Holonomic Techniques for Feynman Integrals

MPI for Physics, Munich, Germany, October 17, 2024

Symbolic summation and integration techniques to simplify Feynman integrals

Carsten Schneider

DESY-cooperation: J. Bluemlein, P. Marquard

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

Holonomic Techniques for Feynman Integrals

MPI for Physics, Munich, Germany, October 17, 2024

Symbolic summation and integration techniques to simplify Feynman integrals

Carsten Schneider

DESY-cooperation: J. Bluemlein, P. Marquard

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

JOHANNES KEPLER UNIVERSITY LINZ

Der Wissenschaftsfonds.

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

there exist $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with $$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Definition: sequence $(F(n))_{n \geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called holonomic if

there exist $a_0(x), \ldots, a_\delta(x) \in \mathbb{K}[x]$ (not all zero) with $$a_0(n) F(n) + \cdots + a_\delta(n) F(n+\delta) = 0 \qquad \text{(RE)}$$

symbolic summation

symbolic integration

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

> there exist $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with
>
> $$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Definition: sequence $(F($ … called holonomic if

Feynman integrals

> there exist $a_0(x), ..$ … (not all zero) with
>
> $$a_0(n)F(n) + \cdots + a_\delta(n)F(n+\delta) = 0 \qquad \text{(RE)}$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz\, \underbrace{x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

The integrand is

- hyperexponential in x,y,z:

$$\frac{D_x f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon, n, x, y, z)}$$

The integrand is

- hyperexponential in x,y,z:

$$\frac{D_y f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

The integrand is

- hyperexponential in x,y,z:

$$\frac{D_z f(\varepsilon, n, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

The integrand is

- hyperexponential in x,y,z:
- hypergeometric in n:

$$\frac{f(\varepsilon,n+1,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

The integrand is

- hyperexponential in x,y,z:
- hypergeometric in n:

$$\frac{f(\varepsilon, n+1, x, y, z)}{f(\varepsilon, n, x, y, z)} \in \mathbb{Q}(\varepsilon, n, x, y, z)$$

Holonomic theory shows that there exists a holonomic recurrence!

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

$$\Big\downarrow \quad \text{Ablinger's MultiIntegrate.m (9 hours)}$$

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

$$
\begin{aligned}
a_0(n,\varepsilon) = {} & (n+1)(n+2)\big(8\varepsilon^{10} + 104\varepsilon^9(n+3) + 4\varepsilon^8\big(96n^2+601n+887\big) \\
& + 4\varepsilon^7\big(12n^3+414n^2+1583n+1393\big) \\
& - 8\varepsilon^6\big(264n^4+2436n^3+8643n^2+14518n+9947\big) \\
& - 16\varepsilon^5\big(156n^5+1690n^4+6847n^3+12661n^2+9537n+717\big) \\
& + 32\varepsilon^4\big(68n^6+1158n^5+8155n^4+30114n^3+61712n^2+67616n+31693\big) \\
& + 64\varepsilon^3\big(40n^7+560n^6+2755n^5+3729n^4-14194n^3-61920n^2-89140n-46600\big) \\
& - 128\varepsilon^2(n+2)\big(12n^7+254n^6+2249n^5+10758n^4+30173n^3+50610n^2 \\
& + 49122n+22706\big) \\
& + 256\varepsilon(n+2)^2(n+3)(n+4)\big(44n^4+501n^3+2044n^2+3455n+1976\big) \\
& - 512(n+1)(n+2)^3(n+3)^2(n+4)\big(6n^2+47n+95\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_1(n,\varepsilon) = (n+2)\Big(& -22\varepsilon^{11} - 2\varepsilon^{10}(157n+435) - \varepsilon^{9}\big(1500n^2+8611n+11745\big) \\
& - \varepsilon^{8}\big(2548n^3+22936n^2+63597n+54229\big) \\
& + 4\varepsilon^{7}\big(266n^4+1857n^3+6065n^2+14351n+15987\big) \\
& + 8\varepsilon^{6}\big(994n^5+12961n^4+67246n^3+174692n^2+226821n+116092\big) \\
& + 16\varepsilon^{5}\big(336n^6+5348n^5+33569n^4+104918n^3+165290n^2+108259n+6100\big) \\
& - 16\varepsilon^{4}\big(404n^7+7578n^6+61778n^5+284762n^4+802660n^3+1382074n^2 \\
& + 1340455n+560287\big) \\
& - 64\varepsilon^{3}\big(94n^8+1823n^7+14305n^6+55870n^5+96299n^4-37256n^3 \\
& - 447044n^2-704959n-379338\big) \\
& + 128\varepsilon^{2}(n+3)\big(30n^8+715n^7+7667n^6+48253n^5+194086n^4+507439n^3 \\
& + 835393n^2+785327n+320382\big) \\
& - 256\varepsilon(n+2)(n+3)^2\big(107n^6+2070n^5+16342n^4+67226n^3+151557n^2 \\
& + 176932n+83196\big) \\
& + 256(n+2)^3(n+3)^3(n+4)\big(30n^3+331n^2+1193n+1386\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
a_2(n,\varepsilon) = \big(&12\varepsilon^{12} + 12\varepsilon^{11}(17n+45) + 2\varepsilon^{10}\big(620n^2 + 3553n + 4795\big) \\
&+ 2\varepsilon^9\big(1504n^3 + 14190n^2 + 41901n + 38907\big) \\
&+ 4\varepsilon^8\big(172n^4 + 4983n^3 + 30942n^2 + 69119n + 50850\big) \\
&- 4\varepsilon^7\big(1996n^5 + 24056n^4 + 113313n^3 + 269119n^2 + 337198n + 185290\big) \\
&- 16\varepsilon^6\big(450n^6 + 8210n^5 + 59749n^4 + 227386n^3 + 486841n^2 + 563176n + 275664\big) \\
&+ 16\varepsilon^5\big(340n^7 + 4314n^6 + 19137n^5 + 25532n^4 - 55105n^3 - 206516n^2 - 191528n \\
&- 23458\big) \\
&+ 32\varepsilon^4\big(140n^8 + 2940n^7 + 26550n^6 + 139926n^5 + 493839n^4 + 1240186n^3 \\
&+ 2161699n^2 + 2304248n + 1100084\big) \\
&+ 64\varepsilon^3\big(4n^9 + 506n^8 + 8651n^7 + 63510n^6 + 236215n^5 + 395334n^4 - 105413n^3 \\
&- 1551017n^2 - 2362944n - 1217770\big) \\
&- 128\varepsilon^2(n+3)\big(12n^9 + 314n^8 + 3782n^7 + 29105n^6 + 160727n^5 + 640273n^4 \\
&+ 1750874n^3 + 3052505n^2 + 3017094n + 1276604\big) \\
&+ 256\varepsilon(n+2)(n+3)^2(n+4)\big(26n^6 + 825n^5 + 8967n^4 + 46529n^3 + 125411n^2 \\
&+ 168628n + 88652\big) \\
&- 512(n+1)(n+2)^2(n+3)^3(n+4)^2\big(6n^3 + 98n^2 + 459n + 655\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_3(n,\varepsilon) = \big(&-64\varepsilon^{12} - 8\varepsilon^{11}(113n+298) - 8\varepsilon^{10}\big(519n^2+2948n+3896\big)\\
&- 4\varepsilon^9\big(1444n^3+13839n^2+39746n+34305\big)\\
&+ 4\varepsilon^8\big(1948n^4+17868n^3+63837n^2+112966n+84655\big)\\
&+ 16\varepsilon^7\big(1456n^5+20460n^4+112365n^3+304963n^2+412258n+221769\big)\\
&- 8\varepsilon^6\big(320n^6+2050n^5+4192n^4+27408n^3+174901n^2+411759n+324872\big)\\
&- 16\varepsilon^5\big(1756n^7+33154n^6+265889n^5+1186719n^4+3218059n^3+5349388n^2\\
&+ 5071913n+2113696\big)\\
&+ 32\varepsilon^4\big(188n^8+4802n^7+59527n^6+439922n^5+2025336n^4+5813984n^3\\
&+ 10076450n^2+9621283n+3878602\big)\\
&+ 64\varepsilon^3\big(140n^9+2768n^8+22500n^7+99545n^6+287700n^5+723136n^4\\
&+ 1854572n^3+3714620n^2+4272517n+2031600\big)\\
&- 128\varepsilon^2\big(24n^{10}+830n^9+14362n^8+152630n^7+1053620n^6+4834279n^5\\
&+ 14824351n^4+29964399n^3+38244797n^2+27875896n+8824032\big)\\
&+ 256\varepsilon(n+2)(n+3)(n+4)\big(118n^7+2639n^6+24247n^5+118311n^4+329565n^3\\
&+ 520306n^2+426076n+136854\big)\\
&- 512(n+1)(n+2)^2(n+3)^2(n+4)^2(n+5)\big(12n^3+97n^2+230n+144\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_4(n,\varepsilon) = \big(&64\varepsilon^{12} + 192\varepsilon^{11}(5n+14) + 16\varepsilon^{10}\big(297n^2 + 1769n + 2451\big)\\
&+ 16\varepsilon^9\big(453n^3 + 4462n^2 + 13094n + 11244\big)\\
&- 8\varepsilon^8\big(1084n^4 + 11117n^3 + 47258n^2 + 103981n + 94650\big)\\
&- 8\varepsilon^7\big(3304n^5 + 51138n^4 + 311957n^3 + 948722n^2 + 1440105n + 858544\big)\\
&+ 16\varepsilon^6\big(420n^6 + 5507n^5 + 36275n^4 + 169650n^3 + 536911n^2 + 952507n + 694370\big)\\
&+ 16\varepsilon^5\big(1828n^7 + 38868n^6 + 353301n^5 + 1801014n^4 + 5604391n^3 + 10664390n^2\\
&+ 11433064n + 5260048\big)\\
&- 32\varepsilon^4\big(316n^8 + 8356n^7 + 105800n^6 + 802421n^5 + 3836854n^4 + 11588223n^3\\
&+ 21401558n^2 + 22066744n + 9745752\big)\\
&- 64\varepsilon^3\big(116n^9 + 2424n^8 + 19923n^7 + 82966n^6 + 208191n^5 + 530980n^4 + 1847484n^3\\
&+ 4687014n^2 + 6120858n + 3111104\big)\\
&+ 128\varepsilon^2\big(24n^{10} + 826n^9 + 14897n^8 + 172000n^7 + 1314686n^6 + 6710299n^5\\
&+ 22873183n^4 + 51298261n^3 + 72551278n^2 + 58573022n + 20544948\big)\\
&- 256\varepsilon(n+2)(n+3)\big(106n^8 + 3278n^7 + 42903n^6 + 310942n^5 + 1366350n^4\\
&+ 3729418n^3 + 6173159n^2 + 5657732n + 2191212\big)\\
&+ 512(n+1)(n+2)^2(n+3)^2(n+4)(n+5)(n+6)\big(12n^3 + 121n^2 + 396n + 431\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_5(n,\varepsilon) = (n+5)\big(&-128\varepsilon^{11} - 128\varepsilon^{10}(11n+26) - 32\varepsilon^9\big(115n^2+592n+647\big)\\
&+32\varepsilon^8\big(63n^3+430n^2+1665n+2384\big)\\
&+16\varepsilon^7\big(714n^4+7881n^3+33802n^2+66225n+47654\big)\\
&-16\varepsilon^6\big(234n^5+2444n^4+13989n^3+50862n^2+104083n+87848\big)\\
&-16\varepsilon^5\big(580n^6+10181n^5+76586n^4+319207n^3+772120n^2+1012046n+547832\big)\\
&+16\varepsilon^4\big(244n^7+5456n^6+61605n^5+401216n^4+1536277n^3+3408574n^2\\
&+4066436n+2026928\big)\\
&+64\varepsilon^3\big(26n^8+357n^7+583n^6-11139n^5-65193n^4-120264n^3+11864n^2\\
&+272830n+222624\big)\\
&-64\varepsilon^2(n+3)\big(12n^8+298n^7+4684n^6+49024n^5+306907n^4+1122441n^3\\
&+2350650n^2+2607576n+1185072\big)\\
&+256\varepsilon(n+2)(n+3)\big(25n^7+743n^6+8856n^5+55358n^4+197497n^3+404131n^2\\
&+439902n+196128\big)\\
&-256(n+1)(n+2)^2(n+3)^2(n+4)(n+6)(n+7)\big(6n^2+35n+54\big)\big).
\end{aligned}
$$

symbolic summation

symbolic integration
RE finding

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

> there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with
>
> $$b_0(x) f(x) + \dots + b_\lambda(x) D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Definition: sequence $(F($ … called holonomic if

Feynman integrals

> there exist $a_0(x), \dots$ (not all zero) with
>
> $$a_0(n) F(n) + \dots + a_\delta(n) F(n + \delta) = 0 \qquad \text{(RE)}$$

symbolic summation

recurrence solving

symbolic integration

RE finding

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x) f(x) + \dots + b_\lambda(x) D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Feynman integrals

Definition: sequence $(F($ … called holonomic if

there exist $a_0(x), .$ … (not all zero) with

$$a_0(n) F(n) + \dots + a_\delta(n) F(n + \delta) = 0 \qquad \text{(RE)}$$

A recurrence solver (`Sigma.m`)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n) + \cdots + a_\delta(n)F(n+\delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta-1) \in \mathbb{K}$

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n) + \cdots + a_\delta(n)F(n+\delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta-1) \in \mathbb{K}$

DECIDE constructively if $F(n)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

S.A. Abramov, M. Bronstein, M. Petkovšek, CS. J. Symb. Comput. 107, pp. 23-66. 2021.

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n), \ldots, a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n) + \cdots + a_\delta(n)F(n+\delta) = h(n);$$

together with initial values $F(0), \ldots, F(\delta-1) \in \mathbb{K}$

DECIDE constructively if $F(n)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(n) = \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j} \qquad \text{(harmonic sums)}$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n) + \dots + a_\delta(n)F(n+\delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta-1) \in \mathbb{K}$

DECIDE constructively if $F(n)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{n} \frac{2^k}{k} \sum_{i=1}^{k} \frac{2^{-i}}{i} \sum_{j=1}^{i} \frac{S_1(j)}{j}$$

(generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n) + \dots + a_\delta(n)F(n+\delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta-1) \in \mathbb{K}$

DECIDE constructively if $F(n)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{n} \frac{1}{(1+2k)^2} \sum_{j=1}^{k} \frac{1}{j^2} \sum_{i=1}^{j} \frac{1}{1+2i} \qquad \text{(cyclotomic harmonic sums)}$$

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n),\dots,a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n)+\dots+a_\delta(n)F(n+\delta)=h(n);$$

together with initial values $F(0),\dots,F(\delta-1)\in\mathbb{K}$

DECIDE constructively if $F(n)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{j=1}^{n}\frac{4^jS_1(j-1)}{\binom{2j}{j}j^2}\qquad\text{(binomial sums)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n) + \dots + a_\delta(n)F(n+\delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta - 1) \in \mathbb{K}$

DECIDE constructively if $F(n)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{h=1}^{n} 2^{-2h} (1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2 \binom{2k}{k}}$$ (generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

A recurrence solver (`Sigma.m`)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n) + \dots + a_\delta(n)F(n+\delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta-1) \in \mathbb{K}$

DECIDE constructively if $F(n)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

A more general example:

$$\sum_{k=1}^{n} \left(\prod_{i=1}^{k} \frac{1+i+i^2}{i+1} \right) \sum_{j=1}^{k} \frac{1}{j \binom{4j}{3j}^2}$$

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m ↓ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

recurrence solver ↓

$F(\varepsilon,n)=$ expression in terms of special functions

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\dots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

refined recurrence solver $\downarrow$

$$F(\varepsilon,n)=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\dots$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F(\varepsilon,n)\Big]\\
&+a_1(\varepsilon,n)\Big[F(\varepsilon,n+1)\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,n)\Big[F(\varepsilon,n+\delta)\Big]\\
&\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F(\varepsilon,n+1)\Big]\\
+&\\
&\vdots\\
+&a_\delta(\varepsilon,n)\Big[F(\varepsilon,n+\delta)\Big]
\end{aligned}
$$

$$
= h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+& a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+& \\
\vdots& \\
+& a_\delta(\varepsilon,n)\Big[F(\varepsilon,n+\delta)\Big]\\
& \qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,n)\Big[F_0(n+\delta)+F_1(n+\delta)\varepsilon+F_2(n+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,n)\Big[F_0(n+\delta)+F_1(n+\delta)\varepsilon+F_2(n+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_\delta(0,n)F_0(n+\delta)=h_0(n)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_\delta(\varepsilon,n)\Big[F_0(n+\delta)+F_1(n+\delta)\varepsilon+F_2(n+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad = h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_\delta(0,n)F_0(n+\delta)=h_0(n)}
$$

REC solver: Given the initial values $F_0(1), F_0(2), \dots, F_0(\delta)$, **decide** if $F_0(n)$ can be written in terms of indefinite nested sums and products.

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_\delta(\varepsilon,n)\Big[F_0(n+\delta)+F_1(n+\delta)\varepsilon+F_2(n+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_\delta(0,n)F_0(n+\delta)=h_0(n)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_0(n)+F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_0(n+1)+F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+&\\
&\vdots\\
+&a_\delta(\varepsilon,n)\Big[F_0(n+\delta)+F_1(n+\delta)\varepsilon+F_2(n+\delta)\varepsilon^2+\dots\Big]\\
&\qquad=h_0(n)+h_1(n)\varepsilon+h_1(n)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$\boxed{a_0(0,n)F_0(n)+a_1(0,n)F_0(n+1)+\dots+a_\delta(0,n)F_0(n+\delta)=h_0(n)}$$

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)\varepsilon + F_2(n)\varepsilon^2 + \dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n{+}1)\varepsilon + F_2(n{+}1)\varepsilon^2 + \dots\Big]\\
+&\\
&\vdots\\
+&a_\delta(\varepsilon,n)\Big[F_1(n{+}\delta)\varepsilon + F_2(n+\delta)\varepsilon^2 + \dots\Big]\\
&\qquad = h_0'(n) + h_1'(n)\varepsilon + h_2'(n)\varepsilon^2 + \dots
\end{aligned}
$$

$$
\begin{aligned}
& a_0(\varepsilon,n)\Big[F_1(n)\varepsilon+F_2(n)\varepsilon^2+\dots\Big]\\
+& a_1(\varepsilon,n)\Big[F_1(n+1)\varepsilon+F_2(n+1)\varepsilon^2+\dots\Big]\\
+& \\
\vdots & \\
+& a_\delta(\varepsilon,n)\Big[F_1(n+\delta)\cancel{\varepsilon}+F_2(n+\delta)\varepsilon^2+\dots\Big]\\
&\qquad = \underbrace{h'_0(n)}_{\mathbf{=0}}+h'_1(n)\cancel{\varepsilon}+h'_2(n)\varepsilon^2+\dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&a_0(\varepsilon,n)\Big[F_1(n)+F_2(n)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,n)\Big[F_1(n+1)+F_2(n+1)\varepsilon+\dots\Big]\\
+&\\
&\vdots\\
+&a_\delta(\varepsilon,n)\Big[F_1(n+\delta)+F_2(n+\delta)\varepsilon+\dots\Big]=h'_1(n)+h'_2(n)\varepsilon+\dots
\end{aligned}
$$

Repeat to get $F_1(n), F_2(n), \dots$

Remark: Works the same for Laurent series.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

A refined recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(\varepsilon,n),\dots,a_\delta(\varepsilon,n)$: polynomials in ε,n
$h_l(n),h_{l+1}(n),\dots,h_\lambda(n)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$a_0(\varepsilon,n)F(\varepsilon,n)+\dots+a_\delta(\varepsilon,n)F(\varepsilon,n+\delta)$$
$$=h_l(n)\varepsilon^l+h_{l+1}(n)\varepsilon^{l+1}+\dots+h_\lambda(n)\varepsilon^r+O(\varepsilon^{r+1});$$

together with ε-expansions of $F(0),\dots,F(\delta-1)$ up to a certain order.

A refined recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(\varepsilon,n),\dots,a_\delta(\varepsilon,n)$: polynomials in ε, n
$h_l(n), h_{l+1}(n),\dots,h_\lambda(n)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$a_0(\varepsilon,n)F(\varepsilon,n)+\dots+a_\delta(\varepsilon,n)F(\varepsilon,n+\delta)$$
$$=h_l(n)\varepsilon^l+h_{l+1}(n)\varepsilon^{l+1}+\dots+h_\lambda(n)\varepsilon^r+O(\varepsilon^{r+1});$$

together with ε-expansions of $F(0),\dots,F(\delta-1)$ up to a certain order.

DECIDE constructively if the coefficients $F_i(n)$ of

$$F(\varepsilon,n)=F_l(n)\,\varepsilon^l+F_{l+1}(n)\,\varepsilon^{l+1}+\dots+F_r(n)\,\varepsilon^r+O(\varepsilon^{r+1})$$

can be given in terms of indefinite nested sums defined over hypergeometric products.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}\underbrace{(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

`Sigma.m` $\downarrow$ (2 hours)

$$F(\varepsilon,n)=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\cdots+F_4(n)\varepsilon^4+O(\varepsilon^5)$$

We get
$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

We get

$$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$$

$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

$$\begin{aligned} F_{-1}(n) = (-1)^n &\left(\frac{2\left(9n^5+81n^4+295n^3+533n^2+500n+204\right)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)}\right) \\ &+ \frac{2\left(18n^5+150n^4+490n^3+755n^2+536n+132\right)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)} \\ &+ \left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)}\right) S_2(n) \\ &+ \left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)}\right) S_{-2}(n) \end{aligned}$$

Find a recurrence and solve it for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \ldots$$

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

ε-recurrence solver

$$a_0(\varepsilon, n)D_\varepsilon(n) + \ldots + a_d(\varepsilon, n)D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence and solve it for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

MultiSum Package
(K. Wegschaider)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence and solve it for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

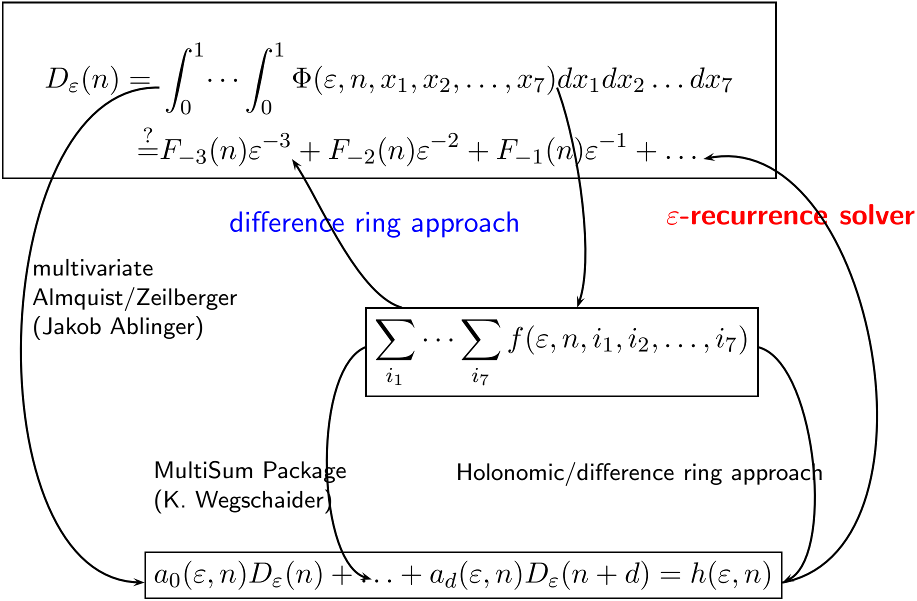
$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

MultiSum Package
(K. Wegschaider)

Holonomic/difference ring approach

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

Find a recurrence and solve it for the integral/sum



symbolic summation

recurrence solving

symbolic integration
RE finding

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with
$b_0(x)f(x) + \dots + b_\lambda(x)D^\lambda f(x) = 0$ (DE)

Feynman integrals

Definition: sequence $(F($ … called holonomic if

there exist $a_0(x), \dots$ … (not all zero) with
$a_0(n)F(n) + \dots + a_\delta(n)F(n+\delta) = 0$ (RE)

symbolic summation
RE finding

recurrence solving

symbolic integration
RE finding

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Feynman integrals

Definition: sequence $(F($ ... called holonomic if

there exist $a_0(x), ..$ ... (not all zero) with

$$a_0(n)F(n) + \cdots + a_\delta(n)F(n+\delta) = 0 \qquad \text{(RE)}$$

A warm up example:

$$\text{GIVEN } F(\varepsilon,n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$
$$\times\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$
$$\underbrace{+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(\varepsilon,n,k,j)}\Bigg).$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm up example:

$$\text{GIVEN } F(\varepsilon,n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}\right.
+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(\varepsilon,n,k,j)}.$$

FIND the first coefficients of the ϵ-expansion

$$F(\varepsilon,n)=F_0(n)+\varepsilon F_1(n)+\dots$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm up example:

GIVEN $F(\varepsilon,n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}\right.
+\left.\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)}_{f(\varepsilon,n,k,j)}.$$

Step 1: Compute the first coefficients of the ϵ-expansion

$$f(\varepsilon,n,k,j)=\qquad f_0(n,k,j)+\qquad \varepsilon f_1(n,k,j)+\dots$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm up example:

$$\text{GIVEN } F(\varepsilon,n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\Bigg(\underbrace{\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(\varepsilon,n,k,j)}\Bigg).$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(\varepsilon,n,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}\quad (=H_n)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

↑ summation package Sigma

$$g(j)=\frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}f(j)=\frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) \Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n)\, f(n,k) + \sum_{k=1}^{a} c_1(n)\, f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\sum_{k=1}^{a} f(n,k) + c_1(n)\sum_{k=1}^{a} f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN
$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n)+c_1(n)\,\mathsf{A}(n+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \quad = \quad -n\mathsf{A}(n)+(2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R}\right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} = 0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(n,k,j)}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \dots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, JSC 2021)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$A(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $A(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)A(n) + \cdots + a_d(n)A(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, JSC 2021)

3. Find a "closed form"

A(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Bigg(\frac{(-1)^{r}(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Bigg)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left[\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)\right]$$

$$||$$

$$\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\left(\left(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\right)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\right)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= $\mathbf{EvaluateMultiSum[mySum, \{\}, \{n\}, \{1\}]}$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

Sigma.m is based on difference ring/field theory

1. S.A. Abramov. The rational component of the solution of a first-order linear recurrence relation with a rational right-hand side. U.S.S.R. Comput. Maths. Math. Phys., 15:216–221, 1975.
2. M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
3. S.A. Abramov. Rational solutions of linear differential and difference equations with polynomial coefficients. U.S.S.R. Comput. Math. Math. Phys., 29(6):7–12, 1989.
4. S.A. Abramov and M. Petkovšek. D'Alembertian solutions of linear differential and difference equations. In ISSAC'94, pages 169–174. 1994.
5. P. Paule. Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)
6. M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.
7. M. van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. J. Pure Appl. Algebra, 139(1-3):109–131, 1999.
8. P. A. Hendriks and M. F. Singer. Solving difference equations in finite terms. *J. Symbolic Comput.*, 27(3):239–259, 1999.
9. M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, 2000.
10. CS. Symbolic summation in difference fields. J. Kepler University, May 2001. PhD Thesis.
11. CS. A collection of denominator bounds to solve parameterized linear difference equations in $\Pi\Sigma$-extensions. *An. Univ. Timişoara Ser. Mat.-Inform.*, 42(2):163–179, 2004.
12. CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
13. CS. Degree bounds to find polynomial solutions of parameterized linear difference equations in $\Pi\Sigma$-fields. *Appl. Algebra Engrg. Comm. Comput.*, 16(1):1–32, 2005.
14. CS. Product representations in $\Pi\Sigma$-fields. *Ann. Comb.*, 9(1):75–99, 2005.
15. CS. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *J. Differ. Equations Appl.*, 11(9):799–821, 2005.
16. CS. Finding telescopers with minimal depth for indefinite nested sums and product expressions. In *Proc. ISSAC'05*, pages 285–292. ACM Press, 2005.
17. M. Kauers and C. Schneider. Indefinite summation with unspecified summands. Discrete Math., 306(17):2021–2140, 2006.
18. CS. Simplifying Sums in $\Pi\Sigma$-Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
19. CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008.
20. CS. A Symbolic Summation Approach to Find Optimal nested Sum Representations. In A. Carey, D. Ellwood, S. Paycha, and S. Rosenberg, editors, *Motives, Quantum Field Theory, and Pseudodifferential Operators*, pages 285–308. 2010.
21. CS. Parameterized Telescoping Proves Algebraic Independence of Sums. *Ann. Comb.*, 14(4):533–552, 2010.
22. CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
23. CS. Simplifying Multiple Sums in Difference Fields. In: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, J. Blümlein, C. Schneider (ed.), Texts and Monographs in Symbolic Computation, pp. 325-360. Springer, 2013.
24. CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. In *Computer Algebra and Polynomials*, Lecture notes in Computer Science (LnCS), Springer, 2014.
25. CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
26. CS. Summation Theory II: Characterizations of $R\Pi\Sigma$-extensions and algorithmic aspects. J. Symb. Comput. 80(3), pp. 616-664. 2017.
27. E.D. Ocansey, CS. Representing (q-)hypergeometric products and mixed versions in difference rings. In: Advances in Computer Algebra, C. Schneider, E. Zima (ed.), Springer Proceedings in Mathematics & Statistics 226. 2018.
28. S.A. Abramov, M. Bronstein, M. Petkovšek, CS. On Rational and Hypergeometric Solutions of Linear Ordinary Difference Equations in $\Pi\Sigma^*$-field extensions. J. Symb. Comput. 107, pp. 23-66. 2021.
29. CS. Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation. In: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computuation. 2021. Springer.
30. J. Ablinger, CS, Solving linear difference equations with coefficients in rings with idempotent representations. In: Proc. ISSAC'21, pp. 27-34. 2021.
31. CS. Refined telescoping algorithms in $R\Pi\Sigma$-extensions to reduce the degrees of the denominators. In: Proc. ISSAC '23, pp. 498-507. 2023.
32. E.D. Ocansey, CS, Representation of hypergeometric products of higher nesting depths in difference rings. J. Symb. Comput. 120, pp. 1-50. 2024.
33. P. Paule, CS. Creative Telescoping for Hypergeometric Double Sums. J. Symb. Comput., 28, 102394, 2025.

symbolic summation
RE finding

recurrence solving

symbolic integration
RE finding

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

there exist $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Definition: sequence $(F($ … called holonomic if

Feynman integrals

there exist $a_0(x), .$ … (not all zero) with

$$a_0(n)F(n) + \cdots + a_\delta(n)F(n+\delta) = 0 \qquad \text{(RE)}$$

symbolic summation
RE&DE finding

recurrence solving

DE solving

symbolic integration
RE&DE finding

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

> there exist $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with
>
> $$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Definition: sequence $(F($ … called holonomic if

Feynman integrals

> there exist $a_0(x), \ldots$ … (not all zero) with
>
> $$a_0(n) F(n) + \cdots + a_\delta(n) F(n + \delta) = 0 \qquad \text{(RE)}$$

Motivation: solving recurrences and D-equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\dots,\hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x),\dots,\hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x\begin{pmatrix}\hat{I}_1(x)\\ \dots\\ \hat{I}_\lambda(x)\end{pmatrix}=A(x)\begin{pmatrix}\hat{I}_1(x)\\ \dots\\ \hat{I}_\lambda(x)\end{pmatrix}+\begin{pmatrix}\hat{R}_1(x)\\ \dots\\ \hat{R}_\lambda(x)\end{pmatrix}$$

given

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} = A(x) \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} + \begin{pmatrix}\hat{R}_1(x)\\ \dots \\ \hat{R}_\lambda(x)\end{pmatrix}$$

$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \ldots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \ldots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \ldots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \ldots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \ldots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \cdots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

2. For $i = 2, \ldots, r$ we get

$$\hat{I}_i(x) = \mathsf{LinCom}(\hat{I}_1(x), \ldots, D_x^{\lambda-1}\hat{I}_1(x)) + \mathsf{LinCom}(..., D^i\hat{R}_i(x), \ldots)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

$\downarrow$ uncoupling algorithms (Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \mathrm{LinCom}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \mathrm{LinCom}(..., D^i\hat{R}_i(x), \dots)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x),\dots,\hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x),\dots,\hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} = A(x) \begin{pmatrix}\hat{I}_1(x)\\ \dots \\ \hat{I}_\lambda(x)\end{pmatrix} + \begin{pmatrix}\hat{R}_1(x)\\ \dots \\ \hat{R}_\lambda(x)\end{pmatrix}$$

↓ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

DE-solver

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0;$$

together with initial values $f(0), \ldots, D^{\lambda-1} f(x)|_{x=0} \in \mathbb{K}$

I. A differential equation solver (`HarmonicSums.m`)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x),\dots,b_\lambda(x)\in\mathbb{K}[x]$

$$b_0(x)f(x)+\cdots+b_\lambda(x)D^\lambda f(x)=0;$$

together with initial values $f(0),\dots,D^{\lambda-1}f(x)|_{x=0}\in\mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$H_{1,-1}(x)=\int_0^x\frac{1}{1-\tau_1}\int_0^{\tau_1}\frac{1}{1+\tau_2}d\tau_2d\tau_1$$ (harmonic polylogarithm)

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \ldots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$H_{2,-2}(x) = \int_0^x \frac{1}{2-\tau_1} \int_0^{\tau_1} \frac{1}{2+\tau_2} d\tau_2 d\tau_1$$

(generalized polylogarithms)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x),\dots,b_\lambda(x)\in\mathbb{K}[x]$

$$b_0(x)f(x)+\cdots+b_\lambda(x)D^\lambda f(x)=0;$$

together with initial values $f(0),\dots,D^{\lambda-1}f(x)|_{x=0}\in\mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{1+\tau_1+\tau_1^2}\int_0^{\tau_1}\frac{1}{1+\tau_2^2}d\tau_2 d\tau_1$$ (cyclotomic polylogarithms)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{\sqrt{1+\tau_1}} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1 \qquad \text{(radical integrals)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

Special cases of iterated integrals over hyperexponential functions:

$$\int_0^x \frac{1}{1-\tau_1+\eta\tau_1} \int_0^{\tau_1} \sqrt{1-\tau_2}\sqrt{1-\tau_2+\eta\tau_2}\,d\tau_2 d\tau_1$$

(generalized radical integrals)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

I. A differential equation solver (HarmonicSums.m)

GIVEN a linear differential equation $b_0(x),\ldots,b_\lambda(x)\in\mathbb{K}[x]$

$$b_0(x)f(x)+\cdots+b_\lambda(x)D^\lambda f(x)=0;$$

together with initial values $f(0),\ldots,D^{\lambda-1}f(x)|_{x=0}\in\mathbb{K}$

DECIDE constructively if $f(x)$ can be expressed in terms of iterated integrals defined over hyperexponential functions.

A more general example:

$$\int_0^x e^{\int_1^{\tau_1}\frac{1}{1+y+y^2}dy}\int_0^{\tau_1}\frac{1}{1+\tau_2}d\tau_2 d\tau_1$$

HarmonicSums can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11+20x)f'(x)+(1+x)(35+134x)f''(x) + 3(1+x)^2(4+37x)f^{(3)}(x)+18x(1+x)^3f^{(4)}(x)=0$$

$$\downarrow$$

$$\Big\{c_1+c_2\int_0^x\frac{1}{1+\tau_1}\,d\tau_1+c_3\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1+\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 + c_4\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1-\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 \mid c_1,c_2,c_3,c_4\in\mathbb{K}\Big\}$$

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

there exist $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Definition: sequence $(F(n))_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called holonomic if

there exist $a_0(x), \ldots, a_\delta(x) \in \mathbb{K}[x]$ (not all zero) with

$$a_0(n)F(n) + \cdots + a_\delta(n)F(n+\delta) = 0 \qquad \text{(RE)}$$

Connection: DE $\longleftrightarrow$ REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

be a (formal) power series. Then:

there exist $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0 \qquad \text{(DE)}$$

$$\Updownarrow$$

there exist $a_0(x), \ldots, a_\delta(x) \in \mathbb{K}[x]$ (not all zero) with

$$a_0(n) F(n) + \cdots + a_\delta(n) F(n+\delta) = 0 \qquad \text{(RE)}$$

Connection: DE $\longleftrightarrow$ REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

be a (formal) power series. Then:

there exist $b_0(x), \ldots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x) f(x) + \cdots + b_\lambda(x) D^\lambda f(x) = 0 \qquad \text{(DE)}$$

$\Updownarrow$ algorithmic

there exist $a_0(x), \ldots, a_\delta(x) \in \mathbb{K}[x]$ (not all zero) with

$$a_0(n) F(n) + \cdots + a_\delta(n) F(n+\delta) = 0 \qquad \text{(RE)}$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda \hat{I}_1(x) = \hat{r}(x)$$

DE-solver

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

↓ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

DE-solver REC-solver

Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x)$$
$$-\left(25x^2 - 208x\right) f''(x) - (15x - 8) f'(x) - f(x) = 0$$

Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x) - \left(25x^2 - 208x\right) f''(x) - (15x - 8) f'(x) - f(x) = 0$$

$$\downarrow$$

$$8(n+1)(2n+1)^3 F(n+1) - (n+1)^4 F(n) = 0$$

Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x) \\ - \left(25x^2 - 208x\right) f''(x) - (15x - 8) f'(x) - f(x) = 0$$

$$\downarrow$$

$$8(n+1)(2n+1)^3 F(n+1) - (n+1)^4 F(n) = 0$$

$$\downarrow$$

$$F(n) = \frac{1}{\binom{2n}{n}^3} = \frac{(1)_n (1)_n (1)_n (1)_n}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n n!} \frac{1}{64^n}$$

Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}^3} = {}_4F_3\left[\begin{matrix} 1,1,1,1 \\ \frac{1}{2},\frac{1}{2},\frac{1}{2} \end{matrix}; \frac{x}{64}\right]$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x)$$
$$- \left(25x^2 - 208x\right) f''(x) - (15x-8)f'(x) - f(x) = 0$$

$$\downarrow$$

$$8(n+1)(2n+1)^3 F(n+1) - (n+1)^4 F(n) = 0$$

$$\downarrow$$

$$F(n) = \frac{1}{\binom{2n}{n}^3} = \frac{(1)_n(1)_n(1)_n(1)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n n!} \frac{1}{64^n}$$

Example 1: Find a power series solution

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{\binom{2n}{n}^3}={}_4F_3\left[\begin{matrix}1,1,1,1\\ \frac{1}{2},\frac{1}{2},\frac{1}{2}\end{matrix};\frac{x}{64}\right]$$

for further transormations see [arXiv:1706.01299]

for

$$-\left(x^4-64x^3\right)f^{(4)}(x)-2\left(5x^3-144x^2\right)f^{(3)}(x)$$
$$-\left(25x^2-208x\right)f''(x)-(15x-8)f'(x)-f(x)=0$$

$$\downarrow$$

$$8(n+1)(2n+1)^3F(n+1)-(n+1)^4F(n)=0$$

$$\downarrow$$

$$F(n)=\frac{1}{\binom{2n}{n}^3}=\frac{(1)_n(1)_n(1)_n(1)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n n!}\frac{1}{64^n}$$

Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$\left(x^6 - 32x^5 + 256x^4\right) f^{(6)}(x) + \left(23x^5 - 528x^4 + 2560x^3\right) f^{(5)}(x)$$
$$+ \left(171x^4 - 2552x^3 + 6272x^2\right) f^{(4)}(x) + 2\left(245x^3 - 2002x^2 + 1728x\right) f^{(3)}(x)$$
$$+ 2\left(253x^2 - 786x + 72\right) f''(x) + 4(35x - 12) f'(x) + 4f(x) = 0$$

Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

for

$$\left(x^6 - 32x^5 + 256x^4\right) f^{(6)}(x) + \left(23x^5 - 528x^4 + 2560x^3\right) f^{(5)}(x)$$
$$+ \left(171x^4 - 2552x^3 + 6272x^2\right) f^{(4)}(x) + 2\left(245x^3 - 2002x^2 + 1728x\right) f^{(3)}(x)$$
$$+ 2\left(253x^2 - 786x + 72\right) f''(x) + 4(35x - 12)f'(x) + 4f(x) = 0$$

$$\downarrow$$

$$(n+2)(n+1)^3F(n) - 4(n+2)(2n+1)^2(2n+3)F(n+1) + 16(2n+1)^2(2n+3)^2F(n+2) = 0$$

Example 2: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n)x^n$$

for

$$\left(x^6 - 32x^5 + 256x^4\right) f^{(6)}(x) + \left(23x^5 - 528x^4 + 2560x^3\right) f^{(5)}(x) + \left(171x^4 - 2552x^3 + 6272x^2\right) f^{(4)}(x) + 2\left(245x^3 - 2002x^2 + 1728x\right) f^{(3)}(x) + 2\left(253x^2 - 786x + 72\right) f''(x) + 4(35x - 12)f'(x) + 4f(x) = 0$$

$$\downarrow$$

$$(n+2)(n+1)^3 F(n) - 4(n+2)(2n+1)^2(2n+3)F(n+1) + 16(2n+1)^2(2n+3)^2 F(n+2) = 0$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$F(n) = \frac{1}{\binom{2n}{n}^2}\Big(c_1 + c_2 S_1(n)\Big) = \frac{(1)_n(1)_n(1)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n n!}\frac{1}{16^n}\Big(c_1 + c_2 S_1(n)\Big)$$

Example 2: Find a power series solution

$$f(x) = c_1 \cdot {}_3F_2\left[\begin{matrix}1,1,1\\ \frac{1}{2},\frac{1}{2}\end{matrix};\frac{x}{16}\right] + c_2\sum_{n=0}^{\infty}\frac{S_1(n)}{\binom{2n}{n}^2}x^n$$

for

$$\left(x^6-32x^5+256x^4\right)f^{(6)}(x)+\left(23x^5-528x^4+2560x^3\right)f^{(5)}(x)$$
$$+\left(171x^4-2552x^3+6272x^2\right)f^{(4)}(x)+2\left(245x^3-2002x^2+1728x\right)f^{(3)}(x)$$
$$+2\left(253x^2-786x+72\right)f''(x)+4(35x-12)f'(x)+4f(x)=0$$

$$\downarrow$$

$$(n+2)(n+1)^3F(n)-4(n+2)(2n+1)^2(2n+3)F(n+1)+16(2n+1)^2(2n+3)^2F(n+2)=0$$

$$\downarrow \texttt{Sigma.m}$$

$$F(n)=\frac{1}{\binom{2n}{n}^2}\Big(c_1+c_2S_1(n)\Big)=\frac{(1)_n(1)_n(1)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n n!}\frac{1}{16^n}\Big(c_1+c_2S_1(n)\Big)$$

Example 3: A partial linear DE-solver

Find a power series solution

$$f(x_1,\dots,x_r)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}F(n_1,\dots,n_r)x_1^{n_1}\dots x_r^{n_r}$$

for

$$\sum_{(s_1,\dots,s_r)\in T}\underbrace{b_{(s_1,\dots,s_r)}(x_1,\dots,x_r)}_{\in\mathbb{K}[x_1,\dots,x_r]}D_{x_1}^{s_1}\dots D_{x_r}^{s_r}f(x_1,\dots,x_r)=0 \qquad \underset{\text{finite}}{T\subset\mathbb{N}^r}$$

Example 3: A partial linear DE-solver

Find a power series solution

$$f(x_1,\dots,x_r)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}F(n_1,\dots,n_r)x_1^{n_1}\dots x_r^{n_r}$$

for

$$\sum_{(s_1,\dots,s_r)\in T}\underbrace{b_{(s_1,\dots,s_r)}(x_1,\dots,x_r)}_{\in\mathbb{K}[x_1,\dots,x_r]}D_{x_1}^{s_1}\dots D_{x_r}^{s_r}f(x_1,\dots,x_r)=0 \qquad T\subset\mathbb{N}^r \text{ finite}$$

Note:

$\boxed{\exists \text{ a solution } f\in\mathbb{K}[x_1,\dots,x_r]}$ is algrithmically undecidable

[otherwise Hilbert's 10th problem would be algorithmically decidable...]

Example 3: A partial linear DE-solver

Find a power series solution

$$f(x_1,\dots,x_r)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}F(n_1,\dots,n_r)x_1^{n_1}\dots x_r^{n_r}$$

for

$$\sum_{(s_1,\dots,s_r)\in T}\underbrace{b_{(s_1,\dots,s_r)}(x_1,\dots,x_r)}_{\in\mathbb{K}[x_1,\dots,x_r]}D_{x_1}^{s_1}\dots D_{x_r}^{s_r}f(x_1,\dots,x_r)=0 \quad T\subset\mathbb{N}^r \text{ finite}$$

$$\downarrow$$

$$\sum_{(s_1,\dots,s_r)\in S}\underbrace{a_{(s_1,\dots,s_r)}(n_1,\dots,n_r)}_{\in\mathbb{K}[n_1,\dots,n_r]}F(n_1+s_1,\dots,n_r+s_r)=0 \quad S\subset\mathbb{Z}^r \text{ finite}$$

Example 3: A partial linear DE-solver

Find a power series solution

$$f(x_1,\dots,x_r)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}F(n_1,\dots,n_r)x_1^{n_1}\dots x_r^{n_r}$$

for

$$\sum_{(s_1,\dots,s_r)\in T}\underbrace{b_{(s_1,\dots,s_r)}(x_1,\dots,x_r)}_{\in\mathbb{K}[x_1,\dots,x_r]}D_{x_1}^{s_1}\dots D_{x_r}^{s_r}f(x_1,\dots,x_r)=0 \qquad T\subset\mathbb{N}^r \text{ finite}$$

$$\downarrow$$

$$\sum_{(s_1,\dots,s_r)\in S}\underbrace{a_{(s_1,\dots,s_r)}(n_1,\dots,n_r)}_{\in\mathbb{K}[n_1,\dots,n_r]}F(n_1+s_1,\dots,n_r+s_r)=0 \qquad S\subset\mathbb{Z}^r \text{ finite}$$

Note:

$\boxed{\exists \text{ a solution } F\in\mathbb{K}[n_1,\dots,n_r]}$ is algrithmically undecidable

[otherwise Hilbert's 10th problem would be algorithmically decidable...]

Example 3: A partial linear DE-solver

Find a power series solution

$$f(x_1,\dots,x_r)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}F(n_1,\dots,n_r)x_1^{n_1}\dots x_r^{n_r}$$

for

$$\sum_{(s_1,\dots,s_r)\in T}\underbrace{b_{(s_1,\dots,s_r)}(x_1,\dots,x_r)}_{\in\mathbb{K}[x_1,\dots,x_r]}D_{x_1}^{s_1}\dots D_{x_r}^{s_r}f(x_1,\dots,x_r)=0 \qquad T\subset\mathbb{N}^r \text{ finite}$$

$$\downarrow$$

$$\sum_{(s_1,\dots,s_r)\in S}\underbrace{a_{(s_1,\dots,s_r)}(n_1,\dots,n_r)}_{\in\mathbb{K}[n_1,\dots,n_r]}F(n_1+s_1,\dots,n_r+s_r)=0 \qquad S\subset\mathbb{Z}^r \text{ finite}$$

But: there are methods to hunt for solutions based on

M. Kauers, CS, *Partial denominator bounds for partial linear difference equations*, in: Proc. ISSAC'10 (2010)

M. Kauers, CS, *A refined denominator bounding algorithm for multivariate linear difference equations*, in: Proc. ISSAC'11 (2011)

J. Blümlein, M. Saragnese, CS, *Hypergeometric Structures in Feynman Integrals*, arXiv:2111.15501 [math-ph]

$$(n+1)^2 \left(k + n^2 + 2\right) \left(3kn^2 - 4k^2 - 5kn - 12k + 2n^3 + 2n^2 - 8n - 8\right) F(n, k+1)$$
$$+ (n+1)^2 \left(k + n^2 + 3\right) \left(2k^2 - 2kn^2 + 2kn + 6k - n^3 - n^2 + 4n + 4\right) F(n, k+2)$$
$$+ (n+1)^2 (k+n+1) \left(2k - n^2 + n + 4\right) \left(k + n^2 + 1\right) F(n, k)$$
$$- (k+1) n^2 (n+2)^2 \left(k + n^2 + 2n + 2\right) F(n+1, k)$$
$$+ k n^2 (n+2)^2 \left(k + n^2 + 2n + 3\right) F(n+1, k+1) = 0$$

$$
\begin{aligned}
&(n+1)^2\left(k+n^2+2\right)\left(3kn^2-4k^2-5kn-12k+2n^3+2n^2-8n-8\right)F(n,k+1)\\
&+(n+1)^2\left(k+n^2+3\right)\left(2k^2-2kn^2+2kn+6k-n^3-n^2+4n+4\right)F(n,k+2)\\
&+(n+1)^2(k+n+1)\left(2k-n^2+n+4\right)\left(k+n^2+1\right)F(n,k)\\
&-(k+1)n^2(n+2)^2\left(k+n^2+2n+2\right)F(n+1,k)\\
&+kn^2(n+2)^2\left(k+n^2+2n+3\right)F(n+1,k+1)=0
\end{aligned}
$$

$$\Big\downarrow \; W=\{S_1(k),S_1(n+k),S_{2,1}(n+k)\}$$

degree bound 5

$$
\begin{aligned}
&(n+1)^2\left(k+n^2+2\right)\left(3kn^2-4k^2-5kn-12k+2n^3+2n^2-8n-8\right)F(n,k+1)\\
&+(n+1)^2\left(k+n^2+3\right)\left(2k^2-2kn^2+2kn+6k-n^3-n^2+4n+4\right)F(n,k+2)\\
&+(n+1)^2(k+n+1)\left(2k-n^2+n+4\right)\left(k+n^2+1\right)F(n,k)\\
&-(k+1)n^2(n+2)^2\left(k+n^2+2n+2\right)F(n+1,k)\\
&+kn^2(n+2)^2\left(k+n^2+2n+3\right)F(n+1,k+1)=0
\end{aligned}
$$

$\downarrow$ $W=\{S_1(k), S_1(n+k), S_{2,1}(n+k)\}$
degree bound 5

37 solutions $\dfrac{p}{(1+n)^2(1+k+n^2)}$ with

$$(n+1)^2\left(k+n^2+2\right)\left(3kn^2-4k^2-5kn-12k+2n^3+2n^2-8n-8\right)F(n,k+1)$$
$$+(n+1)^2\left(k+n^2+3\right)\left(2k^2-2kn^2+2kn+6k-n^3-n^2+4n+4\right)F(n,k+2)$$
$$+(n+1)^2(k+n+1)\left(2k-n^2+n+4\right)\left(k+n^2+1\right)F(n,k)$$
$$-(k+1)n^2(n+2)^2\left(k+n^2+2n+2\right)F(n+1,k)$$
$$+kn^2(n+2)^2\left(k+n^2+2n+3\right)F(n+1,k+1)=0$$

$$\Big\downarrow\; W=\{S_1(k),S_1(n+k),S_{2,1}(n+k)\}$$

degree bound 5

37 solutions $\dfrac{p}{(1+n)^2(1+k+n^2)}$ with

$$p\in\Big\{1+\frac{1}{2}nS_1(k+n),k,n,kn,kn^2,kn^3,kn^4,kS_1(n),knS_1(n),kn^2S_1(n),kn^3S_1(n),kS_1(n)^2,$$
$$knS_1(n)^2,kn^2S_1(n)^2,kS_1(n)^3,knS_1(n)^3,kS_1(n)^4,kS_{2,1}(n),knS_{2,1}(n),kn^2S_{2,1}(n),kn^3S_{2,}$$
$$kS_1(n)S_{2,1}(n),knS_1(n)S_{2,1}(n),kn^2S_1(n)S_{2,1}(n),kS_1(n)^2S_{2,1}(n),knS_1(n)^2S_{2,1}(n),$$
$$kS_1(n)^3S_{2,1}(n),kS_{2,1}(n)^2,knS_{2,1}(n)^2,kn^2S_{2,1}(n)^2,kS_1(n)S_{2,1}(n)^2,knS_1(n)S_{2,1}(n)^2,$$
$$kS_1(n)^2S_{2,1}(n)^2,kS_{2,1}(n)^3,knS_{2,1}(n)^3,kS_1(n)S_{2,1}(n)^3,kS_{2,1}(n)^4\Big\}$$

symbolic summation
RE&DE finding

recurrence solving

DE solving

symbolic integration
RE&DE finding

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x) f(x) + \dots + b_\lambda(x) D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Definition: sequence $(F($ … called holonomic if

Feynman integrals

there exist $a_0(x), \dots$ … (not all zero) with

$$a_0(n) F(n) + \dots + a_\delta(n) F(n+\delta) = 0 \qquad \text{(RE)}$$

DE&RE guessing

symbolic summation RE&DE finding

recurrence solving

DE solving

symbolic integration RE&DE finding

Definition: A function $f : \mathbb{K} \to \mathbb{K}$ is called holonomic if

there exist $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$ (not all zero) with

$$b_0(x) f(x) + \dots + b_\lambda(x) D^\lambda f(x) = 0 \qquad \text{(DE)}$$

Definition: sequence $(F($ … called holonomic if

Feynman integrals

there exist $a_0(x), .$ … (not all zero) with

$$a_0(n) F(n) + \dots + a_\delta(n) F(n+\delta) = 0 \qquad \text{(RE)}$$

coupled systems

for $f(x) = \sum_{n=0}^{\infty} P(n) x^n$

`SolveCoupledSystem.m` $\downarrow$

large no. of moments,
say $P(0), \ldots, P(10000)$

coupled systems

for $f(x) = \sum_{n=0}^{\infty} P(n) x^n$

`SolveCoupledSystem.m` ↓

large no. of moments,
say $P(0), \ldots, P(10000)$

↙

numerics

coupled systems

for $f(x) = \sum_{n=0}^{\infty} P(n)x^n$

`SolveCoupledSystem.m` ↓

large no. of moments,
say $P(0), \ldots, P(10000)$

↓ guessing (`ore_algebra` in Sage)

numerics

recurrence

coupled systems

for $f(x) = \sum_{n=0}^{\infty} P(n)x^n$

`SolveCoupledSystem.m`

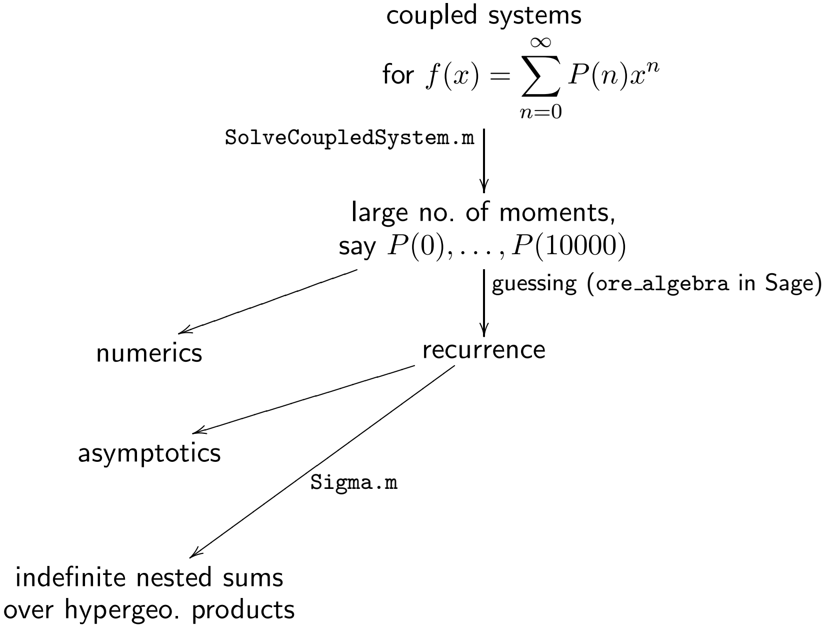
large no. of moments,
say $P(0), \ldots, P(10000)$

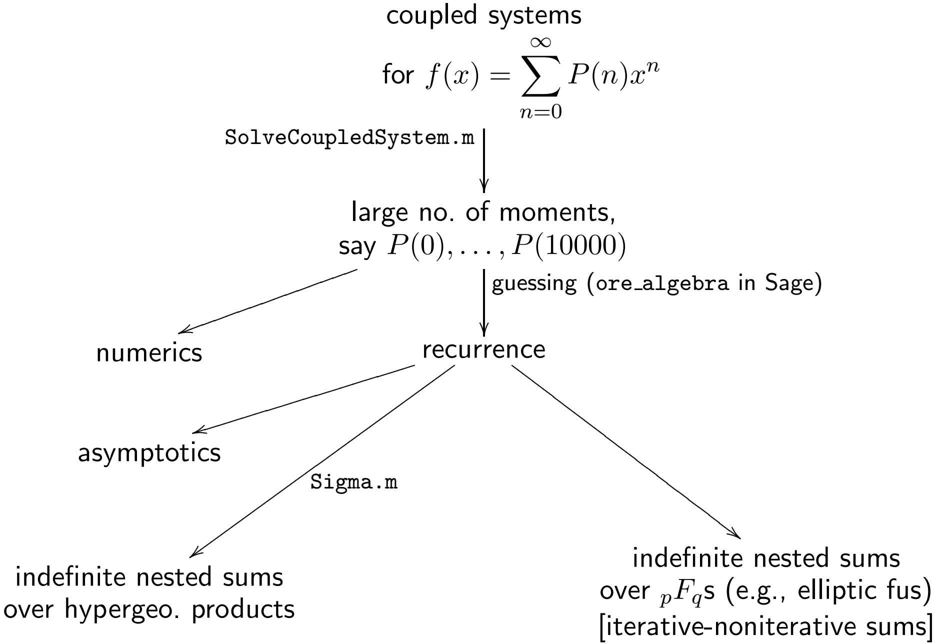
guessing (`ore_algebra` in Sage)

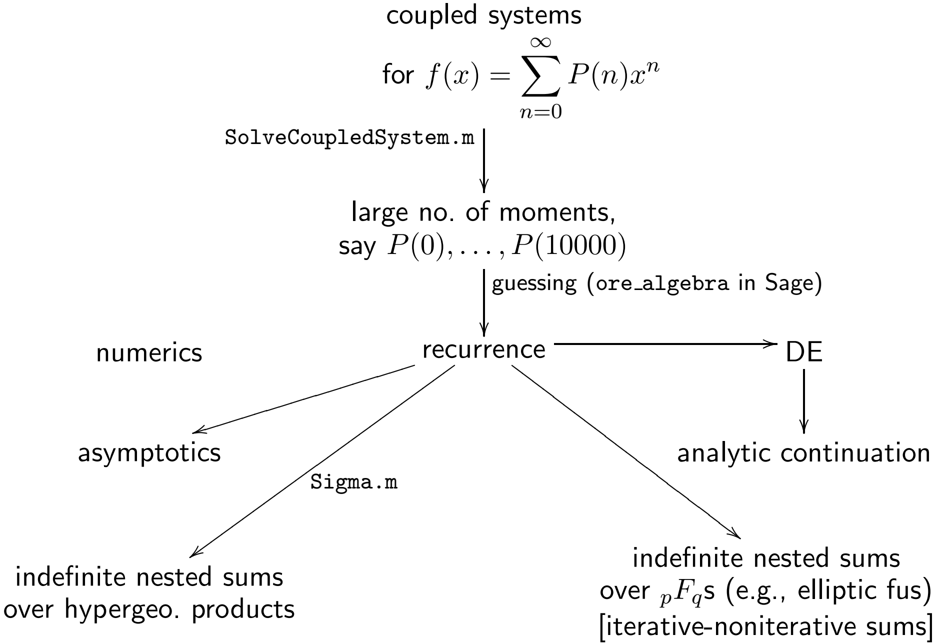
numerics

recurrence

asymptotics


coupled systems
for $f(x) = \sum_{n=0}^{\infty} P(n)x^n$
SolveCoupledSystem.m
large no. of moments,
say $P(0), \ldots, P(10000)$
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products


coupled systems
for $f(x) = \sum_{n=0}^{\infty} P(n)x^n$
SolveCoupledSystem.m
large no. of moments,
say $P(0), \ldots, P(10000)$
guessing (ore_algebra in Sage)
numerics
recurrence
asymptotics
Sigma.m
indefinite nested sums
over hypergeo. products
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]


coupled systems
for $f(x) = \sum_{n=0}^{\infty} P(n)x^n$
SolveCoupledSystem.m
large no. of moments,
say $P(0), \ldots, P(10000)$
guessing (ore_algebra in Sage)
numerics
recurrence
DE
asymptotics
analytic continuation
Sigma.m
indefinite nested sums
over hypergeo. products
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]

coupled systems

for $f(x) = \sum_{n=0}^{\infty} P(n) x^n$

`SolveCoupledSystem.m` ↓

large no. of moments,
say $P(0), \ldots, P(10000)$

↓ guessing (`ore_algebra` in Sage)

recurrence

`Sigma.m` ↙

indefinite nested sums
over hypergeo. products

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

In[6]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= **initial =<< iFile16**

Example (J. Blümlein, P. Marquard, CS, K. Schönwald. Nucl. Phys. B 971, pp. 1-44. 2021)

```
In[6]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[7]:= initial =<< iFile16

Out[7]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,
  475840076183/7501410000, 1432950323678333/21965628762000,
  21648380901382517/328583783127600,
  52869784323778576751/802218994536960000,
  49422862094045523994231/753773992230616156800,
  33131879832907935920726113/50955794398529996976000 0,
  520927472183675516844877 7/8094998411185418045913 6,
  56143711997344769021041145213/88258926638358645638435366 4,
  453500433353845628194790025124807/721722804887946855688695000000 0,
  140615433741204798861101598988693 87/2266431675903503264356560360000 00,
  7155865226664919033249057851786199365711683703077002228078114958950300000000,
  1628672904635927389284127125741885405683641 3/269396588055480390401343344736943104000000,
  1428729642632302467951426905844691837805299/2394075957503412282786131596157367360000 0,
  4989386902195952945051028091991545507830807 67/846888366785297981317126230405400272000000 0,
```

In[8]:= **rec =<< rFile16**

Out[8]=

$$(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32}+38256884318208n^{31}+$$

$$2282100271087616n^{30}+87428170197762048n^{29}+2417273990256001024n^{28}+51388547929265405952n^{27}+$$

$$873862324676687036416n^{26}+12209268055143308328960n^{25}+142860861222820240162816n^{24}+$$

$$1419883954103469621510144n^{23}+12115561235109256405319680n^{22}+89479384946084038000803840n^{21}+$$

$$575561340618928527623274496n^{20}+3239547818363227419971647488n^{19}+16009805333085271423330779136n^{18}+$$

$$69631814641718655426881659392n^{17}+266892117418348771052573667328n^{16}+$$

$$901901113782416884441719270144n^{15}+2685821385767154471801366647296n^{14}+$$

$$7038702625583766161604414471744n^{13}+16195069575749412648646633248128n^{12}+$$

$$32602540883321212533013752639288n^{11}+57154680141624618025310553466704n^{10}+$$

$$86710462147941775492301231896818n^{9}+112917328975807075881545543668548n^{8}+$$

$$124873767581470867343743078943772n^{7}+115624836314544572769501784072647n^{6}+$$

$$87938536330971046886456627610048n^{5}+53481897815980319933589323279298n^{4}+$$

$$25000430622737750756669804052204n^{3}+8430930497463933665464836129855n^{2}+$$

$$1825177817831282261293155379650n+190428196025667395685609855000\Big)(2n+1)^4\,\mathrm{P}[n]$$

$$-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 161315106167193 6n^{37} +$$

$$101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} +$$

$$2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} +$$

$$9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} +$$

$$7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} +$$

$$356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} +$$

$$9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} +$$

$$170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} +$$

$$1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} +$$

$$13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} +$$

$$59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} +$$

$$167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} +$$

$$284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} +$$

$$279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} +$$

$$149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} +$$

$$38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} +$$

$$3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} +$$

$$101701393436276172443717692853400n + 6204709909986751913151675960000\Big)\,P[n+1]$$

$$+2(n+3)^2(2n+5)^3(2n+9)(2n+11)\Big(24739011624960n^{40}+3317836466356224n^{39}+215508170284466176n^{38}+9032884062187945984n^{37}+$$
$$274636134389959884800n^{36}+6455501959255126179840n^{35}+122094572934385260036096n^{34}+1909387225793663151898624n^{33}+$$
$$25180108291969215434326016n^{32}+284171960705270647479074816n^{31}+2775794400720227034854326272n^{30}+$$
$$23677622163992853854566219776n^{29}+177624312783583749157935120384n^{28}+1178515602115604757944201871360n^{27}+$$
$$6947091965313419323781358354432n^{26}+36515023100308314818702129258496n^{25}+171621148571344894953594594017280n^{24}+$$
$$722837793013976317556258102507520n^{23}+2732534027077907914497042720534528n^{22}+9281028665970648470895368668485120n^{21}+$$
$$28337819215557708948254385336117248n^{20}+77786125749274632150536464583130752n^{19}+191877161455672780973502244537632256n^{18}+$$
$$424953221702140663089937921965135648n^{17}+843818276409975584824720931649555264n^{16}+$$
$$1499359936674956711935311062995422344n^{15}+2378007025570977662661938772843220240n^{14}+$$
$$3355671771434535852147325502571953770n^{13}+4196375762867184563407432891655585484n^{12}+$$
$$4627675779563752366067861596232781096n^{11}+4473175960511956000526499430851993603n^{10}+$$
$$3761696365025837909581516781307249585n^{9}+2726553473467254373993685951699145492n^{8}+$$
$$1683383212304999468664293798012773485n^{7}+871926653651504419744271839781064837n^{6}+$$
$$371307437598003570058538796122994147n^{5}+126427972742886389602285855482966072n^{4}+33048762330145623969058704448697313n^{3}+$$
$$6217924746857741077419160100404560n^{2}+748298077423337427195946099994100n+43181089548034246077698611794000\Big)\mathsf{P}[n+2]$$

$$-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40}+3322784268681216n^{39}+216160919414112256n^{38}+9074528155284275200n^{37}+$$
$$276348048819456311296n^{36}+6506479077331107315712n^{35}+123266585640616142569472n^{34}+1931040885785102661976064n^{33}+$$
$$25510503383281445462081536n^{32}+288418124175428279391485952n^{31}+2822442799033603081019326464n^{30}+$$
$$24120717233320712351821332480n^{29}+181295944719289040999116701696n^{28}+1205246297785423925076555694080n^{27}+$$
$$7119049557560114436136213413888n^{26}+37496933571993839665392189775872n^{25}+176616172467048982234270428880896n^{24}+$$
$$745539218875020737621728364206080n^{23}+2824909633156578132652259733712896n^{22}+9618101958268071244680677589035520n^{21}+$$
$$29441860528446423517613263360742912n^{20}+81033563306363873505877563416477312n^{19}+200454769103641040142838133702338304n^{18}+$$
$$445286624972461749049425309485328992n^{17}+887028447418790661018847407251573152n^{16}+$$
$$1581538101499869694224895701784875304n^{15}+2517550244392724509968791166585362672n^{14}+$$
$$3566593026520465155504695877897282630n^{13}+4479066125207404898722179511912639638n^{12}+$$
$$4962006990874351800791769650243464872n^{11}+4819992643914265990647887896664485209n^{10}+$$
$$4074895386694182240941538222230233221n^{9}+2970477229398746689186622534784613554n^{8}+$$
$$1845274131994015990683957902602775337n^{7}+962091291302144537393228847830431614n^{6}+$$
$$412595107814836563208757757032740146n^{5}+141540723940232563767779647013785485n^{4}+37292931812630561528276365992452010n^{3}+$$
$$7074865777225416725452872895397100n^{2}+858794112392644074221312049837000n+49997386738260112603615104780000\Big)P[n+3]$$

$$
\begin{aligned}
&+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} + \\
&3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} + \\
&38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} + \\
&69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} + \\
&32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} + \\
&1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} + \\
&20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} + \\
&260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} + \\
&2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} + \\
&10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} + \\
&34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} + \\
&69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} + \\
&82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} + \\
&55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} + \\
&19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} + \\
&3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} + \\
&168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n + \\
&1274120732351744651125603886400\Big)\,P[n+4]
\end{aligned}
$$

$$-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32} + 28361279668224n^{31} +$$
$$1249518729297920n^{30} + 35220794552352768n^{29} + 713726163159089152n^{28} + 11076866026783113216n^{27} +$$
$$136959486138712588288n^{26} + 1385658801437173350400n^{25} + 11691772665924577918976n^{24} +$$
$$83438339505976242995200n^{23} + 508989054278115477684224n^{22} + 2675508113418826174332928n^{21} +$$
$$12193213796145039633072128n^{20} + 48399020537651722726242304n^{19} + 167881257973769248139515904n^{18} +$$
$$510012482113388176546187776n^{17} + 1358662126092561923541267968n^{16} + 3174925021159974655053814528n^{15} +$$
$$6504205668151125355938798848n^{14} + 11663792381020901870157176128n^{13} +$$
$$18263581057905911985340656960n^{12} + 24881010123632244515458585528n^{11} +$$
$$29346856353503020415409305704n^{10} + 29775859546803351930591002266n^{9} + 25770328899499991754425455738n^{8} +$$
$$18817114309842270306167785140n^{7} + 11424980760825630752861027739n^{6} + 5656051955667821083952617134n^{5} +$$
$$2221448212382554437709999491n^{4} + 664859653803075491350122060n^{3} + 142190920852333874895041748n^{2} +$$
$$19313175036907229252501700n + 1248723341516324359641600\Big)\,P[n+5]==0$$

In[9]:= **recSol = SolveRecurrence[rec, P[n]]**

In[9]:= **recSol = SolveRecurrence[rec, P[n]]**

Out[9]=

$$\Big\{\Big\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\Big\}$$

$$\Big\{0, -\frac{(3+2n)\big(-8-9n+2n^2\big)}{(1+n)^2(1+2n)^2}\Big\}$$

$$\Big\{0, -\frac{(3+2n)\big(-5+8n^2\big)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big\}$$

$$\Big\{0, \frac{(3+2n)\big(-513-2184n-2416n^2+768n^4\big)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \Big(-\frac{2(3+2n)\big(3+44n+48n^2\big)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} + \frac{12(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^2}{(1+n)(1+2n)} + \frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} - \frac{4(3+2n)\big(3+44n+48n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big\}$$

$$
\begin{aligned}
&\{0, \frac{1}{16(1+n)^4(1+2n)^4}\big(72519+572343n+1814716n^2+2918100n^3+2442240n^4+912896n^5+24576n^6-\\
&49152n^7\big)+\frac{16(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{3(1+n)(1+2n)}+\Big(-\frac{(3+2n)\big(29+307n+322n^2\big)}{4(1+n)^2(1+2n)^2}+\frac{44(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i^2}+\\
&\Big(\frac{(3+2n)\big(91+259n+974n^2+1784n^3+1024n^4\big)}{4(1+n)^3(1+2n)^3}+\frac{22(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)}+\frac{24(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}-\\
&\frac{4(3+2n)\big(-13-4n+16n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2}+\frac{16(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i}+\Big(-\\
&\frac{(3+2n)\big(19+92n+80n^2\big)}{(1+n)^2(1+2n)^2}+\frac{40(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)^2+\frac{20(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^3}{3(1+n)(1+2n)}+\\
&\frac{64(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)}-\frac{3(3+2n)\big(63+209n+150n^2\big)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2}+\\
&\Big(\frac{(3+2n)\big(347+1795n+4302n^2+4856n^3+2048n^4\big)}{2(1+n)^3(1+2n)^3}+\frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{-1+2i}-\\
&\frac{4(3+2n)\big(19+92n+80n^2\big)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2}+\frac{32(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)}-\\
&\frac{8(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{i}}{(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{j}\big)^2}{-1+2i}}{(1+n)(1+2n)}\\
&\quad-\frac{32(3+2n)\sum_{i=1}^{n}\frac{\Big(\sum_{j=1}^{i}\frac{1}{j}\Big)\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\displaystyle\sum_{i=1}^{n}\frac{\Big(\displaystyle\sum_{j=1}^{i}\frac{1}{j}\Big)\displaystyle\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)}+\\
&\frac{32(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{\big(\sum_{j=1}^{i}\frac{1}{-1+2j}\big)^2}{-1+2i}}{(1+n)(1+2n)}\}, \{1, 0\}\}
\end{aligned}
$$

In[10]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \rightarrow 1]}$

In[10]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

Out[10]=

$$\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-$$

$$\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+$$

$$\frac{44(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\frac{(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{i})^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^3}{9(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\frac{16(3+2n)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+$$

$$\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-$$

$$\frac{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\frac{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)^2(1+2n)^2}+$$

$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{i})(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)(1+2n)}+\frac{64(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^3}{9(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{i}}{3(1+n)(1+2n)}-$$

$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{3(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{i}}{3(1+n)(1+2n)}+$$

$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)}$$

In[11]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[12]:= **sol = TransformToSSums[sol];**

In[13]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[11]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[12]:= **sol = TransformToSSums[sol];**

In[13]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

Out[13]=
$$\frac{1}{3(1+n)^4(1+2n)^4}\left(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\right)+\frac{64(3+2n)^2S[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)S[1,n]^2}{3(1+n)(1+2n)^2}+\Big(-\frac{2(3+2n)\left(147+985n+1871n^2+1268n^3+212n^4\right)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)S[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)S[-2,2n]}{3(1+n)(1+2n)}\Big)S[1,2n]-\frac{4(3+2n)\left(23+123n+114n^2\right)S[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)S[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\left(53+229n+190n^2\right)S[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[3,2n]}{3(1+n)(1+2n)}+\Big(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)S[1,2n]}{3(1+n)(1+2n)^2}\Big)S[-1,2n]-\frac{64(3+2n)(2+3n)S[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\left(1+8n+8n^2\right)S[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)S[-2,1,2n]}{3(1+n)(1+2n)}$$

In[11]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[12]:= **sol = TransformToSSums[sol];**

In[13]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[14]:= **SExpansion[sol, n, 2]**

Out[14]=

$$\ln 2^2\left(\frac{64\text{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\text{LG}[n]+\frac{64\text{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\text{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\text{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\text{LG}[n]+\frac{64\text{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

In[11]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[12]:= **sol = TransformToSSums[sol];**

In[13]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[14]:= **SExpansion[sol, n, 2]**

Out[14]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

Special function algorithms

- `HarmonicSums` package

  Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]
  Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]
  Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]
- `RICA` package

  Blümlein, Fadeev, CS. ACM Communications in Computer Algebra 57(2), pp. 31-34. 2023.

Conclusion

1. Various holonomic tools have been developed at RISC:
    - multi-summation and integration packages
    - up-to-date solvers for linear recurrences and DEs (within `Sigma.m` and `HarmonicSums.m`)
    - a prototype method to solve partial linear DE/RE equations in QCD
2. Interplay: DE solver $\longleftrightarrow$ RE solver
3. Finding (generalized) hypergeometric structures from DEs
4. Guessing methods open up new applications in QCD
5. Results are contained in about 100 articles produced jointly within the RISC–DESY cooperation