# One-loop calculation and Monte Carlo simulation of the process $e^+e^- \to \pi^+\pi^-$

Marco Ghilardi

Relatore: Prof. Fulvio Piccinini
Correlatore: Dott. Francesco Pio Ucci

July 24,2024

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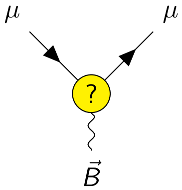
# The muon $g-2$




**Magnetic moment**

$$\vec{\mu} = -g_\mu \frac{e}{2m_\mu}\vec{S}$$

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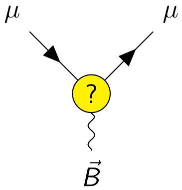


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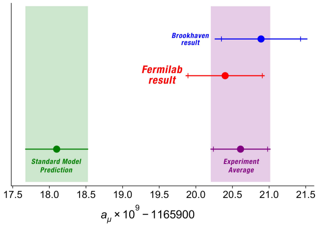

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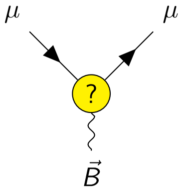


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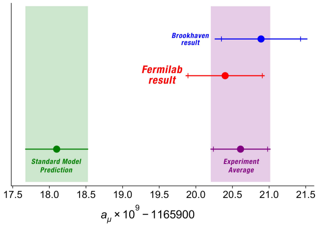
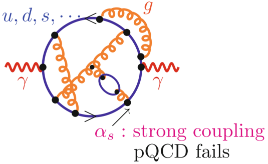



In the SM:

$$a_\mu = \underbrace{a_\mu^{\text{QED}}}_{>99.99\%} + a_\mu^{\text{EW}} + \underbrace{a_\mu^{\text{had}}}_{Non-perturbative}$$



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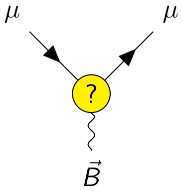


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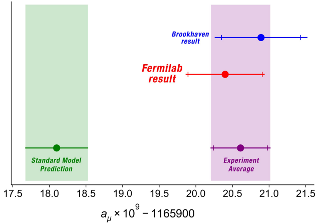


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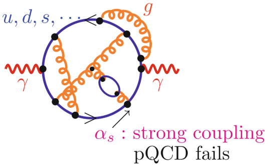

# Hadronic contributions

$$a_\mu^{\text{had}} = \underbrace{a_\mu^{\text{HVP-LO}}}_{\mathcal{O}(7\cdot 10^{-8})} + \underbrace{a_\mu^{\text{HLbL}}}_{\mathcal{O}(10^{-9})} + \underbrace{a_\mu^{\text{HVP-NLO}}}_{\mathcal{O}(10^{-9})} + \underbrace{a_\mu^{\text{HVP-NNLO}}}_{\mathcal{O}(10^{-10})}$$

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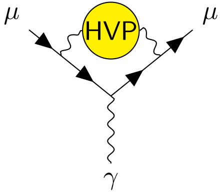

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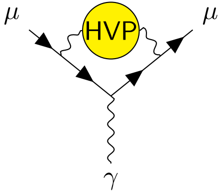


$$a_\mu^{\text{HVP-LO}} = \left(\frac{\alpha\, m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds\, \frac{1}{s^2} \mathcal{R}_0^{\text{had}}(s) \hat{\mathcal{K}}(s)$$

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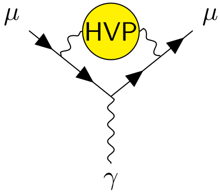


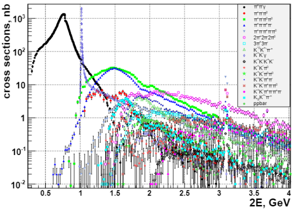


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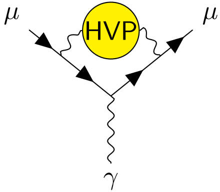


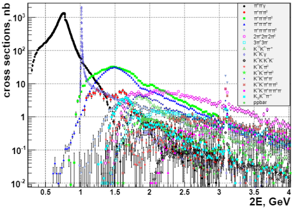


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$$\underbrace{\text{sQED}}_{\text{pert}} \oplus \underbrace{F_\pi(q^2)}_{\text{non-pert}}$$

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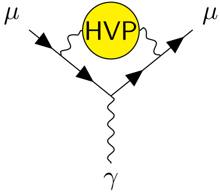


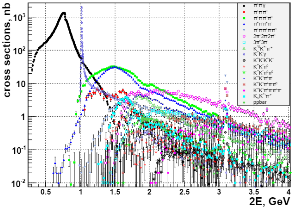


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- **Model**

$$\underbrace{\text{sQED}}_{\text{pert}} \oplus \underbrace{F_\pi(q^2)}_{\text{non-pert}}$$

$$\frac{\delta F_\pi}{F_\pi} \sim 10^{-3} \Rightarrow \text{NLO}$$

# Form factor determination

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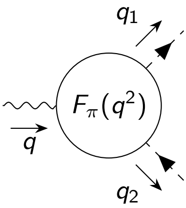


$$\langle \pi^+(q_2)\pi^-(q_1)|J^\mu_\pi(0)|0\rangle = e(q_1 - q_2)^\mu F_\pi\left(q^2\right)$$

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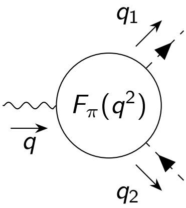


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## Energy scan

- SND, CMD-2 and CMD-3
- $\mathcal{R}^{\text{had}}_{\text{exp}} = \dfrac{\sigma(e^+e^- \to \pi^+\pi^-)}{\sigma(e^+e^- \to \mu^+\mu^-)}$

$$\Downarrow$$

$$F_\pi(q^2)$$

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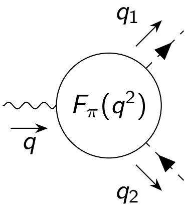


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$$\between$$

$$F_\pi(q^2)$$

## Radiative return

- KLOE, BABAR, and BELLE(II)
- $$\left(\frac{d\sigma}{dM_{\pi\pi}}\right)_{\text{ISR}}$$

$$\between$$

$$F_\pi(q^2)$$

# sQED

$$\mathcal{L} = \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{Spinor}} + \mathcal{L}_{\text{Scalar}} + \mathcal{L}_{\text{GF}}$$

where:

$$\mathcal{L}_{\text{Scalar}} = \mathcal{L}_{\text{Scalar}}^{\text{free}} + ieA_\mu\left(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right) + e^2 A_\mu A^\mu \phi^*\phi$$

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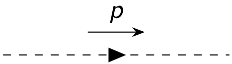


$$\frac{i}{p^2 - m_\pi^2 + i\varepsilon}$$

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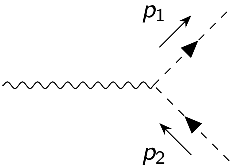


$$+ie(p_1^\mu + p_2^\mu)$$

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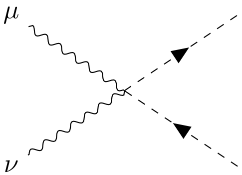


$$2ie^2 g^{\mu\nu}$$

# NLO calculation

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$$\sigma_{\text{NLO}} = \sigma_0(\alpha^2) + \sigma_1(\alpha^3)$$

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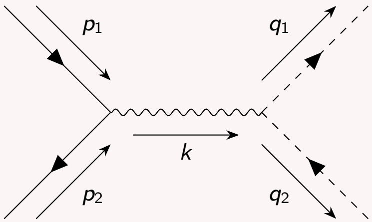

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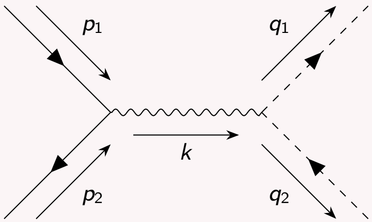

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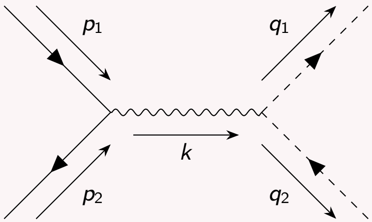
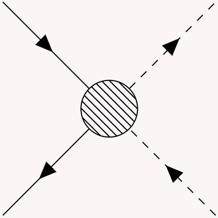



## NLO



- Virtual $\mathcal{M} \propto \alpha^2$
- Real $\mathcal{M} \propto \alpha^{\frac{3}{2}}$

# NLO contributions

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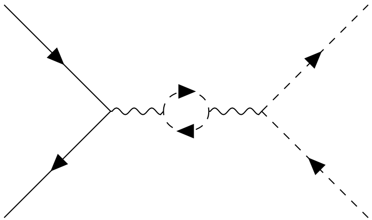
**Virtual corrections**



- $\epsilon$ regularizer (UV)
- $\lambda^2$ regularizer (IR)
- $\sigma_{\mathrm{V}} \propto \mathcal{M}_0 \mathcal{M}_V$

# NLO contributions

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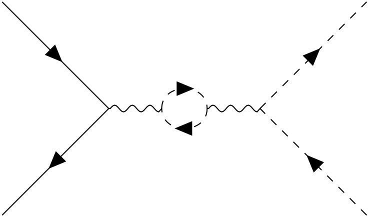

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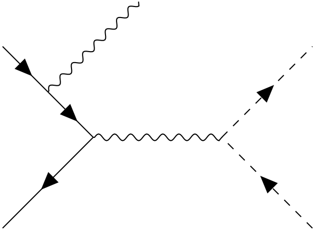
## Real corrections



- $\lambda^2$ regularizer (IR)
- $\sigma_R \propto |\mathcal{M}_R|^2$

# $e^+e^- \to \pi^+\pi^-$ in sQED

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$$\sigma_{\text{NLO}} = \sigma_{2\to 2} + \sigma_{2\to 3}$$

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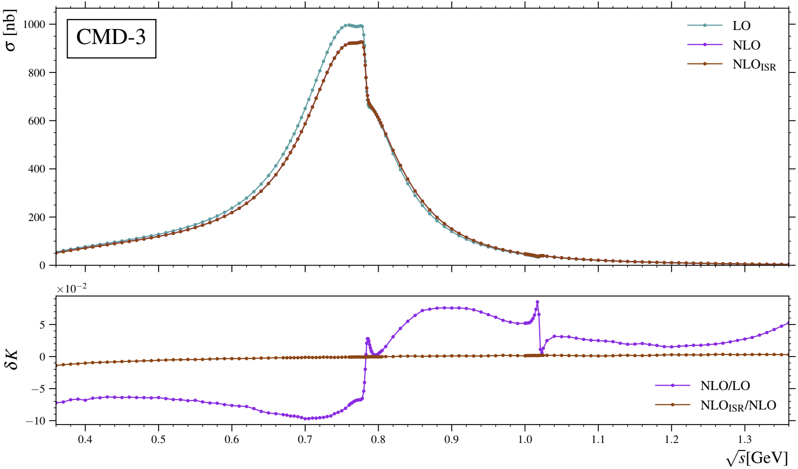
Explicit IR cancellation

# Energy scan


CMD-3
$\sigma$ [nb]
1000
800
600
400
200
0
LO
NLO
$\mathrm{NLO_{ISR}}$
$\times 10^{-2}$
$\delta K$
5
0
−5
−10
NLO/LO
$\mathrm{NLO_{ISR}/NLO}$
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
$\sqrt{s}$[GeV]

# Factorized approach

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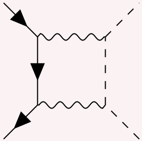
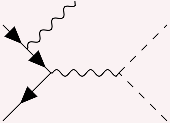
- $\sigma_{\mathrm{LO}}(s) = \sigma^0_{\mathrm{LO}}(s) \times F_\pi(s)$
- $\delta K = \frac{\sigma_{\mathrm{NLO}}}{\sigma_{\mathrm{LO}}} - 1$
- NLO



$\times\ F_\pi(s)$



$\times\ F_\pi(M^2_{\pi\pi})$

# Summary and outlook

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- NLO calculation necessary to reach desired precision.

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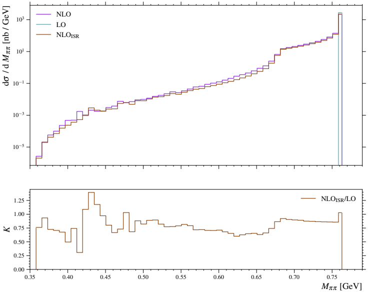
## Future developments

- Higher Order (HO) corrections.
- Introduction of the Form Factor into loop integration.

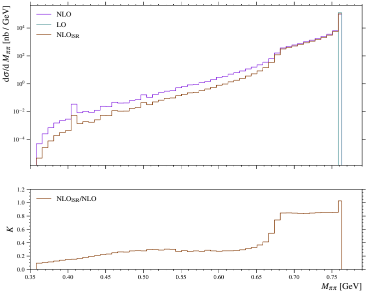
# $M_{\pi\pi}$ distribution

$$M_{\pi\pi} = \sqrt{(q_1 + q_2)^2}$$

**Point-like**




**Form factor**



# Angular distribution

$$\frac{d\sigma}{d(\cos\theta_{\pi^-})}$$

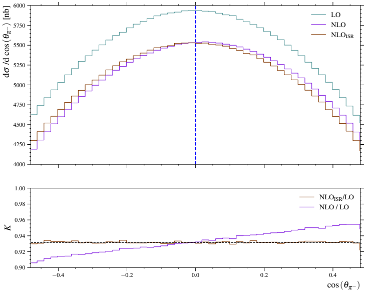
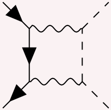



Figure: Differential cross section w.r.t. $\cos\theta_{\pi^-}$.

- $\sqrt{s} = 0.77\,\text{GeV}$
- LO

  $d\sigma \propto (1-\beta_e^2\cos^2\theta) \times F_\pi(s)$
- NLO



$\times\ F_\pi(s)$

$d\sigma_{\text{NLO}} \neq d\sigma^0_{\text{NLO}} \times F_\pi(s)$

# Angular distribution

$$\frac{d\sigma}{d(\cos\theta_{\pi^-})}$$

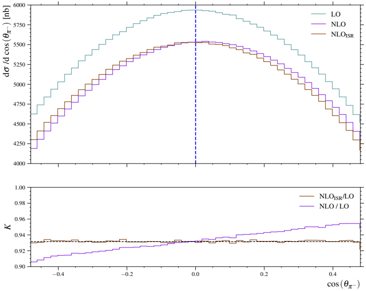



Figure: Differential cross section w.r.t. $\cos\theta_{\pi^-}$.

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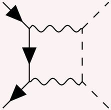
   $\times\ F_\pi(s)$

  $d\sigma_{\text{NLO}} \neq d\sigma^0_{\text{NLO}} \times F_\pi(s)$

# $M_{\pi\pi}$ distribution

$$\frac{d\sigma}{dM_{\pi\pi}}$$

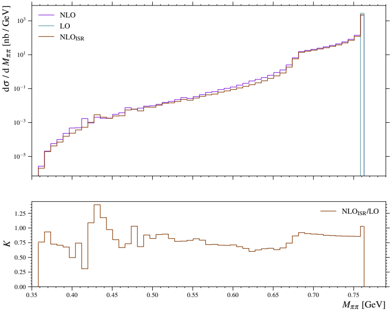
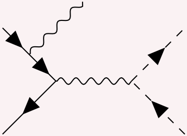



Figure: Differential cross section w.r.t. $M_{\pi\pi}$.

- $\sqrt{s} = 0.77$ GeV
- $M_{\pi\pi} = \sqrt{(q_1 + q_2)^2}$
- LO

  $M_{\pi\pi} = \sqrt{s}$
- NLO



# $M_{\pi\pi}$ distribution

$$\frac{d\sigma}{dM_{\pi\pi}}$$

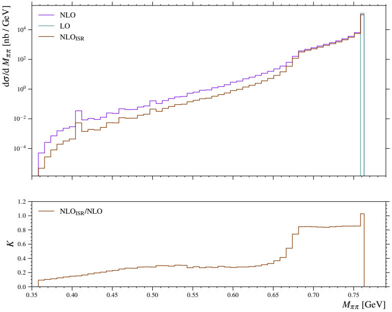
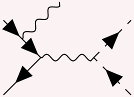



Figure: Differential cross section w.r.t. $M_{\pi\pi}$.

- $\sqrt{s} = 0.77$ GeV
- NLO



$\times F_\pi(M^2_{\pi\pi})$