2a_3+2a_4-d)}{2(s_{12}+s_{14})(a_1+a_2+a_3+a_4-d-1)(a_1+a_2+a_3+a_4-d)}\,1$$

Special IBPs

- Look for IBP relations that do not increase the propagator powers, e.g. for the one-loop bubble

$$\int \frac{d^d\ell}{i\pi^{d/2}} \, \frac{\partial}{\partial \ell^\mu} \, \frac{v_i^\mu}{D_1^{a1}\, D_2^{a2}} = 0$$



- Vector in numerator can be a linear combination of loop and external momenta

$$v_i^\mu = \underset{\uparrow}{C_1^{(i)}}(D_1, D_2)\, \ell^\mu + \underset{\uparrow}{C_2^{(i)}}(D_1, D_2)\, p^\mu \quad = \sum_j C_j^{(i)}\, q_j^\mu$$

polynomial dependence

- Derivative acting on $D_k^{-a_k}$ increases propagator power

$$v_i^\mu \frac{\partial}{\partial \ell^\mu} \frac{1}{D_1^{a1}\, D_2^{a2}} = \sum_k v_i^\mu \frac{\partial D_k}{\partial \ell^\mu} \boxed{\frac{\partial}{\partial D_k} \frac{1}{D_1^{a1}\, D_2^{a2}}} = \sum_{k,j} C_j^{(i)} \underbrace{q_j^\mu \frac{\partial D_k}{\partial \ell^\mu}}_{:=\mathcal{E}_{kj}^{(i)}} \frac{\partial}{\partial D_k} \frac{1}{D_1^{a1}\, D_2^{a2}}$$

$\longleftarrow$ IBP-generating matrix

Special IBPs

$$v_i^\mu \frac{\partial}{\partial \ell^\mu} \frac{1}{D_1^{a_1}\, D_2^{a_2}} = \sum_k \boxed{v_i^\mu \frac{\partial D_k}{\partial \ell^\mu}} \boxed{\frac{\partial}{\partial D_k} \frac{1}{D_1^{a_1}\, D_2^{a_2}}} = \sum_{k,j} C_j^{(i)} \underbrace{q_j^\mu \frac{\partial D_k}{\partial \ell^\mu}}_{:= \mathcal{E}_{kj}^{(i)}} \frac{\partial}{\partial D_k} \frac{1}{D_1^{a_1}\, D_2^{a_2}}$$

⟵ IBP-generating matrix

- Increase of propagator power survives unless

[Kosower et al.’10’18; Schabinger et al.’11’20; Ita’15; Böhm et al.’17]

$$v_i^\mu \frac{\partial D_k}{\partial \ell^\mu} \propto D_k \qquad \forall\, k$$

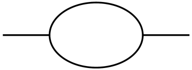
Amounts to computing column syzygy matrix S of $\mathcal{E}$ modulo $\mathrm{diag}(D_1, D_2)$

$$\sum_j \mathcal{E}_{kj}^{(i)}\, S_{ji} \,\propto\, D_k \qquad \forall k, i$$

Baikov represenation

- Basic idea: use propagators as integration variables [Baikov'96]

$$I_{a_1 a_2} = \text{prefactor} \times \int dD_1\, dD_2 \, \frac{\mathcal{B}^\gamma}{D_1^{a_1}\, D_2^{a_2}}$$



- Exponent: $\gamma = \dfrac{d - L - E - 1}{2}$

- Baikov polynomial $\mathcal{B}$ is the Gram determinant of loop and external momenta

$$\mathcal{B} = \begin{vmatrix} \ell \cdot \ell & \ell \cdot p \\ p \cdot \ell & p \cdot p \end{vmatrix} = \frac{1}{4}\left(-D_1^2 + 2 D_1\, D_2 - D_2^2 - 2\, D_1\, p^2 - 2\, D_2\, p^2 - p^4\right)$$

- IBP relations from [Zhang, Larsen; Lee; Frellesvig,Papadopoulos]

$$\int dD_1\, dD_2 \sum_k \frac{\partial}{\partial D_k}\left(\boxed{u_k(D_1, D_2)}\, \frac{\mathcal{B}^\gamma}{D_1^{a_1}\, D_2^{a_2}}\right) = \int dD_1\, dD_2 \left[\gamma\, \boxed{\mathcal{B}^{\gamma-1}} \left(\sum_k \frac{\partial \mathcal{B}}{\partial D_k}\, u_k\right) \frac{1}{D_1^{a_1}\, D_2^{a_2}} + \mathcal{B}^\gamma \sum_k \boxed{\frac{\partial}{\partial D_k} \frac{u_k}{D_1^{a_1}\, D_2^{a_2}}}\right] = 0$$

↑ polynomial

↑ dimension shift

↑ additional dots

Special IBPs

$$\int dD_1\, dD_2 \left[\gamma\, \boxed{\mathcal{B}^{\gamma-1}} \boxed{\left(\sum_k \frac{\partial \mathcal{B}}{\partial D_k}\, u_k \right)} \frac{1}{D_1^{a_1}\, D_2^{a_2}} + \mathcal{B}^{\gamma} \sum_k \boxed{\frac{\partial}{\partial D_k} \frac{\boxed{u_k}}{D_1^{a_1}\, D_2^{a_2}}} \right] = 0$$

$\uparrow$ dimension shift

$\uparrow$ additional dots

- Avoid additional dots if $u_k = \tilde{u}_k(D_1, D_2) \times D_k \xrightarrow{\text{solutions}}$ module $M_1 = \langle D_1, D_2 \rangle$
- Avoid dimension shift if $\sum_k \frac{\partial \mathcal{B}}{\partial D_k}\, u_k = \text{poly}(D_1, D_2) \times \mathcal{B} \xrightarrow{\text{solutions}}$ module M_2 [Böhm et al.'17]
- Obtain special IBPs via module intersection $M_1 \cap M_2$ [Bendle et al.'20]

Special IBPs

$$\int dD_1\, dD_2 \left[\gamma\, \boxed{\mathcal{B}^{\gamma-1}}\, \boxed{\left(\sum_k \frac{\partial\mathcal{B}}{\partial D_k}\, u_k\right)} \frac{1}{D_1^{a_1}\, D_2^{a_2}} + \mathcal{B}^{\gamma} \sum_k \boxed{\frac{\partial}{\partial D_k}\, \frac{\boxed{u_k}}{D_1^{a_1}\, D_2^{a_2}}}\right] = 0$$

↑ dimension shift

↑ additional dots

- Avoid additional dots if $u_k = \tilde{u}_k(D_1, D_2) \times D_k$ $\underrightarrow{\text{solutions}}$ module $M_1 = \langle D_1, D_2 \rangle$
- Avoid dimension shift if $\displaystyle\sum_k \frac{\partial\mathcal{B}}{\partial D_k}\, u_k = \mathrm{poly}(D_1, D_2) \times \mathcal{B}$ $\underrightarrow{\text{solutions}}$ module M_2 [Böhm et al.'17]
- Obtain special IBPs via module intersection $M_1 \cap M_2$ [Bendle et al.'20]

Important observation

Computation is identical in Baikov and in momentum space representation!

$$M_1 \cap M_2 \quad \overleftrightarrow{\text{equivalent}} \quad \sum_j \mathcal{E}_{kj}^{(i)}\, S_{ji} \propto D_k$$

Special IBPs

- In particular, have $M_2 = \langle \mathcal{E} \rangle$, i.e.

$$\sum_k \frac{\partial \mathcal{B}}{\partial D_k} \mathcal{E}_{kj}^{(i)} \propto \mathcal{B}$$

$$\uparrow$$

no dimension-shift

$$\sum_{k,j} \frac{\partial \mathcal{B}}{\partial D_k} \mathcal{E}_{kj}^{(i)} S_{ji} \propto \mathcal{B}$$

$$\uparrow$$

no dim.-shift and no dots

- I.e. we don't have to compute M_2
- M_1 is generated by our special IBPs $\sum_j \mathcal{E}_{kj}^{(i)} S_{ji}$

Special IBPs

- In particular, have $M_2 = \langle \mathcal{E} \rangle$, i.e.

$$\sum_k \frac{\partial \mathcal{B}}{\partial D_k} \mathcal{E}^{(i)}_{kj} \propto \mathcal{B}$$

$$\uparrow$$

no dimension-shift

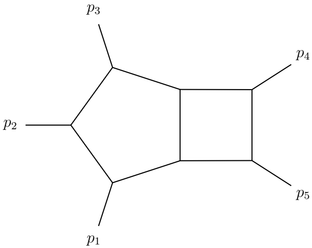
$$\sum_{k,j} \frac{\partial \mathcal{B}}{\partial D_k} \mathcal{E}^{(i)}_{kj} \, S_{ji} \propto \mathcal{B}$$

$$\uparrow$$

no dim.-shift and no dots

- I.e. we don't have to compute M_2
- M_1 is generated by our special IBPs $\sum_j \mathcal{E}^{(i)}_{kj} \, S_{ji}$

- Example: Two-loop massless planar penta-box



[from 2305.08783]

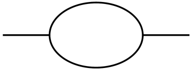
- Compute Gröbner basis of syzygies w.r.t. deg.rev.lex.
- 4h runtime
- $<$ 10 GB of memory
- no degree-bound

Linear algebra ansatz

- We observed that $R_i = a_i D_i^- - \mathrm{NF}_G(a_i D_i^-)$ are well-suited for reduction
  - in all considered problems normal-form IBPs generate the left ideal I_{IBP} [most likely not true for arbitrary problems]

Idea: Use linear algebra to compute R_i when Gröbner basis is not available.

- Consider again one-loop bubble



$$\widehat{r}_1 = (d - \widehat{a}_2 - 2\widehat{a}_1) - p^2 \widehat{a}_2 \widehat{D}_2^- - \widehat{a}_2 \widehat{D}_1 \widehat{D}_2^-$$

$$\widehat{r}_2 = -\widehat{a}_1 \widehat{D}_1^- \widehat{D}_2 + \widehat{a}_2 \widehat{D}_1 \widehat{D}_2^- - p^2 \widehat{a}_1 \widehat{D}_2^- + (\widehat{a}_1 - \widehat{a}_2)$$

$$\widehat{D}_1^- \widehat{r}_1 = (d - \widehat{a}_2 - 2\widehat{a}_1 - 2)\widehat{D}_1^- - p^2 \widehat{a}_2 \widehat{D}_1^- \widehat{D}_2^- - \widehat{a}_2 \widehat{D}_2^-$$

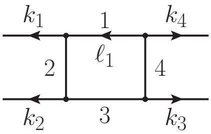
$$\vdots \longleftarrow (\widehat{D}_1^-)^{j_1} (\widehat{D}_2^-)^{j_2} \widehat{r}_i$$

treat as unknowns

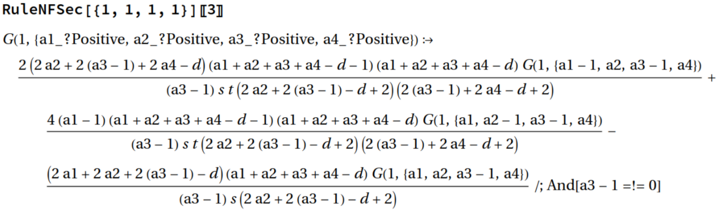
Solve for $\widehat{D}_1^-$ and $\widehat{D}_2^-$

- Question: Which values for j_1, j_2 ??
- Observation: Special IBPs require lower max. values of $j_{1,2}$ than standard IBPs

One-loop box with linear algebra



- Create seeds with values of j_i
- Use FiniteFlow to reconstruct multivariate rational functions from finite fields [Peraro’19]



```
RuleNFSec[{1, 1, 1, 1}][[3]]
```

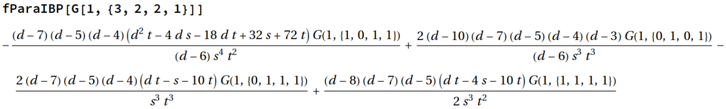
$$G(1,\{\text{a1\_?Positive}, \text{a2\_?Positive}, \text{a3\_?Positive}, \text{a4\_?Positive}\}) :\to \frac{2\,(2\,\text{a2}+2\,(\text{a3}-1)+2\,\text{a4}-d)\,(\text{a1}+\text{a2}+\text{a3}+\text{a4}-d-1)\,(\text{a1}+\text{a2}+\text{a3}+\text{a4}-d)\,G(1,\{\text{a1}-1,\text{a2},\text{a3}-1,\text{a4}\})}{(\text{a3}-1)\,s\,t\,(2\,\text{a2}+2\,(\text{a3}-1)-d+2)\,(2\,(\text{a3}-1)+2\,\text{a4}-d+2)} + \frac{4\,(\text{a1}-1)\,(\text{a1}+\text{a2}+\text{a3}+\text{a4}-d-1)\,(\text{a1}+\text{a2}+\text{a3}+\text{a4}-d)\,G(1,\{\text{a1},\text{a2}-1,\text{a3}-1,\text{a4}\})}{(\text{a3}-1)\,s\,t\,(2\,\text{a2}+2\,(\text{a3}-1)-d+2)\,(2\,(\text{a3}-1)+2\,\text{a4}-d+2)} - \frac{(2\,\text{a1}+2\,\text{a2}+2\,(\text{a3}-1)-d)\,(\text{a1}+\text{a2}+\text{a3}+\text{a4}-d)\,G(1,\{\text{a1},\text{a2},\text{a3}-1,\text{a4}\})}{(\text{a3}-1)\,s\,(2\,\text{a2}+2\,(\text{a3}-1)-d+2)}\ /;\ \text{And}[\text{a3}-1 =!= 0]$$

- Runtime depends on order of seed equations [cf. LiteRed]

$$\Big\{\{a_1 \to a_1 - 2, a_2 \to a_2, a_3 \to a_3, a_4 \to a_4\}\,,\ \{a_1 \to a_1 - 1, a_2 \to a_2 - 1, a_3 \to a_3, a_4 \to a_4\}\,, \ldots\Big\}$$

One-loop box with linear algebra

- Constraints are such that result still contains more masters



```
fParaIBP[G[1, {3, 2, 2, 1}]]
```

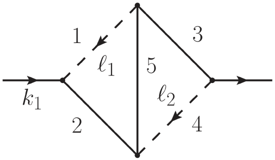
$$-\frac{(d-7)(d-5)(d-4)\left(d^2\,t-4\,d\,s-18\,d\,t+32\,s+72\,t\right)G(1,\{1,0,1,1\})}{(d-6)\,s^4\,t^2}+\frac{2\,(d-10)(d-7)(d-5)(d-4)(d-3)\,G(1,\{0,1,0,1\})}{(d-6)\,s^3\,t^3}-\frac{2\,(d-7)(d-5)(d-4)\left(d\,t-s-10\,t\right)G(1,\{0,1,1,1\})}{s^3\,t^3}+\frac{(d-8)(d-7)(d-5)\left(d\,t-4\,s-10\,t\right)G(1,\{1,1,1,1\})}{2\,s^3\,t^2}$$

- Inclusion of dimensional shift operator finds additional relations

$$G(1,\{0,1,0,1\})=\frac{(d-4)t}{2(d-3)}\,G(1,\{0,1,1,1\})\,, \qquad G(1,\{1,0,1,0\})=\frac{(d-4)s}{2(d-3)}\,G(1,\{1,0,1,1\})$$

With this, every integral from te one-loop massless box can be reduced to three master integrals with the normal-form IBPs that originate from the linear algebra ansatz

Two-loop on-shell kite



$$P_1 = -\ell_1^2,$$
$$P_2 = m^2 - (\ell_1 + p)^2,$$
$$P_3 = -\ell_2^2,$$
$$P_4 = m^2 - (\ell_2 + p)^2,$$
$$P_5 = m^2 - (\ell_1 + \ell_2 + p)^2$$

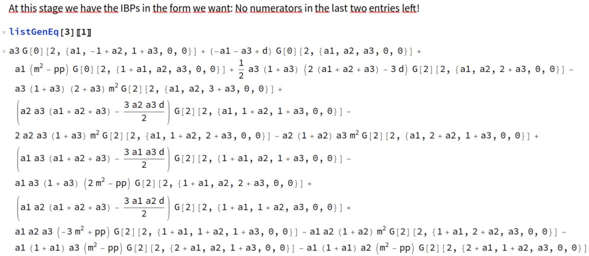
- On-shell kite, $p^2 = m^2 \equiv s$
  - Gröbner basis not yet available
  - Simulating, using linear algebra, the computation of the normal form of $a_i D_i^-$ w.r.t. a Gröbner basis

$$\mathrm{NF}(a_1 D_1^-) = \frac{p_{10}}{4 d_1 d_2 d_3 d_4 d_7 d_8 s} + \frac{p_{12} D_2 + p_{13} D_3 + p_{14} D_4 + p_{15} D_5}{16 d_1 d_2 d_3 d_4 d_7 d_8 d_9 s^2}, \ \ldots$$

  - p_i and d_i are expressions in the field $\mathbb{Q}(d, s_{ij}, m_i^2)(a_1, \ldots, a_n)$.
  - The p_i are too large for printing, the d_i are small, e.g. $d_1 = 2a_1 + a_2 + a_3 + 2a_4 + a_5 - 2d + 1$.
  - Allows for reduction of top-level sector

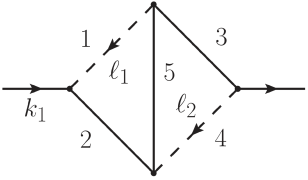
Two-loop off-shell kite

- Off-shell kite, $p^2 \neq m^2$
- Solution including dimensional shift operator allows to trade numerators for dim.-shifted integrals



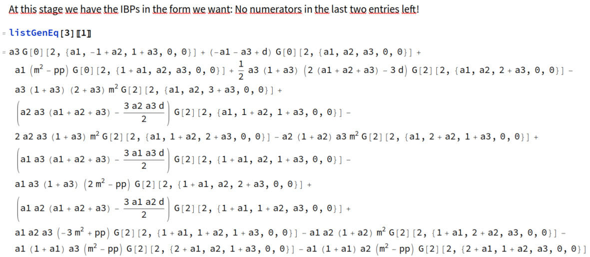
At this stage we have the IBPs in the form we want: No numerators in the last two entries left!

```
listGenEq[3]⟦1⟧

a3 G[0][2, {a1, -1 + a2, 1 + a3, 0, 0}] + (-a1 - a3 + d) G[0][2, {a1, a2, a3, 0, 0}] +
 a1 (m^2 - pp) G[0][2, {1 + a1, a2, a3, 0, 0}] + 1/2 a3 (1 + a3) (2 (a1 + a2 + a3) - 3 d) G[2][2, {a1, a2, 2 + a3, 0, 0}] -
 a3 (1 + a3) (2 + a3) m^2 G[2][2, {a1, a2, 3 + a3, 0, 0}] +
 (a2 a3 (a1 + a2 + a3) - 3 a2 a3 d/2) G[2][2, {a1, 1 + a2, 1 + a3, 0, 0}] -
 2 a2 a3 (1 + a3) m^2 G[2][2, {a1, 1 + a2, 2 + a3, 0, 0}] - a2 (1 + a2) a3 m^2 G[2][2, {a1, 2 + a2, 1 + a3, 0, 0}] +
 (a1 a3 (a1 + a2 + a3) - 3 a1 a3 d/2) G[2][2, {1 + a1, a2, 1 + a3, 0, 0}] -
 a1 a3 (1 + a3) (2 m^2 - pp) G[2][2, {1 + a1, a2, 2 + a3, 0, 0}] +
 (a1 a2 (a1 + a2 + a3) - 3 a1 a2 d/2) G[2][2, {1 + a1, 1 + a2, a3, 0, 0}] +
 a1 a2 a3 (-3 m^2 + pp) G[2][2, {1 + a1, 1 + a2, 1 + a3, 0, 0}] - a1 a2 (1 + a2) m^2 G[2][2, {1 + a1, 2 + a2, a3, 0, 0}] -
 a1 (1 + a1) a3 (m^2 - pp) G[2][2, {2 + a1, a2, 1 + a3, 0, 0}] - a1 (1 + a1) a2 (m^2 - pp) G[2][2, {2 + a1, 1 + a2, a3, 0, 0}]
```



Two-loop off-shell kite

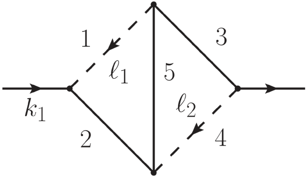
- Off-shell kite, $p^2 \neq m^2$
- Solution including dimensional shift operator allows to trade numerators for dim.-shifted integrals



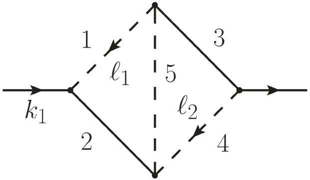
At this stage we have the IBPs in the form we want: No numerators in the last two entries left!

```
listGenEq[3][[1]]
```

$$
\begin{aligned}
&a3\,G[0][2,\{a1,-1+a2,1+a3,0,0\}]+(-a1-a3+d)\,G[0][2,\{a1,a2,a3,0,0\}]+\\
&a1\left(m^2-pp\right)G[0][2,\{1+a1,a2,a3,0,0\}]+\frac{1}{2}\,a3\,(1+a3)\left(2\,(a1+a2+a3)-3\,d\right)G[2][2,\{a1,a2,2+a3,0,0\}]-\\
&a3\,(1+a3)\,(2+a3)\,m^2\,G[2][2,\{a1,a2,3+a3,0,0\}]+\\
&\left(a2\,a3\,(a1+a2+a3)-\frac{3\,a2\,a3\,d}{2}\right)G[2][2,\{a1,1+a2,1+a3,0,0\}]-\\
&2\,a2\,a3\,(1+a3)\,m^2\,G[2][2,\{a1,1+a2,2+a3,0,0\}]-a2\,(1+a2)\,a3\,m^2\,G[2][2,\{a1,2+a2,1+a3,0,0\}]+\\
&\left(a1\,a3\,(a1+a2+a3)-\frac{3\,a1\,a3\,d}{2}\right)G[2][2,\{1+a1,a2,1+a3,0,0\}]-\\
&a1\,a3\,(1+a3)\left(2\,m^2-pp\right)G[2][2,\{1+a1,a2,2+a3,0,0\}]+\\
&\left(a1\,a2\,(a1+a2+a3)-\frac{3\,a1\,a2\,d}{2}\right)G[2][2,\{1+a1,1+a2,a3,0,0\}]+\\
&a1\,a2\,a3\left(-3\,m^2+pp\right)G[2][2,\{1+a1,1+a2,1+a3,0,0\}]-a1\,a2\,(1+a2)\,m^2\,G[2][2,\{1+a1,2+a2,a3,0,0\}]-\\
&a1\,(1+a1)\,a3\left(m^2-pp\right)G[2][2,\{2+a1,a2,1+a3,0,0\}]-a1\,(1+a1)\,a2\left(m^2-pp\right)G[2][2,\{2+a1,1+a2,a3,0,0\}]
\end{aligned}
$$



- Off-shell kite with vertical line massless
- Solution to NF IBP from LA ansatz in sector with 2 masters (sunrise)



Loose ends

- Derive IBPs from parametrization-independent representation of loop-integral ($Z = z_1 + \cdots + z_n$), e.g.

$$I(d, z_1, \ldots, z_n) \, \propto \, \frac{\Gamma(Z - L\,\frac{d}{2})}{\Gamma(z_1)\cdots\Gamma(z_n)} \int_0^\infty dx_1 \, x_1^{z_1-1} \cdots \int_0^\infty dx_n \, x_n^{z_n-1} \, \delta(1 - \textstyle\sum_i x_i) \, \frac{\mathcal{U}^{Z-(L+1)\,\frac{d}{2}}}{\mathcal{F}^{Z-L\,\frac{d}{2}}}$$

  - Schwinger representation

$$I(d, z_1, \ldots, z_n) \, \propto \, \frac{1}{\Gamma(z_1)\cdots\Gamma(z_n)} \int_0^\infty dx_1 \, x_1^{z_1-1} \cdots \int_0^\infty dx_n \, x_n^{z_n-1} \, \mathcal{U}^{-\frac{d}{2}} \, \exp\left(-\frac{\mathcal{F}}{\mathcal{U}}\right)$$

  - Lee-Pomeransky representation [Lee,Pomeransky'13]

$$I(d, z_1, \ldots, z_n) \, \propto \, \frac{\Gamma(\frac{d}{2})}{\Gamma((L+1)\frac{d}{2} - Z)\Gamma(z_1)\cdots\Gamma(z_n)} \int_0^\infty dx_1 \, x_1^{z_1-1} \cdots \int_0^\infty dx_n \, x_n^{z_n-1} \, (\mathcal{U} + F)^{-\frac{d}{2}}$$

  - Baikov representation: See separate slide [Baikov'96]

- Find Gröbner basis of sector which has more than one master integrals

Conclusion

Conclusion

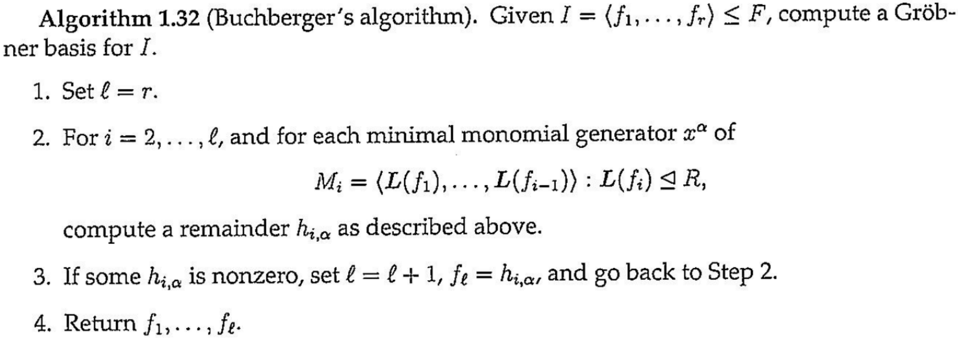
- We established IBP relations as a left ideal in the rational double-shift algebra
- For simple problems, the reduction is completely solved using the Gröbner basis
  - Fast reduction, easy parallelization
  - However: Derivation of GB computationally expensive for more complicated problems.
- Module intersection corresponds to column syzygies of IBP-generating matrix $\mathcal{E}$
- Derivation of normal-form IBPs possible with linear algebra if GB is not available

Outlook / wishlist

- In general: want more loops, more legs, more scales
  - However: New ideas required to deal with enormous expression swell with increasing complexity.
- Combine normal-form IBPs with existing reduction algorithms
- Establish a database to store results of Gröbner bases or normal forms.

Backup slides

Buchberger’s algorithm



Algorithm 1.32 (Buchberger’s algorithm). Given $I = \langle f_1, \dots, f_r \rangle \leq F$, compute a Gröbner basis for I.

1. Set $\ell = r$.
2. For $i = 2, \dots, \ell$, and for each minimal monomial generator x^α of
$$M_i = \langle L(f_1), \dots, L(f_{i-1}) \rangle : L(f_i) \trianglelefteq R,$$
compute a remainder $h_{i,\alpha}$ as described above.
3. If some $h_{i,\alpha}$ is nonzero, set $\ell = \ell + 1$, $f_\ell = h_{i,\alpha}$, and go back to Step 2.
4. Return $f_1, \dots, f_\ell$.