Differential Equations and Dispersion Relations for Feynman Integrals – An Application to QED Two-Point Functions

Master's Thesis Project with Professor Lorenzo Tancredi

Felix Forner

MPP, July 24, 2024

Technical University of Munich

Two-Point Functions in QED

- important building blocks in QFT
- simplest correlation functions
- new classes of functions beyond polylogarithms already at two loops

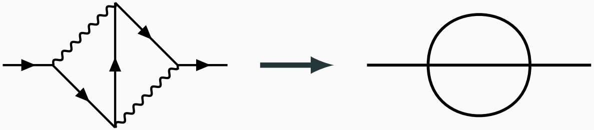
Two-Point Functions in QED

- important building blocks in QFT
- simplest correlation functions
- new classes of functions beyond polylogarithms already at two loops
  $\rightarrow$ playground to study perturbative structure of QFT
  $\rightarrow$ interesting from both physical and mathematical view

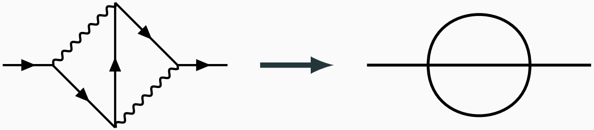
Two-Point Functions in QED

- important building blocks in QFT
- simplest correlation functions
- new classes of functions beyond polylogarithms already at two loops
  $\to$ playground to study perturbative structure of QFT
  $\to$ interesting from both physical and mathematical view
- today: electron and photon self-energy in QED up to three loops

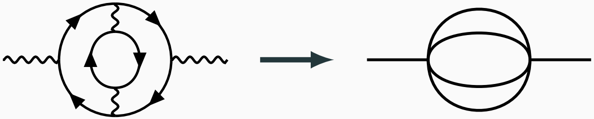
- **electron self energy at 2 loops** (first studied by Sabry in 1962): elliptic integrals, associated to a torus



- **electron self energy at 2 loops** (first studied by Sabry in 1962):
  elliptic integrals, associated to a torus



- **photon self energy at 3 loops**:
  integrals associated to a 2-(complex)-dimensional Calabi-Yau variety (K3)



⋮

Differential Equations and Dispersion Relations

Differential Equations and Dispersion Relations

- amplitude $\rightarrow$ reduction to scalar integrals
- integration by parts identities (IBPs)
  $\rightarrow$ reduction to master integrals

Differential Equations and Dispersion Relations

- amplitude $\to$ reduction to scalar integrals
- integration by parts identities (IBPs)
  $\to$ reduction to master integrals
  $\to$ differential equations for master integrals: $\frac{\mathrm{d}\vec{\mathcal{I}}}{\mathrm{d}s_i} = A(\{s_j\}, \varepsilon)\,\vec{\mathcal{I}}$
  [Gehrmann, Remiddi; arXiv: 9912329]

Differential Equations and Dispersion Relations

- amplitude $\rightarrow$ reduction to scalar integrals
- integration by parts identities (IBPs)
  $\rightarrow$ reduction to master integrals
  $\rightarrow$ differential equations for master integrals: $\frac{\mathrm{d}\vec{\mathcal{I}}}{\mathrm{d}s_i} = A(\{s_j\},\varepsilon)\,\vec{\mathcal{I}}$
  [Gehrmann, Remiddi; arXiv: 9912329]
- transform $\vec{\mathcal{I}}$ to the *canonical basis*: $\frac{\mathrm{d}\vec{\mathcal{I}}_C}{\mathrm{d}s_i} = \varepsilon A_C(\{s_j\})\,\vec{\mathcal{I}}_C$
  [Henn; arXiv: 1304.1806]

Differential Equations and Dispersion Relations

- amplitude $\rightarrow$ reduction to scalar integrals
- integration by parts identities (IBPs)
  $\rightarrow$ reduction to master integrals
  $\rightarrow$ differential equations for master integrals: $\frac{\mathrm{d}\vec{\mathcal{I}}}{\mathrm{d}s_i} = A(\{s_j\}, \varepsilon)\,\vec{\mathcal{I}}$
  [Gehrmann, Remiddi; arXiv: 9912329]
- transform $\vec{\mathcal{I}}$ to the *canonical basis*: $\frac{\mathrm{d}\vec{\mathcal{I}}_C}{\mathrm{d}s_i} = \varepsilon A_C(\{s_j\})\,\vec{\mathcal{I}}_C$
  [Henn; arXiv: 1304.1806]
- immediately gives formal solution at order ε^n in terms of iterated integrals,

$$\frac{\mathrm{d}\vec{\mathcal{I}}_C^{(n)}}{\mathrm{d}s} = A_C\,\vec{\mathcal{I}}_C^{(n-1)} \qquad \text{(up to boundary constants)}$$

Differential Equations and Dispersion Relations

- amplitude $\to$ reduction to scalar integrals
- integration by parts identities (IBPs)

  $\to$ reduction to master integrals

  $\to$ differential equations for master integrals: $\frac{\mathrm{d}\vec{\mathcal{J}}}{\mathrm{d}s_i} = A(\{s_j\},\varepsilon)\,\vec{\mathcal{J}}$

  [Gehrmann, Remiddi; arXiv: 9912329]
- transform $\vec{\mathcal{J}}$ to the *canonical basis*: $\frac{\mathrm{d}\vec{\mathcal{J}}_C}{\mathrm{d}s_i} = \varepsilon A_C(\{s_j\})\,\vec{\mathcal{J}}_C$

  [Henn; arXiv: 1304.1806]
- immediately gives formal solution at order ε^n in terms of iterated integrals,

$$\frac{\mathrm{d}\vec{\mathcal{J}}_C^{(n)}}{\mathrm{d}s} = A_C\,\vec{\mathcal{J}}_C^{(n-1)} \qquad \text{(up to boundary constants)}$$

- alternatively, rewrite with dispersion relation: $\mathcal{J}(s) = \frac{1}{2\pi i}\int_{\text{branch cut}} \mathrm{d}\sigma \frac{\mathrm{Disc}(\mathcal{J}(\sigma))}{\sigma - s}$

Differential Equations and Dispersion Relations

- amplitude $\to$ reduction to scalar integrals
- integration by parts identities (IBPs)
  $\to$ reduction to master integrals
  $\to$ differential equations for master integrals: $\frac{\mathrm{d}\vec{\mathcal{J}}}{\mathrm{d}s_i} = A(\{s_j\},\varepsilon)\,\vec{\mathcal{J}}$
  [Gehrmann, Remiddi; arXiv: 9912329]
- transform $\vec{\mathcal{J}}$ to the *canonical basis*: $\frac{\mathrm{d}\vec{\mathcal{J}}_C}{\mathrm{d}s_i} = \varepsilon A_C(\{s_j\})\,\vec{\mathcal{J}}_C$
  [Henn; arXiv: 1304.1806]
- immediately gives formal solution at order ε^n in terms of iterated integrals,

$$\frac{\mathrm{d}\vec{\mathcal{J}}_C^{(n)}}{\mathrm{d}s} = A_C\,\vec{\mathcal{J}}_C^{(n-1)} \qquad \text{(up to boundary constants)}$$

- alternatively, rewrite with dispersion relation: $\mathcal{J}(s) = \frac{1}{2\pi i}\int_{\text{branch cut}} \mathrm{d}\sigma \frac{\mathrm{Disc}(\mathcal{J}(\sigma))}{\sigma - s}$

The QED Electron and Photon Self-Energy

The QED Self-Energies up to Three Loops

	electron self-energy	photon self-energy
1 loop	polylogs	polylogs
2 loop	polylogs + elliptic integrals	polylogs
3 loop	polylogs + elliptic integrals	polylogs + integrals of K3

The QED Self-Energies up to Three Loops

	electron self-energy	photon self-energy
1 loop	polylogs	polylogs
2 loop	polylogs + elliptic integrals	polylogs
3 loop	polylogs + elliptic integrals	polylogs + integrals of K3

- No new geometries here!

The QED Self-Energies up to Three Loops

	electron self-energy	photon self-energy
1 loop	polylogs	polylogs
2 loop	polylogs + elliptic integrals	polylogs
3 loop	polylogs + elliptic integrals	polylogs + integrals of K3

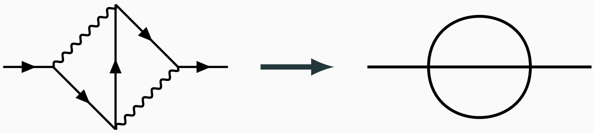
- No new geometries here!
- Let's focus on these!

The QED Self-Energies up to Three Loops

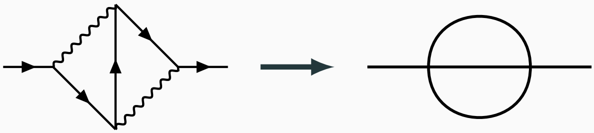
	electron self-energy	photon self-energy
1 loop	polylogs	polylogs
2 loop	polylogs + elliptic integrals	polylogs
3 loop	polylogs + elliptic integrals	polylogs + integrals of K3

- No new geometries here!
- Let's focus on these!
- not computed yet

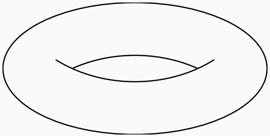
Electron Self-Energy at 2 Loops – Elliptic Integrals



Electron Self-Energy at 2 Loops – Elliptic Integrals

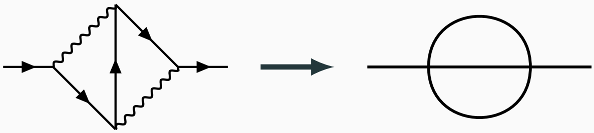


Elliptic curve
$y^2 = R_4(x)$

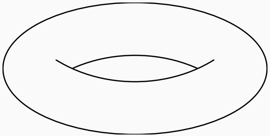


where $R_4(x) = x(x-4)((\sqrt{s}-1)^2 - x)((\sqrt{s}+1)^2 - x)$

Electron Self-Energy at 2 Loops – Elliptic Integrals



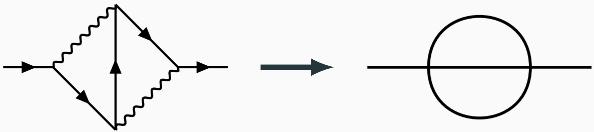
Elliptic curve
$y^2 = R_4(x)$



Elliptic integral
$$\varpi_i = \int_{\text{root } i}^{\text{root } i+1} \mathrm{d}x \frac{1}{\sqrt{R_4(x)}}$$

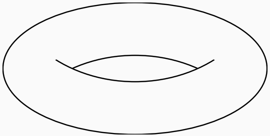
where $R_4(x) = x(x-4)((\sqrt{s}-1)^2 - x)((\sqrt{s}+1)^2 - x)$

Electron Self-Energy at 2 Loops – Elliptic Integrals



Elliptic curve

$y^2 = R_4(x)$



Elliptic integral

$$\varpi_i = \int_{\text{root } i}^{\text{root } i+1} \mathrm{d}x \frac{1}{\sqrt{R_4(x)}}$$

where $R_4(x) = x(x-4)((\sqrt{s}-1)^2 - x)((\sqrt{s}+1)^2 - x)$

- satisfy a 2nd order deq $\to$ two independent solutions $\varpi_0(s)$, $\varpi_1(s)$

Electron Self-Energy at 2 Loops – Canonical Basis

$$A_C =$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{s} & -\frac{(s+1)}{(s-1)s} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{4}{(s-1)s} & \frac{1}{s} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{2(s-6)}{(s-1)s} & -\frac{3}{s} & 0 & 0 & 0 & 0 \\
0 & \frac{2}{s} & 0 & 0 & -\frac{2(s+1)}{(s-1)s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{(3s^2-10s-9)}{2(s-9)(s-1)s} & \frac{1}{(s-9)(s-1)s\varpi_0(s)^2} & 0 \\
6\varpi_0(s) & 0 & 0 & 0 & 0 & \frac{(s+3)^4\varpi_0(s)^2}{4(s-9)(s-1)s} & -\frac{(3s^2-10s-9)}{2(s-9)(s-1)s} & 0 \\
-\frac{(s+1)}{4(s-1)s} & \frac{1}{2s} & -\frac{5}{6(s-1)} & -\frac{1}{3(s-1)} & \frac{1}{4s} & -\frac{(s-9)\varpi_0(s)}{6(s-1)} & 0 & -\frac{(s+1)}{(s-1)s}
\end{pmatrix}$$

Electron Self-Energy at 2 Loops – Canonical Basis

$$A_C =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{s} & -\frac{(s+1)}{(s-1)s} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{(s-1)s} & \frac{1}{s} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2(s-6)}{(s-1)s} & -\frac{3}{s} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{s} & 0 & 0 & -\frac{2(s+1)}{(s-1)s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{(3s^2-10s-9)}{2(s-9)(s-1)s} & \frac{1}{(s-9)(s-1)s\varpi_0(s)^2} & 0 \\ 6\varpi_0(s) & 0 & 0 & 0 & 0 & \frac{(s+3)^4\varpi_0(s)^2}{4(s-9)(s-1)s} & -\frac{(3s^2-10s-9)}{2(s-9)(s-1)s} & 0 \\ -\frac{(s+1)}{4(s-1)s} & \frac{1}{2s} & -\frac{5}{6(s-1)} & -\frac{1}{3(s-1)} & \frac{1}{4s} & -\frac{(s-9)\varpi_0(s)}{6(s-1)} & 0 & -\frac{(s+1)}{(s-1)s} \end{pmatrix}$$

- purely polylogarithmic
- sunrise
- kite

Electron Self-Energy at 2 Loops – Results

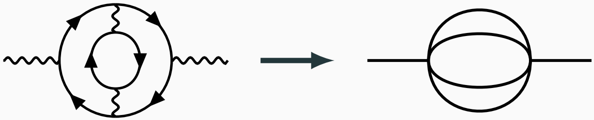
$$\Sigma_{\text{bare}} = \Sigma_{\text{bare},V}\,\not{p} + \Sigma_{\text{bare},S}\, m,$$

Electron Self-Energy at 2 Loops – Results

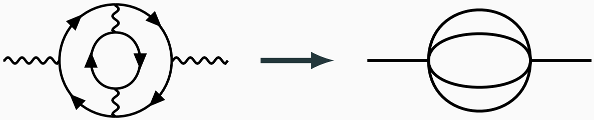
$$\Sigma_{\text{bare}} = \Sigma_{\text{bare},V}\,\not{p} + \Sigma_{\text{bare},S}\, m,$$

$$\begin{aligned}
\Sigma^{\xi=0,\varepsilon^0}_{\text{bare},V} \sim & -\frac{80\pi^2 s^2+527s^2-160\pi^2 s-2611s+80\pi^2+548}{1920\pi^2(s-1)s}+\frac{\left(4s^3-177s^2+432s+20\pi^2-259\right)G(1,s)}{480\pi^2 s^2}\\
& -\frac{(s-1)\left(s^2-3s-3\right)G(1,1,s)}{8\pi^2 s^3}+\frac{(s-1)G(0,1,s)}{4\pi^2 s}+\frac{G(0,1,1,s)}{2\pi^2 s^2}-\frac{G(1,0,1,s)}{4\pi^2 s^2}\\
& +\frac{Cl_2\left(\frac{2\pi}{3}\right)(s-9)\left(2s^3-103s^2+260s-111\right)\varpi_0(s)}{240\sqrt{3}\pi^2(s-1)s}+\frac{Cl_2\left(\frac{2\pi}{3}\right)(s-9)\left(2s^4-139s^3+425s^2-249s+153\right)\varpi_0'(s)}{240\sqrt{3}\pi^2(s-1)s}\\
& -\frac{\left(-20\sqrt{3}Cl_2\left(\frac{2\pi}{3}\right)\varpi_0(s)(1-s)^2+2s^4-139s^3+425s^2-249s+153\right)I(\varpi_0(s))}{240\pi^2(s-1)^2s^2\varpi_0(s)}-\frac{2Cl_2\left(\frac{2\pi}{3}\right)I\left(\frac{\varpi_0(s)}{s-1}\right)}{\sqrt{3}\pi^2 s^2}\\
& -\frac{1}{2160\pi^2(s-1)s}(s-9)\Big(2s^4\varpi_0'(s)-139s^3\varpi_0'(s)+2s^3\varpi_0(s)+425s^2\varpi_0'(s)-103s^2\varpi_0(s)-249s\varpi_0'(s)\\
& +153\varpi_0'(s)+260s\varpi_0(s)-111\varpi_0(s)\Big)\left(I\left(\frac{1}{s\varpi_0(s)^2},\varpi_0(s)\right)-\frac{9}{8}I\left(\frac{1}{(s-1)\varpi_0(s)^2},\varpi_0(s)\right)+\frac{1}{8}I\left(\frac{1}{(s-9)\varpi_0(s)^2},\varpi_0(s)\right)\right)\\
& +\frac{I\left(\frac{\varpi_0(s)}{s-1},\frac{1}{(s-9)\varpi_0(s)^2},\varpi_0(s)\right)}{36\pi^2 s^2}+\frac{2I\left(\frac{\varpi_0(s)}{s-1},\frac{1}{s\varpi_0(s)^2},\varpi_0(s)\right)}{9\pi^2 s^2}-\frac{I\left(\frac{\varpi_0(s)}{s-1},\frac{1}{(s-1)\varpi_0(s)^2},\varpi_0(s)\right)}{4\pi^2 s^2}\\
& +\frac{I\left(\varpi_0(s),\frac{1}{(s-1)\varpi_0(s)^2},\varpi_0(s)\right)}{32\pi^2 s^2}-\frac{I\left(\varpi_0(s),\frac{1}{s\varpi_0(s)^2},\varpi_0(s)\right)}{36\pi^2 s^2}-\frac{I\left(\varpi_0(s),\frac{1}{(s-9)\varpi_0(s)^2},\varpi_0(s)\right)}{288\pi^2 s^2}
\end{aligned}$$

Photon Self-Energy at 3 Loops



Photon Self-Energy at 3 Loops



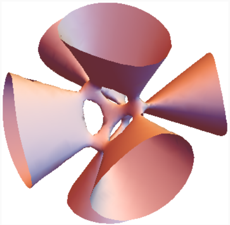
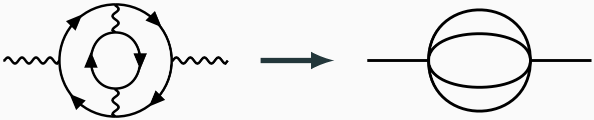


Image taken from wikipedia.org

New functions:
two-(real)-dimensional integrals associated to a two-(complex)-dimensional Calabi-Yau variety (K3 surface)

Photon Self-Energy at 3 Loops



New functions:
two-(real)-dimensional integrals associated to a two-(complex)-dimensional Calabi-Yau variety (K3 surface)

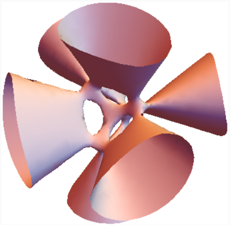
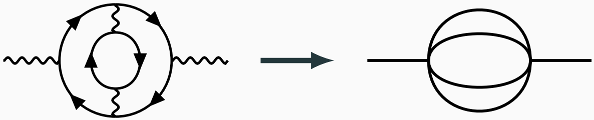


Image taken from wikipedia.org

- satisfy a 3rd order deq $\rightarrow$ three independent solutions $\omega_0(s)$, $\omega_1(s)$, $\omega_2(s)$

Photon Self-Energy at 3 Loops



New functions:
two-(real)-dimensional integrals associated to a two-(complex)-dimensional Calabi-Yau variety (K3 surface)

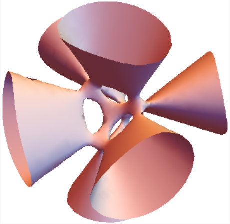


Image taken from wikipedia.org

- satisfy a 3rd order deq $\rightarrow$ ~~three~~ independent solutions $\omega_0(s)$, $\omega_1(s)$, $\omega_2(s)$
- special symmetry two

Photon Self-Energy at 3 Loops – Overview and Outlook

- canonical basis (36 masters) ✔
  - use Baikov representation, leading singularities, Feynman parameters,...
  - canonical deq is not fully in dlog-form here!

($\rightarrow$ backup slides)

Photon Self-Energy at 3 Loops – Overview and Outlook

- canonical basis (36 masters) ✔
    - use Baikov representation, leading singularities, Feynman parameters,...
    - canonical deq is not fully in dlog-form here!

  ($\rightarrow$ backup slides)
- compute boundary constants ✔

Photon Self-Energy at 3 Loops – Overview and Outlook

- canonical basis (36 masters) ✔
  - use Baikov representation, leading singularities, Feynman parameters,...
  - canonical deq is not fully in dlog-form here!

  ($\rightarrow$ backup slides)
- compute boundary constants ✔
- analytic continuation

Photon Self-Energy at 3 Loops – Overview and Outlook

- canonical basis (36 masters) ✔
  - use Baikov representation, leading singularities, Feynman parameters,...
  - canonical deq is not fully in dlog-form here!

  ($\rightarrow$ backup slides)
- compute boundary constants ✔
- analytic continuation
- compare results with and without dispersive representations

Photon Self-Energy at 3 Loops – Overview and Outlook

- canonical basis (36 masters) ✔
  - use Baikov representation, leading singularities, Feynman parameters,...
  - canonical deq is not fully in dlog-form here!

  ($\rightarrow$ backup slides)
- compute boundary constants ✔
- analytic continuation
- compare results with and without dispersive representations
- compute full, renormalized propagator

Conclusion

- differential equations as state-of-the-art method for precision calculations

Conclusion

- differential equations as state-of-the-art method for precision calculations
- QED self-energies as a playground to study the mathematical structures in perturbative QFT

Conclusion

- differential equations as state-of-the-art method for precision calculations
- QED self-energies as a playground to study the mathematical structures in perturbative QFT
- interesting already at two loops

Conclusion

- differential equations as state-of-the-art method for precision calculations
- QED self-energies as a playground to study the mathematical structures in perturbative QFT
- interesting already at two loops
- even more interesting at three loops (new functions beyond modular forms)

Conclusion

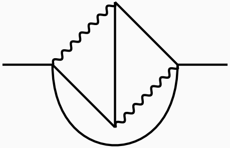
- differential equations as state-of-the-art method for precision calculations
- QED self-energies as a playground to study the mathematical structures in perturbative QFT
- interesting already at two loops
- even more interesting at three loops (new functions beyond modular forms)
- dispersion relations $\rightarrow$ alternative analytic representations for iterated integrals

Thank you!

Backup Slides

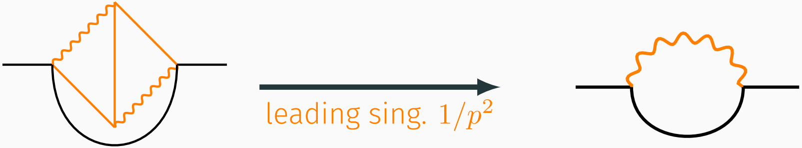
Canonical Basis – Polylogarithmic Case

- crucial to start with ’good’ master integrals
- use Baikov representation to find masters …
  - with constant leading singularities
  - that are integrals of dlog-forms on the maximal cut
- sometimes similar analysis with Feynman parameters is useful
- ’Building block method’



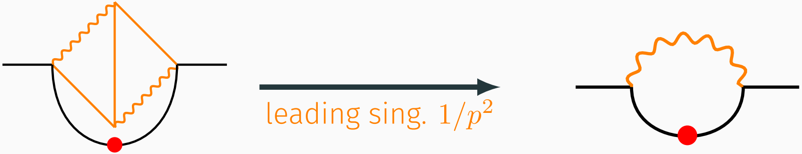
Canonical Basis – Polylogarithmic Case

- crucial to start with 'good' master integrals
- use Baikov representation to find masters ...
  - with constant leading singularities
  - that are integrals of dlog-forms on the maximal cut
- sometimes similar analysis with Feynman parameters is useful
- 'Building block method'



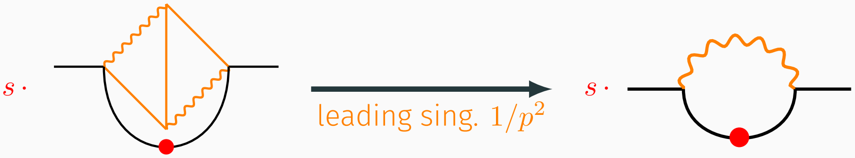
Canonical Basis – Polylogarithmic Case

- crucial to start with ’good’ master integrals
- use Baikov representation to find masters ...
    - with constant leading singularities
    - that are integrals of dlog-forms on the maximal cut
- sometimes similar analysis with Feynman parameters is useful
- ’Building block method’



Canonical Basis – Polylogarithmic Case

- crucial to start with 'good' master integrals
- use Baikov representation to find masters ...
  - with constant leading singularities
  - that are integrals of dlog-forms on the maximal cut
- sometimes similar analysis with Feynman parameters is useful
- 'Building block method'



Canonical Basis Beyond Polylogarithms

Example: 3-loop banana on the maximal cut [Görges et al.; arXiv: 2305.14090]

- split the fundamental solution matrix

$$\begin{pmatrix} \omega_0(s) & \omega_1(s) & \omega_2(s) \\ \omega_0'(s) & \omega_1'(s) & \omega_2'(s) \\ \omega_0''(s) & \omega_1''(s) & \omega_2''(s) \end{pmatrix} = W_{ss} \cdot W_u = W_{ss} \cdot \begin{pmatrix} 1 & \frac{\omega_1}{\omega_0} & \frac{1}{2}\left(\frac{\omega_1}{\omega_0}\right)^2 \\ 0 & 1 & \frac{\omega_1}{\omega_0} \\ 0 & 0 & 1 \end{pmatrix}$$

- rotate with W_{ss}^{-1}
- remove derivatives $\omega_0'(s), \omega_0''(s)$
- extra rotations with iterated integrals necessary:

$$G_1(s) \equiv -\int_0^s \mathrm{d}u \frac{2(u-8)(u+8)^3 \omega_0(u)^2}{(u^2-20u+64)^2}, \qquad G_2(s) \equiv \int_0^s \mathrm{d}u \frac{G_1(u)}{u\sqrt{4-u}\sqrt{16-u}\,\omega_0(u)}$$

One more of them for the couplings to the banana!

Dispersion Relations

$$f(z_*) = \frac{1}{2\pi i}\int\limits_{\gamma_{z_*}} \mathrm{d}z \frac{f(z)}{z - z_*} = \frac{1}{2\pi i}\int\limits_{\Gamma} \mathrm{d}z \frac{f(z)}{z - z_*}$$

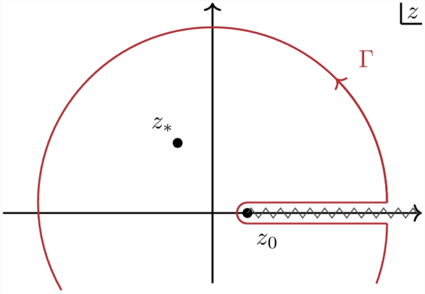


Figure taken from [Britto et al.; arXiv: 2402.19415]

Dispersion Relations

$$f(z_*) = \frac{1}{2\pi i}\int_{\gamma_{z_*}} \mathrm{d}z \frac{f(z)}{z - z_*} = \frac{1}{2\pi i}\int_{\Gamma} \mathrm{d}z \frac{f(z)}{z - z_*} = \frac{1}{2\pi i}\int_{z_0}^{\infty} \mathrm{d}z \frac{\mathrm{Disc} f(z)}{z - z_*}$$

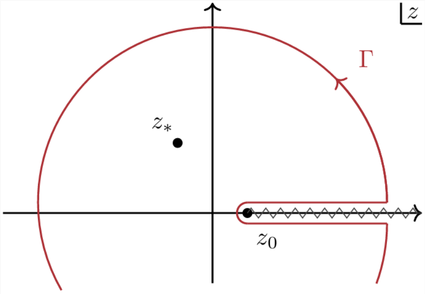
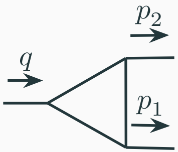


Figure taken from [Britto et al.; arXiv: 2402.19415]

The One-Loop Triangle – A Simple Example

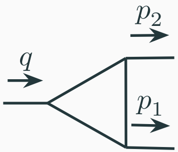
Consider



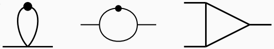
with internal lines of equal mass $m \equiv 1, \quad p_1^2 = p_2^2 = 0, \quad q^2 \equiv s$ and $d = 4 - 2\varepsilon$

The One-Loop Triangle – A Simple Example

Consider

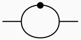
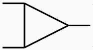


with internal lines of equal mass $m \equiv 1, \quad p_1^2 = p_2^2 = 0, \quad q^2 \equiv s$ and $d = 4 - 2\varepsilon$

- define $\vec{\mathcal{J}}(s) \equiv \Big($  $\Big)^\top$

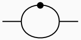
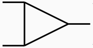
$$\rightarrow \frac{\mathrm{d}\vec{\mathcal{J}}}{\mathrm{d}s} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\varepsilon}{(s-4)s} & -\frac{s\varepsilon+s-2}{(s-4)s} & 0 \\ 0 & \frac{1}{s} & -\frac{1}{s} \end{pmatrix} \vec{\mathcal{J}} \equiv A\,\vec{\mathcal{J}}$$

The One-Loop Triangle – A Simple Example

- canonical basis: $\vec{\mathcal{J}}_C(s) \equiv \Big($  $\sqrt{-s}\sqrt{4-s}$  εs  $\Big)^\top$

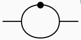
$$\rightarrow \frac{\mathrm{d}\vec{\mathcal{J}}_C}{\mathrm{d}s} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{-s}\sqrt{4-s}} & \frac{1}{4-s} & 0 \\ 0 & \frac{1}{\sqrt{-s}\sqrt{4-s}} & 0 \end{pmatrix} \vec{\mathcal{J}}_C \equiv \varepsilon A_C \, \vec{\mathcal{J}}_C$$

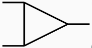
The One-Loop Triangle – A Simple Example

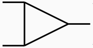
- canonical basis: $\vec{\mathcal{J}}_C(s) \equiv \Big($  $\sqrt{-s}\sqrt{4-s}$  εs  $\Big)^\top$

$$\rightarrow \frac{\mathrm{d}\vec{\mathcal{J}}_C}{\mathrm{d}s} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{-s}\sqrt{4-s}} & \frac{1}{4-s} & 0 \\ 0 & \frac{1}{\sqrt{-s}\sqrt{4-s}} & 0 \end{pmatrix} \vec{\mathcal{J}}_C \equiv \varepsilon A_C \, \vec{\mathcal{J}}_C$$

- define $\mathrm{Tad}_{\bullet} \equiv 1$ and use regularity conditions $\rightarrow$ to leading order in ε:

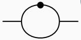
 $$\Big|_C^{(1)}(s) = \int_0^s \mathrm{d}s' \, \frac{1}{\sqrt{-s'}\sqrt{4-s'}},$$

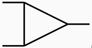
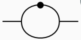
 $$\Big|_C^{(2)}(s) = \int_0^s \mathrm{d}s' \, \frac{1}{\sqrt{-s'}\sqrt{4-s'}}$$  $$\Big|_C^{(1)}(s')$$

- canonical basis: $\vec{\mathcal{J}}_C(s) \equiv \Big($  $\sqrt{-s}\sqrt{4-s}$  εs  $\Big)^\top$

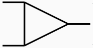
$$\rightarrow \frac{\mathrm{d}\vec{\mathcal{J}}_C}{\mathrm{d}s} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{-s}\sqrt{4-s}} & \frac{1}{4-s} & 0 \\ 0 & \frac{1}{\sqrt{-s}\sqrt{4-s}} & 0 \end{pmatrix} \vec{\mathcal{J}}_C \equiv \varepsilon A_C \, \vec{\mathcal{J}}_C$$

  - define $\mathrm{Tad}_\bullet \equiv 1$ and use regularity conditions $\rightarrow$ to leading order in ε:

 $${}^{(1)}_{C}(s) = \int_0^s \mathrm{d}s' \, \frac{1}{\sqrt{-s'}\sqrt{4-s'}},$$

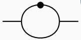
 $${}^{(2)}_{C}(s) = \int_0^s \mathrm{d}s' \, \frac{1}{\sqrt{-s'}\sqrt{4-s'}}$$  $${}^{(1)}_{C}(s') = \int_0^s \mathrm{d}s'$$  $${}^{(1)}(s')$$

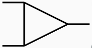
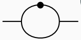
The One-Loop Triangle – A Simple Example

- canonical basis: $\vec{\mathcal{J}}_C(s) \equiv \Big($  $\sqrt{-s}\sqrt{4-s}$  εs  $\Big)^{\top}$

$$\rightarrow \frac{\mathrm{d}\vec{\mathcal{J}}_C}{\mathrm{d}s} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{-s}\sqrt{4-s}} & \frac{1}{4-s} & 0 \\ 0 & \frac{1}{\sqrt{-s}\sqrt{4-s}} & 0 \end{pmatrix} \vec{\mathcal{J}}_C \equiv \varepsilon A_C \vec{\mathcal{J}}_C$$

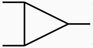
- define $\text{Tad}_{\bullet} \equiv 1$ and use regularity conditions $\rightarrow$ to leading order in ε:

 $$\Big._C^{(1)}(s) = \int_0^s \mathrm{d}s' \, \frac{1}{\sqrt{-s'}\sqrt{4-s'}},$$

 $$\Big._C^{(2)}(s) = \int_0^s \mathrm{d}s' \, \frac{1}{\sqrt{-s'}\sqrt{4-s'}}$$  $$\Big._C^{(1)}(s') = \int_0^s \mathrm{d}s'$$  $$\Big.^{(1)}(s')$$

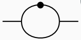
$$= \int_0^s \mathrm{d}s' \, \frac{1}{2\pi i} \int_4^\infty \mathrm{d}\sigma \, \frac{1}{\sigma - s'} \, \mathrm{Disc}\Big($$  $$\Big.^{(1)}(\sigma)\Big)$$

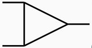
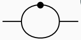
The One-Loop Triangle – A Simple Example

- canonical basis: $\vec{\mathcal{J}}_C(s) \equiv \Big($  $\sqrt{-s}\sqrt{4-s}$  εs  $\Big)^{\top}$

$$\rightarrow \frac{\mathrm{d}\vec{\mathcal{J}}_C}{\mathrm{d}s} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{-s}\sqrt{4-s}} & \frac{1}{4-s} & 0 \\ 0 & \frac{1}{\sqrt{-s}\sqrt{4-s}} & 0 \end{pmatrix} \vec{\mathcal{J}}_C \equiv \varepsilon A_C \vec{\mathcal{J}}_C$$

- define $\text{Tad}_{\bullet} \equiv 1$ and use regularity conditions $\rightarrow$ to leading order in ε:

 $\displaystyle {}^{(1)}_{C}(s) = \int_0^s \mathrm{d}s' \frac{1}{\sqrt{-s'}\sqrt{4-s'}},$

 $\displaystyle {}^{(2)}_{C}(s) = \int_0^s \mathrm{d}s' \frac{1}{\sqrt{-s'}\sqrt{4-s'}}$  $\displaystyle {}^{(1)}_{C}(s') = \int_0^s \mathrm{d}s'$  $\displaystyle {}^{(1)}(s')$

$\displaystyle = \frac{1}{2\pi i} \int_4^{\infty} \mathrm{d}\sigma \log\left(1 - \frac{s}{\sigma}\right) \text{Disc}\Big($  $\displaystyle {}^{(1)}(\sigma)\Big)$

Electron Self-Energy at 2 Loops – Results

$$
\begin{aligned}
\Sigma^{\xi=0,\varepsilon^0}_{\text{bare},V} \supset & -\frac{\left(-20\sqrt{3}Cl_2\left(\frac{2\pi}{3}\right)(1-s)^2\right)I(\varpi_0(s))}{240\pi^2(s-1)^2s^2} - \frac{2Cl_2\left(\frac{2\pi}{3}\right)I\left(\frac{\varpi_0(s)}{s-1}\right)}{\sqrt{3}\pi^2s^2} \\
& + \frac{I\left(\frac{\varpi_0(s)}{s-1}, \frac{1}{(s-9)\varpi_0(s)^2}, \varpi_0(s)\right)}{36\pi^2s^2} + \frac{2I\left(\frac{\varpi_0(s)}{s-1}, \frac{1}{s\varpi_0(s)^2}, \varpi_0(s)\right)}{9\pi^2s^2} - \frac{I\left(\frac{\varpi_0(s)}{s-1}, \frac{1}{(s-1)\varpi_0(s)^2}, \varpi_0(s)\right)}{4\pi^2s^2} \\
& + \frac{I\left(\varpi_0(s), \frac{1}{(s-1)\varpi_0(s)^2}, \varpi_0(s)\right)}{32\pi^2s^2} - \frac{I\left(\varpi_0(s), \frac{1}{s\varpi_0(s)^2}, \varpi_0(s)\right)}{36\pi^2s^2} - \frac{I\left(\varpi_0(s), \frac{1}{(s-9)\varpi_0(s)^2}, \varpi_0(s)\right)}{288\pi^2s^2}
\end{aligned}
$$

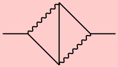
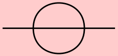
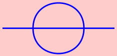
Electron Self-Energy at 2 Loops – Results

$$\Sigma^{\xi=0,\varepsilon^0}_{\text{bare},V} \supset -\frac{\left(-20\sqrt{3}Cl_2\left(\frac{2\pi}{3}\right)(1-s)^2\right)I(\varpi_0(s))}{240\pi^2(s-1)^2s^2} - \frac{2Cl_2\left(\frac{2\pi}{3}\right)I\left(\frac{\varpi_0(s)}{s-1}\right)}{\sqrt{3}\pi^2s^2}$$

$$+\frac{I\left(\frac{\varpi_0(s)}{s-1}, \frac{1}{(s-9)\varpi_0(s)^2}, \varpi_0(s)\right)}{36\pi^2s^2} + \frac{2I\left(\frac{\varpi_0(s)}{s-1}, \frac{1}{s\varpi_0(s)^2}, \varpi_0(s)\right)}{9\pi^2s^2} - \frac{I\left(\frac{\varpi_0(s)}{s-1}, \frac{1}{(s-1)\varpi_0(s)^2}, \varpi_0(s)\right)}{4\pi^2s^2}$$

$$+\frac{I\left(\varpi_0(s), \frac{1}{(s-1)\varpi_0(s)^2}, \varpi_0(s)\right)}{32\pi^2s^2} - \frac{I\left(\varpi_0(s), \frac{1}{s\varpi_0(s)^2}, \varpi_0(s)\right)}{36\pi^2s^2} - \frac{I\left(\varpi_0(s), \frac{1}{(s-9)\varpi_0(s)^2}, \varpi_0(s)\right)}{288\pi^2s^2}$$

$=$

 $\supset \int \mathrm{d}s\, f(s)$  $(s) = \frac{1}{2\pi i}\int \mathrm{d}\sigma\, \text{Disc}\Big($  $(\sigma)\Big) \int \mathrm{d}s \frac{f(s)}{\sigma - s}$

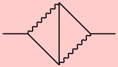
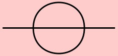
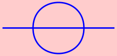
Electron Self-Energy at 2 Loops – Results

$$\Sigma^{\xi=0,\varepsilon^0}_{\text{bare},V} \supset -\frac{\left(-20\sqrt{3}Cl_2\left(\frac{2\pi}{3}\right)(1-s)^2\right)I(\varpi_0(s))}{240\pi^2(s-1)^2s^2} - \frac{2Cl_2\left(\frac{2\pi}{3}\right)I\left(\frac{\varpi_0(s)}{s-1}\right)}{\sqrt{3}\pi^2s^2}$$

$$+\frac{I\left(\frac{\varpi_0(s)}{s-1},\frac{1}{(s-9)\varpi_0(s)^2},\varpi_0(s)\right)}{36\pi^2s^2} + \frac{2I\left(\frac{\varpi_0(s)}{s-1},\frac{1}{s\varpi_0(s)^2},\varpi_0(s)\right)}{9\pi^2s^2} - \frac{I\left(\frac{\varpi_0(s)}{s-1},\frac{1}{(s-1)\varpi_0(s)^2},\varpi_0(s)\right)}{4\pi^2s^2}$$

$$+\frac{I\left(\varpi_0(s),\frac{1}{(s-1)\varpi_0(s)^2},\varpi_0(s)\right)}{32\pi^2s^2} - \frac{I\left(\varpi_0(s),\frac{1}{s\varpi_0(s)^2},\varpi_0(s)\right)}{36\pi^2s^2} - \frac{I\left(\varpi_0(s),\frac{1}{(s-9)\varpi_0(s)^2},\varpi_0(s)\right)}{288\pi^2s^2}$$

$$= -\frac{1}{24\sqrt{3}\pi s^2}\int_9^\infty \mathrm{d}\sigma\,\tilde{\varpi}_0(\sigma)\,\frac{\sigma G(\sigma,s)-9G(\sigma,s)+8G(1,s)}{\sigma-1}$$

 $\supset \int \mathrm{d}s\, f(s)$  $(s) = \frac{1}{2\pi i}\int \mathrm{d}\sigma\,$ Disc $\Big($  $(\sigma)\Big) \int \mathrm{d}s\,\frac{f(s)}{\sigma-s}$

Very compact!

Modular Forms

For the electron self-energy at two loops:

- introduce the *modular parameter*

$$\tau(s) \equiv \frac{\varpi_1(s)}{\varpi_0(s)} \quad \to \quad s = s(\tau)$$

- $\varpi_0(s)$, $\varpi_1(s)$ only defined up to *modular transformations*

$$\begin{pmatrix} \varpi_0 \\ \varpi_1 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \varpi_0 \\ \varpi_1 \end{pmatrix}, \qquad \tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathrm{Z})$$

- $s(\tau)$ is invariant for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6) \subset \mathrm{SL}(2, \mathrm{Z})$
- a *modular form* $f(\tau)$ *of weight* n *for* $\Gamma_1(6)$ is defined by

$$f(\tau) \longrightarrow f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6)$$

- basis functions for space of modular forms for $\Gamma_1(6)$:

$$f_{n,p}(\tau) \equiv \varpi_0(s(\tau))^n s(\tau)^p, \qquad 0 \leq p \leq n$$

- iterated integrals

$$\begin{aligned} \mathcal{J}\left(\begin{smallmatrix} n_1 & \ldots & n_k \\ p_1 & \ldots & p_k \end{smallmatrix}; \tau\right) &\equiv \int_{i\infty}^{\tau} \mathrm{d}\tau' \, f_{n_1,p_1}(\tau') \, \mathcal{J}\left(\begin{smallmatrix} n_2 & \ldots & n_k \\ p_2 & \ldots & p_k \end{smallmatrix}; \tau'\right) \\ &= \int_0^s \mathrm{d}s' \, \frac{\varpi_0(s')^{n_1-2}(s')^{p_1}}{s'(s'-1)(s'-9)} \, \mathcal{J}\left(\begin{smallmatrix} n_2 & \ldots & n_k \\ p_2 & \ldots & p_k \end{smallmatrix}; \tau'(s')\right) \end{aligned}$$

- integration kernels can only have poles at $s' = 0,\ 1,\ 9,$
  i.e., the singular points of the geometry!

The 3-Loop Banana as a Symmetric Square

- homogeneous solutions of the 3-loop banana satisfy a 3rd order deq,

$$\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3} + c_2(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2} + c_1(s)\frac{\mathrm{d}}{\mathrm{d}s} + c_0(s)\right)\omega_{0,1,2}(s) \equiv \mathcal{L}_3(s)\,\omega_{0,1,2}(s) = 0$$

- but they are the different products of just two functions,

$$\omega_0(s) = \varpi_0^2(s), \quad \omega_1(s) = \varpi_1^2(s), \quad \omega_2(s) = \varpi_0(s)\varpi_1(s)$$

- these two functions obey a second order differential equation,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2} + a_1(s)\frac{\mathrm{d}}{\mathrm{d}s} + a_0(s)\right)\varpi_{0,1}(s) \equiv \mathcal{L}_2(s)\,\varpi_{0,1}(s) = 0,$$

  $\rightarrow \mathcal{L}_3(s)$ is the *symmetric square* of $\mathcal{L}_2(s)$
- $\mathcal{L}_2(s)$ is related to the 2-loop sunrise by a change of variables

  [Broedel et al.; arXiv: `1907.03787`]