# Concepts of Experiments at Future Colliders II

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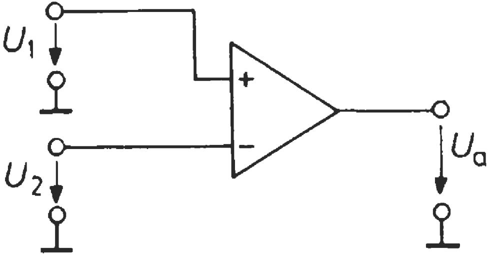
17.05.2024

# From analog to digital signals

## Operational amplifiers as comparators

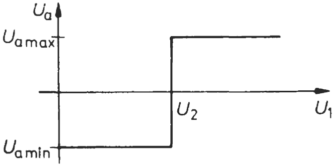
- An operational amplifier saturates when $|U_P - U_N|$ exceeds a small range of values.
- Comparators are operational amplifiers where this range has been chosen very small.




In the ideal case:

$$U_a = \begin{cases} U_{a,\text{max}} & \text{for } U_1 > U_2, \\ U_{a,\text{min}} & \text{for } U_1 < U_2. \end{cases}$$

Characteristic curve:



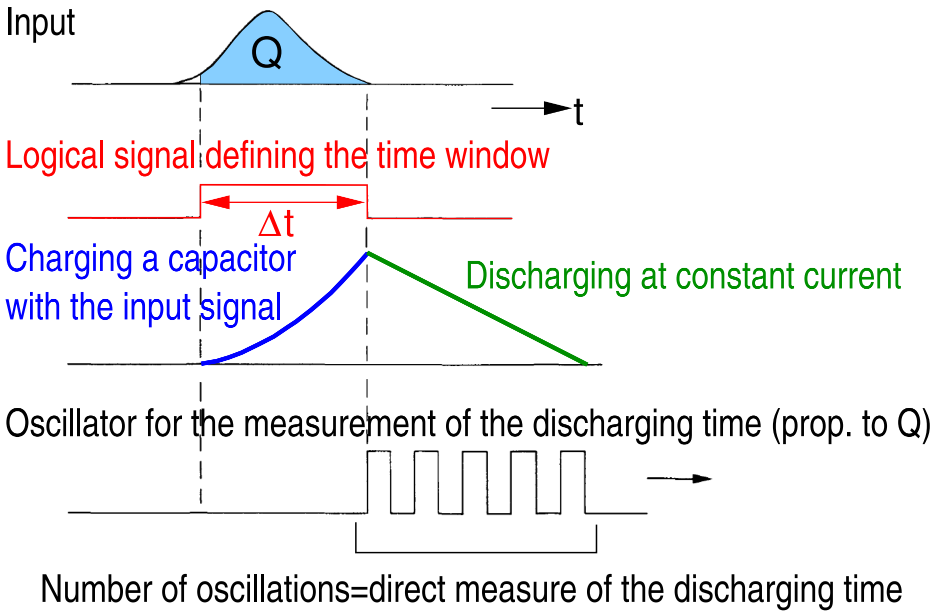
# Recapitulation of the previous lecture

Analog-to-Digital Converter (ADC)

Two basic types of analog-to-digital converters are distinguished.

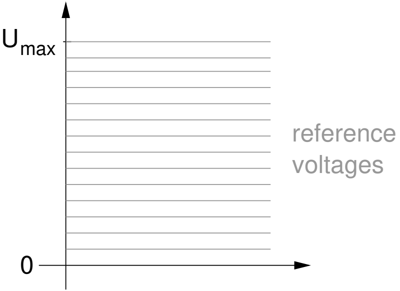
- Charge-sensing analog-to-digital converter
  Measurement of
  $$Q := \int_{t_0}^{t_0+\Delta t} I(t)\,dt$$
  and conversion of the measured value into an integer.
- Amplitude sensing analog-to-digital converter
  Measurement of the peak value of a signal $U(t)$ in the interval $[t_0, t_0 + \Delta t]$ and conversion of the measured value into an integer.

## Wilkinson's method for charge measurement



# Recapitulation of the previous lecture

## Weighing method for signal amplitude measurement




Division of the dynamic range of the analog-to-digital converter into a series of comparison voltages.
Conversion of the results of the voltage comparisons into a bit pattern.

# Recapitulation of the previous lecture

Time-to-Digital Converter (TDC)
Analog signal → Comparator → Logic signal → Time measurement

## Simplest approach to time measurement

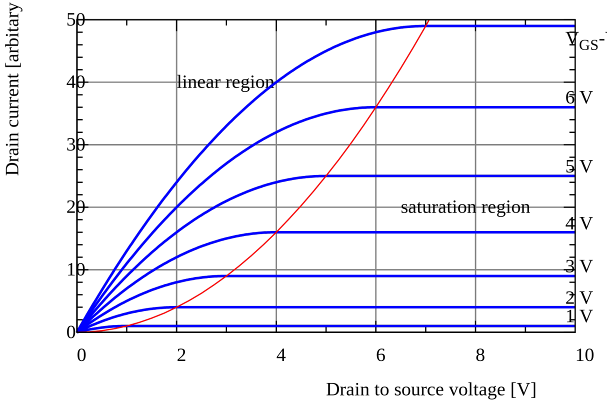
- Clock generator with a period $T$ smaller than the desired time measurement accuracy.
- Continuous counting of clock cycles. Use a counter with $n$ bits such that $2^n \cdot T >$(time intervals to be measured).
- Record at which clock cycles $n_{Start}$ and $n_{Stop}$ the start and stop signals have arrived.
  $t_{Start} - t_{Stop}$ is then measured as $n_{Start} - n_{Stop}$.
  If the counter overflows, one must use $n_{Start} - n_{Stop} + 1$.

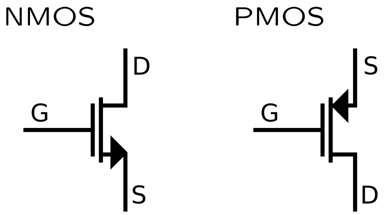
# Components for processing digital/logical signals

## Logic families

- As mentioned earlier there are different definitions of logic signal levels related to different so-called "logic families".
- Still in use today (or "popular"):
  - Transistor-transistor logic (TTL) using bipolar transistors.
  - Emitter coupled logic (ECL) using bipolar transistors.
  - Complementary metal oxide semicondutor logic (CMOS) using MOSFETs.

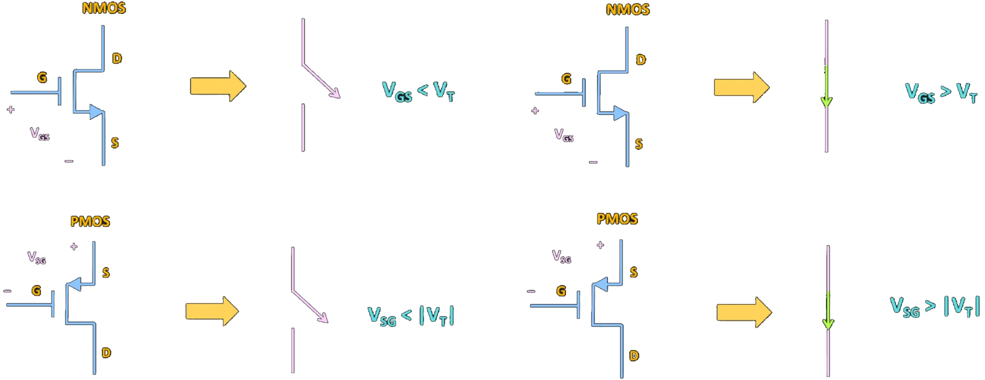
## Characteristic curve of a MOSFET



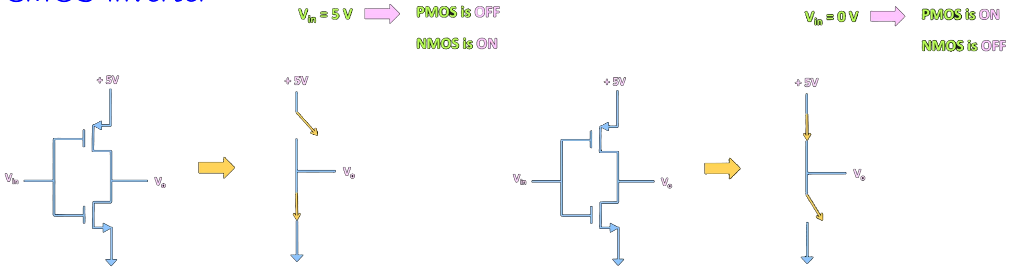




- MOSFETs are operated in saturation mode for logic gates.

## MOSFETs as switches



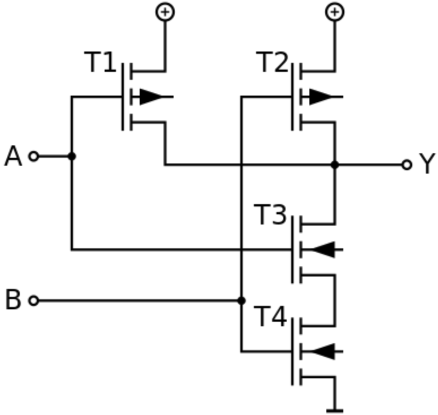
## CMOS inverter

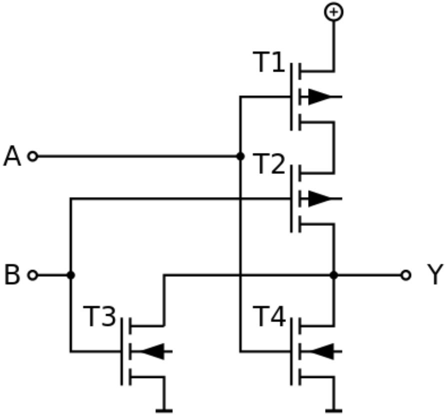


## CMOS NAND and NOR

$$Y = \overline{A\ AND\ B}$$

$$Y = \overline{A\ OR\ B}$$



## Logical basic functions

**Two States:** logical 0 and logical 1.

**Logical Basic Functions**

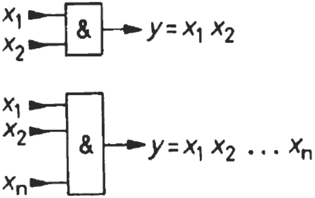
- Conjunction: $y = x_1 \wedge x_2 = x_1 \cdot x_2 = x_1 x_2$.
- Disjunction: $y = x_1 \vee x_2 = x_1 + x_2$.
- Negation: $y = \bar{x}$.

## Rules of Calculation

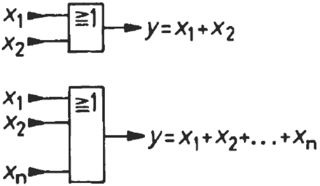
| | |
|---|---|
| Kommutatives Gesetz: | |
| $x_1 x_2 = x_2 x_1$ | $x_1 + x_2 = x_2 + x_1$ |
| Assoziatives Gesetz: | |
| $x_1(x_2 x_3) = (x_1 x_2) x_3$ | $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$ |
| Distributives Gesetz: | |
| $x_1(x_2 + x_3) = x_1 x_2 + x_1 x_3$ | $x_1 + x_2 x_3 = (x_1 + x_2)(x_1 + x_3)$ |
| Absorptionsgesetz: | |
| $x_1(x_1 + x_2) = x_1$ | $x_1 + x_1 x_2 = x_1$ |
| Tautologie: | |
| $xx = x$ | $x + x = x$ |
| Gesetz für die Negation | |
| $x\overline{x} = 0$ | $x + \overline{x} = 1$ |
| Doppelte Negation: | |
| $\overline{(\overline{x})} = x$ | |
| De Morgans Gesetz: | |
| $\overline{x_1 x_2} = \overline{x}_1 + \overline{x}_2$ | $\overline{x_1 + x_2} = \overline{x}_1 \overline{x}_2$ |
| Operationen mit 0 und 1: | |
| $x \cdot 1 = x$ | $x + 0 = x$ |
| $x \cdot 0 = 0$ | $x + 1 = 1$ |

Switching elements for logical basic functions

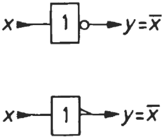
Conjunction
AND Gate




Disjunction
OR Gate




Negation
NOT Gate



Method of disjunctive normal form

To establish more complex logical functions, one can use the so-called disjunctive normal form.

$n$ input variables $x_1, \ldots, x_n$. 1 output variable $y$.

1. Set up a table listing all possible input values along with the desired output value. This table is also called a truth table.
2. Identify all rows in the truth table where $y = 1$.
3. For each of these rows, form the conjunction of all input variables; for $x_k = 1$, substitute $x_k$, otherwise $\bar{x}_k$.
4. The sought function is obtained by forming the disjunction of all found product terms.

# Recapitulation of the previous lecture

## Example of exclusive OR

### Truth Table

| Row | $x_1$ | $x_2$ | $y$ | |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | |
| 2 | 1 | 0 | 1 | $\rightarrow x_1 \cdot \bar{x}_2 =: K_2$ |
| 3 | 0 | 1 | 1 | $\rightarrow \bar{x}_1 \cdot x_2 =: K_3$ |
| 4 | 0 | 0 | 0 | |

### Result

$$y = K_2 + K_3 = (x_1 \cdot \bar{x}_2) + (\bar{x}_1 \cdot x_2).$$

# Example of a 1-of-4 decoder

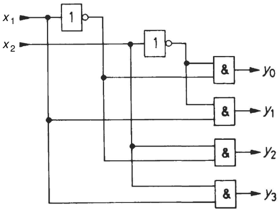
## Truth table

| Row | $x_1$ | $x_2$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 1 | 0 | 0 | 1 | 0 |
| 3 | 1 | 0 | 0 | 1 | 0 | 0 |
| 4 | 1 | 1 | 1 | 0 | 0 | 0 |

## Result

$$y_0 = \bar{x}_1 \cdot \bar{x}_2. \; y_1 = \bar{x}_1 \cdot x_2. \; y_2 = x_1 \cdot \bar{x}_2. \; y_3 = x_1 \cdot x_2.$$

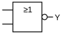
## Circuit



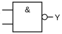
# Flip-flop as storage element

## Derived Basic Functions

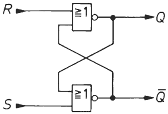
$$x_1 \text{ NOR } x_2 := \overline{x_1 + x_2} = \bar{x}_1 \cdot \bar{x}_2.$$




$$x_1 \text{ NAND } x_2 := \overline{x_1 \cdot x_2} = \bar{x}_1 + \bar{x}_2.$$




## Flip-Flop




$$Q = \overline{\bar{Q} + R}.$$

$$\bar{Q} = \overline{S + Q}.$$

| $S$ | $R$ | $Q$ | $\bar{Q}$ | |
|---|---|---|---|---|
| 0 | 0 | $Q_{-1}$ | $\bar{Q}_{-1}$ | (bisheriger Zustand) |
| 0 | 1 | 0 | 1 | (Zurücksetzen: 0\|1) |
| 1 | 0 | 1 | 0 | (Setzen: 1\|0) |
| 1 | 1 | (0) | (0) | |

Setting $S = R = 1$ results in $Q = \overline{\bar{Q} + 1} = \bar{1} = 0$ and $\bar{Q} = \overline{1 + Q} = \bar{1} = 0$.
Subsequently setting $R = 0$ and $S = 0$ simultaneously leaves the output state undefined.
$Q = \overline{\bar{Q} + 0} = \bar{\bar{Q}}$ can be 0 or 1.
$\bar{Q} = \overline{Q + 0} = \bar{Q}$ can be 0 or 1.
$\Rightarrow R = S = 1$ is generally prohibited.

# Fundamentals of statistical treatment of experimental data

# Introductory example: beam energy measurement

Example: Measurement of the energy of a monoenergetic particle beam.

Notations

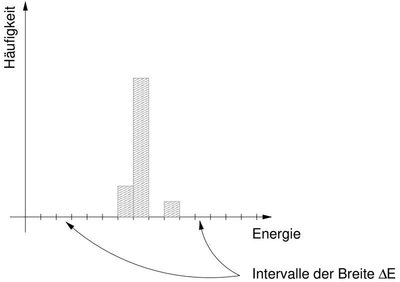
$E_S$: actual beam energy.

$N$: number of measurements of beam energy.

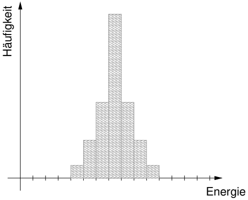
$E_k$: result of the $k$-th measurement of beam energy.

Frequency Distributions

$N$ small

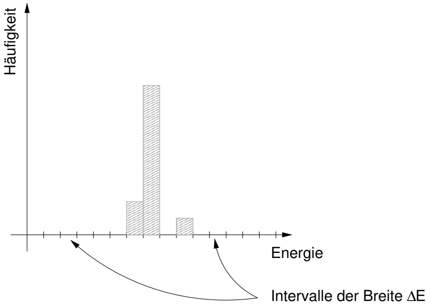



$N$ large

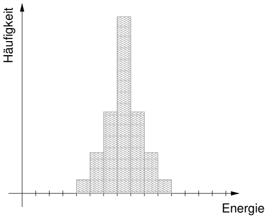


# Introductory example: beam energy measurement

$N$ small



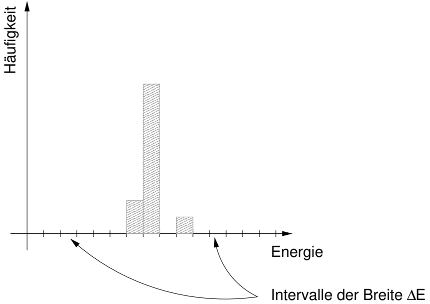

$N$ large



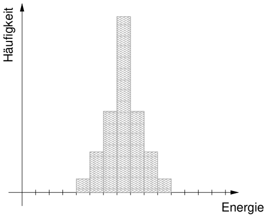

- When $N$ is large, repeating the $N$ measurements yields (nearly) the same frequency distribution.
- In the limit $N \to \infty$, the frequency distribution converges to the probability distribution for the outcome of the measurement.

# Introductory example: beam energy measurement

$N$ small




$N$ large




- The probability of measuring $E_k$ when the beam energy is $E_S$ depends on the value of $E_S$ and the measurement method. If one knows the probability function $p(E_k; E_S)$, one can determine $E_S$ from the measurement of the frequency distribution.
- In practice, $p(E_k; E_S)$ is only partially known, and one tries to infer $p(E_k; E_S)$ from the measured frequency distribution, which provides an estimate of $E_S$. In statistics, methods are employed to infer the underlying probability distributions from frequency distributions.

# Probability distributions

- A physical measurement is a random process.
- A measured quantity $x$, which represents the outcome of a random process, is called a random variable or random quantity.
- Any function of $x$ is also a random variable.
- If the random variable can only take discrete values, there is a probability for the occurrence of each of these values, which is the probability function.
- For random variables with continuous range of values, the probability density $p(x)$ replaces the probability function. Let $\Omega$ be a measurable set of possible values of $x$, whose measure is greater than zero. Then

$$\int_{\Omega} p(x)\,dx$$

is the probability of observing a value $x \in \Omega$.

# Axiomatic Definition of Probability

The mathematical field of probability theory is based on Kolmogorov's Axioms.

**Kolmogorov's Axioms**

Let $\Sigma$ denote a set of events.

1. For every event $A \in \Sigma$, the probability of the occurrence of $A$ is a real number $p(A) \in [0, 1]$.
2. The certain event $S \in \Sigma$ has probability $p(S) = 1$.
3. The probability of the union of countably many incompatible events is equal to the sum of the probabilities of the individual events. Here, events $A_k$ are incompatible if they are pairwise disjoint, i.e., $A_k \cap A_\ell = \emptyset$ for all $k \neq \ell$.

# Characteristics of probability distributions

Remark. In this section, we consider probability densities. Probability functions of discrete variables are also covered if one considers the $\delta$-distribution as a probability density.

Nomenclature. $D$: Range of values of a random variable $x = (x_1, ..., x_n)$.
$p(x)$: Probability density of $x$.
($D$ is the domain of $p$.)

Definitions

The expectation value of $x$, $E(x)$ (also $< x >$), is defined as

$$E(x) := \int_D x \cdot p(x) dx.$$

The covariance matrix $cov(x_k, x_l)$ is defined as

$$cov(x_k, x_l) :=< (x_k - < x_k >) \cdot (x_l - < x_l >) > .$$

The diagonal element $cov(x_k, x_k)$ is called the variance of $x_k$, $Var(x_k)$, and $\sqrt{Var(x_k)}$ is the standard deviation $\sigma(x_k)$.

# Expectation value of a function of a random variable

- A function $f(x)$ is also a random variable.

$$< f >= \int_D f(x)p(x)dx.$$

- If $f(x) = f(x- < x > + < x >)$ is significantly different from 0 only for small values of $|x- < x >|$, one can approximate $f(x)$ by

$$f(< x >) + \left.\frac{df}{dx}\right|_{<x>} \cdot (x- < x >)$$

Then

$$\begin{aligned} {\color{blue}< f >} &\approx \left\langle f(< x >) + \left.\frac{df}{dx}\right|_{<x>} \cdot (x- < x >)\right\rangle \\ &= < f(x) > + \left\langle \left.\frac{df}{dx}\right|_{<x>} \cdot (x- < x >)\right\rangle \\ &= f(< x >) + \left.\frac{df}{dx}\right|_{<x>} \cdot (< x > - < x >) = {\color{blue}f(< x >).} \end{aligned}$$

# Variance of a function of a random variable

Special Case: $f(x) \in |\mathrm{R}$.

$$
\begin{aligned}
Var(f) &= \langle (f - < f >)^2 \rangle = \langle [f - f(< x >)] \rangle \\
&\approx \left\langle \left[ \sum_{k=1}^{n} \frac{df}{dx_k}\bigg|_{<x>} \cdot (x_k - < x_k >) \right]^2 \right\rangle \\
&= \left\langle \left[ \sum_{k,\ell=1}^{n} \frac{df}{dx_k}\bigg|_{<x>} \frac{df}{dx_\ell}\bigg|_{<x>} \cdot (x_k - < x_k >) \cdot (x_\ell - < x_\ell >) \right] \right\rangle \\
&= \sum_{k,\ell=1}^{n} \frac{df}{dx_k}\bigg|_{<x>} \frac{df}{dx_\ell}\bigg|_{<x>} \cdot \langle (x_k - < x_k >) \cdot (x_\ell - < x_\ell >) \rangle \\
&= \sum_{k,\ell=1}^{n} \frac{df}{dx_k}\bigg|_{<x>} \frac{df}{dx_\ell}\bigg|_{<x>} \cdot cov(x_k, x_\ell),
\end{aligned}
$$

which is the well-known error propagation formula.

# Examples of important probability distributions

# The binomial distribution

- The binomial distribution gives the probability of observing $n_k$ events out of a total of $N$ events when $\nu_k$ events are expected:

$$p(n_k;\nu_k) = \binom{N}{n_k}\left(\frac{\nu_k}{N}\right)^{n_k}\left(1-\frac{\nu_k}{N}\right)^{N-\nu_k}.$$

- With $p := \frac{\nu_k}{N}$, one obtains from

$$\begin{aligned}
0 &= \frac{d}{dp}1 = \frac{d}{dp}\sum_{n_k=0}^{N}\binom{N}{n_k}p^{n_k}(1-p)^{N-n_k} \\
&= \sum_{n_k=0}^{N}\binom{N}{n_k}\left[n_k p^{n_k-1}(1-p)^{N-n_k} - (N-n_k)p^{n_k}(1-p)^{N-n_k-1}\right] \\
&= \frac{1}{p} < n_k > - \frac{1}{1-p} < N - n_k >= \left(\frac{1}{p}+\frac{1}{1-p}\right) < n_k > + \frac{N}{1-p} \\
&= \frac{1}{p(1-p)} < n_k > + \frac{N}{1-p} \Leftrightarrow < n_k >= N \cdot p = N \cdot \frac{\nu_k}{N} = \nu_k.
\end{aligned}$$

- Using the same calculation trick, one obtains $Var(n_k) = \nu_k(1-\frac{\nu_k}{N})$.