Holonomic Techniques for Feynman Integrals

Completely monotone functions and applications

Khazhgali Kozhasov (LJAD, UniCA)
based on works with J.-B. Lasserre, M. Michałek and B. Sturmfels

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Outline

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Complete monotonicity and ...

Part I

Hyperbolic polynomials

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(real algebraic geometry, combinatorics, algebraic statistics)

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(real algebraic geometry, polynomial optimization)

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They observed that Taylor coefficients $a_{k,\ell,m}$ of the function

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satisfy the difference equation $(*)$.

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In 1930 Lewy wrote to G. Szegö asking him to prove positivity of Taylor coefficients ($a_{k,\ell,m} > 0$) in general.

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He expressed coefficients as some integrals of products of Bessel functions which are shown to be positive (nonnegative).

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If G is the n-cycle, $T_G(\mathbf{x}) = \sum_{i=1}^n \prod_{j \neq i} x_j$ and Szegö's result follows from Scott and Sokals' theorem with $\mathbf{x} = (1, \ldots, 1)$.

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$$\begin{aligned} f(\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2) \;&=\; \int_{\mathcal{C}^*} \exp\Big(-(\alpha_1\theta^\mathsf{T}\mathbf{x}_1 + \alpha_2\theta^\mathsf{T}\mathbf{x}_2)\Big)\, \mathrm{d}\mu(\theta) \\ &\leq\; \int_{\mathcal{C}^*} \alpha_1 \exp(-\theta^\mathsf{T}\mathbf{x}_1) + \alpha_2 \exp(-\theta^\mathsf{T}\mathbf{x}_2)\, \mathrm{d}\mu(\theta) \;=\; \alpha_1 f(\mathbf{x}_1) + \alpha_2 f(\mathbf{x}_2) \end{aligned}$$

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$\rightsquigarrow$ Conjecture of Michałek et al. holds in this case!

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$\rightsquigarrow$ **Conjecture of Michalek et al. holds for all $E_{d,n}$!**

What is left?

Conj. (Michałek, Sturmfels, Uhler and Zwiernik, 2015)

Let p be a real homogeneous polynomial that is hyperbolic w.r.t. $\mathcal{C}$. Then there is $\alpha = \alpha(p) > 0$ such that $p^{-\alpha}$ is CM on $\mathcal{C}$.

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- Would be enough to prove (or disprove) for multi-affine polynomials

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Open question: is $p^{-\alpha}$ CM for some $\alpha > 0$?

Part II

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$\Sigma_{d,n} \subseteq \overline{\mathcal{C}_{d,n}}$ - closed convex cone of sums of squares.

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Idea: Study "natural" functions on $\overline{\mathcal{C}_{d,n}}$ with the hope of understanding it better.

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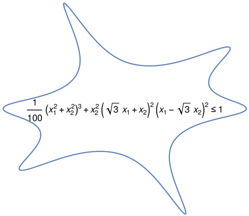

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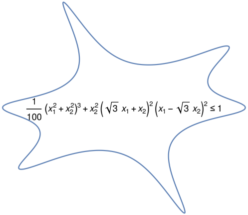

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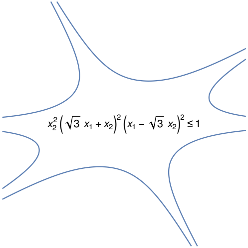

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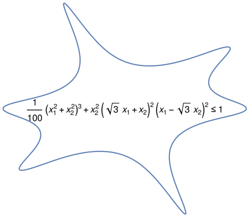

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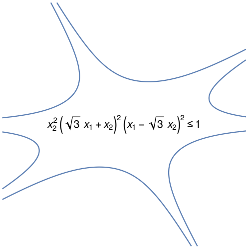

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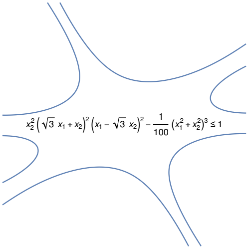

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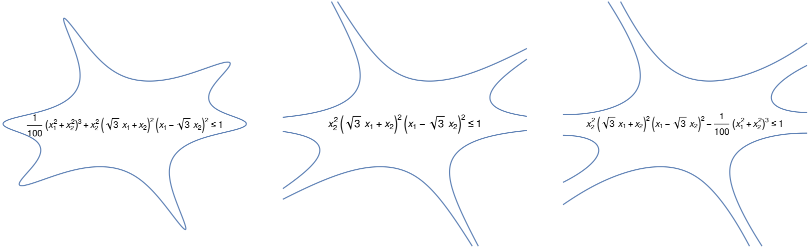

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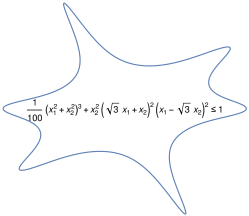

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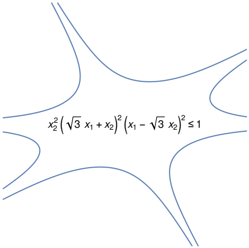

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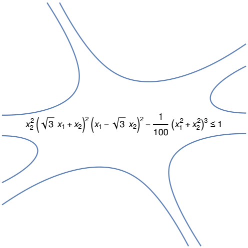

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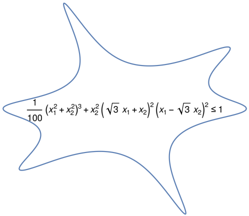

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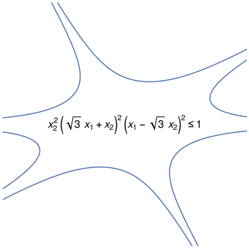

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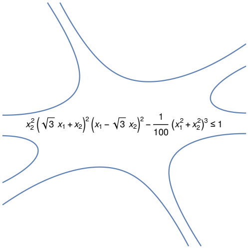

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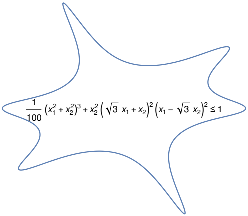

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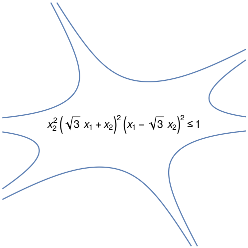

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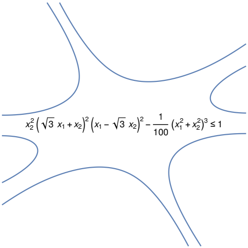

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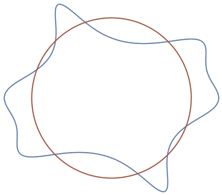

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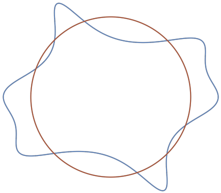

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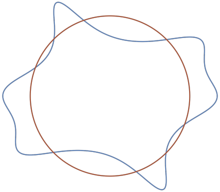

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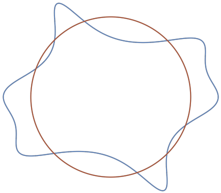

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Theorem (Scott and Sokal, 2014)

The function $G \mapsto \det(G)^{-\alpha}$ on the cone of $n \times n$ positive definite matrices is CM iff $\alpha = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ or $\alpha \geq \frac{n-1}{2}$.

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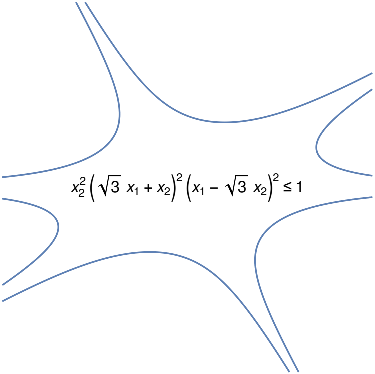

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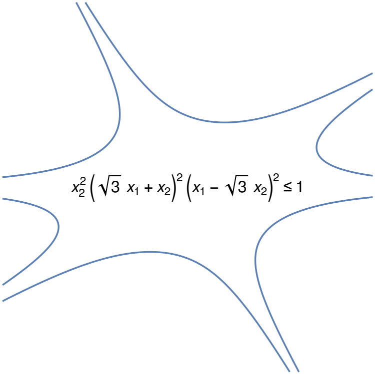

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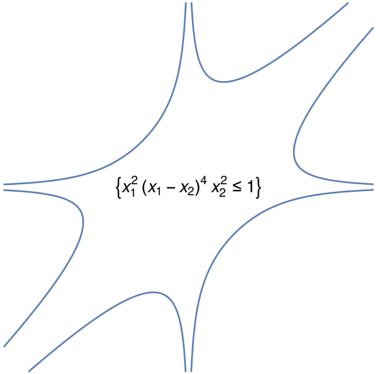

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$$\int_{U_j} \frac{1}{\tilde{g}(\mathbf{y})^{n/d}} \, \mathbf{dy}, \quad j = 1, \ldots, k,$$

over (arbitrary) small neighborhoods of $\mathbf{y}_1, \ldots, \mathbf{y}_k$ converge.

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Vielen Dank für Ihre Aufmerksamkeit!