Triangulations of cosmological polytopes

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Holonomic Techniques for Feynman Diagrams

October 17, 2024

UNIVERSITÄT OSNABRÜCK

Outline

- What and why
- A mathematically nice (and large) triangulation
- Smaller triangulations and secondary polytopes

What is a cosmological polytope?

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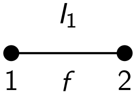
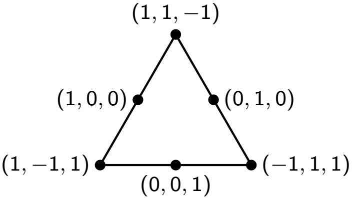


$$\mathcal{C}_{I_1} = \mathrm{conv}\{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$$



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Observation

- $\mathbf{e}_f \in \mathcal{C}_G$ and $\mathbf{e}_i \in \mathcal{C}_G$ for all $f \in E$ and $i \in V$.
- $\dim \mathcal{C}_G = |V| + |E| - 1$.

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$$\Omega_{\mathcal{C}_G} = \frac{g}{f_1 \cdots f_r}\omega,$$

with

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Alternatively, one can compute $\Omega_{\mathcal{C}_G}$ from a subdivison of $\mathcal{C}_G$ into polytopes $Q_1, \ldots, Q_m$ since

$$\Omega_{\mathcal{C}_G} = \Omega_{Q_1} + \cdots + \Omega_{Q_m}.$$

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What is known

- *facet descriptions (Arkani-Hamed-Benincasa-Postnikov; 2017)*
- *face descriptions (Benincasa, Kühne-Monin; 2022)*
- *the normalized volume for trees (Kühne-Monin; 2022)*

The toric ideal

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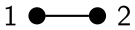
$$\begin{aligned} \varphi_G : R_G &\to K[\mathbf{x}^{\mathbf{p}} \ : \ \mathbf{p} \in \mathcal{C}_G \cap \mathbb{Z}^{V \cup E}] \\ z_k &\mapsto x_k \\ z_e &\mapsto x_e \\ y_{ij} &\mapsto x_i x_j^{-1} x_e \\ y_{ji} &\mapsto x_i^{-1} x_j x_e \\ t_e &\mapsto x_i x_j x_e^{-1} \end{aligned}$$

$I_{\mathcal{C}_G} = \ker(\varphi_G)$ is called the toric ideal of $\mathcal{C}_G$.

Example



$$I_{\mathcal{C}_G} = \langle y_{12}y_{21} - z_e^2, y_{12}t_e - z_1^2, y_{21}t_e - z_2^2, y_{12}z_2 - z_1z_e, y_{21}z_1 - z_2z_e, t_ez_e - z_1z_2\rangle$$

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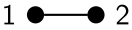

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Order the variables in R_G

$$y_{12} > y_{21} > t_e > z_e > z_1 > z_2$$

and consider the lexicographic order on monomials, i.e.,

$y_{12}^{b_{12}} y_{21}^{b_{21}} t_e^{b_e} z_e^{b_{z_e}} z_1^{b_1} z_2^{b_2} \prec y_{12}^{a_{12}} y_{21}^{a_{21}} t_e^{a_e} z_e^{a_{z_e}} z_1^{a_1} z_2^{a_2} \Leftrightarrow$ The leftmost nonzero entry in $\mathbf{a} - \mathbf{b}$ is positive.

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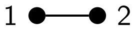

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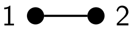

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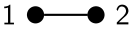

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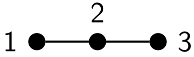


$$z_1y_{32}z_{12} - z_3y_{12}z_{32} \in I_{\mathcal{C}_G}$$

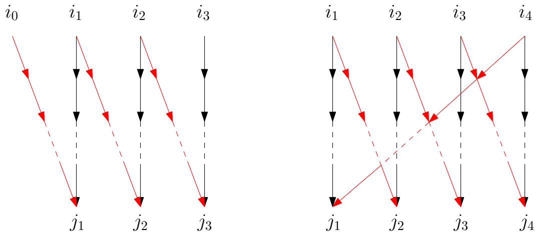
is not divisible by any of the above leading terms.

A Gröbner basis

Theorem (J., Solus, Venturello; 2023)

The following binomials are a squarefree Gröbner basis $\mathcal{B}_G$ of $I_{\mathcal{C}_G}$:

- *fundamental binomials for all $e \in E$,*
- *cycle binomials for all directed cycles,*
- *zig-zag pair resp. cyclic pair binomials for any path resp. cycle.*



From Gröbner bases to triangulations

Corollary (J., Solus, Venturello; 2023)

Let G be a graph. The cosmological polytope $\mathcal{C}_G$ has a regular unimodular triangulation $\mathcal{T}_G$.

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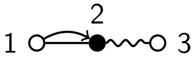
The monomial $z_1 z_3 z_{12} y_{12} t_{23}$ corresponds to



Example

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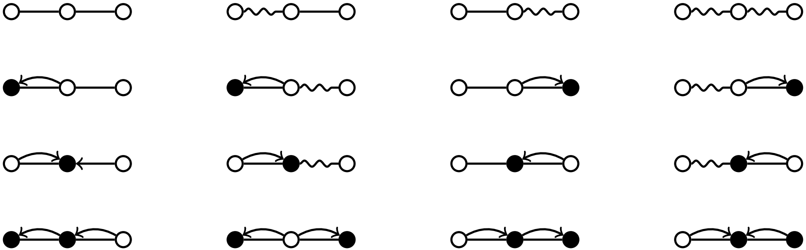

Normalized volumes

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The normalized volume of the cosmological polytope of

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What else?

We can compute the h^*-polynomials for

- trees (with multiple edges)
- cycles (with multiple edges)
- 1-sums of graphs.

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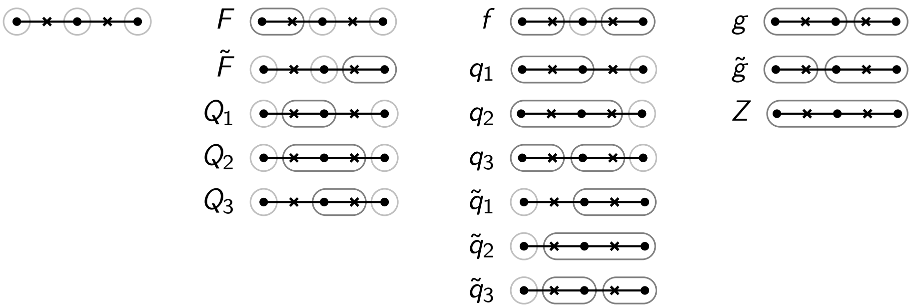

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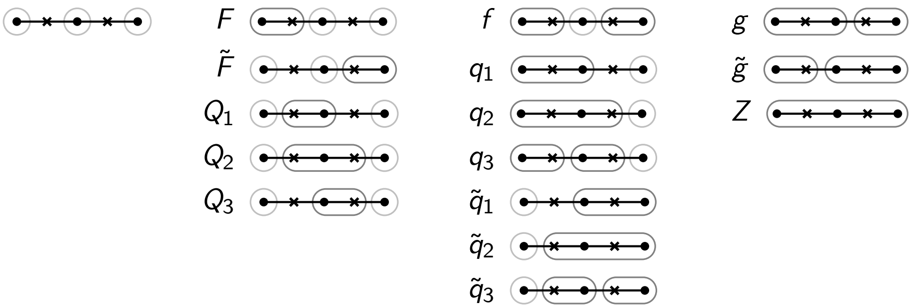

Bernd Sturmfels was attending both talks and asked if there is a relation.

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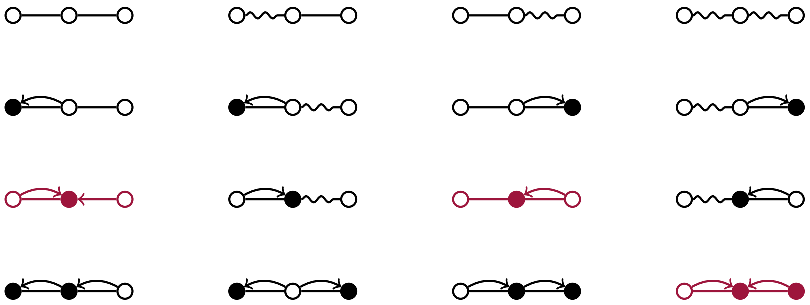

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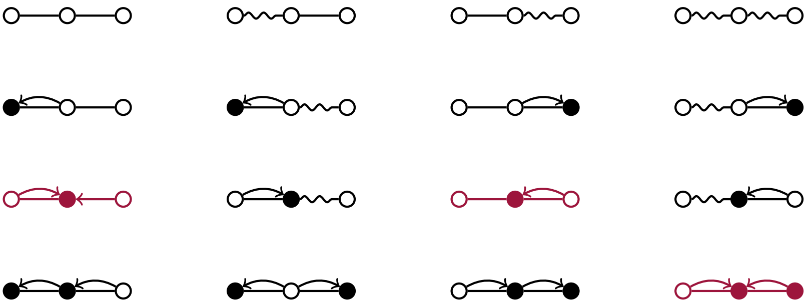
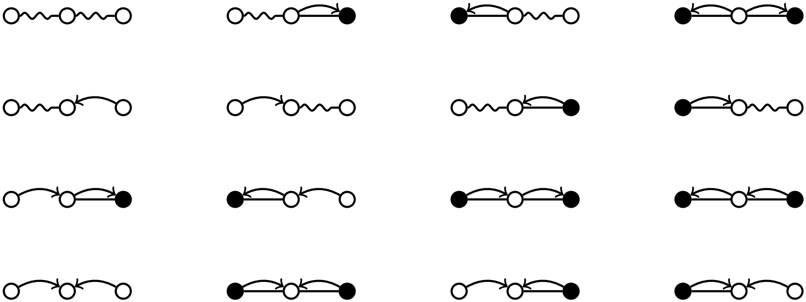


But: This is not the only unimodular triangulation.

A symmetric unimodular triangulation for I_3



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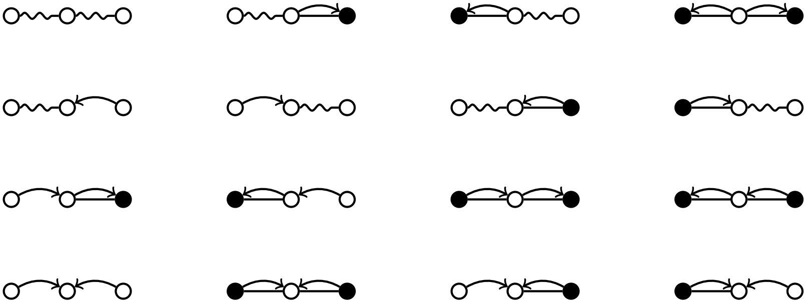
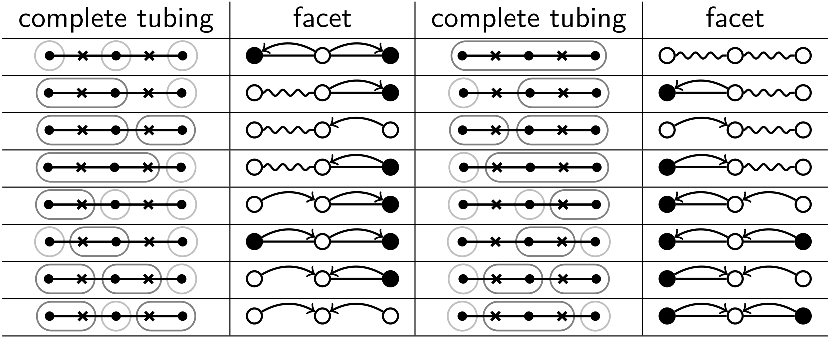


Question

Is there a canonical bijection between facets of this triangulation and tubings?

A bijection



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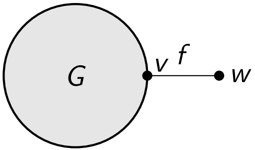
Question

Can we find a smaller and easy to compute triangulation that extends nicely?

Towards a smaller triangulation

Let $G = (V, E)$, $f = \{v, w\}$ with $v \in V$ and $w \notin V$.



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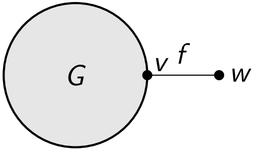

$$\mathcal{C}_{G+f} = \mathrm{conv}\{\mathcal{C}_G, \mathbf{e}_f + \mathbf{e}_v - \mathbf{e}_w, \mathbf{e}_f - \mathbf{e}_v + \mathbf{e}_w, \mathbf{e}_f + \mathbf{e}_v + \mathbf{e}_w\}.$$

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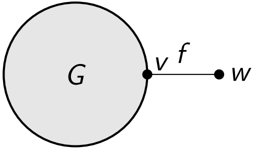

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Geometrically, $\mathcal{C}_{G+f}$ is obtained from $\mathcal{C}_G$ by

- first taking a bipyramid over $\mathcal{C}_G$ with apices $\mathbf{e}_f + \mathbf{e}_v - \mathbf{e}_w$ and $-\mathbf{e}_f + \mathbf{e}_v + \mathbf{e}_w$, and

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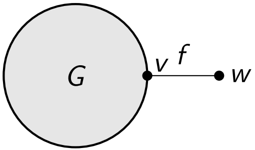

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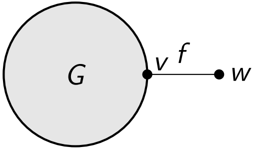

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- then taking a pyramid with apex $\mathbf{e}_f - \mathbf{e}_v + \mathbf{e}_w$.

Hence, given a triangulation $\mathcal{T}_G$ of $\mathcal{C}_G$ there exists a triangulation $\mathcal{T}_{G+f}$ of $\mathcal{C}_{G+f}$ with

$$|\mathcal{T}_{G+f}| = 2|\mathcal{T}_G|.$$

A small triangulation for trees

Observation

Let T_n be a tree on n edges. There exists a triangulation $\mathcal{T}_{T_n}$ of $\mathcal{C}_{T_n}$ with

$$2^{n-1}$$

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Observation

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- The constructed triangulation is extremely easy to describe.
- It extends in a simple way from smaller to larger trees.
- It is by far smaller than the unimodular one: 2^{n-1} vs. 4^n maximal cells.

Secondary polytopes

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Computing the secondary polytope of $\mathcal{C}_{I_n}$ for small n, we get the following results

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3	4	6	1	2
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The secondary polytopes are associahedra.

Associahedra for $n \in \{4, 5\}$

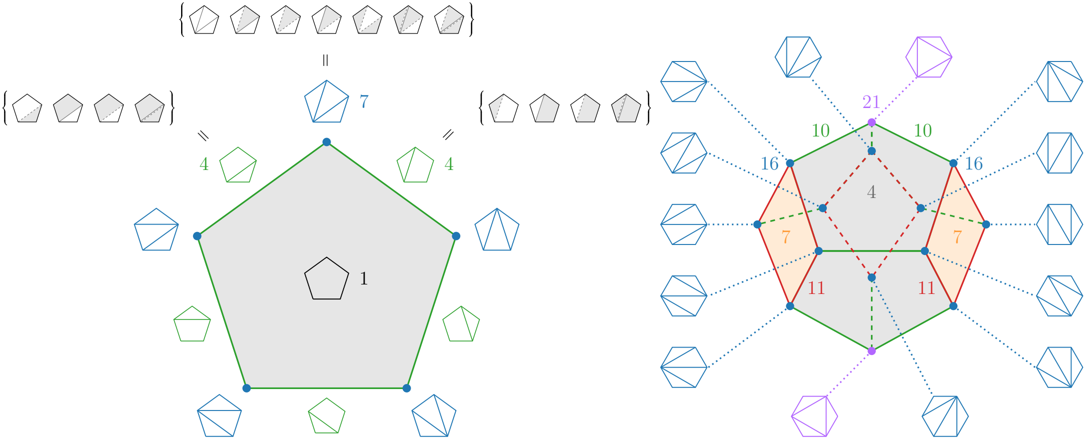

Figure: Pictures from Arkani-Hamed et al., Differential Equations for Cosmological Correlators

TODO list

- Compute the canonical forms via triangulations.
- Can the canonical form in some way read off from a Gröbner basis?
- Understand the connections to tubings.
- Understand the secondary polytopes. Is there a connection to the work of Arkani-Hamed et al.?
- What is the role of the scattering amplitude facet?
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Thank you!