Single-valued Integration or Double Copy on the Elliptic Curve:

Monodromy Relations and Twisted (Co)homology

Stephan Stieberger, MPP München

Holonomic Techniques for Feynman Integral
MPP München
October 14-18, 2024

Single-valued Integration and Double Copy

periods $\int_{\Delta} \omega$ integrals of closed differential forms ω over a domain of integration Δ

algebraic geometry: integration can be interpreted as a canonical pairing between de Rham cohomology ω and cycle Δ (singular homology for algebraic varieties)

varieties depend algebraically on parameters:
one typically obtains multi-valued functions of the parameters

open string amplitudes

consider (specific) pairing between de Rham cohomology and its dual:

periods $$(2\pi i)^{-n} \int_X \omega \wedge \overline{\nu}$$

with parameters:
one typically obtains single-valued functions of the parameters

closed string amplitudes

double copy formula:

$$\int_X \omega \wedge \overline{\nu} = \sum_{\gamma \in H_n} \langle \gamma, \delta \rangle \int_\gamma \omega \int_{\bar{\delta}} \nu$$

Brown, Dupont (2018)
Mizera (2017)

in particular: complex integrals $\prod_{i=1}^{n} d^2 z_i$ on genus g surfaces

well understood for $g = 0$

$n = 1$: $\langle \gamma, \delta \rangle$ intersection number: $H_1(X, \mathscr{L}_\omega^\vee) \times H_1(X, \mathscr{L}_\omega) \to \mathbf{C}$

KLT relation

conformal block decomposition

closed - open string duality

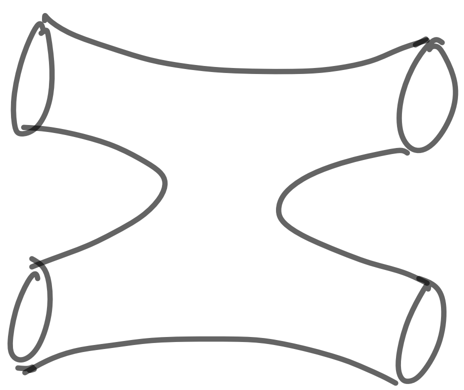
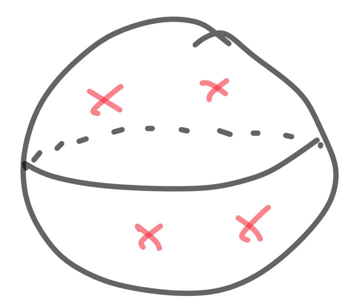
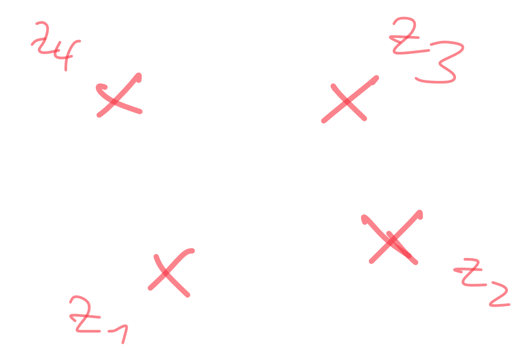
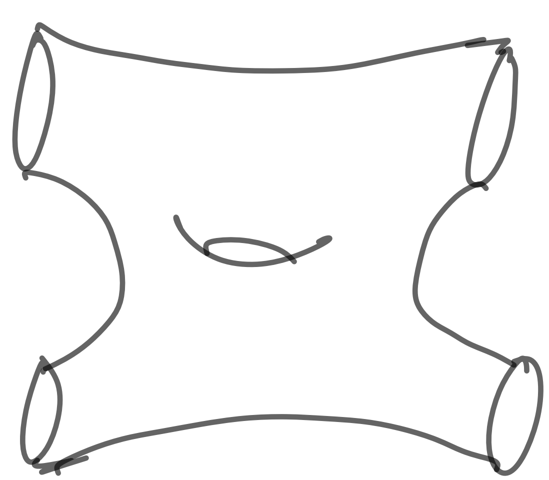
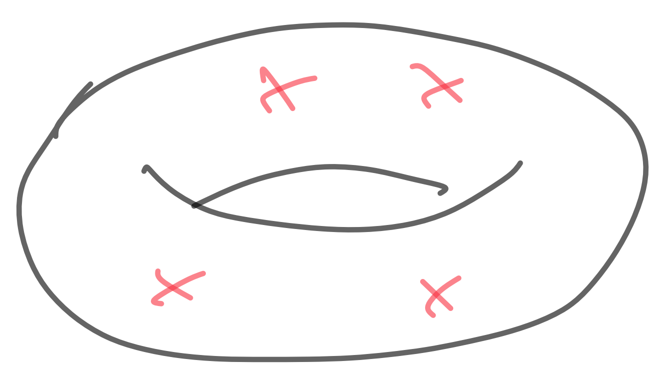
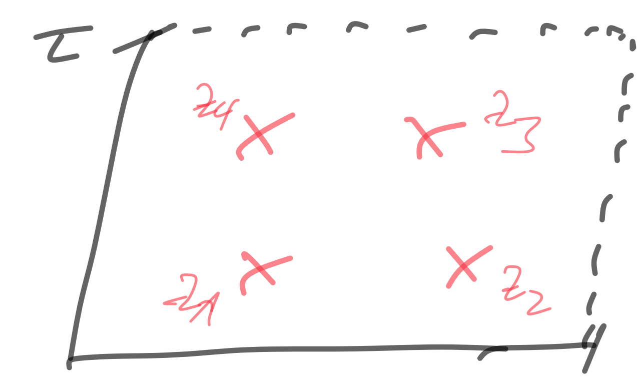
Kawai, Lewellen, Tye (1986)

Dotsenko, Fateev (1984)

String Amplitudes: S-matrix

<u>closed string amplitudes</u>

gravitational amplitudes

asymptotic string scattering	*(oriented) Riemann surface Σ_g of genus g*	*moduli space of m marked points on Σ_g*
	 $g=0$	 z_4 z_3 z_1 z_2 $m=4$
	 $g=1$	 τ z_4 z_3 z_1 z_2 $m=4$ 1

What is KLT ?

Kawai, Lewellen, Tye (1986): tree-level relation between closed and open strings

- abstractly: provides a way of expressing closed string world-sheet integrals in terms of open string integrals

- at the technical level: a way of computing *complex* integrals on S^2 by reducing them to pairs of *real* integrals

- at the physical level: gauge/gravity or open/closed string relation (perturbative tree-level)

- far reaching consequences by elevating it to double copy (DC) conjecture or structure

so far well established at tree-level
enough for higher-loop field-theory (unitary cuts)

Question: how does it work on the elliptic curve (torus)

$\Rightarrow$ *One-Loop Generalization of famous tree-level result*

$\Rightarrow$ *(String) Double Copy structure at one-loop*

based on:

- *A Relation between One-Loop Amplitudes of Closed and Open Strings (One-Loop KLT Relation), arXiv: 2212.1253*
- *One-Loop Double Copy Relation in String Theory, arXiv:2310.07755*
- *One-loop Double Copy Relation from Twisted (Co)homology, arXiv:2403.05208*

with Pouria Mazloumi

KLT

Kawai, Lewellen, Tye (1986)

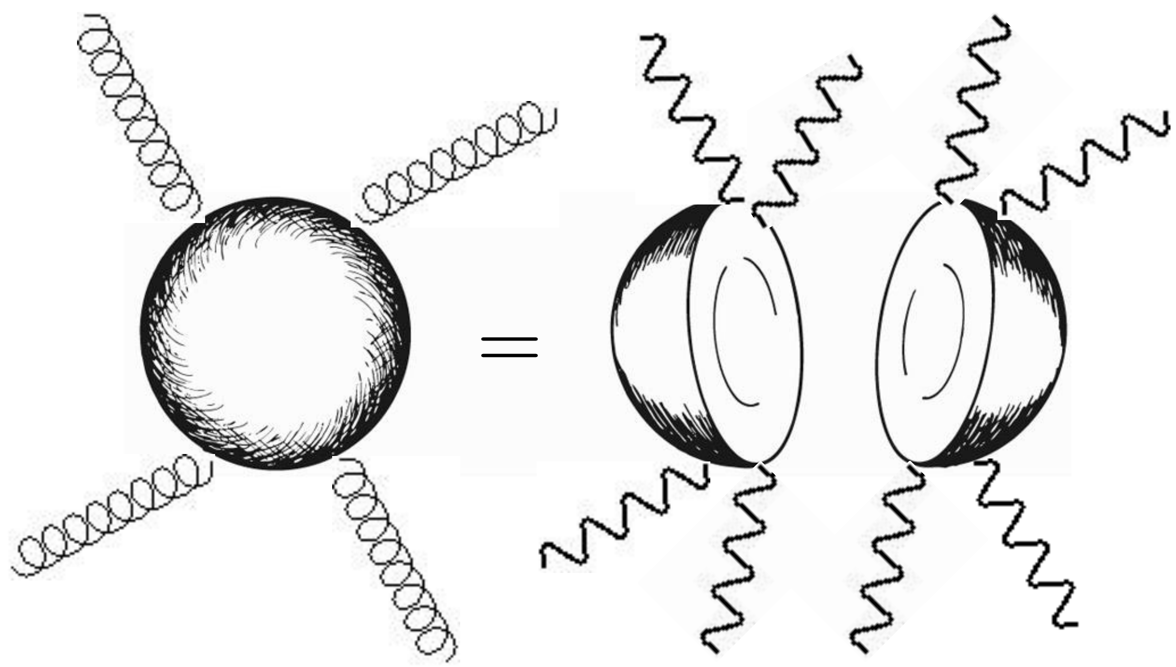
$$M^{closed}_{4;0} := \int_{\mathscr{C}} d^2 z \, |z|^{2\alpha' s-2} \, |1-z|^{2\alpha' u} = \frac{\Gamma(\alpha' s)\Gamma(\alpha' t)\Gamma(\alpha' u)}{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}$$

$$A^{open}_{4;0} := \int_0^1 d\xi \; \xi^{\alpha' s-1} \, (1-\xi)^{\alpha' u} = \frac{\Gamma(\alpha' s)\,\Gamma(\alpha' u+1)}{\Gamma(\alpha' s+\alpha' u+1)}$$

$$\tilde{A}^{open}_{4;0} := \int_1^\infty d\eta \; \eta^{\alpha' t-1} \, (\eta-1)^{\alpha' u} = \frac{\Gamma(\alpha' t)\,\Gamma(\alpha' u+1)}{\Gamma(\alpha' t+\alpha' u+1)}$$

$$M^{closed}_{4;0} = \sin(\pi\alpha' u) \; A^{open}_{4;0} \; \tilde{A}^{open}_{4;0}$$

derived from monodromies on string world-sheet



Homological Splitting on Sphere

KLT

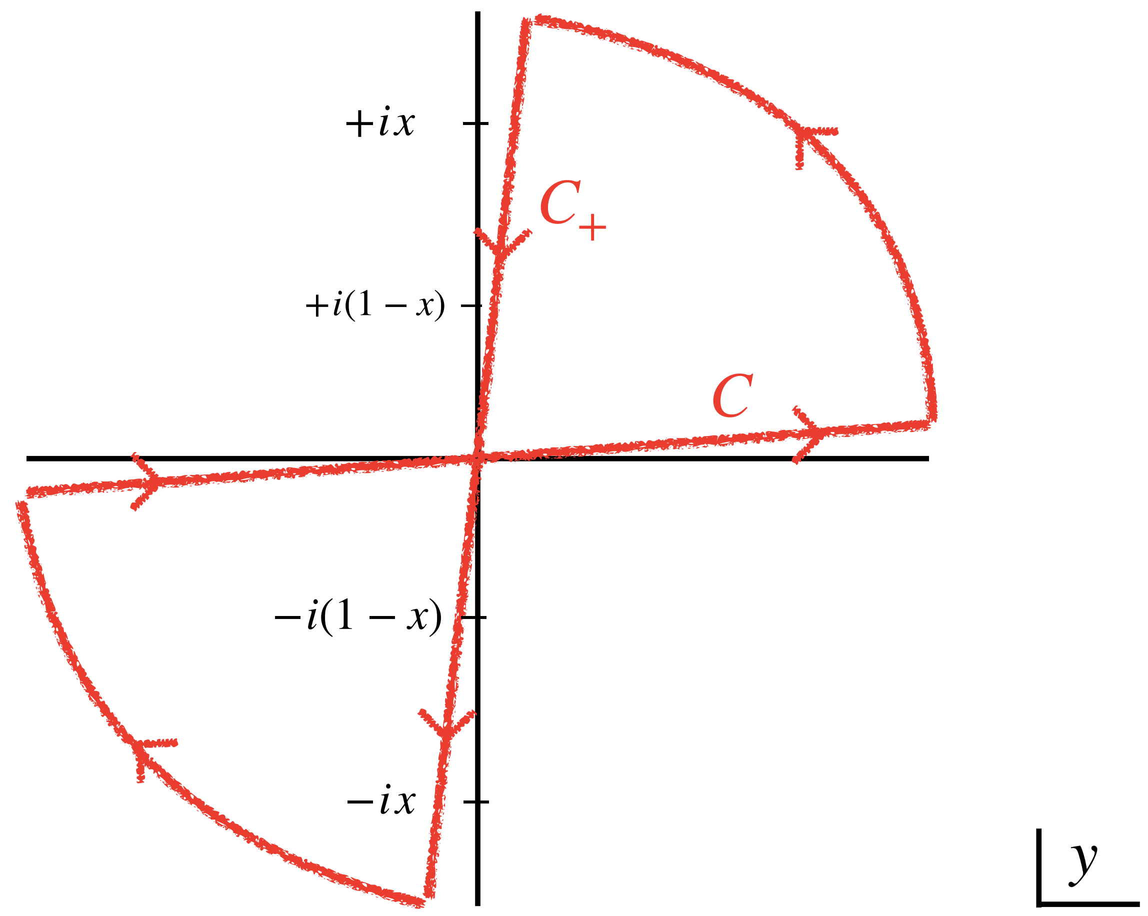
$$\mathcal{I} = \int_{\mathbf{C}} d^2z \; z^{\alpha' s + n_{12}} \; (1-z)^{\alpha' u + n_{23}} \; \bar{z}^{\alpha' s + \bar{n}_{12}} \; (1-\bar{z})^{\alpha' u + \bar{n}_{23}}$$

$$z = x + iy = x - \tilde{y} := \xi$$

$$\bar{z} = x - iy = x + \tilde{y} := \eta$$

$x, y \in (-\infty, \infty)$

$y = i\tilde{y} \;,\; \tilde{y} \in \mathbf{R}$



$$\mathcal{I} = \frac{i}{2} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \; |\xi|^{\alpha' s + n_{12}} \; |1-\xi|^{\alpha' u + n_{23}} \; |\eta|^{\alpha' s + \bar{n}_{12}} \; |1-\eta|^{\alpha' u + \bar{n}_{23}} \; \Pi(\xi, \eta; s, u)$$

n Unintegrated Points

Mazloumi, St.St. (2024)

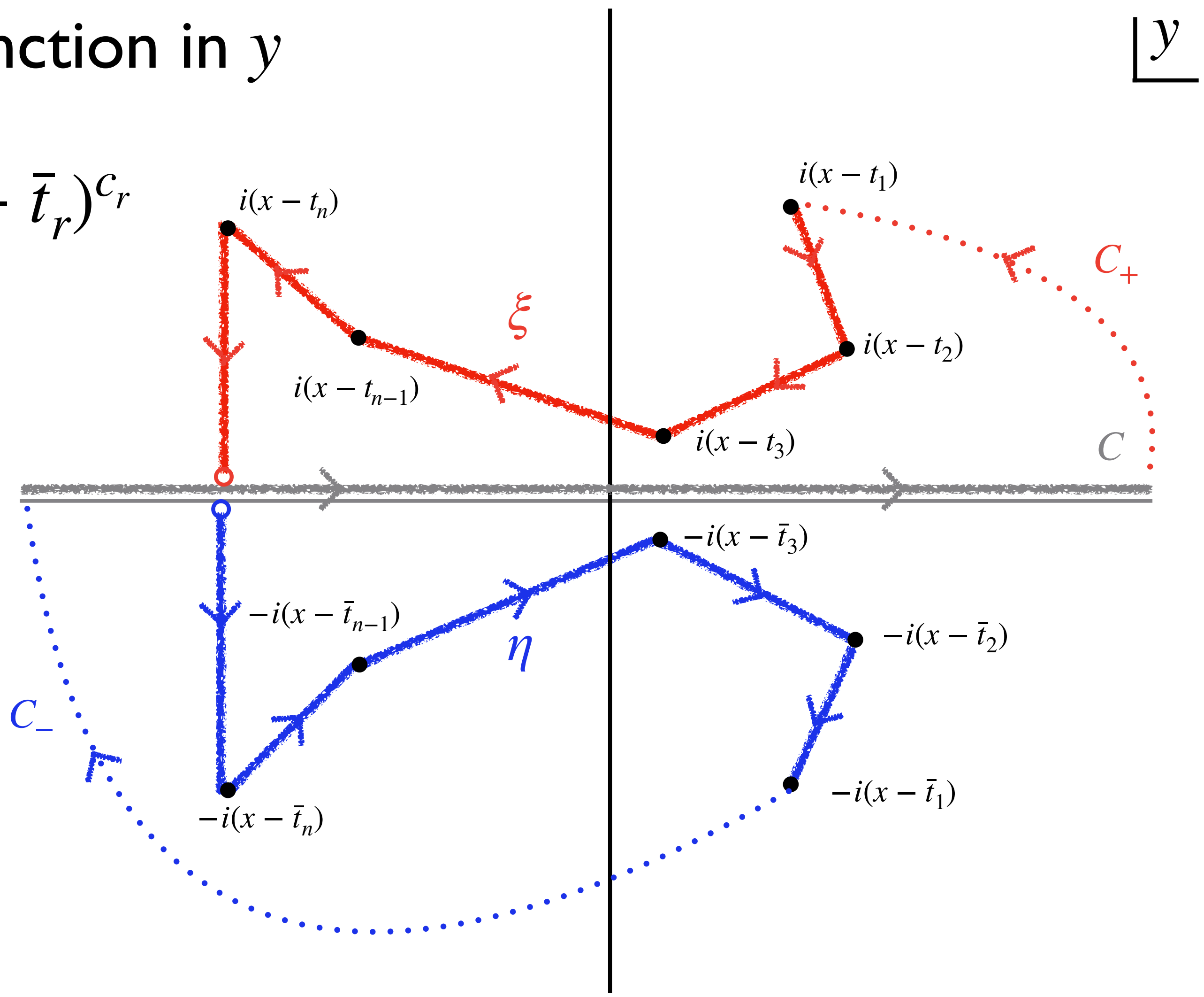
$$\mathcal{M}^{closed}_{1;0} = V^{-1}_{CKG} \int_{\mathbf{C}} d^2z \prod_{r=1}^{n} |z - t_r|^{2c_r}$$

$$z = x + iy$$

consider integrand I as holomorphic function in y

$$I(x, y) = \prod_{r=1}^{n} (x + iy - t_r)^{c_r} \; (x - iy - \bar{t}_r)^{c_r}$$

$$\xi = x + iy$$
$$\eta = x - iy$$



$$\mathcal{M}^{closed}_{1;0} = -\,V^{-1}_{CKG} \oint_{C_\xi} d\xi \prod_{r=1}^{n} (\xi - t_r)^{c_r} \; \oint_{C_\eta} d\eta \prod_{s=1}^{n} (\eta - \bar{t}_s)^{c_s} \; \Pi(\xi, \eta)$$

$\Longrightarrow$ SVMPLs in t_l

after appropriate choice of c_s

generalization to n closed strings:

$$\mathcal{M}^{closed}_{n;0} = \kappa^{n-2} \sum_{\sigma,\rho \in S_{n-3}} A^{open}_{n;0}(1,\sigma(2,3,\ldots,n-2),n-1,n) \times \mathcal{S}[\rho\,|\,\sigma]_{p_1}\, \tilde{A}^{open}_{n;0}(1,\rho(2,3,\ldots,n-2),n,n-1)$$

KLT kernel (intersection matrix following e.g. from twisted de Rham theory):

$$\mathcal{S}[\sigma\,|\,\rho]_{p_0} := \mathcal{S}[\,\sigma(1,\ldots,k)\,|\,\rho(1,\ldots,k)\,]_{p_0} = \prod_{t=1}^{k} \sin\left(\pi\alpha'\left[p_0 p_{t_\sigma} + \sum_{r<t} p_{r_\sigma} p_{t_\sigma} \theta(r_\sigma, t_\sigma)\right]\right)$$

has received a lot of interest on its own: derive from first principles

$$\mathcal{M}^{closed}_{n;0} = \kappa^{n-2} \sum_{\sigma,\rho \in S_{n-3}} A^{open}_{n;0}(1,\sigma(2,3,\ldots,n-2),n-1,n)$$
$$\times \; S^{(0)}[\rho\,|\,\sigma]_{p_1} \; \tilde{A}^{open}_{n;0}(1,\rho(2,3,\ldots,n-2),n,n-1)$$

Question: how are open and closed string amplitudes are related ?

↪ Brown's single-valued projection

St.St. (2013)
St.St., Taylor (2014)

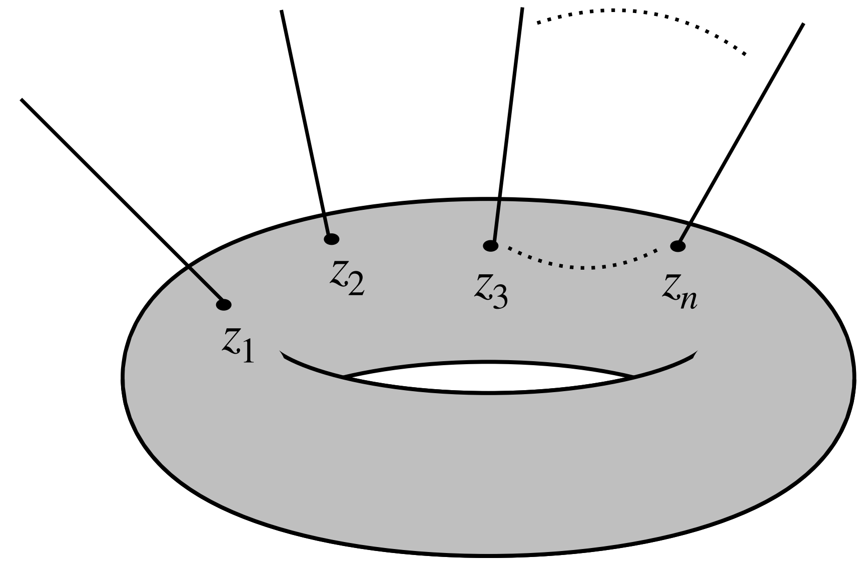
⋮

Brown, Dupont (2018)

Reverse Question: how a single-valued amplitude can be related to a pair amplitudes with multi-valued coefficients ?

Baune, Broedel (2023)

One-loop String Theory



$z_i \in \mathscr{T}$

$$\mathscr{M}_{n;1}^{closed} \sim \int_{\mathscr{F}_1} \frac{d^2\tau}{\tau_2} \left(\int_{\mathscr{T}} \prod_{s=1}^{n} d^2 z_s \right) EI(\{z_s, \bar{z}_s\}, \tau)$$

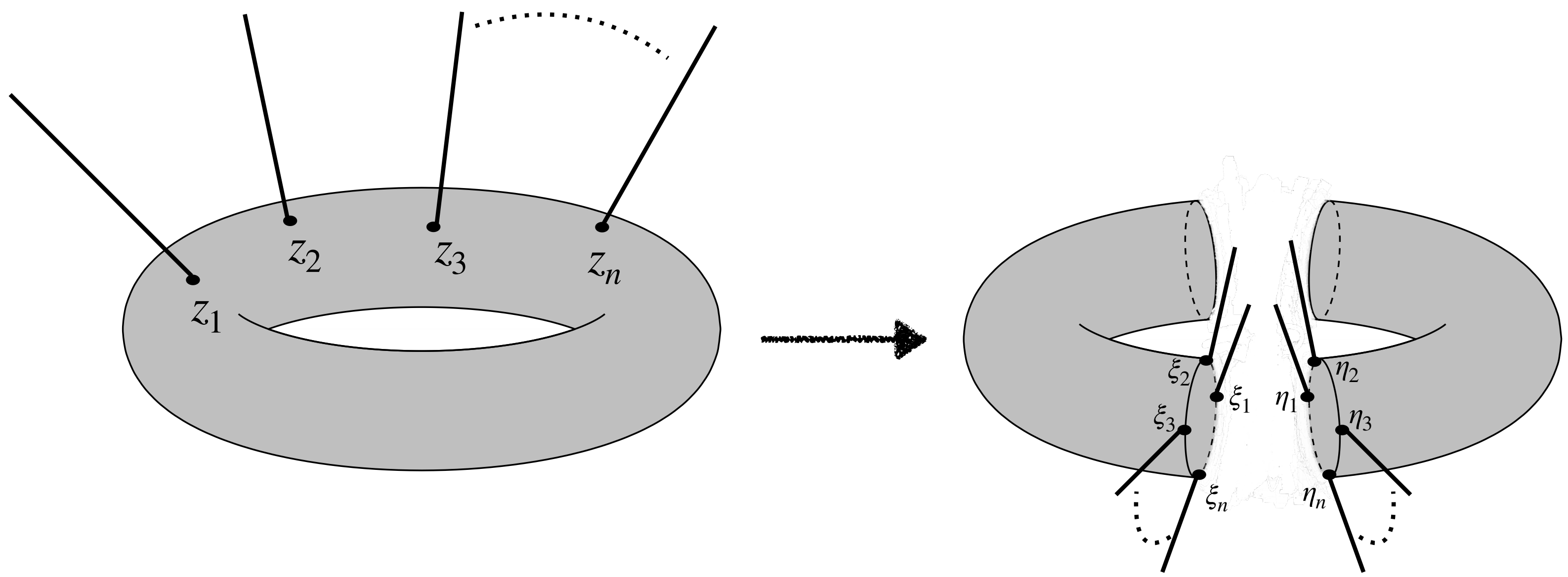
multi-dimensional torus integral over elliptic functions
with branch cuts

Question: to what extent can we factorize this amplitude into real integrals ?

↪ (multi-dimensional) monodromy problem on the elliptic curve

String One-loop Double Copies

We expect the following geometric picture:



Example:

$$\widehat{M}^{closed}_{2;1} := \int_{\mathcal{T}} d^2z\ e^{2G^{(1)}(z,\tau)} = 2\tau_2^{\frac{1}{2}} \left|\frac{\theta_3(2\tau)}{\eta^6}\right|^2 + 2\tau_2^{\frac{1}{2}} \left|\frac{\theta_2(2\tau)}{\eta^6}\right|^2$$

$$\widehat{\widetilde{M}}^{closed}_{2;1} := \int_{\mathcal{T}} d^2z\ e^{G^{(1)}(z,\tau)} = 2\ \tau_2^{\frac{1}{2}} \left|\frac{1}{\eta^3}\right|^2$$

$$G^{(1)}(z,\tau) = \ln\left|\frac{\theta_1(z,\tau)}{\theta_1'(0,\tau)}\right|^2 - 2\pi\frac{(\Im z)^2}{\Im\tau}$$

St.St. (2023)

Corresponding open string amplitudes:

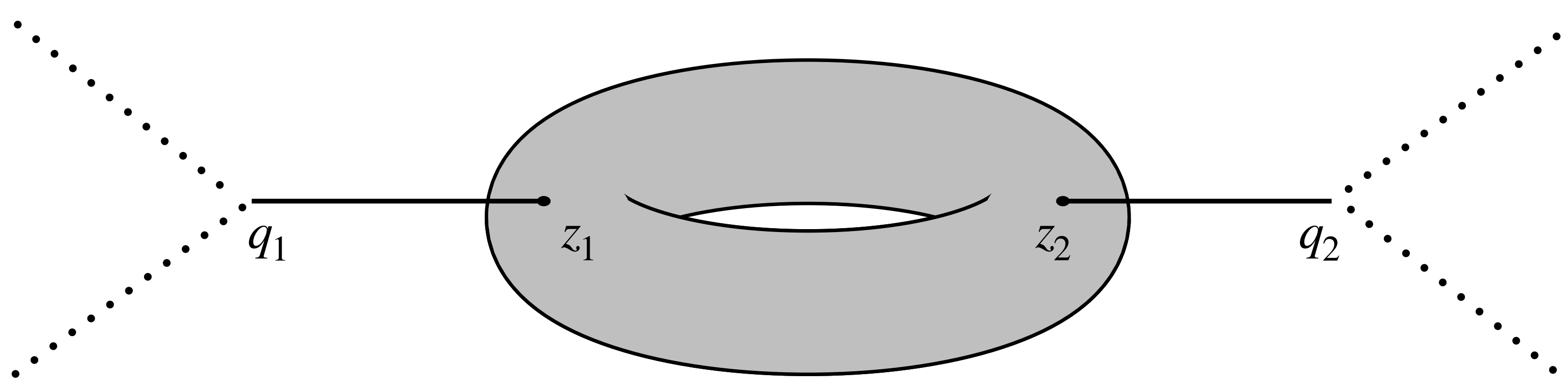
$$A^{p}_{2;1} := \int_0^1 d\xi \, \frac{\theta_1(\xi,\tau)^2}{\eta^6} = -\frac{\theta_2(2\tau)}{\eta^6} \,,$$

$$A^{np}_{2;1} := \int_0^1 d\zeta \, \frac{\theta_4(\zeta,\tau)^2}{\eta^6} = \frac{\theta_3(2\tau)}{\eta^6},$$

$$\widehat{M}^{closed}_{2;1} = 2\tau_2^{1/2}\, |A^{p}_{2;1}|^2 + 2\tau_2^{1/2}\, |A^{np}_{2;1}|^2$$

St.St. (2023)

Actually $\widehat{M}^{closed}_{2;1}$ computes the mass correction δm^2 of the least massive string state in type II superstring:



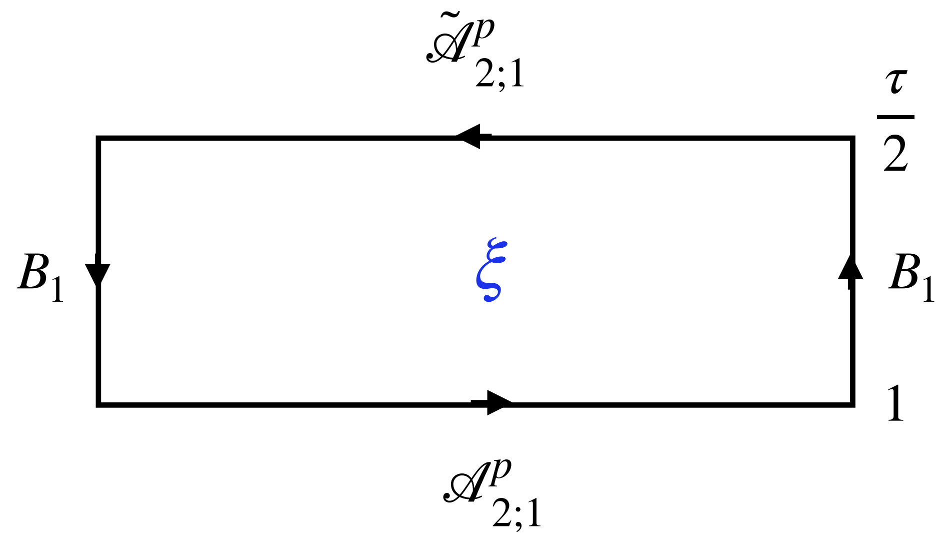
$$q_i^2 = -4/\alpha' \quad , \quad i = 1,2$$

$$\mathscr{M}^{closed}_{2;1} = \delta^{(d)}(q_1+q_2) \int \frac{d^2\tau}{\tau_2}\, \tau_2^{-4} \int_{\mathscr{T}} d^2z \; e^{-\frac{\alpha'}{2} q_1^2 G^{(1)}(z,\tau)}$$

Marcus (1989)

One-loop Monodromy (Bordered Surface)

on cylinder $\Re(\tau) = 0$



open string monodromy relation

$$\oint d\xi \, \frac{\theta_1(\xi,\tau)}{\eta^3} = 0$$

$$\mathscr{A}^{p}_{2;1} - \tilde{\mathscr{A}}^{p}_{2;1} = 2\, B_1$$

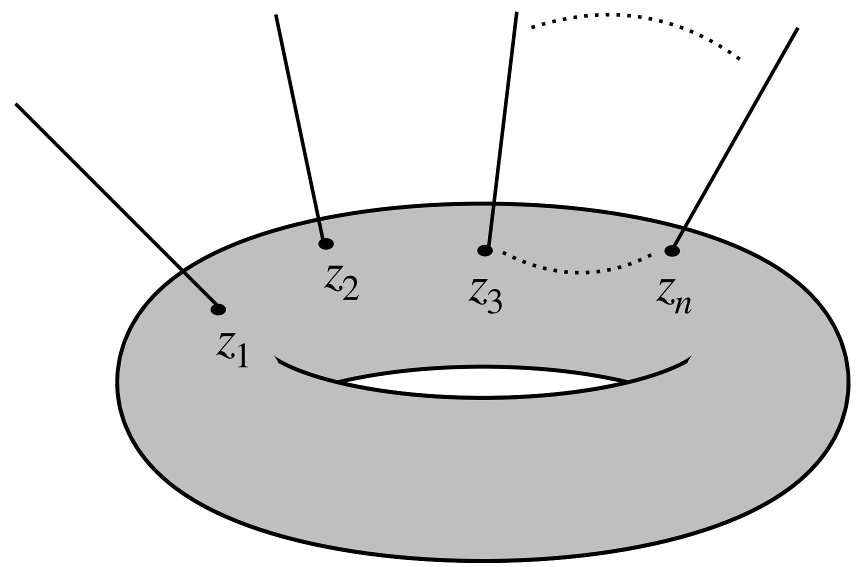
Hohenegger, St.St. (2017)

$$\mathscr{A}^{p}_{2;1} := \int_0^1 d\xi \, \frac{\theta_1\left(\xi,\tau\right)}{\eta^3} = \frac{2}{\pi}\frac{q^{\frac{1}{8}}}{\eta^3}\sum_{n\in\mathbf{Z}}(-1)^n \frac{q^{\frac{1}{2}(n+1)n}}{2n+1} \; ,$$

$$\tilde{\mathscr{A}}^{p}_{2;1} := -\, i\int_0^1 d\zeta \, \frac{\theta_4\left(\zeta,\tau\right)}{\eta^3} \, e^{\pi i\zeta} \, q^{-\frac{1}{8}} = \frac{1}{\pi}\frac{q^{-\frac{1}{8}}}{\eta^3}\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}n^2}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)$$

$$B_1 = \int_0^{\tau/2} dz \, \frac{\theta_1(z,\tau)}{\eta^3}$$

One-loop String Torus Amplitude with n Closed Oriented Strings



$$\mathscr{M}^{closed}_{n;1}(q_1,\ldots,q_n) = \frac{1}{2}\, g_c^n\, \delta^{(d)}\left(\sum_{r=1}^{n} q_r\right) \int_{\mathscr{F}_1} \frac{d^2\tau}{\tau_2} V^{-1}_{CKG}(\mathscr{T})$$
$$\times \left(\int_{\mathscr{T}} \prod_{s=1}^{n} d^2 z_s \right) I(\{z_s, \bar{z}_s\})\; Q(\{z_s, \bar{z}_s\}; \tau)$$

Q = some doubly-periodic function comprising possible kinematical factors

$$I(\{z_s, \bar{z}_s\}) = \prod_{1\le r<s\le n} \left[\frac{\theta_1(z_s - z_r, \tau)}{\theta_1'(0,\tau)}\right]^{\frac{1}{2}\alpha' q_s q_r} \left[\frac{\bar{\theta}_1(\bar{z}_s - \bar{z}_r, \bar{\tau})}{\bar{\theta}_1'(0,\bar{\tau})}\right]^{\frac{1}{2}\alpha' q_s q_r} \prod_{\substack{r,s=1 \\ r<s}}^{n} e^{-\frac{\pi\alpha'}{\tau_2} q_r q_s \Im(z_r - z_s)^2}$$

Comments:

- lack of double periodic function
  $\Longrightarrow$ deal with quasi-periodic functions with non-harmonic contributions

- no holomorphic- anti-holomorphic factorization
  in contrast to Virasoro-Shapiro amplitude

Introduce loop momentum ℓ to holomorphically factorize the amplitude:

D'Hoker, Phong (1988)

$$(\alpha'\tau_2)^{-d/2}\ I(\{z_s,\bar{z}_s\}) = \int_{-\infty}^{\infty} d^d\ell\ \exp\left\{-\pi\alpha'\tau_2\ell^2 - \pi i\alpha'\ell\sum_{r=1}^{n} q_r(z_r-\bar{z}_r)\right\}$$

$$\times \prod_{1\leq r<s\leq n} \theta_1(z_s-z_r,\tau)^{\frac{1}{2}\alpha' q_s q_r}\ \theta_1(\bar{z}_s-\bar{z}_r,\bar{\tau})^{\frac{1}{2}\alpha' q_s q_r}$$

$$z_t = \sigma_t^1 + i\sigma_t^2 \quad , \quad \sigma_t^1 \in (0,1)\ ,\ \sigma_t^2 \in \left(-\frac{\tau_2}{2},\frac{\tau_2}{2}\right)$$

Riemann-Wirtinger Integral

Goto (2022)
Mano, Watanabe (2012)

multi-valued function on $E_\tau = \mathbf{C}/\Lambda$

lattice Λ generated by 1 and τ

$$T(z) = e^{2\pi i c_0 z} \prod_{k=1}^{n} \theta_1(z - z_k; \tau)^{c_k}$$

$c_0 \in \mathbf{C}, \quad c_i \in \mathbf{C}$

later $\mathbf{C} - \mathbf{Z}$ for intersection numbers

$$\sum_{i=1}^{n} c_i = 0$$

$$T(z \pm 1) = e^{\pm 2\pi i c_0}\, T(z)$$

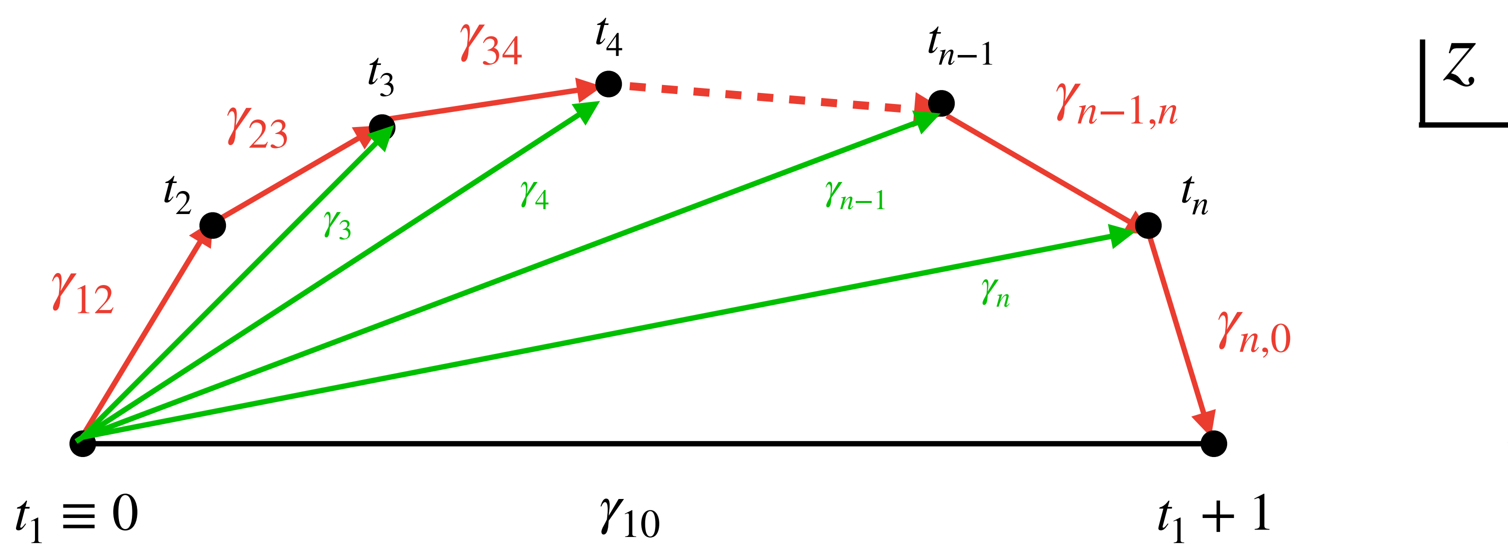
$$T(z + \tau) = e^{2\pi i c_\infty}\, T(z)$$

$$c_\infty = c_0 \tau + \sum_{i=1}^{n} c_i z_i$$

$\hookrightarrow$ periods $\displaystyle\int_\gamma T(z)\, dz$

Veech's full holonomy map

$n+1$ cycles $\gamma_{1\infty}, \gamma_{10}, \gamma_{12}, \ldots, \gamma_{1n}$



$\gamma_j \equiv \gamma_{1j} = \mathrm{reg}(t_1, t_j)$

C-linear monodromy relation on E_τ

$$\sum_{j=2}^{n} e^{-2\pi i(c_1+\ldots+c_j)}\,(1-e^{2\pi i c_j})\,\gamma_{1j} + (1-e^{2\pi i c_0})\,\gamma_{1\infty} = (1-e^{-2\pi i c_\infty})\,\gamma_{10}\ .$$

$\Longrightarrow$ basis of H_1 with n cycles (twisted cycles)

similar to monodromy relations
on cylinder (bordered surface)
$\Leftrightarrow$ monodromy relations on doubled surface

consider single-valued integration: $$M = \int_{E_\tau} d^2z \; T(z) \; \overline{T(z)}$$

$$d^2z \simeq A\text{-cycle} \otimes B\text{-cycle}$$

invariance under B-cycle shift:

- $\Im(c_\infty) = 0$ *Ghazouani, Pirio (2016)*

  $$\Im(c_\infty) = c_0 \Im(\tau) + \sum_{i=1}^{n} c_i \Im(z_i) = 0$$

- introduce loop momentum ℓ

  $$c_0 = -\frac{1}{2}\alpha' \, \ell q_{n+1} \quad , \quad c_k = \frac{1}{2}\alpha' \, q_k q_{n+1}$$

  $$z \to z + \tau \quad , \quad \ell \to \ell + q_{n+1}$$

Task: deform B-cycle integration to A-cycle
by means of Cauchy theorem:

$$M = \sum \int_{\gamma_{ij}} d\xi \int_{\gamma_{kl}} d\eta \; \Omega(\xi, \eta) \; T(\xi) \; T(\eta)$$

$\hookrightarrow$ double copy formula

Homological Splitting on Torus

Assume $\Re(\tau) = 0$ rectangular torus

technically: surprising
physically: cylinder $\otimes$ cylinder

generalization to $\Re(\tau) \neq 0$ in progress

$$z = \sigma^1 + i\sigma^2 \quad , \quad \begin{array}{l} \sigma^1 \in (0,1) \\ \sigma^2 \in (0,\tau_2) \end{array}$$

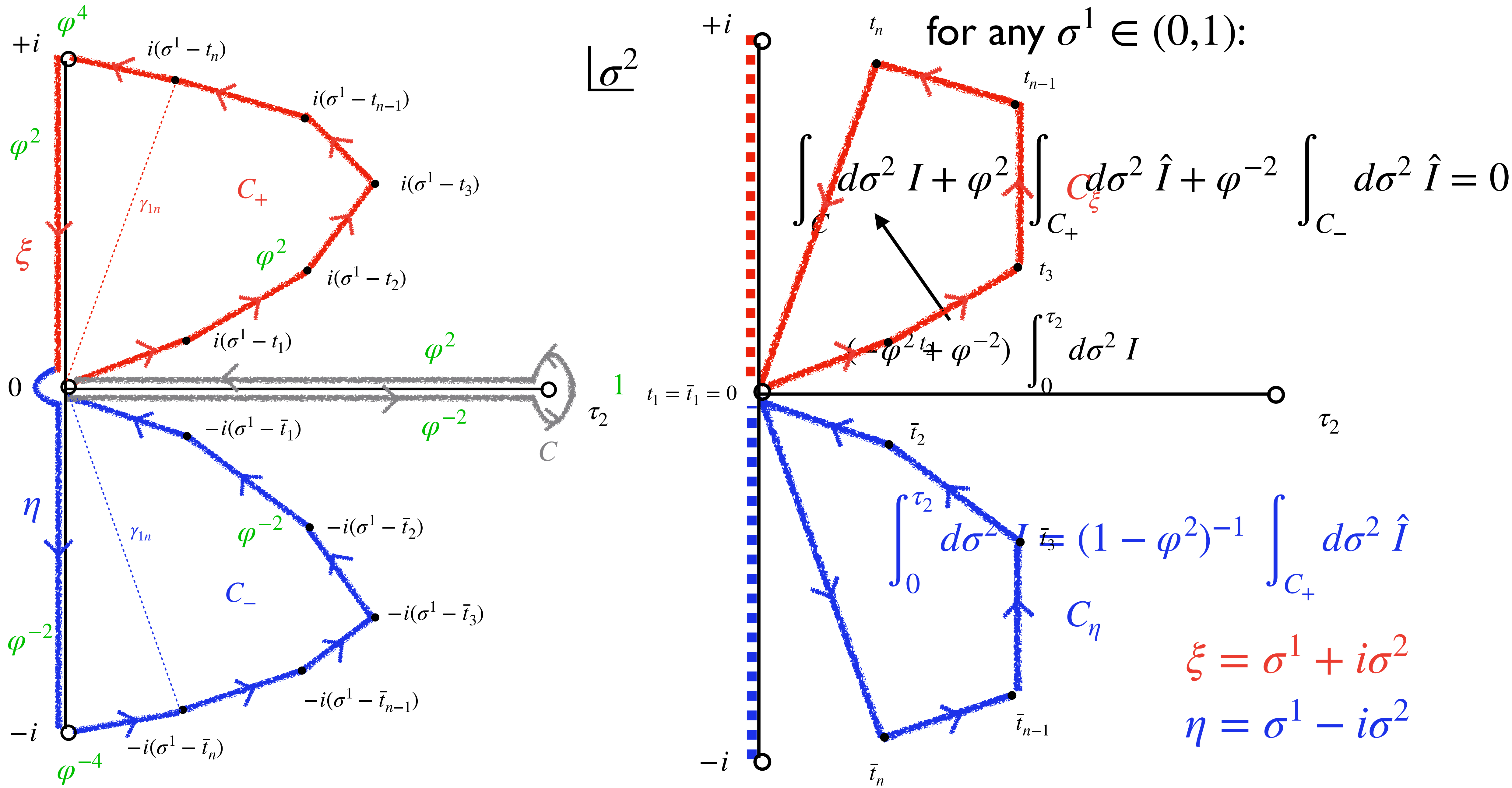
$$e^{2\pi i c_0 (z-\bar{z})} = e^{-4\pi c_0 \sigma^2} \sim \left(\frac{\theta_1(i\sigma^2 - \tau)}{\theta_1(i\sigma^2)} \right)^{2c_0}$$

$$\begin{array}{ll} (\sigma^2)^{-2c_0} & \sim \varphi^2 \\ (\sigma^2 - \tau_2)^{-2c_0} & \sim \varphi^{-2} \end{array}$$

$$\varphi = e^{-\pi i c_0}$$

consider integrand I as holomorphic function in σ^2:

$$I = T(z)\,\overline{T(z)} = \left(\frac{\theta_1(i\sigma^2 - \tau)}{\theta_1(i\sigma^2)} \right)^{2c_0} \prod_{l=1}^{n} \theta_1(\sigma^1 + i\sigma^2 - t_l)^{c_l}\, \theta_1(\sigma^1 - i\sigma^2 - \bar{t}_l)^{c_l}$$



$$M = \left(1 - e^{2\pi i c_0}\right)^{-1} \oint_{C_\xi} d\xi\, T(\xi) \oint_{C_\eta} d\eta\, \bar{T}(\eta)\, \Psi'(\xi,\eta)\, \Pi(\xi,\eta)$$

$$= \left(1 - e^{2\pi i c_0}\right)^{-1} \oint_{C_\xi} d\xi \prod_{r=1}^{n} \theta_1(\xi - t_r; \tau)^{c_r} \oint_{C_\eta} d\eta\, e^{-2\pi i c_0(\eta - \xi)}\, \Psi'(\xi,\eta) \prod_{s=1}^{n} \theta_1(\eta - \bar{t}_s; -\bar{\tau})^{c_s}\, \Pi(\xi,\eta)$$

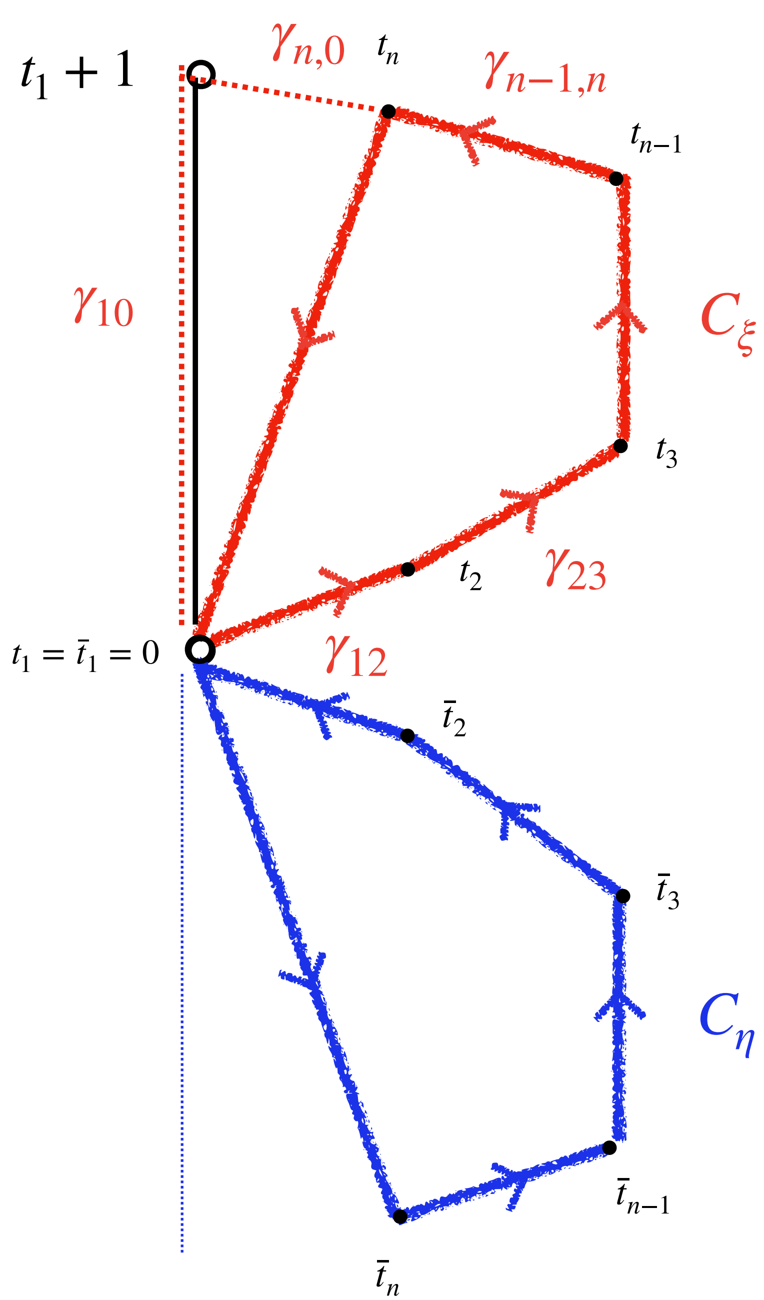
holomogical (holomorphic) splitting into two open string sectors

$$\Omega(\xi,\eta) \simeq \Psi(\xi,\eta)\ \Pi(\xi,\eta)$$

split function
describes <u>global</u> properties
of cutting torus into two cylinders

<u>local</u> properties
can be described by
intersection numbers

<u>choose n cycles:</u>



decompose cycles C_ξ, C_η w.r.t. basis of twisted cycles

$$\Pi(\xi,\eta) \simeq \langle \gamma_{ij} \otimes KN_z \,|\, \gamma_{kl}^\vee \otimes KN_z^{-1} \rangle_\omega$$

complex bulk integrals over E_τ
and <u>localize</u> near the boundary of the moduli space
describing configurations in which two or more points coalesce

$$M = \sum_{\gamma\in\Gamma} \sum_{\tilde\gamma\in\Gamma} \langle \gamma \otimes KN_z \,|\, \tilde\gamma^\vee \otimes \overline{KN_{\bar z}} \rangle \int_\gamma dz \prod_{r=1}^{n} \theta_1(z-t_r;\tau)^{c_r}$$
$$\times \int_{\tilde\gamma} d\bar z\, e^{-2\pi i c_0(\bar z - z)}\, \Psi(z,\bar z) \prod_{s=1}^{n} \overline{\theta_1(z-t_s;\tau)}^{c_s}$$

cf. double copy formula of Brown, Dupont

Twisted (co)homology associated with RW integral

generically consider periods on $X = \mathscr{M}_{0,n},\ E_{1,n}$

open string-amplitudes

periods $\int_\Delta KN_z\ \varphi$

pairing between cycle Δ and differential form φ

defined up to

$$KN^{-1}\ d(KN\ \vartheta) = d\ \ln(KN\ \vartheta) + d\vartheta$$

$$=:\ \nabla_\omega \vartheta$$

covariant derivative

local system $\mathscr{L}_\omega(c_0, c_1, \ldots, c_n) \equiv KN_z$

dual local system $\mathscr{L}^\vee_{-\omega}(c_0, c_1, \ldots, c_n) \equiv KN_z^{-1}$

Mano, Watanbe (2008)
Goto (2023)

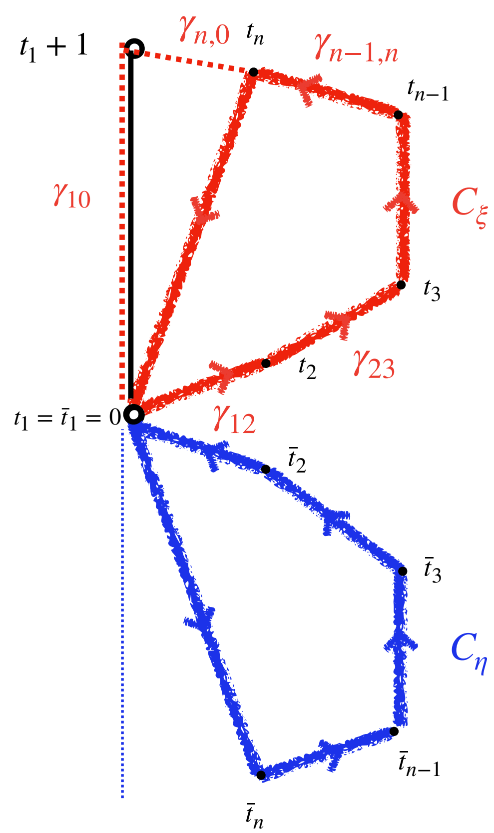
Δ: twisted homology group $H_1(X, \mathscr{L}_\omega)$ and dual $H_1(X, \mathscr{L}^\vee_{-\omega})$

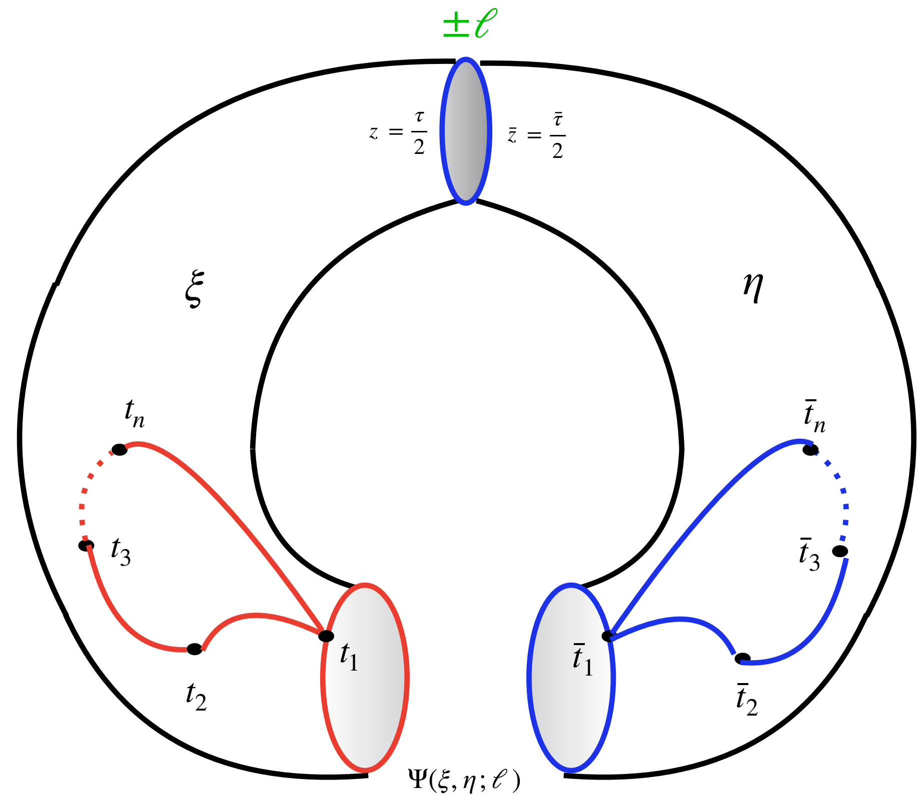
φ: twisted cohomology group $H^1(X, \nabla_\omega)$ and dual $H^1(X, \nabla^\vee_{-\omega})$

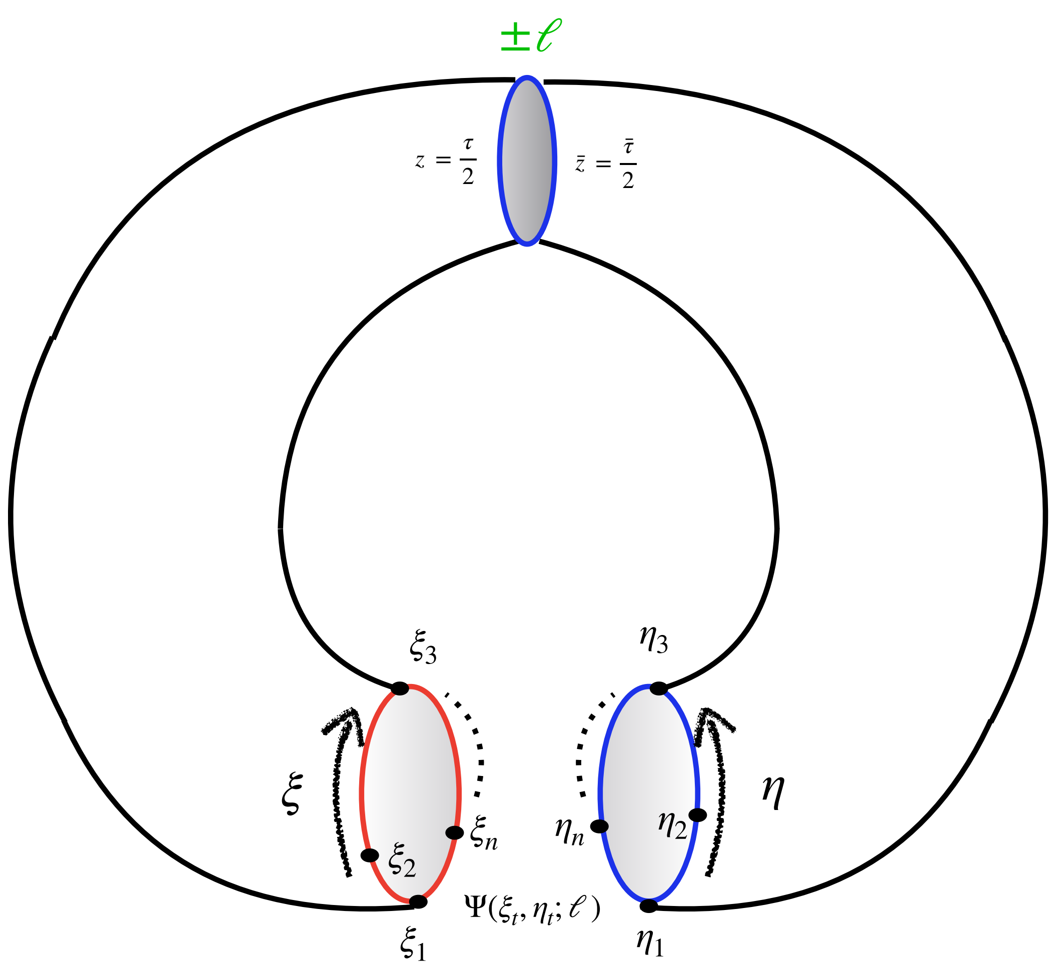
space of twisted differential forms, which are closed but not exact w.r.t. ∇_ω

intersection form $\langle\ \ \rangle:\ \ H_1(X, \mathscr{L}^\vee_\omega) \times H_1^{lf}(X, \mathscr{L}_\omega) \longrightarrow \mathbf{C}$

Multi-dimensional Complex Integrations

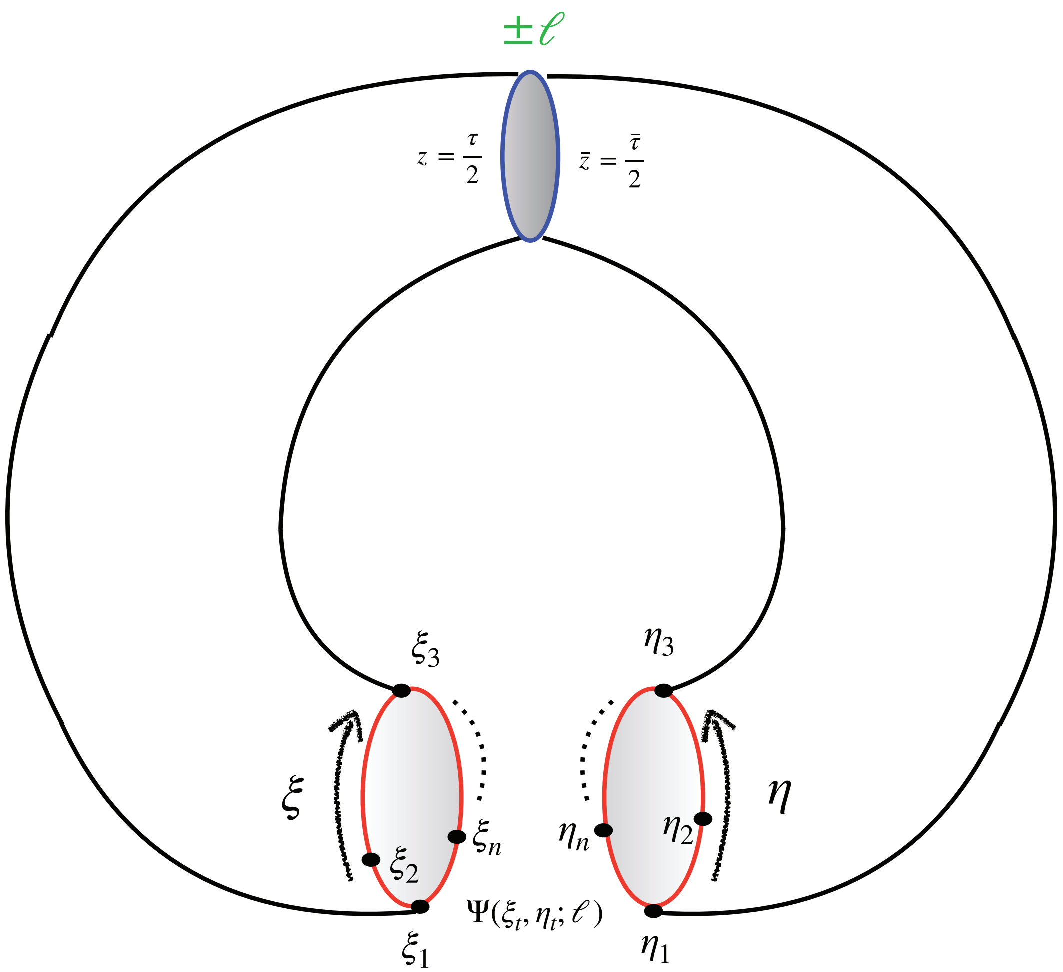






cutting along A-cycle
$Re(\tau) = 0$

Geometric Picture of One-loop KLT Relation



"non-planar (semi off-shell)
open string configuration"

torus sliced
along A-cycle

$$z = \frac{\tau}{2} + ia \quad , \quad \bar{z} = \frac{\bar{\tau}}{2} + a \ , \ a \in \mathbf{R}$$

closed string along B-cycle:

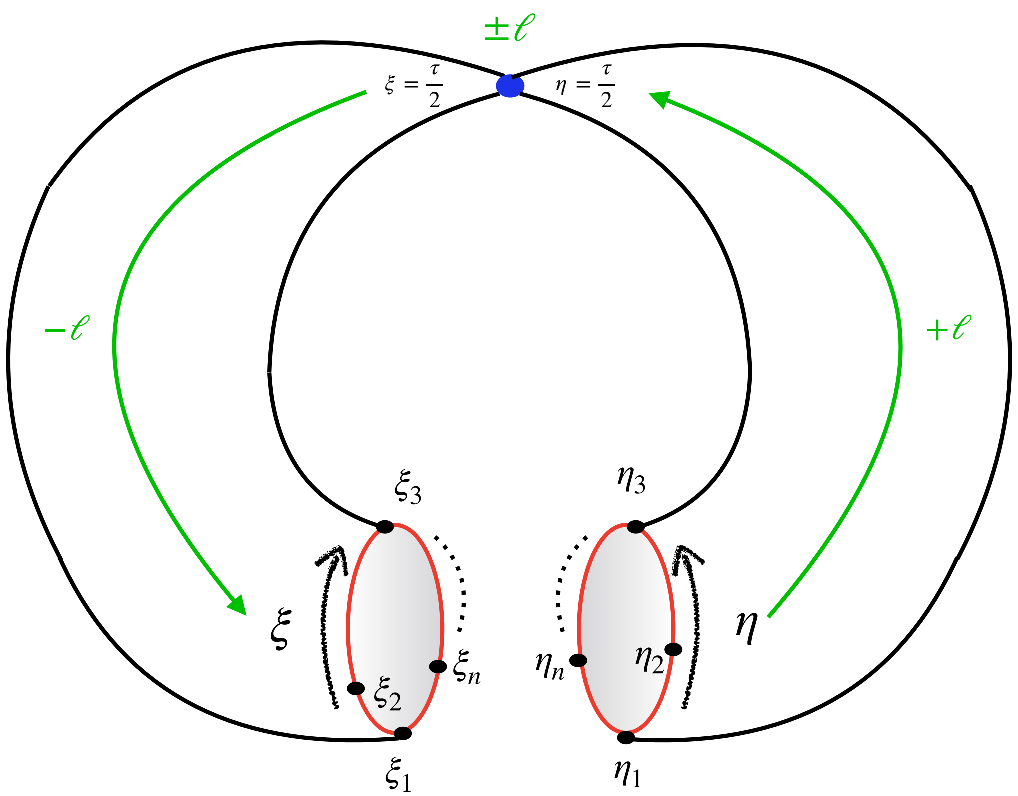
$$q_L = -\frac{1}{2}\ell \quad , \quad q_R = +\frac{1}{2}\ell$$

(Dirichlet boundary conditions)

$\Psi(\xi_t, \eta_t; l)$ = splitting function

St.St. (2022)

large complex structure limit $\tau_2 \to \infty$



closed string becomes node
connecting <u>two</u> $(n+2)-$point disk diagrams
(n open and one closed string)

fully fledged (n+2)-point
tree-level string amplitudes

$$\mathcal{M}_{n;1} = \frac{1}{2}\delta^{(d)}\Big(\sum_{i=1}^{n} q_i\Big)\int \frac{d^d\ell}{\ell^2}$$

$$\sum_{\sigma,\rho\in S_{n-1}} A_{n+2;0}(+\ell, \sigma(1,\ldots,n-1), n, -\ell)$$

$$\times\, S^{(0)}[\sigma|\rho]_\ell\, \tilde{A}_{n+2;0}(+\ell, \rho(1,\ldots,n-1), -\ell, n)$$

One-loop double-copy including α'

field-theory $\alpha' \to 0$

$$\mathcal{M}^{grav}_{n;1} = \frac{1}{2}\delta^{(d)}\Big(\sum_{i=1}^{n} q_i\Big)\int \frac{d^d\ell}{\ell^2}$$
$$\sum_{\sigma,\rho\in S_{n-1}} A^{FT}_{n+2;0}(+\ell,\sigma(1,\ldots,n-1),n,-\ell)$$
$$\times\, S^{(0)}_{FT}[\sigma|\rho]_\ell\; \tilde{A}^{FT}_{n+2;0}(+\ell,\rho(1,\ldots,n-1),-\ell,n)$$

Field-theory one-loop double-copy

involving the $n+2$-point tree-level gluon amplitudes

and the field-theory kernel

$$S_{FT}[\sigma|\rho]_\ell := \lim_{\alpha'\to 0}(\pi\alpha')^{1-n}\,\mathcal{S}[\sigma|\rho]_\ell$$

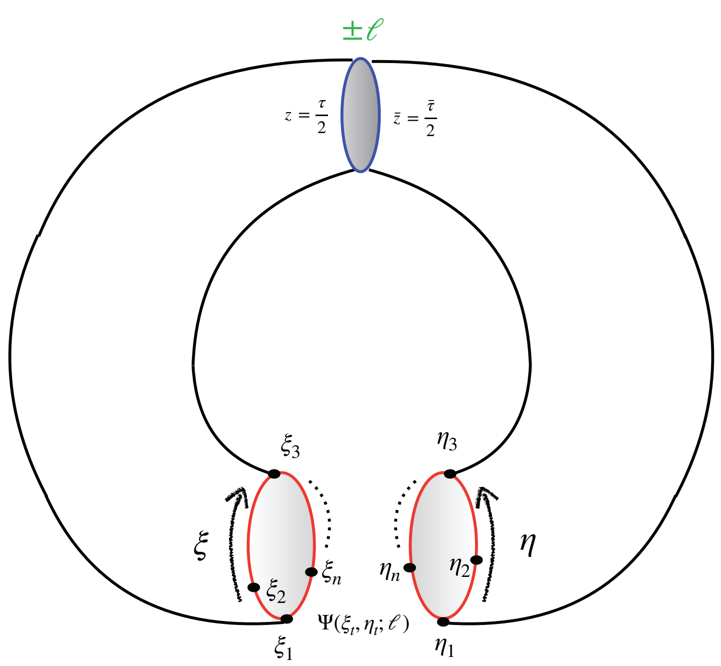
Actually: n–point open string amplitudes at genus one can be obtained from (n+2)–point open–string amplitudes at tree level by introducing auxiliary point.

Broedel, Kaderli (2019)

Remarks

$$c_0 = -\frac{1}{2}\alpha' \, \ell \, q_{n+1} \quad , \quad c_k = \frac{1}{2}\alpha' \, q_k q_{n+1}$$

$$\Re(\tau) = 0$$



cut along A-cycle

all z_i are aligned, i.e. $\Im(z_i) = a$

Bhardwaj, Pokraka, Ren, Rodriguez (2023)

invariance under B-cylce shifts

$$\Im(c_\infty) = c_0 \Im(\tau) + \sum_{i=1}^{n} c_i \Im(z_i) = 0$$

generic integration of all z_i ?

$$\Rightarrow c_0 = 0 \Rightarrow l = 0$$

then true double copy: $\Psi = 1$

$\hookrightarrow$ appears to describe a slicing condition

Concluding Remarks

- The one-loop generalization of the KLT relations for $Re(\tau) = 0$:
geometric picture: cutting torus into two cylinders

- gluing procedure for generic complex structure ?

- closed/open string relations at one-loop for
any supersymmetry, any states, any spin: massless or massive

- string theory: closed/open string correspondence
linking closed and open string amplitudes at one-loop

- field theory: proof of the field-theory DC conjecture

- going beyond $g \geq 2$?