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Geometrizing Landau analysis

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Intro: Singularity/Symbology study for scattering amplitudes/Feynman Integrals

Perturbative QFT: scattering amplitudes= Sum of Feynman Integrals

After integration (at integer dimension): transcendental functions

(But most cases are rather difficult; Advanced methods as canonical differential equations, etc.)

Multi-polylogarithmic (MPL), elliptic, hyperelliptic, Calabi-Yau…

Topic today: Study singularity/symbology structures for amplitudes/integrals

- Singularity:

1. important analytic information of special functions involved before integration
2. Possible starting point for bootstrapping analytic result combined with physical conditions

Intro: Singularity/Symbology study for scattering amplitudes/Feynman Integrals

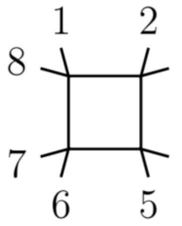
N=4 SYM theory

- Our focus: 4D, dual conformal invariant (DCI), internal massless , MPL integrals [N. Arkani-Hamed et al. 1012.6032] (can be extended to more physical background)

$$p_i = x_{i+1} - x_i \qquad x_{n+1} := x_1$$

$$\ell = y - x_1$$

Conformal invariant in dual space



$$= \int \frac{\mathrm{d}^4 y}{(2\pi)^4} \frac{(x_2 - x_6)^2 (x_4 - x_8)^2}{(y - x_2)^2 (y - x_4)^2 (y - x_6)^2 (y - x_8)^2}$$

- MPL functions, their symbol letters and alphabet

$$G(a_1, a_2, \cdots, a_w; t_0) := \int_0^{t_0} \frac{\mathrm{d}t_1}{t_1 - a_1} G(a_2, \cdots, a_w; t_1)$$

$$= \int_0^{t_0} \frac{\mathrm{d}t_1}{t_1 - a_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2 - a_2} \cdots \int_0^{t_{w-1}} \frac{\mathrm{d}t_w}{t_w - a_w}$$

weight w

$$\mathrm{d}G^{(w)} = \sum_i G_i^{(w-1)} \mathrm{d}\log x_i \implies \mathcal{S}[G^{(w)}] = \sum_i \mathcal{S}[G_i^{(w-1)}] \otimes \log x_i$$

$$\mathcal{S}(G^{(w)}) = \sum_I x_1^I \otimes x_2^I \otimes \cdots \otimes x_w^I$$

Intro: Landau's story for one-loop box integral

$$u = \frac{x_{2,4}^2 x_{6,8}^2}{x_{2,6}^2 x_{4,8}^2} = z\bar{z}, \quad v = \frac{x_{2,8}^2 x_{4,6}^2}{x_{2,6}^2 x_{4,8}^2} = (1-z)(1-\bar{z}),$$

$$\Delta_{2,4,6,8} := \sqrt{(1-u-v)^2 - 4uv}$$

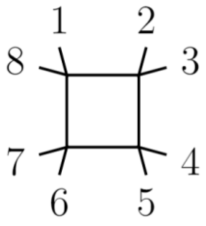
- If an integrand is known, Landau's equations offer us method to derive its singularities without really integrating it.

$$\int \prod_{i=1}^{L} \mathrm{d}^D \ell_i \int_{\mathcal{C}} \mathrm{d}^\nu \alpha \frac{\mathcal{N}}{\mathcal{D}^\nu},$$

$$\mathcal{D} = \sum_{i=1}^{\nu} \alpha_i (q_i^2 - m_i^2)$$

$$\alpha_i (q_i^2 - m_i^2) = 0 \qquad \text{(cut condition)},$$

$$\sum_{i \in \text{each loop}} \alpha_i q_i^\mu = 0 \quad \text{(pinch condition)},$$



$$= \frac{1}{2\Delta_{2,4,6,8}} \left(\mathrm{Li}_2(1-z) - \mathrm{Li}_2(1-\bar{z}) + \log(uv) \log\left(\frac{z}{\bar{z}}\right) \right)$$

$$\alpha_1 (y - x_2)^2 = \alpha_2 (y - x_4)^2 = \alpha_3 (y - x_6)^2 = \alpha_4 (y - x_8)^2 = 0$$

$$\alpha_1 (y - x_2)^\mu + \alpha_2 (y - x_4)^\mu + \alpha_3 (y - x_6)^\mu + \alpha_4 (y - x_8)^\mu = 0$$

Cut solutions $y_+, \quad y_-$

- On the support of cut condition, the pinching condition reads

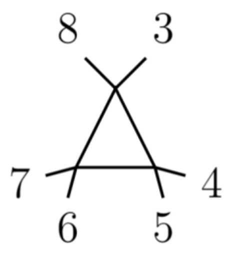
$$\begin{pmatrix} 0 & x_{24}^2 & x_{26}^2 & x_{28}^2 \\ x_{24}^2 & 0 & x_{46}^2 & x_{68}^2 \\ x_{26}^2 & x_{46}^2 & 0 & x_{68}^2 \\ x_{28}^2 & x_{48}^2 & x_{68}^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} := Q \cdot \alpha = 0$$

$$\Longrightarrow \det Q \propto \Delta_{2,4,6,8}^2 \to 0$$

Leading Landau locus $\mathbf{Gram}_{2,4,6,8}$
(four point Gram determinant)

Intro: Landau's story for one-loop box integral

- Then going to sub-leading branch (Trivial singularities as Mandelstam to 0)



Bubbles, tadpoles···
Landau system: integrals and all its sub-topologies

For DCI purpose, we do not consider second-type Landau loci, where loop momenta go to infinity

$$\alpha_1 = 0, \quad \alpha_2(y-x_4)^2 = \alpha_3(y-x_6)^2 = \alpha_4(y-x_8)^2 = 0$$

$$\alpha_2(y-x_4)^\mu + \alpha_3(y-x_6)^\mu + \alpha_4(y-x_8)^\mu = 0$$

- However, singularities are not enough for symbol letters···

Landau loci

Odd letters/algebraic letters

$$c_0 \prod_i W_i = (a+\sqrt{\Delta})(a-\sqrt{\Delta}) \longrightarrow \frac{a+\sqrt{\Delta}}{a-\sqrt{\Delta}}$$

Q:How to uplift singularities to letters?

A:Geometrizing them (Schubert analysis)

$$\mathcal{S}(G^{(w)}) = \sum_I x_1^I \otimes x_2^I \otimes \cdots \otimes x_w^I$$

Symbology study: Constructing all symbol letters for MPL function from singularities and any principle

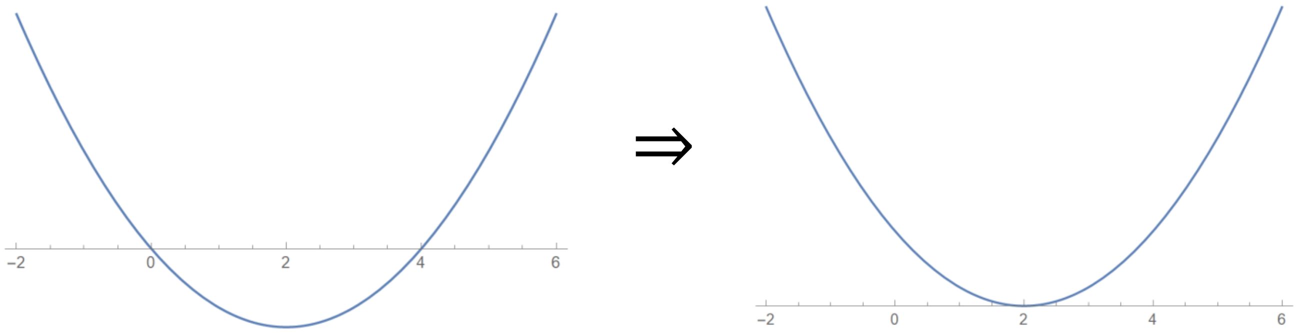
Intro: Landau's story for one-loop box integral

- A more geometrical way to see this singularities?
- Pinching of two cut solutions!

$$(y_\pm - x_2)^2 = (y_\pm - x_4)^2 = (y_\pm - x_6)^2 = (y_\pm - x_8)^2 = 0$$

$$(y_+ - y_-)^2 \propto \Delta^2_{2,4,6,8} = \mathbf{Gram}_{2,4,6,8}$$

- Leading Landau locus= discriminant for cut condition= square of distance of two solutions

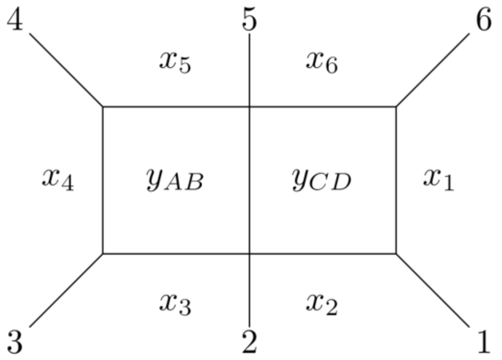


Pinching of two solutions yields leading Landau locus

- Beginning point for geometrizing Landau loci!

Geometrizing Landau loci and Schubert analysis

6-point double-box and its Landau system



$$
\begin{aligned}
\mathcal{D} =& \alpha_1 (y_{CD}-x_1)^2 + \alpha_2 (y_{CD}-x_2)^2 + \alpha_6 (y_{CD}-x_6)^2 \\
& + \alpha_3 (y_{AB}-x_3)^2 + \alpha_4 (y_{AB}-x_4)^2 + \alpha_5 (y_{AB}-x_5)^2 + \beta (y_{AB}-y_{CD})^2,
\end{aligned}
$$

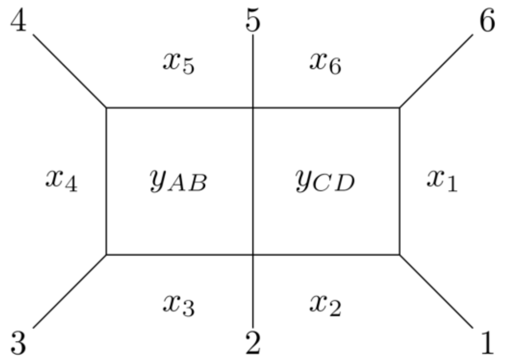
$$
\text{(cuts):}\ \alpha_i \frac{\partial \mathcal{D}}{\partial \alpha_i} = 0; \qquad \beta \frac{\partial \mathcal{D}}{\partial \beta} = 0.
$$

$$
\text{(pinch):}\ \frac{1}{2}\frac{\partial \mathcal{D}}{\partial y_{AB}} = \alpha_3 (y_{AB}-x_3) + \alpha_4 (y_{AB}-x_4) + \alpha_5 (y_{AB}-x_5) + \beta (y_{AB}-y_{CD}) = 0,
$$

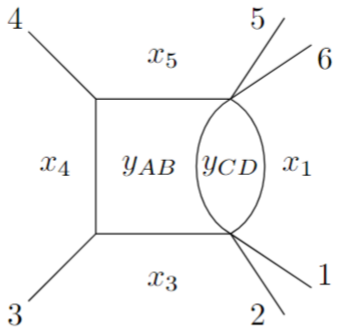
$$
\text{(pinch):}\ \frac{1}{2}\frac{\partial \mathcal{D}}{\partial y_{CD}} = \alpha_1 (y_{CD}-x_1) + \alpha_2 (y_{CD}-x_2) + \alpha_6 (y_{CD}-x_6) + \beta (y_{CD}-y_{AB}) = 0.
$$

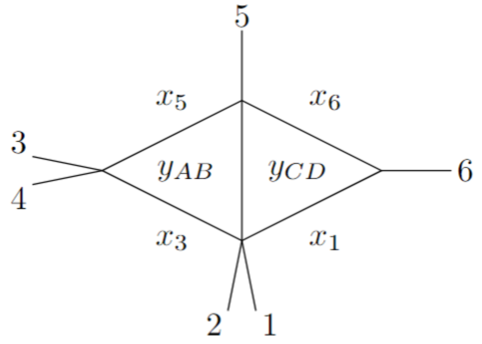
- To explore its Landau loci, we need to look into all its sub-topologies
- We begin with first non-trivial solution branch with most $\alpha_i = 0$

6-point double-box and its Landau system



First non-trivial solution branch: Sub^2-leading: box-bubble & double-triangles yielding boxes equations (loop-by-loop approach for Landau equation)





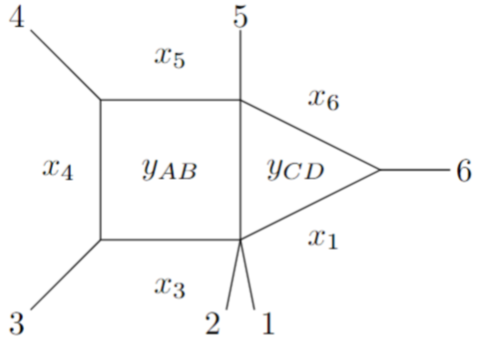
cut problems

$$\mathrm{L}_{3451} : (y_{AB} - x_1)^2 = (y_{AB} - x_3)^2 = (y_{AB} - x_4)^2 = (y_{AB} - x_5)^2 = 0$$

$$\mathrm{L}_{3561} : (y_{AB} - x_1)^2 = (y_{AB} - x_3)^2 = (y_{AB} - x_5)^2 = (y_{AB} - x_6)^2 = 0$$

- We have **cut solutions** and box leading singularities
- For two-loop integrals, we should consider all "box" sub-topologies and corresponding loci (four-point Gram determinants)

6-point double-box and its Landau system



- Sub-leading sectors: box-triangle

Constraining problems

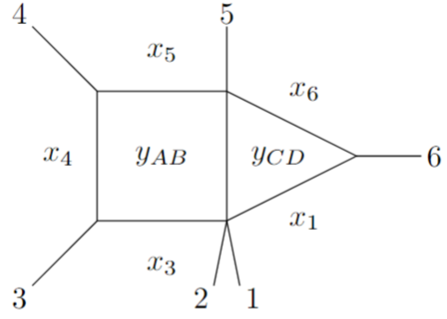
$$\mathrm{L}_{34561} : (y_{AB} - x_1)^2 = (y_{AB} - x_3)^2 = (y_{AB} - x_4)^2 = (y_{AB} - x_5)^2 = (y_{AB} - x_6)^2 = 0$$

- No solutions for loop momentum in 4D, unless there are **extra constraints** on external kinematics
- Choosing any four conditions, determining the loop momentum and get the constraint from the last condition
- For instance, we can solve $\mathrm{L}_{3451} : (y_{AB} - x_1)^2 = (y_{AB} - x_3)^2 = (y_{AB} - x_4)^2 = (y_{AB} - x_5)^2 = 0$ first and the final condition reads

$$(y_{AB}^{(1)} - x_6)^2 (y_{AB}^{(2)} - x_6)^2 = 0$$

$$\mathbf{Gram}_{13456} = \det \begin{pmatrix} 0 & x_{13}^2 & x_{14}^2 & x_{15}^2 & x_{16}^2 \\ x_{13}^2 & 0 & x_{34}^2 & x_{35}^2 & x_{36}^2 \\ x_{14}^2 & x_{34}^2 & 0 & x_{45}^2 & x_{46}^2 \\ x_{15}^2 & x_{35}^2 & x_{45}^2 & 0 & x_{56}^2 \\ x_{16}^2 & x_{36}^2 & x_{46}^2 & x_{56}^2 & 0 \end{pmatrix} \to 0$$

6-point double-box and its Landau system



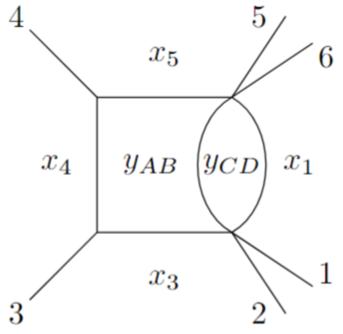
- Sub-leading sectors: box-triangle

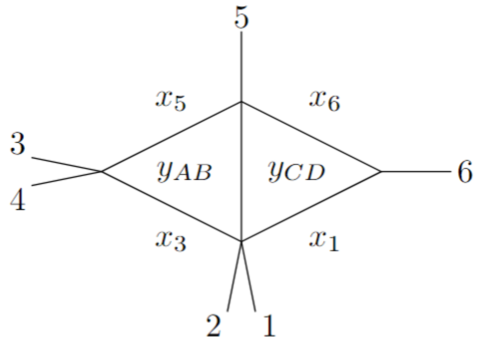
$$(y_{AB}^{(1)}-x_6)^2(y_{AB}^{(2)}-x_6)^2=0 \qquad y_{AB}^{(1,2)} \text{ are also cut solutions for } \mathrm{L}_{3561}$$

- Making the expression more symmetric, we see

$$\mathbf{Gram}^2_{1,3,4,5,6} \propto (y_{AB}^{(1)}-y_{AB}^{(1)})^2(y_{AB}^{(1)}-y_{AB}^{(2)})^2(y_{AB}^{(2)}-y_{AB}^{(1)})^2(y_{AB}^{(2)}-y_{AB}^{(2)})^2$$

where the two pairs of solutions are from cut problems in this system (two "box" problems from the Landau system)



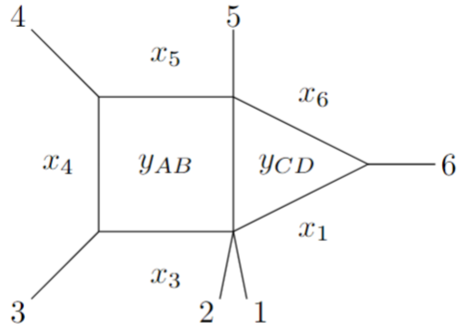


$$\mathrm{L}_{3451}:(y_{AB}-x_1)^2=(y_{AB}-x_3)^2=(y_{AB}-x_4)^2=(y_{AB}-x_5)^2=0 \qquad \mathrm{L}_{3561}:(y_{AB}-x_1)^2=(y_{AB}-x_3)^2=(y_{AB}-x_5)^2=(y_{AB}-x_6)^2=0$$

- In another words, the constraint on external kinematics can be interpreted as pinch of two solutions from two different cut problems

6-point double-box and its Landau system

- Sub-leading sectors: box-triangle

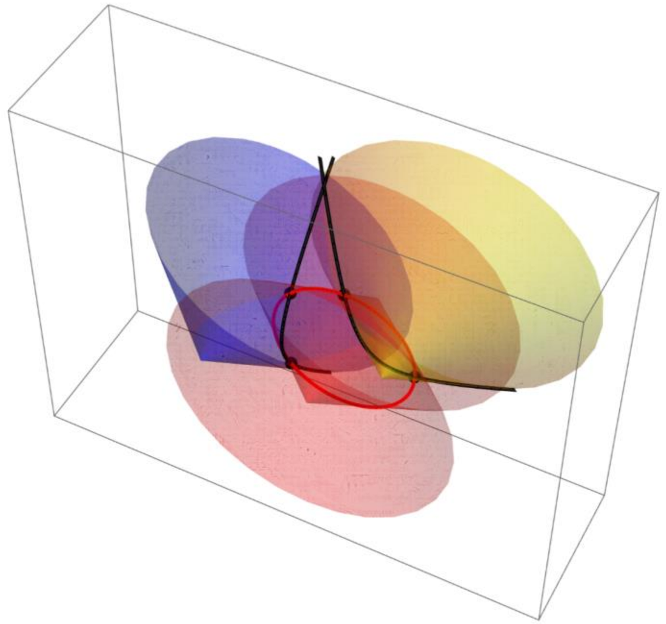


Comments:

1.For general cases, all Landau equations in a Landau system can always be classified into three categories

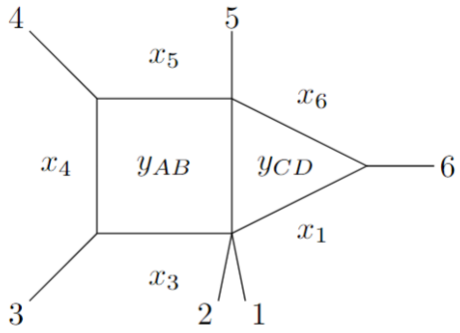
1) Equations with cut conditions less than 4 (triangles, bubbles, etc.) contributing only trivial singularities as Mandelstam to 0
2) Equations with 4 cut conditions exactly (cut problems)
3) Equations with cut conditions more than 4 (constraining problems)

In another word, we have associate all non-trivial Landau loci with their geometrical interpretation.



6-point double-box and its Landau system

- Sub-leading sectors: box-triangle



$$\mathrm{L}_{34561} : (y_{AB}-x_1)^2=(y_{AB}-x_3)^2=(y_{AB}-x_4)^2=(y_{AB}-x_5)^2=(y_{AB}-x_6)^2=0$$

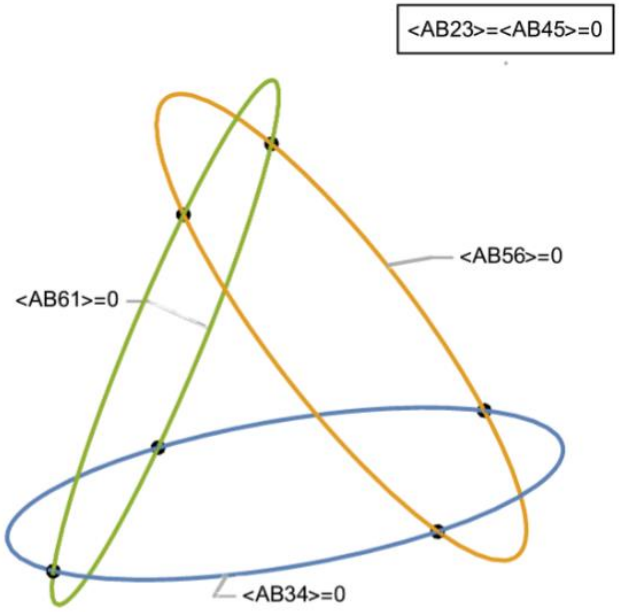
Comments

2.Of course, we have more choices to divide the five conditions into two cut problems from sub^2-leading, for instance

$$\mathrm{L}_{3451} : (y_{AB}-x_1)^2=(y_{AB}-x_3)^2=(y_{AB}-x_4)^2=(y_{AB}-x_5)^2=0$$

$$\mathrm{L}_{3456} : (y_{AB}-x_3)^2=(y_{AB}-x_4)^2=(y_{AB}-x_5)^2=(y_{AB}-x_6)^2=0$$

Another box-bubble by shrinking $(y_{CD}-x_1)^2$

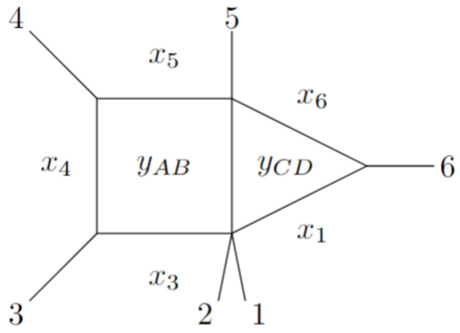


$$\mathbf{Gram}^2_{1,3,4,5,6} \propto (y_{AB}^{(1)}-y_{AB}^{(1)})^2(y_{AB}^{(1)}-y_{AB}^{(2)})^2(y_{AB}^{(2)}-y_{AB}^{(1)})^2(y_{AB}^{(2)}-y_{AB}^{(2)})^2$$

This singularity has different decompositions following each of these divisions

6-point double-box and its Landau system

- Sub-leading sectors: box-triangle



$$L_{34561}: (y_{AB}-x_1)^2 = (y_{AB}-x_3)^2 = (y_{AB}-x_4)^2 = (y_{AB}-x_5)^2 = (y_{AB}-x_6)^2 = 0$$

3. Distances of these cut solutions should be viewed as the most basic building blocks for symbol letters!

$$\mathbf{Gram}^2_{1,3,4,5,6} \propto (y^{(1)}_{AB}-y^{(1)}_{AB})^2(y^{(1)}_{AB}-y^{(2)}_{AB})^2(y^{(2)}_{AB}-y^{(1)}_{AB})^2(y^{(2)}_{AB}-y^{(2)}_{AB})^2$$

$$\frac{(y^{(1)}_{AB}-y^{(2)}_{AB})^2(y^{(2)}_{AB}-y^{(1)}_{AB})^2}{(y^{(1)}_{AB}-y^{(1)}_{AB})^2(y^{(2)}_{AB}-y^{(2)}_{AB})^2}$$

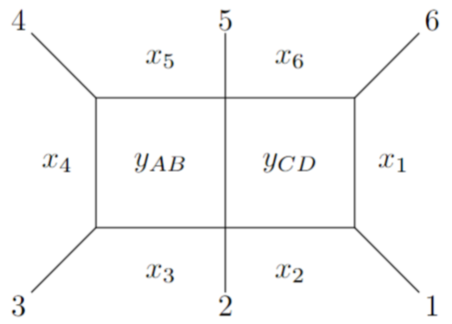
$$c_0\prod_i W_i = (a+\sqrt{\Delta})(a-\sqrt{\Delta}) \longrightarrow \frac{a+\sqrt{\Delta}}{a-\sqrt{\Delta}}$$

Conformal Invariants in the dual space

A natural way to uplift the Landau loci into odd letters!

6-point symbology

Leading (top) sector



$$(y_{AB}-x_3)^2=(y_{AB}-x_5)^2=(y_{AB}-x_1)^2=(y_{AB}-x_4)^2=(y_{AB}-x_6)^2=(y_{AB}-x_2)^2=0$$

divided into three problems

- We have triple of boxes and 6 cut solutions after this division
- DCI background: dual conformal invariants from cross-ratios of the distances
- In total, there are 9 independent letters from 6 solutions

$$\frac{(y_1-y_2)^2(y_1-y_2)^2}{(y_1-y_1)^2(y_2-y_2)^2},\quad \frac{(y_1-y_2)^2(y_1-y_2)^2}{(y_1-y_2)^2(y_2-y_1)^2}\qquad \frac{(y_1-y_1)^2(y_1-y_2)^2(y_2-y_1)^2(y_2-y_2)^2}{(y_2-y_1)^2(y_2-y_2)^2(y_1-y_1)^2(y_1-y_2)^2}$$

and all permutation of color (9 independent)

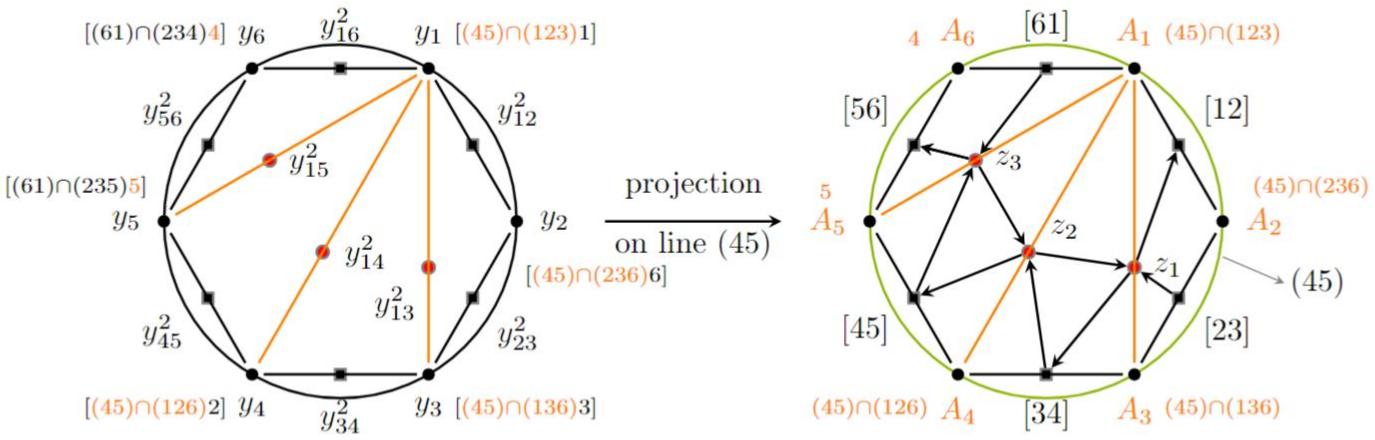
- Exactly read the 9 letters in 6-point bootstrap! (A3 bootstap to L=8 **[L.Dixon et al, 2204.11901]**)

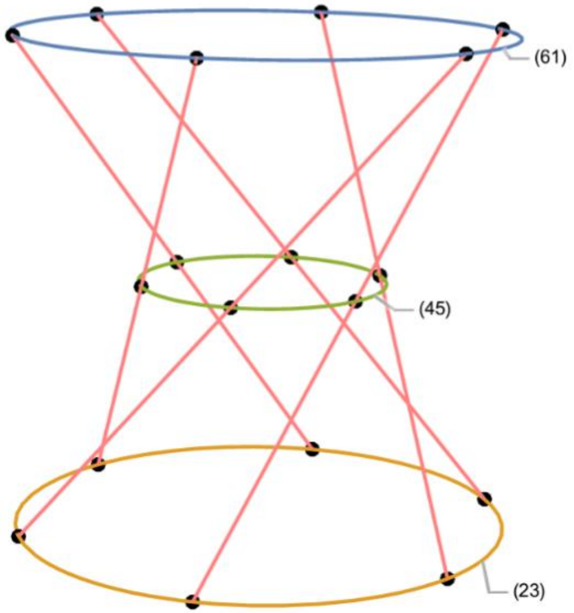
$$\{u,v,w,1-u,1-v,1-w,y_u,y_v,y_w\},\ y_u=\frac{1+u-v-w+\sqrt{(1-u-v-w)^2-4uvw}}{1+u-v-w-\sqrt{(1-u-v-w)^2-4uvw}}\qquad u=\frac{s_{12}s_{45}}{s_{123}s_{345}},\quad v=\frac{s_{23}s_{56}}{s_{234}s_{123}},\quad w=\frac{s_{34}s_{61}}{s_{345}s_{234}},$$

6-point symbology

Original story of Schubert analysis

- Six solutions as bitwistors in momentum twistor space
- Projecting them onto any lines (6 intersections)





$$\{(45)\cap(126),\ (45)\cap(136),\ (45)\cap(236),\ (45)\cap(123), 4, 5\}$$

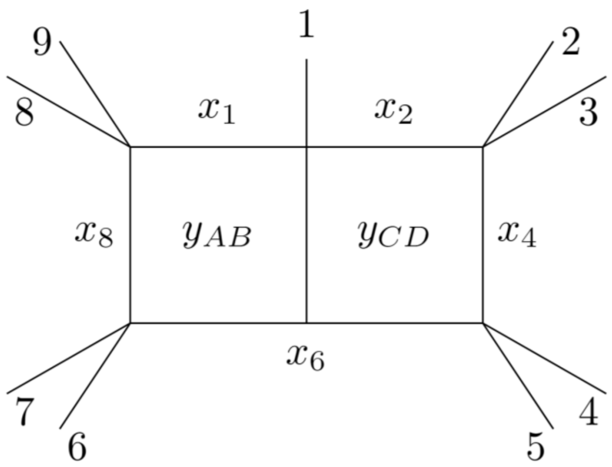
- Ordered in positive Grassmannian region, 9 positive cross-ratios

$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\},\ \ y_u = \frac{1+u-v-w+\sqrt{(1-u-v-w)^2-4uvw}}{1+u-v-w-\sqrt{(1-u-v-w)^2-4uvw}}$$

$$u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle}$$

9-point double box and its last-entries

- In general cases, constraining problems and cut problems can be generalized to more complicated problems



Constraining problem from box-triangle

$$(y_{AB}-x_1)^2=(y_{AB}-x_2)^2=(y_{AB}-x_4)^2=(y_{AB}-x_6)^2=(y_{AB}-x_8)^2=0$$

$$\Downarrow$$

Cut problem for Box-bubble

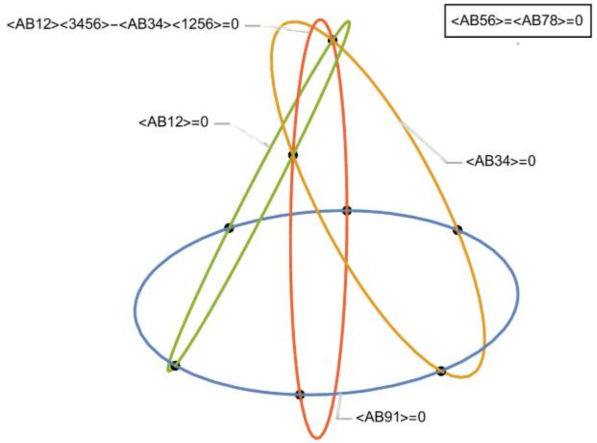
$$(y_{AB}-x_1)^2=(y_{AB}-x_6)^2=(y_{AB}-x_8)^2=(y_{AB}-x_4)^2=0$$

$$(y_{AB}-x_1)^2=(y_{AB}-x_6)^2=(y_{AB}-x_8)^2=(y_{AB}-x_2)^2x_{4,6}^2-(y_{AB}-x_4)^2x_{2,6}^2=0$$

Cut problem for top integral

$$\frac{(y_1-y_3)^2(y_2-y_4)^2}{(y_1-y_4)^2(y_2-y_3)^2}=\left[\frac{A+\sqrt{\Delta_9\mathbf{Gram}_{6,8,1,4}}}{A-\sqrt{\Delta_9\mathbf{Gram}_{6,8,1,4}}}\right]^2$$

Odd letters with respect to four-mass box root and two-loop root (last-entry)



Landau-based Schubert analysis, a summary

All Landau loci for an integral come from pinching of cut solutions from its Landau equation system.

[for Canonical differential equation system, cut/constraining problems are from UTs, 2309.16441]

Pinching of two solutions from an cut problem yield leading Landau loci.

(reasonable) Pinching of two solutions from two different cut problems yield constraint on kinematics

(under DCI background) Symbol letters for the system are from cross-ratios of these distances

Landau diagrams and scattering amplitudes

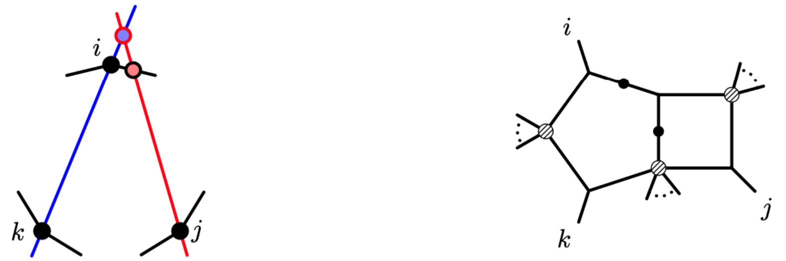
Landau diagrams from Amplituhedra

- Problems: when adding integral basis together, cancellation may happen for singularities from integrals, and some of the letters we got may be spurious.
- Geometrical input: integrand for N=4 SYM amplitudes are from canonical functions on Amplituhedra
- All cut solutions should always localize inside the Amplituhedra!

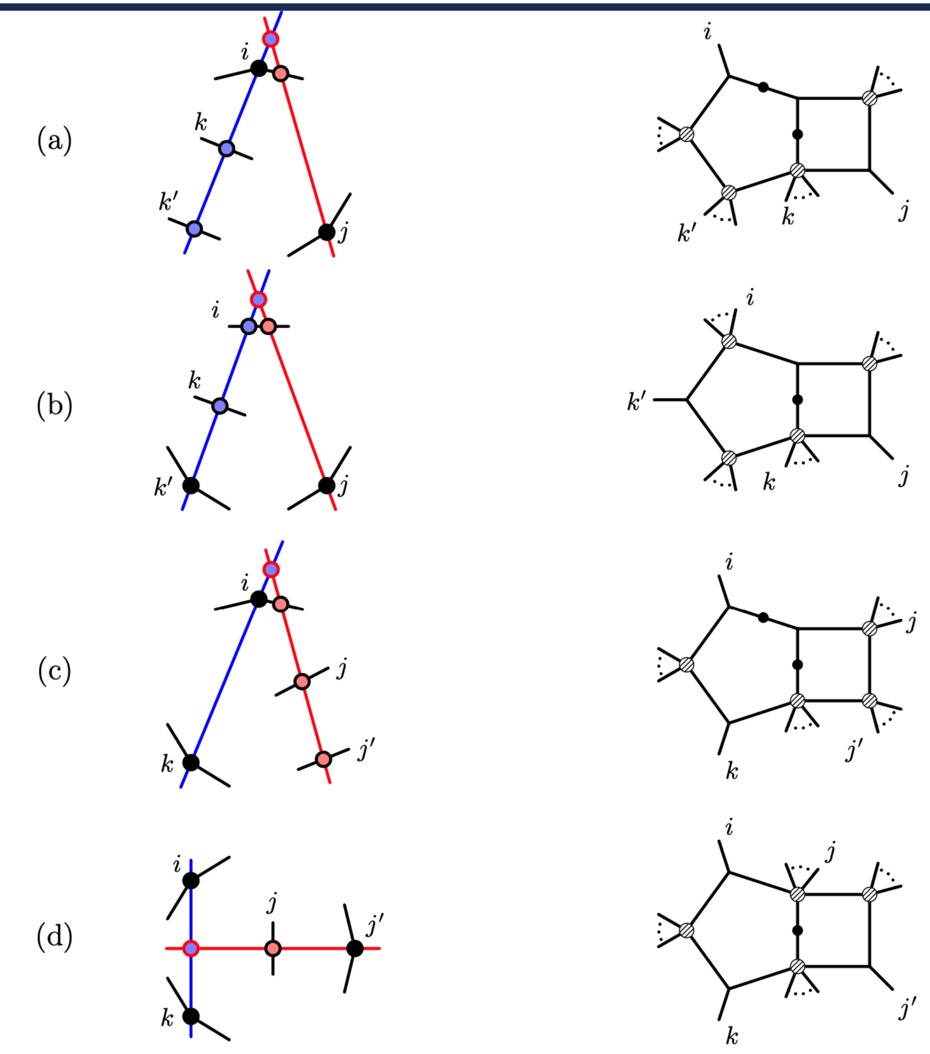
- By classifying all boundaries of amplituhedra, we have instruction for a set of reasonable cut solutions directly by Landau diagrams, instead of checking cut solutions from integral basis **[I.Prlina et al. 1712.08049]**

Landau diagrams from Amplituhedra

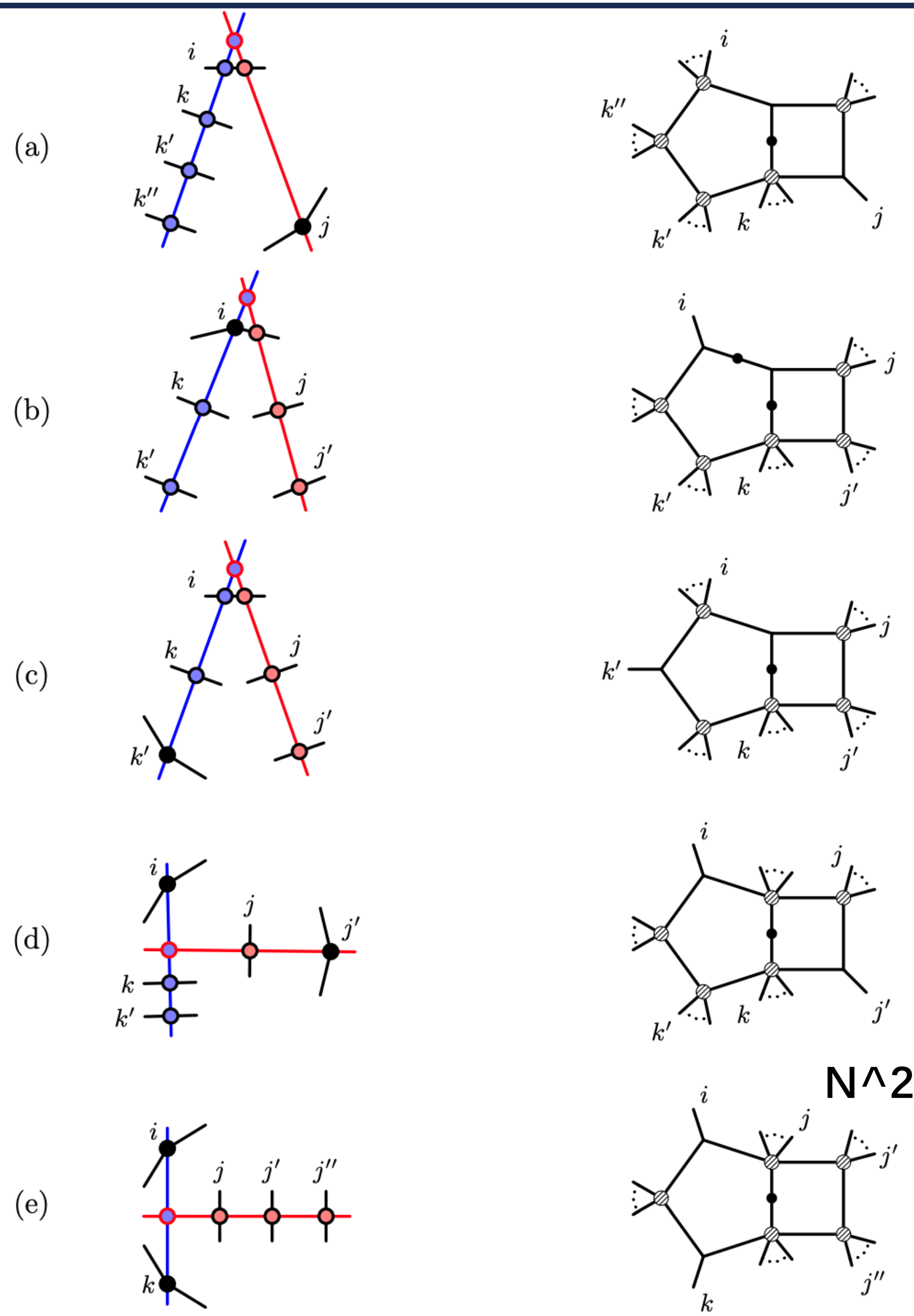
(a)



MHV



NMHV

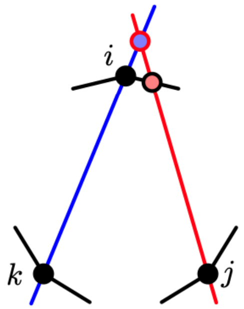


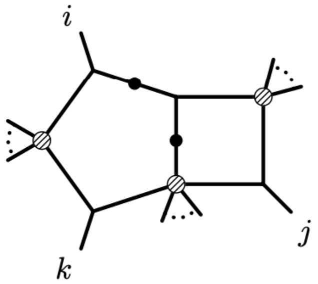
N^2MHV

L=2 MHV amplitudes and its alphabet

- For MHV amplitudes, only one type of maximal cuts and corresponding integrals should be considered

(a)

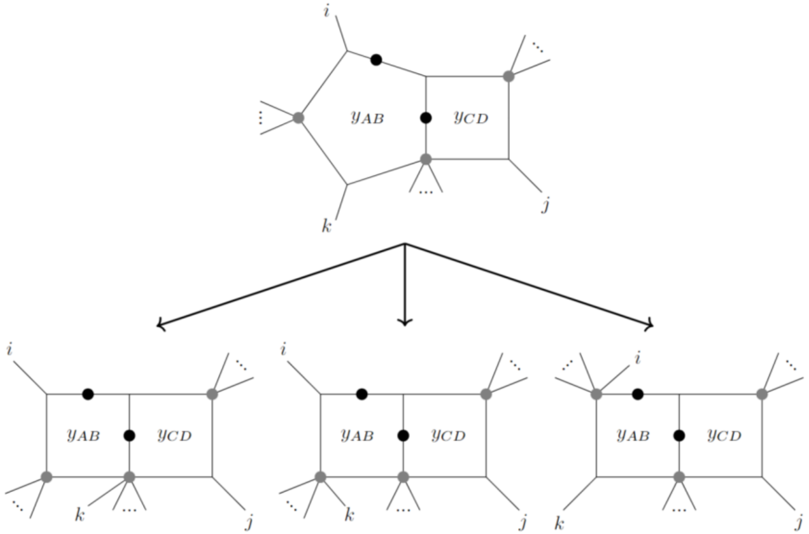


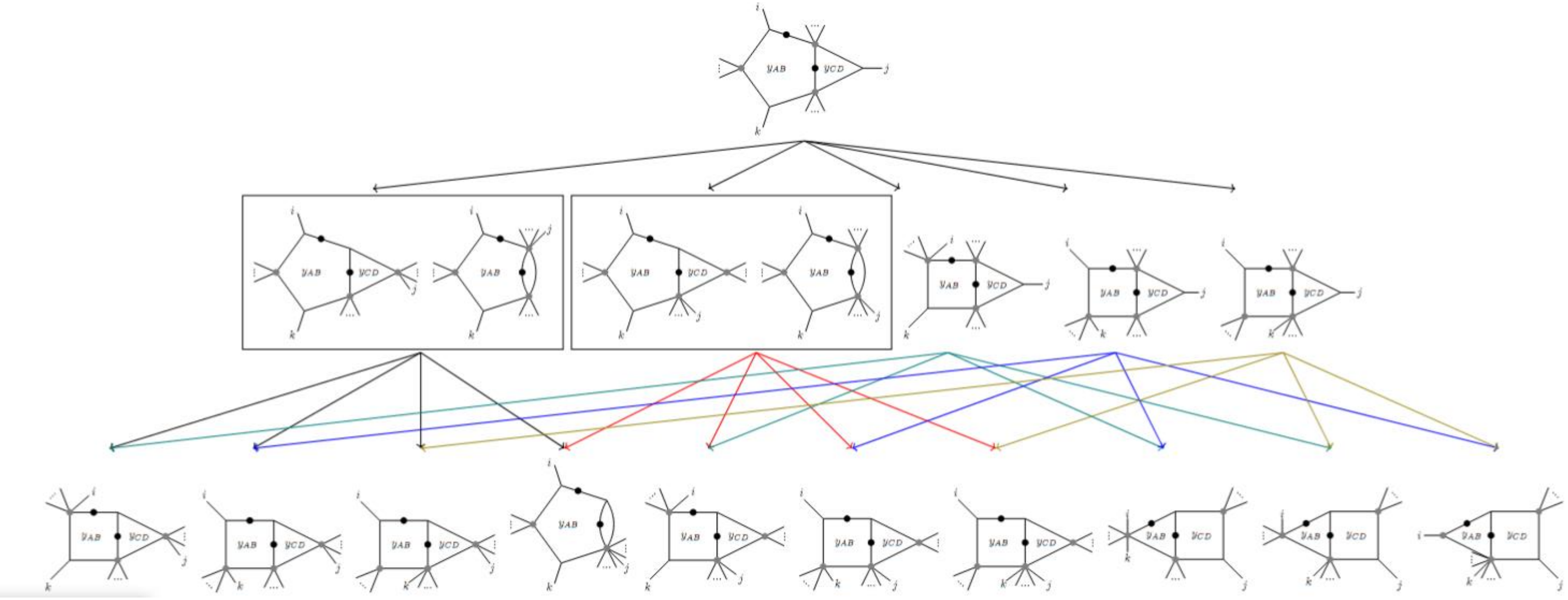


- The procedure is as the following:
- 1. classify all sub-topologies following this integral by shrinking its propagators.
- 2. Write down cut conditions for each of the case. Conditions yielding cut solutions are viewed as cut problems; Conditions imposing constraints on kinematics are viewed as constraining problems, lead to possible combinations of cut problems
- 3.We then have a limit set of problems and all their possible combination, from which alphabet of amplitudes can be computed.

L=2 MHV amplitudes and its alphabet

- All cut problems: all boxes up to three-mass
- All constraining problems: five-point Gram conditions from box-triangles & penta-bubbles

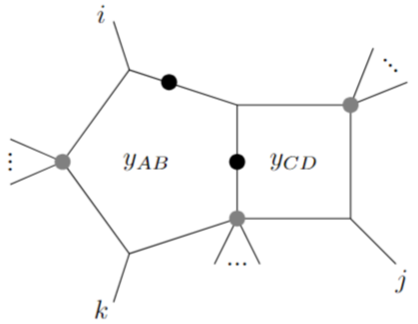




- Singularities:

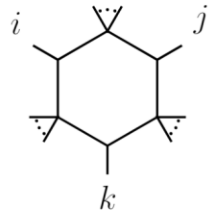
$$\{x_{i,j}^2, \quad \mathbf{Gram}_{a,a+1,b,c}, \quad \mathbf{Gram}_{a,a+1,b,b+1,c}\}$$

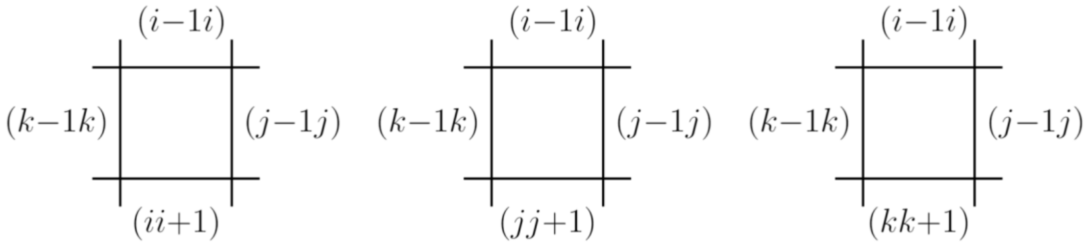
L=2 MHV amplitudes and its alphabet



MHV alphabets are always a union of A3 configuration!

<ins>All box-triples sharing three external points from three-mass-easy hexagon topology</ins>





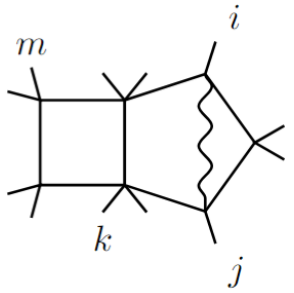
$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\},\ \ y_u = \frac{1+u-v-w+\sqrt{(1-u-v-w)^2-4uvw}}{1+u-v-w-\sqrt{(1-u-v-w)^2-4uvw}}$$

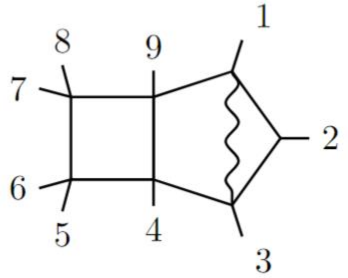
$$u = \frac{\langle \bar{i} j\rangle\langle i\bar{j}\rangle\langle k{-}1\ k\ i{-}1\ i\rangle\langle k{-}1\ k\ j{-}1\ j\rangle}{\langle i(i{-}1\ i{+}1)(j{-}1\ j)(k{-}1\ k)\rangle\langle j(j{-}1\ j{+}1)(i{-}1\ i)(k{-}1\ k)\rangle}$$

L=2 N2MHV amplitudes and a special component

- For N^2MHV, almost no results for n>=8 are known
- Calculation for the full amplitude is rather complicated
- Try to compute a special component amplitude

$$\mathcal{A}_{n,2,2}\Big|_{\chi_i^1\chi_j^2\chi_k^2\chi_{k+1}^3\chi_l^4\chi_{l+1}^3\chi_m^4\chi_{m+1}^1} \propto$$





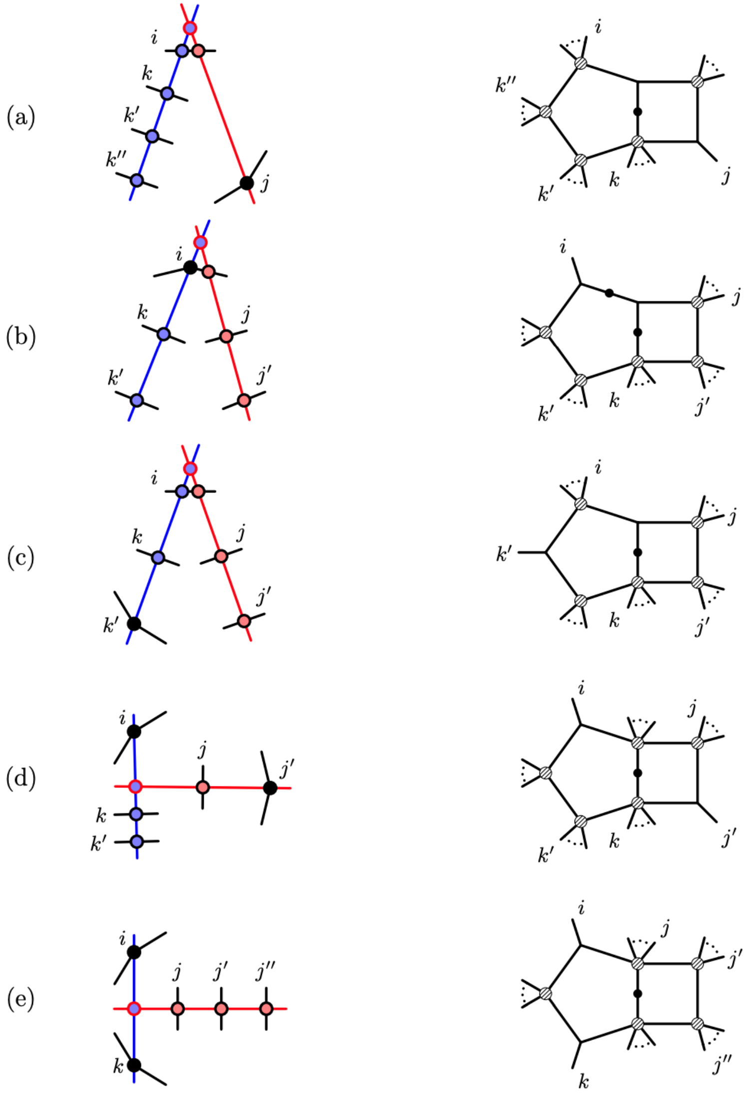
9pt special case

cut: boxes, 2 double-boxes and leading penta-box.

constraining: five-point Gram conditions & 2 double -boxes top constraints.

Symbology analysis: 9 1st + 40 2nd + 131 3rd+10 4th

Very reasonable for bootstrapping (in progress)



Discussion

- We discovered ways to uplifting singularities of MPL functions from Landau loci to their symbol letters through a geometrical viewpoint
- From Landau diagrams of N=4 SYM theory from amplituhedra, we computed symbol letter predictions for two-loop amplitudes

- Future directions:
- Elliptic integrals/non-planar integrals? Schubert analysis works..
- More Landau diagrams for higher-loop amplitudes or other physical quantities?

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Thanks!

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