THE ARITHMETIC OF RESURGENT TOPOLOGICAL STRINGS

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INTRODUCTION

Taming divergences

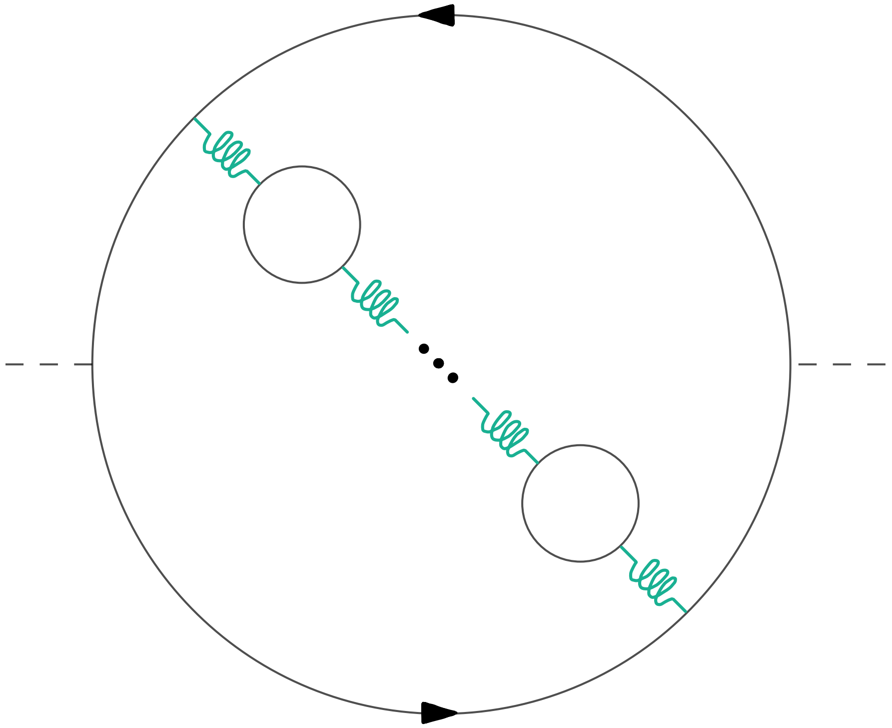
Perturbative computations in quantum theories rely on ***approximation schemes*** in a small parameter—typically, a coupling constant. The probability amplitude of a given interaction process in QFT is an infinite sum of individual Feynman integrals.

The resulting formal power series have ***zero radius of convergence*** and do not determine the original functions uniquely due to the presence of ***non-analytic terms***.
[Dyson, 1952 - Bender, Wu, 1971 - Gross, Periwal, 1988]

There are two non-perturbative physical mechanisms underlying the factorial divergence of the perturbative expansion of observables in QFT.

Instantons arise from the factorial growth in the number of Feynman diagrams at each order.

Renormalons arise from individual Feynman diagrams whose momentum integration diverges.



The divergence of perturbation theory can sometimes be tamed by applying ***resurgence***. Observables in QFT can be written as unambiguous Borel–Laplace resummed ***trans-series***.

Topological strings beyond perturbation theory — I

Let X be a toric Calabi–Yau (CY) threefold. The ***A-model topological string theory*** on X is defined perturbatively by a worldsheet genus expansion
[Shenker, 1990]

$$F^{\mathrm{WS}}(\vec{t}, g_s) = \sum_{g \geq 0} \underbrace{F_g(\vec{t})}_{\substack{\textit{well-defined} \\ \textit{for } \Re(t_i) \gg 1}} g_s^{2g-2}\,, \quad F_g(\vec{t}) \sim (2g)! \quad \textit{for } g \to \infty\,,$$

signaling the presence of exponentially small corrections in g_s.

Local mirror symmetry pairs X with an algebraic curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$ of genus g_Σ, which describes the ***B-model topological string theory*** on the mirror $\tilde{X}$.
[Katz, Klemm, Vafa, 1996 - Chiang, Klemm, Yau, Zaslow, 1999]

The Weyl quantization of the mirror curve Σ leads to ***quantum-mechanical operators***

$$\rho_j\,, \quad j = 1, \ldots, g_\Sigma\,,$$

acting on $L^2(\mathbb{R})$. They are conjectured to be positive-definite and of trace class under some assumptions on the mass parameters $\vec{\xi}$.
[Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015 - Kashaev, Mariño, 2015]

Topological strings beyond perturbation theory — II

Their ***generalized Fredholm determinant*** $\Xi(\vec{\kappa}, \vec{\xi}, \hbar)$ is an entire function of the true complex deformation parameters κ_j. Its local expansion at $\vec{\kappa} = 0$ is

$$\Xi(\vec{\kappa}, \vec{\xi}, \hbar) = \sum_{N_1 \geq 0} \cdots \sum_{N_{g_\Sigma} \geq 0} Z(\overrightarrow{N}, \vec{\xi}, \hbar) \;\; \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}},$$

where the ***fermionic spectral traces*** $Z(\overrightarrow{N}, \vec{\xi}, \hbar)$ are analytic functions of $\hbar \in \mathbb{R}_{>0}$.

The ***Topological String/Spectral Theory (TS/ST)*** correspondence states
[Hatsuda, Moriyama, Okuyama, 2012 - Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

$$Z(\overrightarrow{N}, \vec{\xi}, \hbar) = \frac{1}{(2\pi \mathrm{i})^{g_\Sigma}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_1 \cdots \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu_{g_\Sigma} \, \mathrm{e}^{J(\vec{\mu}, \vec{\xi}, \hbar) - \overrightarrow{N} \cdot \vec{\mu}}, \quad \kappa_j = \mathrm{e}^{\mu_j},$$

where the ***total grand potential*** of the A-model topological string on X

$$J(\overrightarrow{\mu}, \vec{\xi}, \hbar) = J^{\mathrm{WS}}(\overrightarrow{\mu}, \vec{\xi}, \hbar) + J^{\mathrm{WKB}}(\overrightarrow{\mu}, \vec{\xi}, \hbar)$$

encodes the standard and Nekrasov-Shatashvili topological string free energies. These can be regarded as non-perturbative corrections of one another in the appropriate regimes.
[Hatsuda, Mariño, Moriyama, Okuyama, 2013]

Enumerative invariants from resurgence

By the TS/ST correspondence, the fermionic spectral traces $Z(\overrightarrow{N}, \vec{\xi}, \hbar)$ *provide a way to access the non-perturbative effects associated with the factorial divergence of the topological string perturbation series in the spirit of large-N gauge/string dualities.*

The string coupling constant g_s is related to the quantum deformation parameter $\hbar$ by

$$g_s = \frac{4\pi^2}{\hbar} \quad (\textbf{\textit{strong-weak coupling duality}}).$$

Growing evidence indicates that the theory of ***resurgence*** can be applied to obtain a systematic understanding of the hidden non-perturbative sectors of topological string theory.
[Mariño, 2006 - Mariño, Schiappa, Weiss, 2007 - Mariño, 2008 - …]

Resurgence uniquely associates a divergent formal power series with a collection of ***exponential-type corrections*** paired with a set of complex numbers, known as ***Stokes constants***, which capture information about the large-order behavior of the perturbative series and its non-perturbative sectors.

Remarkably, the Stokes constants are (conjecturally) interpreted in terms of ***enumerative invariants*** of the CY based on the counting of BPS states.
[Alim, Saha, Teschner, Tulli, 2021 - Gu, Mariño, 2021 - 2022 - Rella, 2022 - Gu, Kashani-Poor, Klemm, Mariño, 2023 - Alexandrov, Mariño, Pioline, 2023 - Fantini, Rella, 2024]

THE RESURGENCE TOOLBOX

Asymptotic expansions and the non-perturbative ambiguity

Let z be a formal variable. A formal power series $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]]$ is an ***asymptotic series*** if there exists a function $f(z)$ satisfying

$$f(z) - \sum_{n=0}^{N} a_n z^n = o(z^N)\,, \quad z \to 0\,, \quad \forall N \in \mathbb{N}_{>0}\,.$$

We use $f(z) \sim \varphi(z)$ to indicate that $\varphi(z)$ is the ***asymptotic expansion*** of $f(z)$ as $z \to 0$.
[Poincaré, 1886 - Stieltjes, 1886]

We take $\varphi(z)$ to be ***Gevrey-1***, that is, its coefficients behave as

$$|a_n| \sim \mathscr{A}^{-n} n! \quad n \gg 1\,, \quad \mathscr{A} \in \mathbb{R}_{>0}\,.$$

A first-approach best estimate of $f(z)$ is given by ***optimal truncation***—that is, truncating $\varphi(z)$ after the term that is the smallest in absolute value. This implies the choice $N^* = \mathscr{A}/|z|$.

At fixed $|z|$, the error is measured by

$$\epsilon(z) = |a_{N^*+1} z^{N^*+1}| \sim \mathrm{e}^{-\mathscr{A}/|z|}\,, \quad z \to 0 \quad (\textit{\textbf{non-perturbative ambiguity}}),$$

and it cannot be improved upon using conventional perturbation theory only.

Basic notions in resurgence — I

In favorable circumstances, the intrinsic limits of classical asymptotics can be overcome within the framework of ***resurgence***.
[Écalle, 1981]

The ***Borel–Laplace resummation*** of $\varphi(z)$ along the line $\rho_\theta = \mathrm{e}^{\mathrm{i}\theta}\,\mathbb{R}_+$ is the two-step process

$$\varphi(z) \longrightarrow \underbrace{\hat{\varphi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\zeta^k}_{\substack{\text{locally analytic at } \zeta = 0 \text{ with} \\ \text{singularities at } \zeta = \zeta_\omega,\ \omega \in \Omega}} \longrightarrow \underbrace{s_\theta(\varphi)(z) = \int_{\rho_\theta} \mathrm{e}^{-\zeta}\hat{\varphi}(\zeta z)\,\mathrm{d}\zeta}_{\substack{\text{locally analytic in the complex } z\text{-plane with} \\ \text{discontinuities at } \arg(z) = \arg(\zeta_\omega),\ \omega \in \Omega}}$$

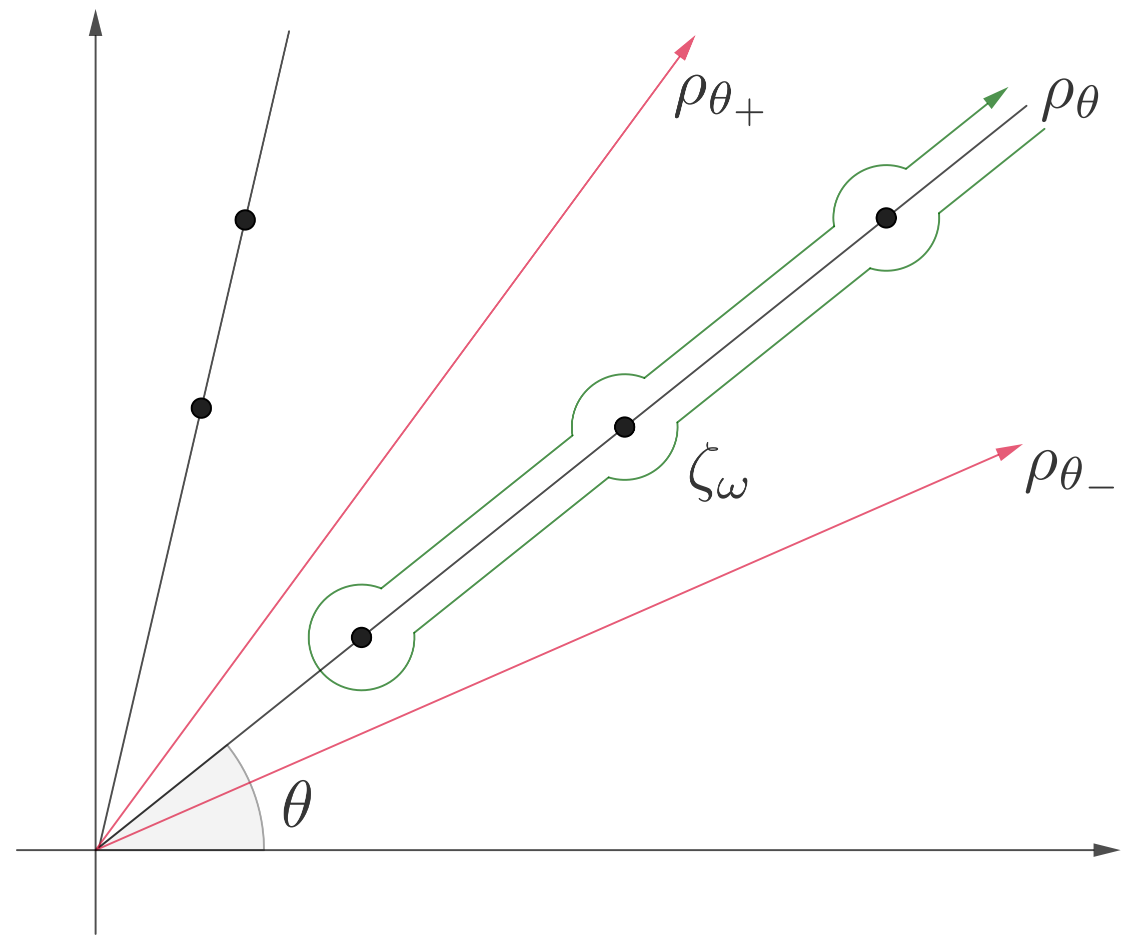
We assume $\varphi(z)$ to be ***resurgent*** ($\hat{\varphi}(\zeta)$ can be endlessly analytically continued) and ***simple*** ($\hat{\varphi}(\zeta)$ has only simple poles and logarithmic branch points).

If the Borel transform $\hat{\varphi}(\zeta)$ has a logarithmic branch point at $\zeta = \zeta_\omega$, then

$$\hat{\varphi}(\zeta) = -\frac{S_\omega}{2\pi\mathrm{i}} \log(\zeta - \zeta_\omega)\hat{\varphi}_\omega(\zeta - \zeta_\omega) + \dots ,$$

where $S_\omega \in \mathbb{C}$ is the ***Stokes constant*** and $\hat{\varphi}_\omega(\zeta - \zeta_\omega)$ is locally analytic at $\zeta = \zeta_\omega$.

When $\theta = \arg(\zeta_\omega)$ for some $\omega \in \Omega$, the line ρ_θ is called a ***Stokes ray***.



The ***discontinuity*** across ρ_θ is given by

$$\begin{aligned} \mathrm{disc}_\theta \varphi(z) &= s_{\theta_+}(\varphi)(z) - s_{\theta_-}(\varphi)(z) \\ &= \sum_\omega S_\omega \, \mathrm{e}^{-\zeta_\omega/z} \, s_{\theta_-}(\varphi_\omega)(z)\,, \end{aligned}$$

where $\theta_\pm = \theta \pm \epsilon$ for $0 < \epsilon \ll 1$ and the sum runs over the indices $\omega \in \Omega$ such that $\arg(\zeta_\omega) = \theta$.

The ***median resummation*** across ρ_θ is given by

$$\mathcal{S}^{\mathrm{med}}_\theta \varphi(z) = \frac{s_{\theta_+}(\varphi)(z) + s_{\theta_-}(\varphi)(z)}{2} = \begin{cases} s_{\theta_-}(\varphi)(z) + \frac{1}{2}\,\mathrm{disc}_\theta \varphi(z)\,, & \Re\left(\mathrm{e}^{-\mathrm{i}\theta_-} z\right) > 0\,, \\ s_{\theta_+}(\varphi)(z) - \frac{1}{2}\,\mathrm{disc}_\theta \varphi(z)\,, & \Re\left(\mathrm{e}^{-\mathrm{i}\theta_+} z\right) > 0\,, \end{cases}$$

which is an analytic function for $\arg(z) \in \left(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right)$.

Basic notions in resurgence — III

We can repeat the procedure with each of the series obtained in this way.

$$\varphi \longrightarrow \{\varphi_\omega, S_\omega\} \longrightarrow \{\varphi_{\omega'}, S_{\omega\omega'}\} \longrightarrow \cdots$$

Each series in this process can be promoted to a ***basic trans-series*** as

$$\Phi_\omega(z) = \mathrm{e}^{-\zeta_\omega/z}\varphi_\omega(z)\,.$$

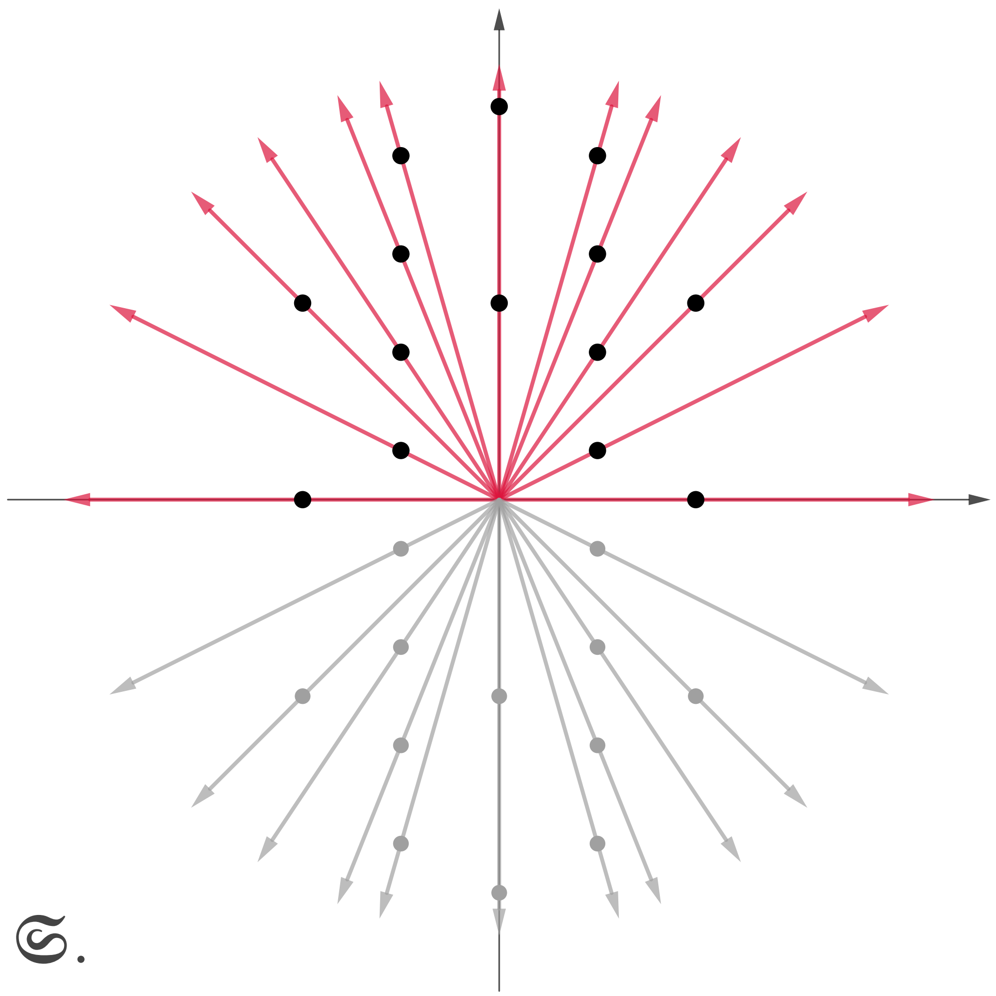
The ***Stokes automorphism*** $\mathfrak{S}_\theta$ across ρ_θ is defined by

$$s_{\theta_+} = s_{\theta_-} \circ \mathfrak{S}_\theta\,.$$

The ***minimal resurgent structure*** and the matrix of Stokes constants of $\varphi(z)$ are

$$\mathfrak{B}_\varphi = \left\{\Phi_\omega(z)\right\}_{\omega\in\bar{\Omega}}\,, \quad \mathcal{S}_\varphi = \{S_{\omega\omega'}\}_{\omega,\omega'\in\bar{\Omega}}\,,$$

where $\bar{\Omega} \subseteq \Omega$ denotes the smallest subset closed under $\mathfrak{S}$.



Peacock patterns are expected in theories controlled by quantum curves.
[Grassi, Gu, Mariño, 2019 - Garoufalidis, Gu, Mariño, 2020 - 2022 - Gu, Mariño, 2021 - Rella, 2022]

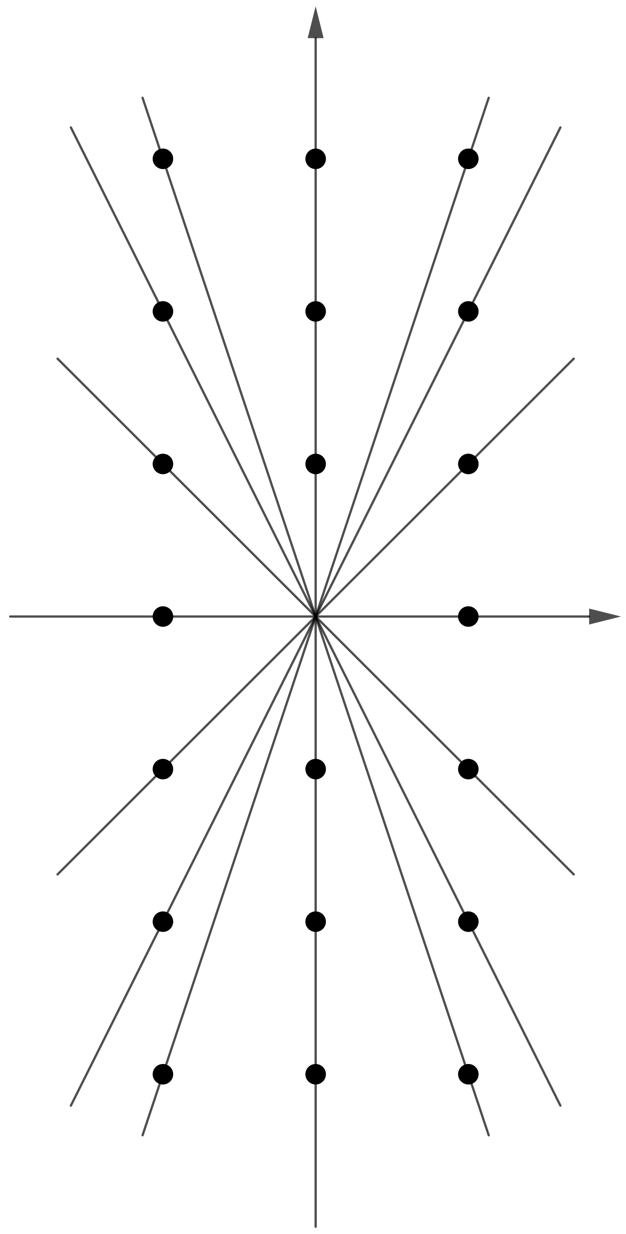
RESURGENCE OF THE SPECTRAL THEORY

Resurgent structures at strong and weak coupling — I

For fixed $\vec{N} \in \mathbb{N}^{g_\Sigma}$, we consider the ***dual asymptotic expansions***

$$\log Z(\vec{N}, \vec{\xi}, \hbar) \sim \phi_{\vec{N}}(\hbar) \qquad \text{for} \;\; \hbar \to 0\,,$$
$$\log Z(\vec{N}, \vec{\xi}, \hbar) \sim \psi_{\vec{N}}(\hbar^{-1}) \quad \text{for} \;\; \hbar \to \infty\,,$$

which are expected to be Gevrey-1 and simple resurgent. We conjecture their ***minimal resurgent structures***.
[Gu, Mariño, 2021 - Rella, 2022]



In the semiclassical limit $\hbar \to 0$,

$$\mathfrak{B}_{\phi_{\vec{N}}} = \{\Phi_{\sigma,n;\vec{N}}(\hbar) = \mathrm{e}^{-n\frac{\mathscr{A}_0}{\hbar}}\, \phi_{\sigma;\vec{N}}(\hbar)\}\,,$$

where $n \in \mathbb{Z}$, $\sigma \in \{0,\ldots,l_0\}$, $l_0 \in \mathbb{N}$, and $\mathscr{A}_0 \in \mathbb{C}$. The Gevrey-1 asymptotic series $\phi_{\sigma;\vec{N}}(\hbar)$ resurge from $\phi_{\vec{N}}(\hbar) = \phi_{0;\vec{N}}(\hbar)$.

After fixing a canonical normalization of $\phi_{\sigma;\vec{N}}(\hbar)$, the ***matrix of Stokes constants*** satisfies

$$\mathcal{S}_{\phi_{\vec{N}}} = \{S_{\sigma,\sigma',n;\vec{N}} \in \mathbb{Q}\} \quad (\textbf{\textit{enumerative invariants}}).$$

Resurgent structures at strong and weak coupling — II

Analogously, in the weakly interacting regime $g_s \propto \hbar^{-1} \to 0$,

$$\mathfrak{B}_{\psi_{\vec{N}}} = \{\Psi_{\sigma,n;\vec{N}}(g_s) = \mathrm{e}^{-n\mathcal{A}_\infty/g_s}\psi_{\sigma;\vec{N}}(g_s)\} \quad (\textbf{\textit{peacock pattern}}),$$

where $n \in \mathbb{Z}$, $\sigma \in \{0,\dots,l_\infty\}$, $l_\infty \in \mathbb{N}$, and $\mathcal{A}_\infty \in \mathbb{C}$. The Gevrey-1 asymptotic series $\psi_{\sigma;\vec{N}}(g_s)$ resurge from $\psi_{\vec{N}}(g_s) = \phi_{0;\vec{N}}(g_s)$.

After fixing a canonical normalization of $\psi_{\sigma;\vec{N}}(g_s)$, the matrix of Stokes constants satisfies

$$\mathcal{S}_{\psi_{\vec{N}}} = \{R_{\sigma,\sigma',n;\vec{N}} \in \mathbb{Q}\}\,.$$

In both limits, the Stokes constants along a tower can be naturally organized into ***q*-series**

$$S_{\sigma,\sigma';\vec{N}}(q) = \sum_{n\in\mathbb{Z}} S_{\sigma,\sigma',n;\vec{N}}q^n\,, \quad R_{\sigma,\sigma';\vec{N}}(q) = \sum_{n\in\mathbb{Z}} R_{\sigma,\sigma',n;\vec{N}}q^n\,,$$

which play a crucial role in decoding their arithmetic and enumerative meaning.

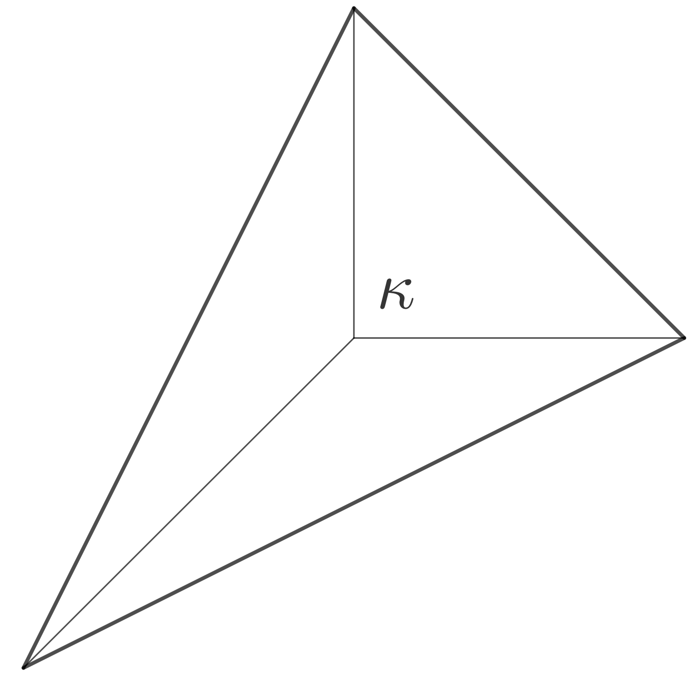
peacock pattern in the Borel plane	$\longrightarrow$	*infinitely many Stokes constants in* $\mathbb{Q}$	$\longrightarrow$	*enumerative invariants of the geometry*

LOCAL $\mathbb{P}^2$ — A CASE STUDY

[Rella, 2022 - Fantini, Rella, 2024]

An exactly solvable example

Local $\mathbb{P}^2$ is the total space of the canonical bundle over $\mathbb{P}^2$, which is a toric del Pezzo CY threefold with one complex modulus κ and no mass parameters.



The spectral trace $Z_{\mathbb{P}^2}(1,\hbar) = \mathrm{Tr}(\rho_{\mathbb{P}^2})$ factorizes into ***holomorphic/anti-holomorphic blocks*** as

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{\sqrt{3}\mathrm{b}} \mathrm{e}^{-\frac{\pi \mathrm{i}}{36}\mathrm{b}^2 + \frac{\pi \mathrm{i}}{12}\mathrm{b}^{-2} + \frac{\pi \mathrm{i}}{4}} \frac{(q^{2/3};\, q)_\infty^2}{(q^{1/3};\, q)_\infty} \frac{(w;\, \tilde{q})_\infty}{(w^{-1};\, \tilde{q})_\infty^2} \,,$$

where $2\pi \mathrm{b}^2 = 3\hbar$, $q = \mathrm{e}^{2\pi \mathrm{i} \mathrm{b}^2}$, $\tilde{q} = \mathrm{e}^{-2\pi \mathrm{i} \mathrm{b}^{-2}}$, and $w = \mathrm{e}^{2\pi \mathrm{i}/3}$.
[Kashaev, Mariño, 2015 - Mariño, Zakany, 2015 - Gu, Mariño, 2021]

The ***all-orders perturbative expansions*** of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ at weak $(\hbar \to 0)$ and strong $(\tau = -\mathscr{A}_\infty/\hbar \to 0)$ coupling give the Gevrey-1 asymptotic series

$$\phi(\hbar) = \sum_{n=1}^{\infty} a_{2n}\hbar^{2n} \in \mathbb{Q}[\![\hbar]\!] \,, \qquad \psi(\tau) = \sum_{n=1}^{\infty} b_{2n}\tau^{2n-1} \in \mathbb{Q}[\pi, \sqrt{3}][\![\tau]\!] \,,$$

$$a_{2n} \sim (-1)^n (2n)!\, \mathscr{A}_0^{-2n} \,, \qquad b_{2n} \sim (-1)^n (2n)!\, \mathscr{A}_\infty^{-2n} \,,$$

where $\mathscr{A}_0 = \frac{4\pi^2}{3}$ and $\mathscr{A}_\infty = \frac{2\pi}{3}$.

Arithmetic properties of the Stokes constants — I

We obtain the ***exact resurgent structures*** at weak and strong coupling.

The Borel transforms $\hat{\phi}(\zeta), \hat{\psi}(\zeta)$ are simple resurgent functions with logarithmic branch points at $\zeta_n = n\mathscr{A}_0 \mathrm{i}$ and $\eta_n = n\mathscr{A}_\infty \mathrm{i}$, $n \in \mathbb{Z}_{\neq 0}$.

The secondary resurgent series are trivial, that is, $\hat{\phi}_n(\zeta) = 1$ and $\hat{\psi}_n(\zeta) = 1$.

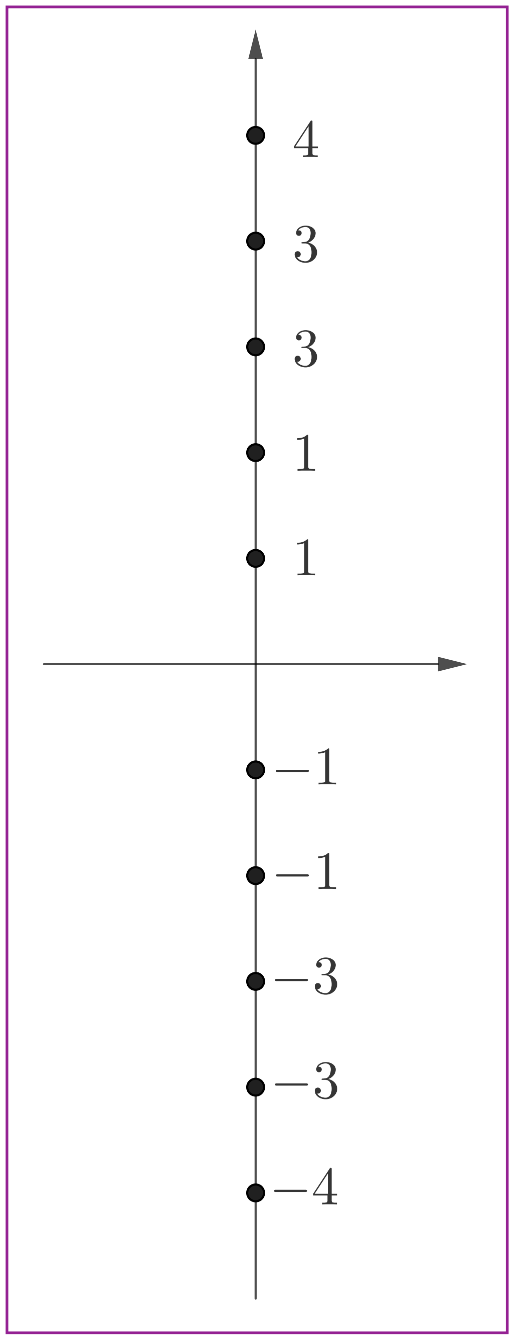
The normalized Stokes constants $S_n/S_1, R_n/R_1$, $n \in \mathbb{Z}_{\neq 0}$, are the ***divisor sum functions***

$$\frac{S_n}{S_1} = \sum_{d|n} \frac{1}{d} \chi_{3,2}(d) \in \mathbb{Q}_{>0}, \quad S_1 = 3\sqrt{3}\mathrm{i},$$

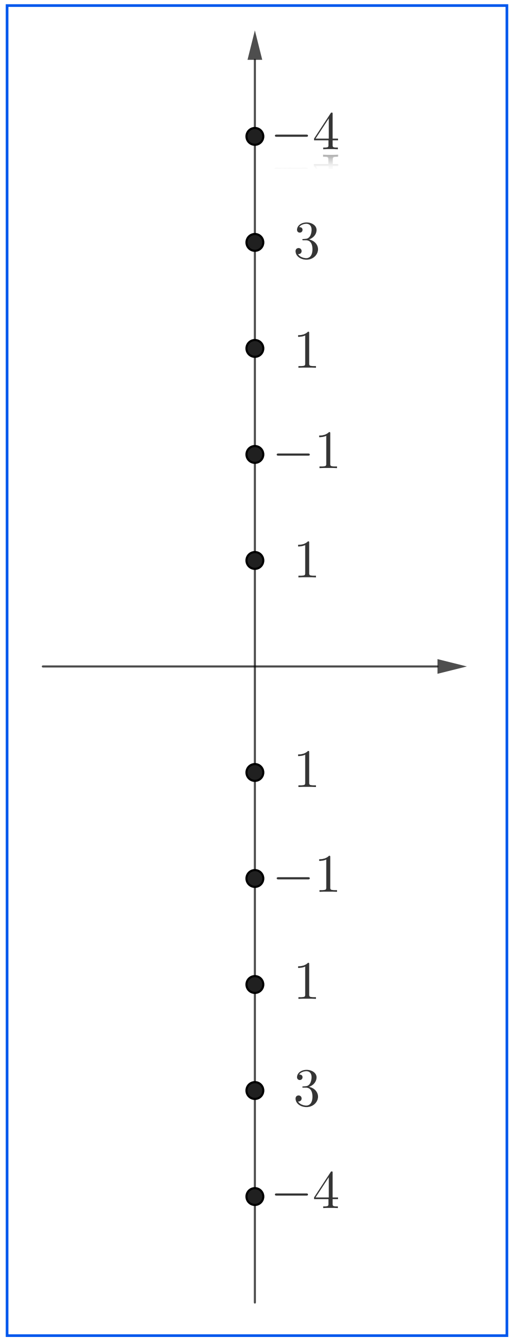
$$\frac{R_n}{R_1} = \sum_{d|n} \frac{d}{n} \chi_{3,2}(d) \in \mathbb{Q}_{\neq 0}, \quad R_1 = 3,$$

where $\chi_{3,2}(n) = [n]_3$ is the non-principal Dirichlet character modulo 3.

$$n\frac{S_n}{S_1} \in \mathbb{Z}_{\neq 0}$$



$$n\frac{R_n}{R_1} \in \mathbb{Z}_{\neq 0}$$



Arithmetic properties of the Stokes constants — II

Theorem: The Dirichlet series of the Stokes constants S_n, R_n, $n \in \mathbb{Z}_{>0}$, can be analytically continued to ***weak and strong coupling L-functions*** $L_0(s)$, $L_\infty(s)$, $s \in \mathbb{C}$, and factorize as

$$L_0(s) = \sum_{m=1}^{\infty} \frac{S_m}{m^s} = S_1 L(s+1, \chi_{3,2})\, \zeta(s)\,, \quad L_\infty(s) = \sum_{m=1}^{\infty} \frac{R_m}{m^s} = R_1 L(s, \chi_{3,2})\, \zeta(s+1)\,.$$

Note that the weak and strong coupling L-functions are simply related by a symmetric unitary shift in the arguments of the factors.

Theorem: The generating series of the Stokes constants S_n, R_n, $n \in \mathbb{Z}_{>0}$, equal the discontinuities and are given by the $\tilde{q}$, ***q-series in the anti-holomorphic/holomorphic blocks*** in the factorized expression for the spectral trace

$$\mathrm{disc}_{\frac{\pi}{2}}\, \phi(\hbar) = \sum_{n=1}^{\infty} S_n\, \tilde{q}^n = 3 \log \frac{(w^{-1}\tilde{q};\, \tilde{q})_\infty}{(w\tilde{q};\, \tilde{q})_\infty}\,, \quad \mathrm{disc}_{\frac{\pi}{2}}\, \psi(\tau) = \sum_{n=1}^{\infty} R_n\, q^{n/3} = 3 \log \frac{(q^{2/3};\, q)_\infty}{(q^{1/3};\, q)_\infty}\,.$$

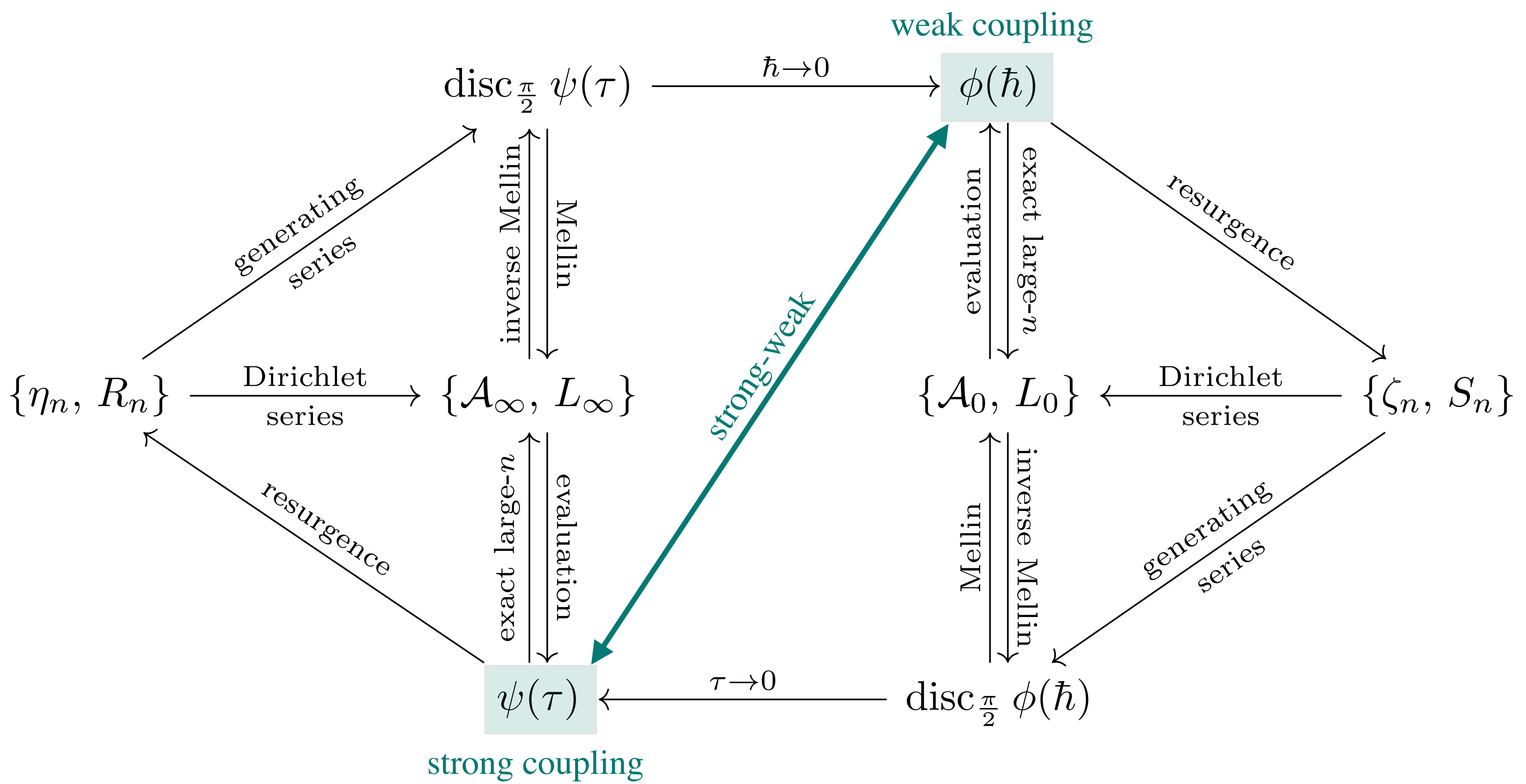
Each of the two blocks determines the perturbative content in one regime and the non-perturbative content in the other.

The relations connecting the exact resurgent structures of the dual perturbative expansions of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$ embed into a global commutative diagram.

A full-fledged analytic number-theoretic symmetry — I

The two-way exchange of perturbative/non-perturbative information between the dual regimes in $\hbar$ takes a mathematically precise form (***strong-weak resurgent symmetry***).

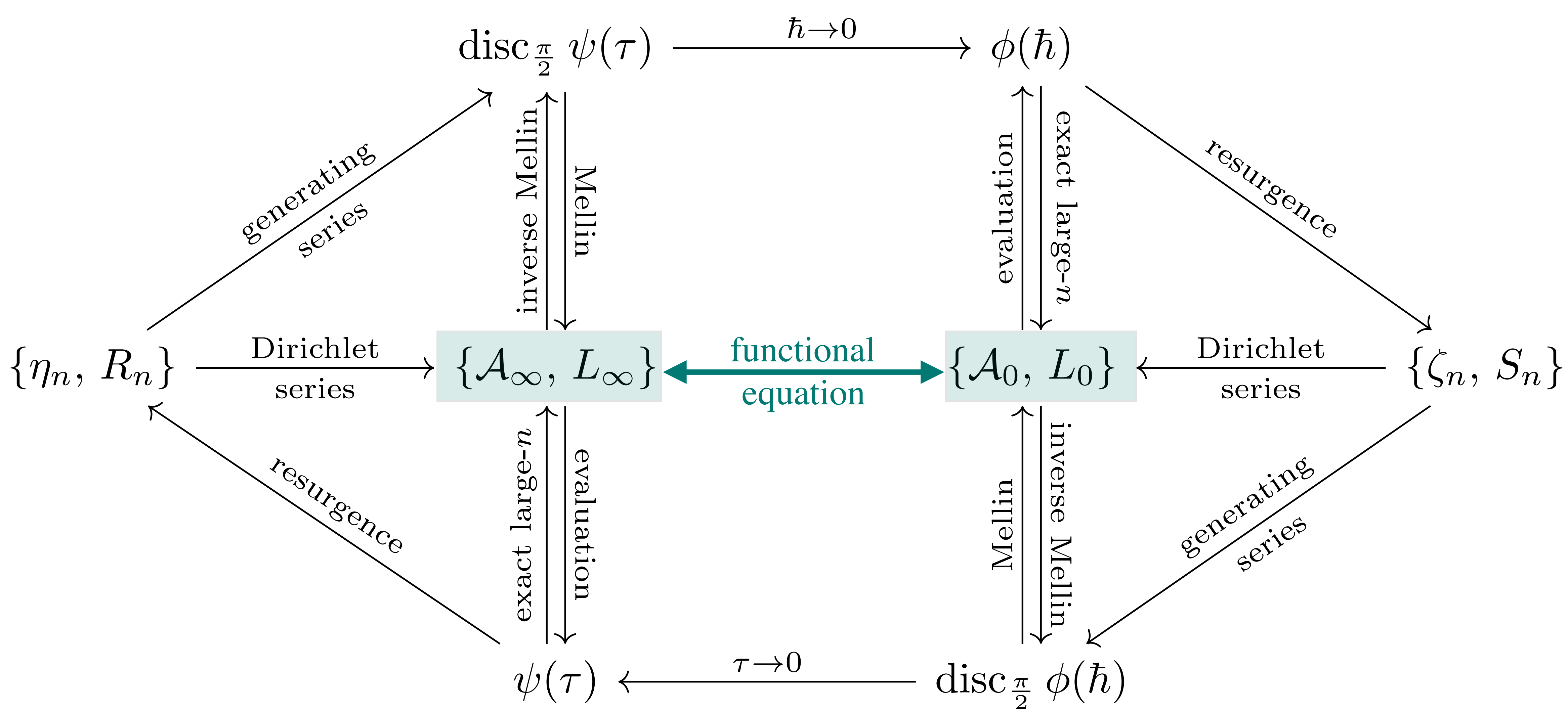
This is a realization of underlying physical mechanisms that can be intuitively traced back to the ***S-type duality*** between the worldsheet and WKB contributions to the total grand potential of the topological string on local $\mathbb{P}^2$.



A full-fledged analytic number-theoretic symmetry — II

The completions of the weak and strong coupling L-functions $\Lambda_0(s)$, $\Lambda_\infty(s)$, $s \in \mathbb{C}$, do not individually satisfy a standard functional equation—e.g., $\Lambda_\zeta(s) = \Lambda_\zeta(1-s)$.

Theorem: The completed weak and string coupling L-functions $\Lambda_0(s)$, $\Lambda_\infty(s)$, $s \in \mathbb{C}$, are analytically continued to the whole complex s-plane through each other as they satisfy the ***combined functional equation*** $\Lambda_0(s) = \Lambda_\infty(-s)$.



A THEORY OF MODULAR RESURGENCE

[Fantini, Rella, 2024]

New perspectives on resurgence via L-functions

<u>*Definition:*</u> A Gevrey-1 asymptotic series $\varphi(y) \in \mathbb{C}[[y]]$ has a ***modular resurgent structure*** if

1. The Borel transform $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ is singular at $\zeta_m = m\mathscr{A}\,,\, m \in \mathbb{Z}_{\neq 0}$, for some $\mathscr{A} \in \mathbb{C}$, and the resurgent series at ζ_m is the Stokes constant $A_m \in \mathbb{C}$;

2. The Stokes constants A_m, $m \in \mathbb{Z}_{\neq 0}$, are the coefficients of two L-functions

$$L_+(s) = \sum_{m>0} \frac{A_m}{m^s}\,, \quad L_-(s) = -\sum_{m>0} \frac{A_{-m}}{m^s}\,.$$

A ***modular resurgent series*** is equivalently characterized by the generating function

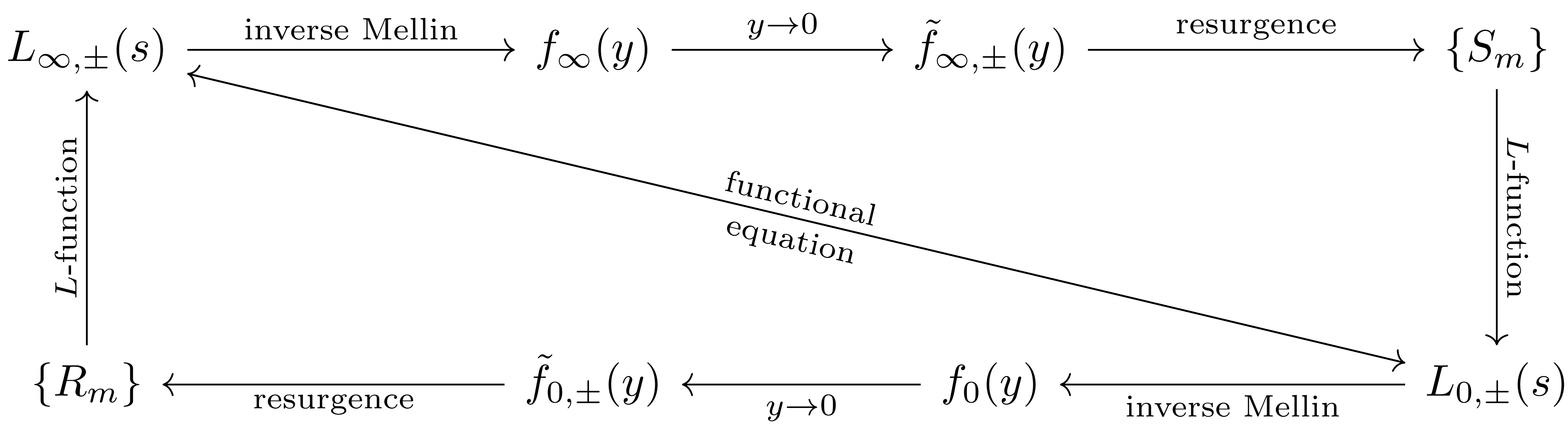
$$f(y) = \begin{cases} \sum_{m>0} A_m \mathrm{e}^{2\pi \mathrm{i} m y}\,, & \Im(y) > 0\,, \\ -\sum_{m<0} A_m \mathrm{e}^{2\pi \mathrm{i} m y}\,, & \Im(y) < 0\,, \end{cases} \qquad y \in \mathbb{C}\backslash\mathbb{R}\,.$$

There is a canonical correspondence between the L-functions $L_\pm$ and the q-series f.

A rich analytic number-theoretic fabric underlies the properties of modular resurgent series and motivates us to present a ***new paradigm of resurgence***.

The modular resurgence paradigm

A network of exact relations connects ***pairs of modular resurgent structures***, forming a commutative diagram that generalizes the strong-weak resurgent symmetry of $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$.



<u>*Conjecture:*</u> Let $f : \mathbb{H} \to \mathbb{C}$ be a q-series where $q = \mathrm{e}^{2\pi \mathrm{i} y}$. If its asymptotic expansion $\varphi(y)$ as $y \to 0$ with $\mathfrak{I}(y) > 0$ has a modular resurgent structure, then $f(y)$ is a ***holomorphic quantum modular form*** for a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and

$$\mathcal{S}_\theta^{\mathrm{med}} \varphi(y) = f(y)\,, \quad y \in \mathbb{H} \cap \{\mathfrak{R}(\mathrm{e}^{-\mathrm{i}\theta} y) > 0\}\,.$$

The conjecture is proven for the generating functions of the Stokes constants of $\log$ *Tr*$(\rho_{\mathbb{P}^2})$ *with respect to* $\Gamma_1(3)$. *More examples from Maass cusp forms, combinatorics, and the quantum invariants of knots and 3-manifolds.*

CONCLUSIONS

Final remarks

Most perturbative series in quantum theories are factorially divergent. Their ***resurgent analysis*** unveils a universal structure of ***non-perturbative sectors*** involving a set of complex numbers called ***Stokes constants***.

The resurgence of the spectral traces of a toric CY threefold shows symmetric patterns of singularities in the Borel plane (***peacock patterns***) and infinitely many, rational Stokes constants (***enumerative invariants***).

The resurgence of the spectral trace of local $\mathbb{P}^2$ in the weak and strong $\hbar$-regimes fits into a unique global number-theoretic construction (***strong-weak resurgent symmetry***), which is deeply related to the traditional notion of S-duality in string theory.

Our results suggest a new paradigm linking the resurgent properties of q-series, the analytic properties of L-functions, and quantum modular forms (***modular resurgent structures***).

For the future…

- Explore the geometric/physical meaning of the non-perturbative sectors and Stokes constants for the spectral traces of toric CY threefolds.

- Use the tools and results of (modular) resurgence to understand non-perturbative effects and dualities in QFT.

THANK YOU!