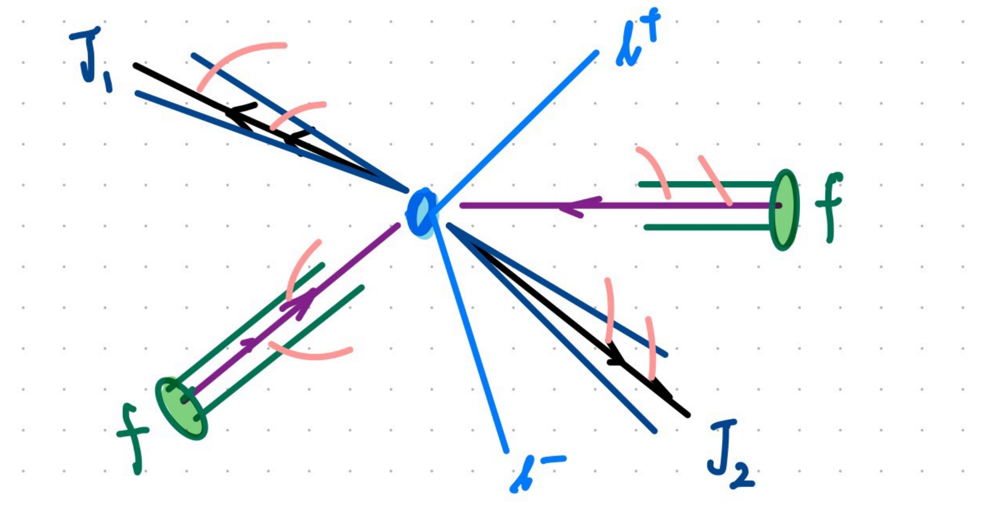
Two-Loop Generalized Splitting Amplitude for N=4 Super-Yang-Mills Theory

Kai Yan

MPP Seminar 07/2024

Factorization of hadronic cross section at LHC



Predicitve power of precision program at LHC relies on universality of parton distribution function

$$\sigma = f \otimes f \otimes \hat{\sigma}.$$

High-multiplicity processes: proof or fixed-order counterexamples are both absent.

What could invalidate factorization?

Space-like collinear limit of scattering amplitudes not strictly factorized.

Are cross-section level factorization invalidated?

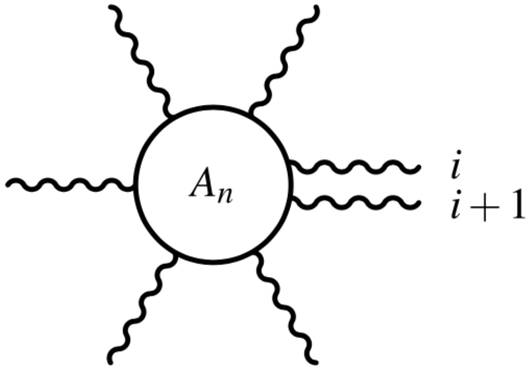
We made a concrete argument at N3LO

Henn,Ma,Xu,Yan,Zhang,Zhu [2406.14604]

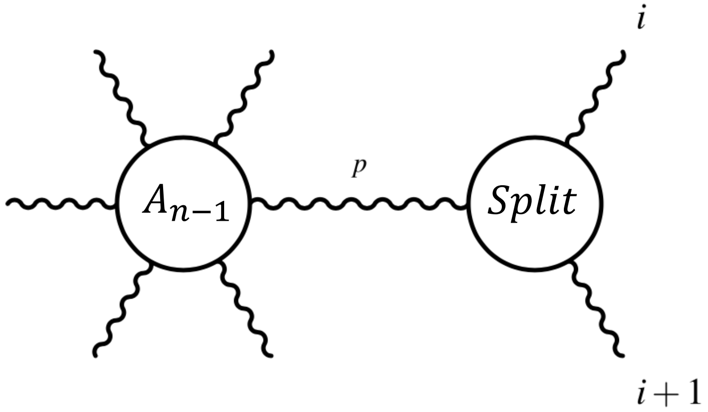
Strict Collinear Factorization

Gauge theory amplitudes factorizes on the two-particle pole $P_{i,i+1}= 0$, when two adjacent external momenta are collinear.

$$A_n(\dots, i, i+1, \dots) \xrightarrow{i \parallel i+1} \sum_{\lambda} Split_{-\lambda}\,(z;\ i, i\ +\ 1)\ A_{n-1}(\dots, P^{\lambda}, \dots)$$



$$\lambda_i = \sqrt{z}\,\lambda_P,$$
$$\lambda_{i+1} = \sqrt{1-z}\,\lambda_P$$



Splitting amplitudes are independent of color or kinematics of non-collinear external legs

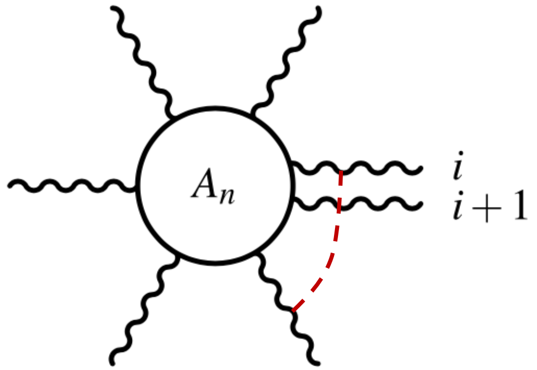
holds to all-loop order for time-like splitting s_{i,i+1} >0 (as a consequence of color coherence).

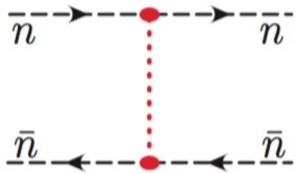
Factorization violation

Generalized splitting amplitude depends on color and kinematics of non-collinear external legs

Space-like splitting i+1 --> i P :

$$|A_n(\dots, i, i+1, \dots)\rangle \xrightarrow{i \parallel i+1} \boldsymbol{Sp}\, |A_{n-1}(\dots, P, \dots)\rangle$$





long range Coulomb interactions responsible for absorptive contributions

The physical origin of the breakdown is causality of the theory

$$\frac{1}{\epsilon}\, \gamma_K \sum \left(T_i \cdot T_j\right) \log\left(\frac{-|s_{ij}| e^{i\pi\lambda_{ij}}}{\mu^2}\right)$$

Such absorptive IR poles cancel at cross section level up to N^3LO [Catani et al, 1112.4405] [Forshaw, Seymour, 1206.6363].

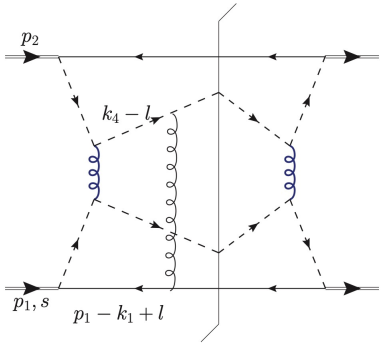
$$\sum T_i \cdot T_j\, \lambda_{ij} = \sum T_P \cdot T_j\, \lambda_{Pj} + 2\, T_j \cdot T_k\, \theta(-z_k)$$

We would like to make a concrete argument about the finite part.

Counter-examples of cross-section level factorization

Spin asymmetry in kT-factorization/beam thrust: [Collins,Qiu, 0705.2141] [Zeng, 1507.01652].

$O(\alpha_s^4)$ Super Leading Logarithms [Forshaw, Kyrieleis, Seymour, 0604094][Becher, Neubert, Shao, 2107.01212]



"Bottom-up" approach for extracting the Glauber contribution at fixed order:

Effective field theory for Glauber gluons: [Rothstein, Stewart, 1601.04695] [Schwartz,KY,Zhu,1703.08572]

"Top-down" approach:

Analyzing the collinear-limit of perturbative amplitudes calculations

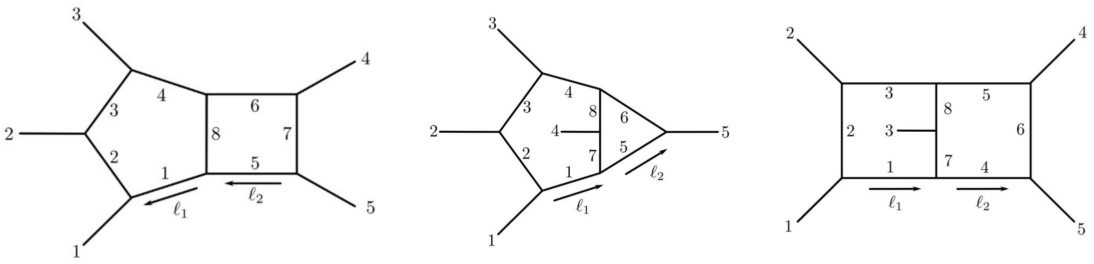
Soft-collinear limit: [Dixon,et al,1912.09370]

Space-like collinear limit of two-loop five-point amplitudes (this talk)

Benefit from remarkable data for high multiplicities amplitudes [Chicherin et al, 1812.11160, Abreu et al, 1812.08941] [2009.07803, 2112.10605, 1807.09812] [Agarwal et al, 2311.09870]

Analyzing the (23) −collinear limit of two-loop five-point amplitudes in full color

$$A_5^{N=4\ sYM}(12 \to 345)$$



Two complementary approaches:

Differential equation: solving asymptotically near the collinear region

Discontinuity: $A_5\Big|_{12\to345} \to A_5\Big|_{1'2\to345}$ $(strictly\ factorized)$

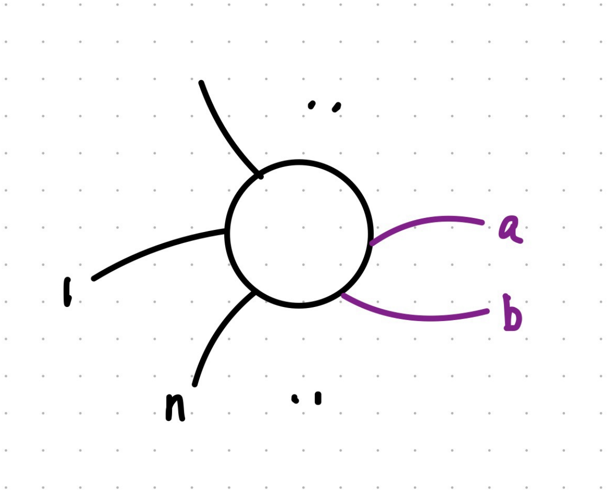
Shed light on the physical mechanism of collinear factorization breaking
and the structure of space-like v.s. timelike splitting amplitudes

Planar Limit

$$\mathcal{A}_n|_{\text{large-}N_c} = g^{n-2}\sum_{\vec{\sigma}_n}\text{tr}(T^{a_{\vec{\sigma}_n}})A_n^{\text{BDS}}(\vec{\sigma}_n) = g^{n-2}\sum_{\vec{\sigma}_n}\text{tr}(T^{a_{\vec{\sigma}_n}})\exp[F_n]\,A_{n,\text{tree}}(\vec{\sigma}_n)$$

$n = 5$

$$F_n = \sum_L \left(\frac{2N_c\,g^2 e^{-\epsilon\gamma_E}}{(4\pi)^{2-\epsilon}}\right)^L f^{(L)}(\epsilon)\,F_n^{(1)}(L\epsilon) + C^{(L)}$$

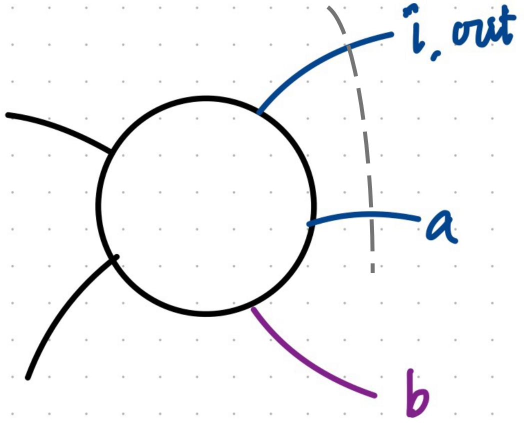


Time-like collinear splitting: $k_a \to \tau P,\ k_b \to (1-\tau)P\ ,\ 0 < \tau < 1.$

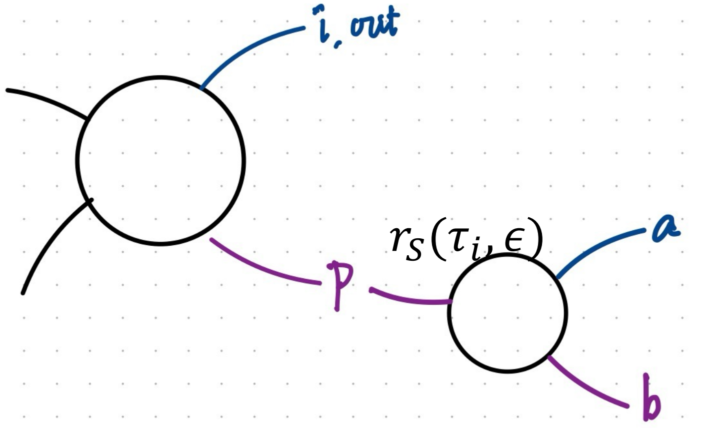
$$F_n^{(1)}(\cdots, a, b, \cdots) \to F_{n-1}^{(1)}(\cdots, P, \cdots) + r_S^{(1)}(\tau, \epsilon)$$

BDS-like iterative formalism for time-like splitting amplitudes [0309040].

Space-like collinear splitting



space-like
regime: $\tau < 0$

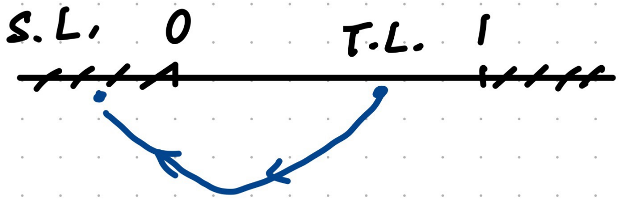


$$\tau_i \coloneqq \frac{s_{ai}}{s_{Pi}} = -|\tau| \exp[-i\pi\,(\lambda_{ai} - \lambda_{Pi})]$$

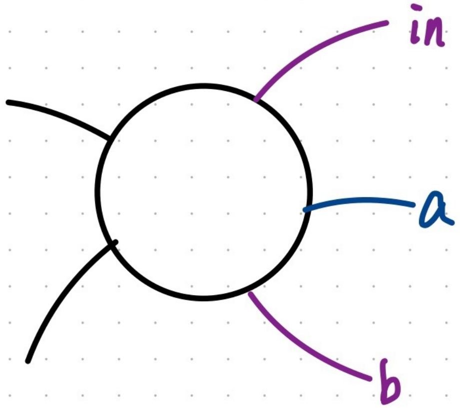
$$\frac{A_5^{\mathrm{BDS}}(a, b, \mathrm{in}, \{I\})}{A_4^{\mathrm{BDS}}(P, \mathrm{in}, \{I\})} = \exp\left[\sum_{L=1} \bar{a}^L f^{(L)}(\epsilon)\, r_S^{(1)}(L\epsilon)\right] \mathrm{Split}^{(0)}$$

$\bar{a} \equiv 2N_c\, a$

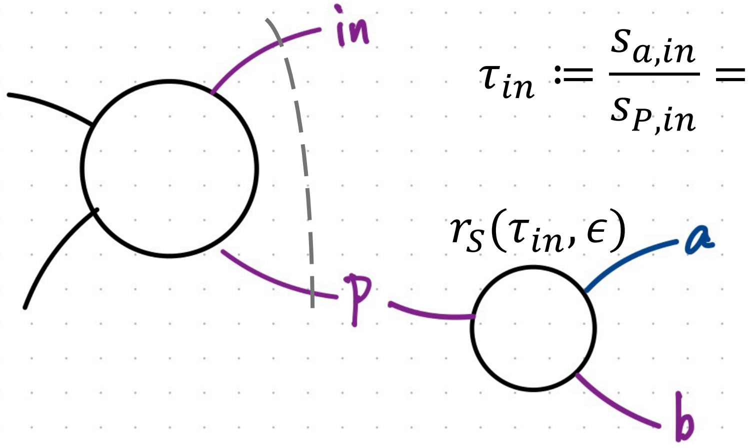
$r_S(\tau - i0, \epsilon)$



Space-like collinear splitting



space-like regime: $\tau < 0$

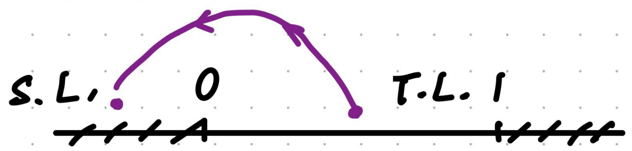


$r_S(\tau_{in}, \epsilon)$

$$\tau_{in} := \frac{s_{a,in}}{s_{P,in}} = -|\tau| \exp[+i\pi]$$

$$\frac{A_5^{\mathrm{BDS}}(a,b,\{I\},\mathrm{in})}{A_4^{\mathrm{BDS}}(P,\{I\},\mathrm{in})} = \exp\left[\sum_{L=1} \bar{a}^L f^{(L)}(\epsilon)\left(r_S^{(1)}(L\epsilon) - (2\pi i)\,\frac{c_\Gamma(L\epsilon)}{L\epsilon}\right)\right] \mathrm{Split}^{(0)}$$

discontinuity across the branch cut starting at $\tau = 0$



Color-space formalism

Dressing with color,

$$\mathbf{Sp}\big|_{\text{large-}N_c} = \exp\left[\frac{1}{2N_c}\sum_L \bar{a}^L f^{(L)}(\epsilon)\ \left[\mathcal{G}^{(1)}(L\epsilon)\big|_{Nc^1}\right]\right]\mathbf{Sp}^{(0)}, \quad \bar{a} \equiv 2N_c\, a\,.$$

$$\mathcal{G}^{(1)}(\epsilon,\tau) = 2N_c\, r_1(\epsilon,\tau - i0) + i\phi_{\mathcal{G}}^{(1)}(\epsilon), \quad \phi_{\mathcal{G}}^{(1)} = -2\pi\frac{c_\Gamma(\epsilon)}{\epsilon}\mathcal{T}_{\text{a,in}}$$

dipole operator

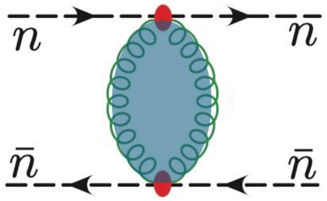
$T_a \cdot T_{in}$

$$(T_a \cdot T_{in})^2 = 2\,N_c\ (T_a \cdot T_{in}) + \text{subleading Nc}$$

To all-loop order,

$Sp|_{large-N_c}$ = color singlet $+(i\pi)\times$ color dipole

violates naïve factorization

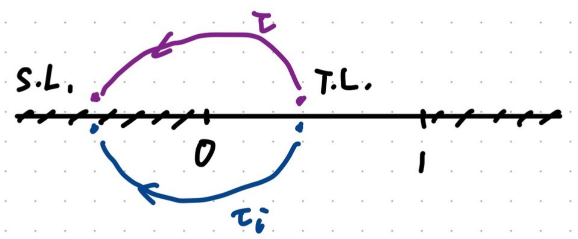


Coulomb/Glauber potential

Short summary

Necessity condition for factorization violation :

presence of at least one incoming and one outgoing non-colinear external leg



On the two-loop five-point amplitudes in full color :

How to perform analytic continuation on the amplitudes into a region where it is strictly factorized?

Structure of the generalized splitting amplitudes?

Generalized splitting amplitudes

Ansatz for the all-loop structure

$$\mathbf{Sp} = \exp[\mathcal{G} + \Delta_{\mathcal{G}}]\ \mathbf{Sp}^{(0)}$$

$$\mathcal{G}(\epsilon;\tau) = \sum_{k=1}^{\infty} a^k\, \mathcal{G}^{(k)}, \quad a := \left[\frac{\mu^2(1-\tau)}{s_{ab}\tau}\right]^{\epsilon} \frac{g^2 e^{-\epsilon\gamma_E}}{(4\pi)^{2-\epsilon}}$$

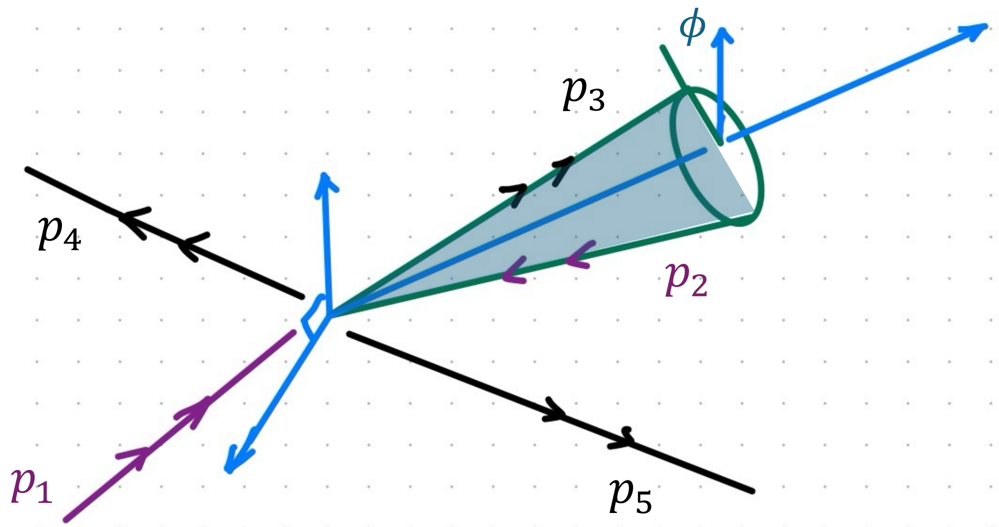
$$\mathcal{G}^{(L)}(\epsilon,\tau) \equiv 2^L N_c^L\, r_L + i\phi_{\mathcal{G}}^{(L)}$$

Color singlet + Color dipole

$$\Delta_{\mathcal{G}} = a^2 \Delta_{\mathcal{G}}^{(2)} + \cdots$$

Multi-pole correlation
First appear at the level of two-loop five point amplitudes (IR poles given in [1112.4405])

Explicit two-loop result



$$\mathbf{Sp} = \exp[\mathcal{G} + \Delta_{\mathcal{G}}]\ \mathbf{Sp}^{(0)}$$

Dipole contributions come from both planar and non-planar topologies

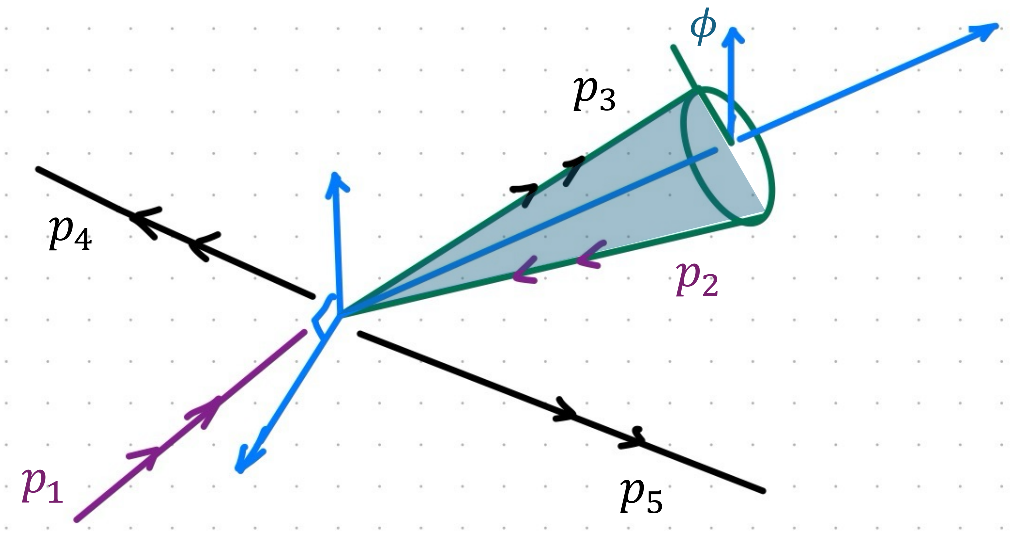
$$\mathcal{G}^{(1)}(\epsilon, \tau) = 2N_c\, r_1(\epsilon, \tau - i0) + i\phi_{\mathcal{G}}^{(1)}(\epsilon), \quad \phi_{\mathcal{G}}^{(1)} = -2\pi \frac{c_\Gamma(\epsilon)}{\epsilon}\, \mathcal{T}_{\mathrm{a,in}}$$

$$\mathcal{G}^{(2)}(\epsilon) = 2N_c\, f(\epsilon)\left[\mathcal{G}^{(1)}(2\epsilon)\right] \qquad f(\epsilon) \equiv (\psi(1-\epsilon) - \psi(1))/\epsilon$$

$$\mathcal{G}^{(L)}(\epsilon) := (2N_c)^{(L-1)}\, f^{(L)}(\epsilon)\left[\mathcal{G}^{(1)}(L\epsilon)\right]$$

Exhibit BDS-like structure at two-loop order

Explicit two-loop result



$$\mathbf{Sp} = \exp[\mathcal{G} + \Delta_{\mathcal{G}}]\, \mathbf{Sp}^{(0)}$$

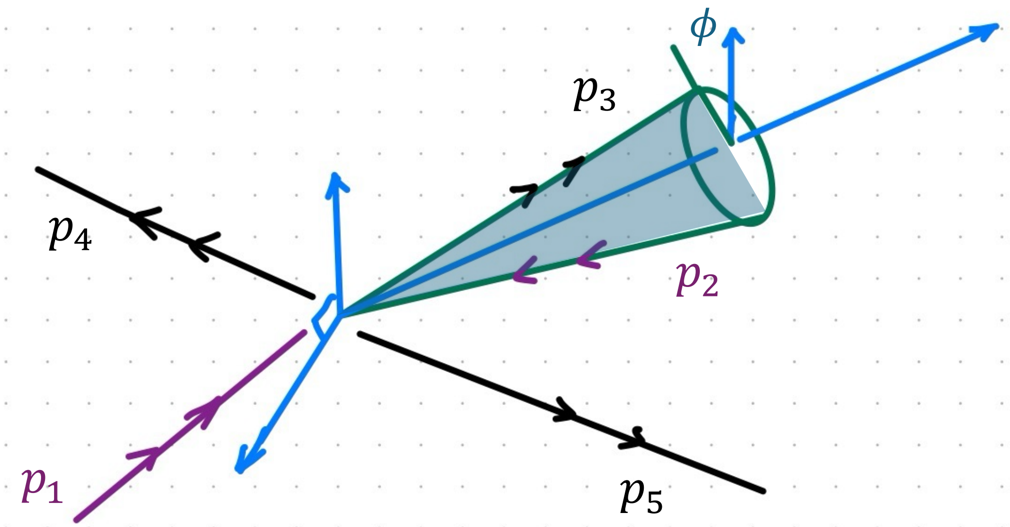
$\Delta_G^{(2)}$ is purely non-planar:

$$\Delta_{\mathcal{G}}^{(2)} = (2\pi i) \sum_I \mathcal{T}_{\mathrm{a,in,I}} \left[\left(\frac{1}{2\epsilon^2} - \frac{1}{2}\zeta_2 \right) (\ln |z_I|^2 + 2\pi i) + 2\zeta_3 + \boxed{\frac{1}{6}\left(\ln^2 \frac{z_I}{\bar{z}_I} + 4\pi^2 \right) \ln \frac{z_I}{\bar{z}_I}} \right]$$

$$\frac{1}{2i} \ln \frac{z_i}{\bar{z}_I} \sim \frac{\pi}{2} - \phi$$

$$z_I = \frac{\langle ab\rangle \langle \mathrm{in}\, I\rangle}{\langle \mathrm{in}\, a\rangle \langle bI\rangle}, \quad \bar{z}_I = \frac{[ab][\mathrm{in}\, I]}{[\mathrm{in}\, a][bI]}, \quad a \parallel b\,.$$

Explicit two-loop result



$$\mathbf{Sp} = \exp[\mathcal{G} + \Delta_{\mathcal{G}}]\ \mathbf{Sp}^{(0)}$$

$\Delta_G^{(2)}$ is purely non-planar:

$$\Delta_{\mathcal{G}}^{(2)} = (2\pi i) \sum_I \boxed{\mathcal{T}_{\mathrm{a,in,I}}} \left[\left(\frac{1}{2\epsilon^2} - \frac{1}{2}\zeta_2 \right) (\ln |z_I|^2 + 2\pi i) + 2\zeta_3 + \frac{1}{6} \left(\ln^2 \frac{z_I}{\bar{z}_I} + 4\pi^2 \right) \ln \frac{z_I}{\bar{z}_I} \right]$$

$[T_a \cdot T_{in}, T_a \cdot T_I]$

$< A_{tree} | \, [T_a \cdot T_{in}, T_a \cdot T_I] | A_{tree} > = 0$

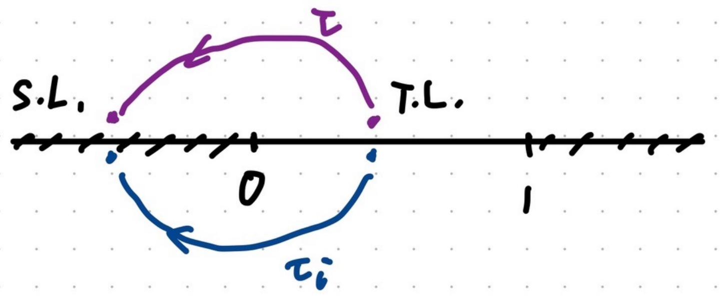
vanished at the level of color summed squared amplitudes at N^3LO

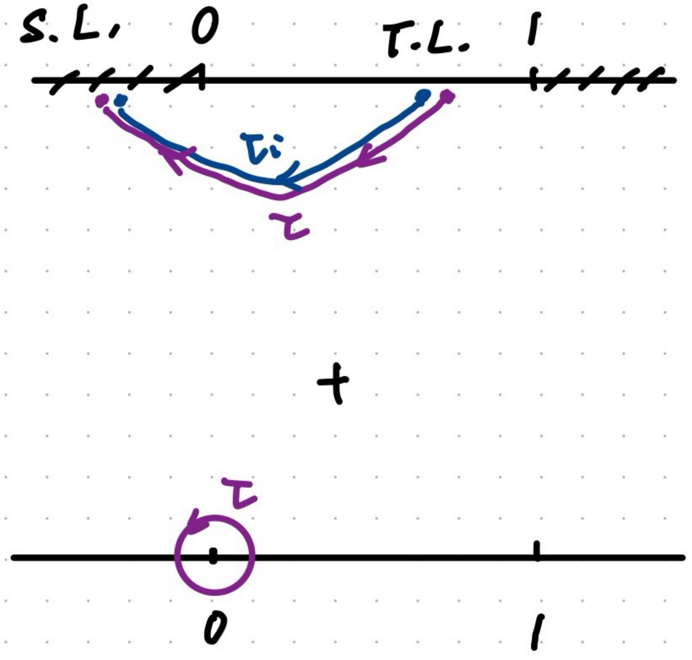
Factorization violation from discontinuity

Factorization restored after bringing variable $\tau = \frac{s_{a,in}}{s_{a,in}+s_{b,in}}$ to the next Riemann sheet

$$\text{s.t.} \quad \ln\left(\frac{s_{aI}}{s_{aI}+s_{bI}}\right) \to \ln|\tau| - i\,\pi, \qquad \forall\, I$$

Difference generated in taking $disc_\tau$ accounts for the factorization breaking terms in the amplitudes





Twistor variables

Fixing the overall scale $\quad s_{12}+s_{13}=s_{45}-s_{23}\equiv 1$

Five-point kinematics parametrized by twistor variables

$$s_{12}=\frac{z_3-z_4}{z_1-z_4}\,,\;\; s_{23}=-\frac{z_2-z_3}{z_2}\,,\;\; s_{34}=-\frac{z_3(1-z_4)(z_1-z_2)}{z_2(z_1-z_4)}\,,\;\; s_{45}=\frac{z_3}{z_2}\,,\;\; s_{15}=-1+z_3$$

$$\left[\,\tau:=\frac{s_{13}}{s_{12}+s_{13}}=\frac{z_1-z_3}{z_1-z_4},\quad z_2\,,\; z_3\,,\; z_4\,\right]$$

$\mathrm{disc} \;:=\; \mathrm{disc}^{\circlearrowleft}_{\tau=0}$ extracts the monodromy around $z_1-z_3=0$

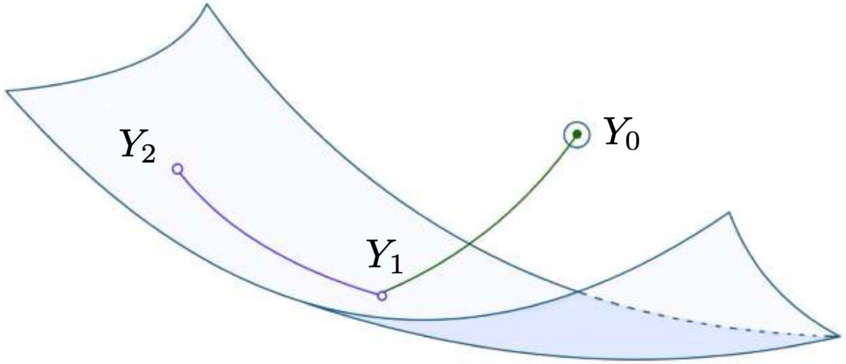
$$disc[\ln W_a]=s_a\,2\pi\, i \qquad\qquad s_{16}=1,\; s_{30}=-1\,.\; s_a=0, otherwise$$

Kinematic space

Y_0 :generic point in the physical scattering region $[\,\tau\,,\ z_2\,,\ z_3\,,\ z_4\,]$

in the neighborhood of collinear region: $s_{23} \sim O(\delta^2)$, $Tr_5 \sim O(\delta)$

$$[\,\tau\,,\ z_2\,,\ c\,,\ \delta\,] \qquad z_4 \to z_2(1+\delta)\,, \quad z_3 \to z_2(1+c\,\delta^2)$$



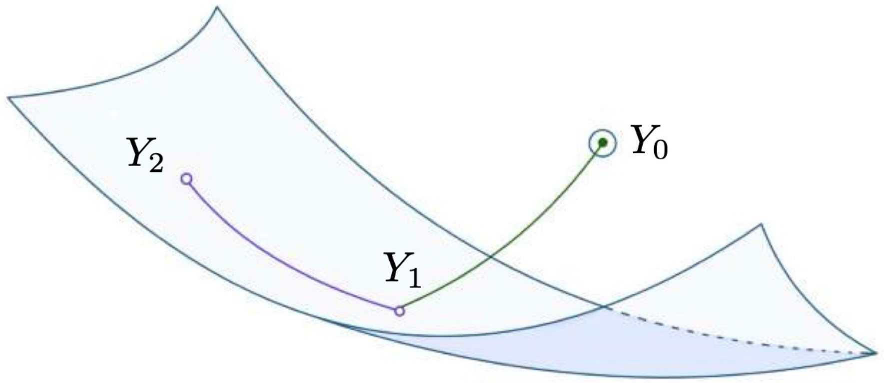
Y_1/Y_2 : in the spacelike/ timelike collinear region

$$s_{12} = 1-\tau\,, \quad s_{23} = 0\,, \quad s_{34} = (-1+z_2)\,\tau\,, \quad s_{45} = 1\,, \quad s_{15} = -1+z_2\,.$$

On the slice of collinear phase-space, no dependence on c

$$\ln\frac{z_i}{\overline{z}_I} \sim \frac{c(1-z_2)(1-\tau)}{\tau\, z_2}$$

Analytic continuation procedure



-Taking on A_5 in the physical region

$$A_5 \to A_5{}' \qquad \text{(unphysical, factorized)}$$

- Taking collinear limit $\delta \to 0$

- Extract the factorization violation terms ϕ_G, Δ_G

$$\mathrm{disc}[A_5] = (\exp[i\phi_{\mathcal{G}} + \Delta_{\mathcal{G}}] - 1)\, A_5'$$

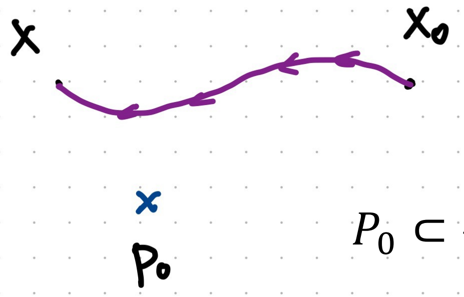
$$A_5' := \mathrm{Split}(\tau - i0) \times A_4$$

Discontinuity of pentagon functions

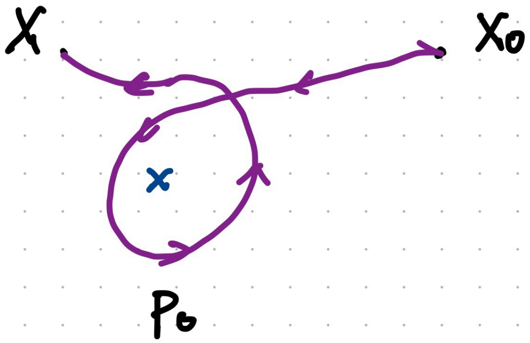
$X_0: \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} = \{3, -1, 1, 1, -1\}$

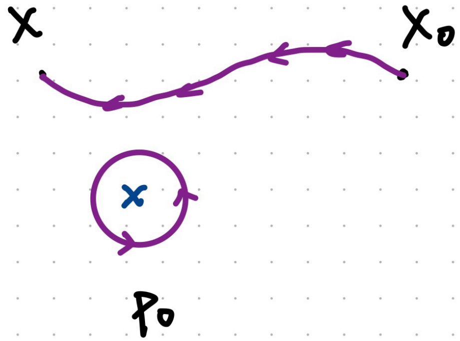
$$F_n = \int_{X_0}^{X} d\omega \otimes F_{n-1} \qquad F_0 = \{\, 1 \,\}$$

$$d\,\omega_{ab} = \sum_{\alpha=1}^{31} d \ln W_\alpha \, m^\alpha_{ab}$$



$P_0 \subset \{\Sigma |\, \tau = 0\}$





$$\mathrm{disc}[F_1] = \mathrm{Res}_{\tau=0}\left[\frac{\partial w}{\partial \tau} \otimes F_0\right]$$

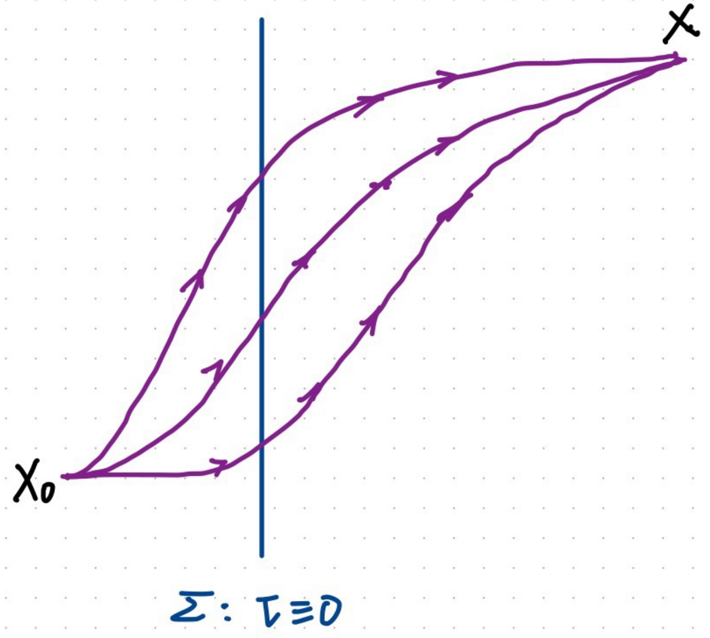
$$d\,F_n = d\,\omega \otimes F_{n-1}$$

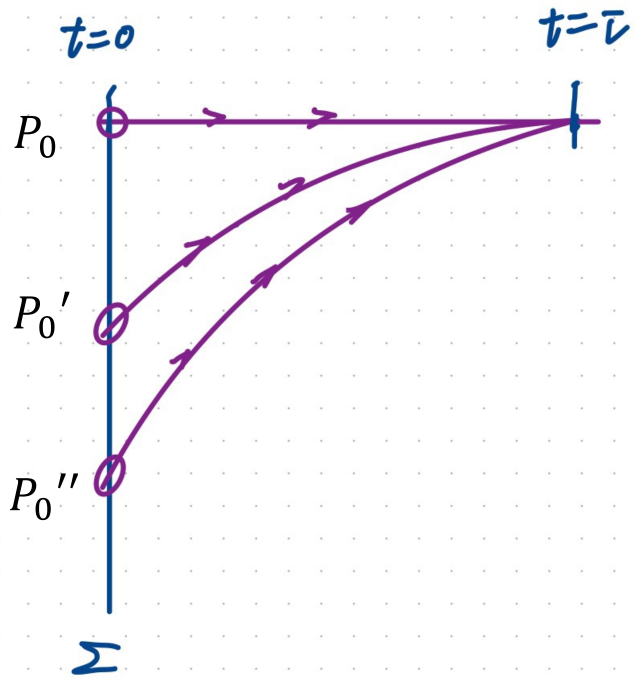
$$F_n = \int_{X_0}^{X} d\omega \otimes F_{n-1}$$

$$d\,\mathrm{disc}[F_n] = d\,\omega \otimes \mathrm{disc}[F_{n-1}]$$

$$\mathrm{disc}[F_n] = \int_{P_0}^{X} d\omega \otimes \mathrm{disc}[F_{n-1}] \quad + \mathrm{Res}_{\tau=0}\left[\frac{\partial w}{\partial \tau} \otimes F_{n-1}\right]\Big|_{P_0}$$

$\forall\, P_0 \subset \{\Sigma |\; \tau = 0\}$





Rules of taking residues at τ=0

$$\begin{aligned}\mathrm{disc}\,[\tau^{-1}\ln^n\tau] &= \mathrm{disc}\,\partial_\epsilon^n[\tau^{-1+\epsilon}]\Big|_{\epsilon=0} = 2i\,\partial_\epsilon^n\left\{\sin(\pi\epsilon)[\omega^{-1+\epsilon}]\right\}\Big|_{\epsilon=0} \\ &= 2i\,\partial_\epsilon^n\left[\sin(\pi\epsilon)\left(\frac{1}{\epsilon}\delta(\omega)+\sum_{k=0}\epsilon^k\left[\omega_+^{-1}\ln^k\omega_+\right]\right)\right]\Bigg|_{\epsilon=0}\end{aligned}$$

Taking discontinuity transforms the singularities at $\tau = 0$ into distributional and contact terms

$$\begin{aligned}\mathrm{disc}\,[\tau^{-1}] &= 2\pi i\;[\delta(\omega)] \\ \mathrm{disc}\,[\tau^{-1}\ln\tau] &= 2\pi i\;[\omega_+] \\ \mathrm{disc}\,[\tau^{-1}\ln^2\tau] &= 2\pi i\;\left[-\frac{\pi^2}{3}\delta(\omega)+2\,\omega_+\ln\omega_+\right]\end{aligned}$$

$disc[F_n]$ = regular part (vanishing at $\tau = 0$) + boundary terms (fixed by $F_{n-1}|_{\tau=0}$)

Results for the discontinuity

Discontinuity of pentagon function of weight-2,3,4: $G_{n-1} = (2\pi\, i)^{-1} \times disc[F_n]$ are given by weight-1,2,3 classical polylogarithms.

Further taking collinear limit, their symbol alphabets are drawn from the set:

$$\mathcal{A} = \{\, \delta, \tau, 1-\tau, z_2, 1-z_2, 1-\tau z_2, 1-\tau+\tau z_2, 1-z_2+\tau z_2, \tau+z_2-\tau z_2, \tau - z_2, 1-\tau-z_2, \underline{\underline{c}}, \underline{(1-z_2)(1-\tau)-z_2\tau c} \,\}$$

$\propto z_I - \bar{z}_I \propto Tr_5$

Alphabets that appear in the final result :

$$\{\, \delta, c, \tau, 1-\tau, z_2, 1-z_2 \,\}$$

Shortcut: symbol-level analysis

Given the symbol-level result for the pentagon functions, the weight-2 integration constants in the discontinuity can be unambiguously determined

$$\Delta_1[\ln(-s_{12} - i\varepsilon)] = [\otimes \ln W_1] - [\otimes (i\pi)], \quad \Delta_1[\ln(-s_{34} - i\varepsilon)] = [\otimes \ln W_{18}] - [\otimes (i\pi)]$$
$$\Delta_1[\ln(-s_{35} - i\varepsilon)] = [\otimes \ln W_3] - [\otimes (i\pi)], \quad \Delta_1[\ln(-s_{45} - i\varepsilon)] = [\otimes \ln W_4] - [\otimes (i\pi)]$$

$$\mathrm{disc}[F_2] = \int_{P_0}^{X} d\omega \otimes \mathrm{disc}[F_1] \quad + \mathrm{Res}_{\tau=0}\left[\frac{\partial w}{\partial \tau} \otimes F_1\right]\Bigg|_{P_0}$$ → π^2constants

Given the symbol of the amplitude A_5,

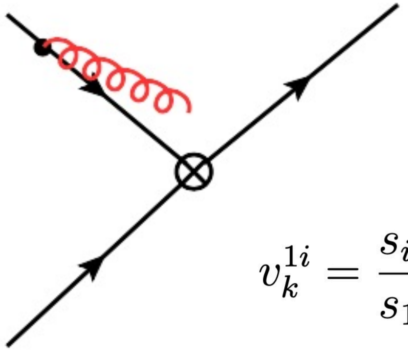
$disc[A_5]$ can be determined up to $i\pi^3\times$single log and weight 4-constants ($i\pi\zeta_3, \pi^4$)

↓

Extracted from the soft-collinear ($\tau \to 0$) limit of the splitting amplitude [1912.09370]

Soft-collinear limit

The limit where $\tau \to 0$ corresponds to the collinear limit of soft emission factor



$$v_k^{1i} = \frac{s_{ik}s_{1q}}{s_{1i}s_{kq}}, \qquad z_k^{1i} = \frac{\langle ki\rangle\langle 1q\rangle}{\langle 1i\rangle\langle kq\rangle}$$

$$\mathbf{Sp}^{(2)}\Big|_{\text{tripole}} \overset{q-\text{soft}}{\simeq} -\frac{1}{4}\sum_{\substack{\text{tripoles}\\ \{i,1,k\}}} \boldsymbol{S}^{+,(2)}_{a,\{i,1,k\}}\Big|_{q\|p_1}$$

$$= \left(\frac{\mu^2}{x_q s_{1q}}\right)^{2\epsilon} \sum_{i\neq k\neq 1} \delta_{0,\lambda_{ik}}\delta_{1,\lambda_{1k}} \Bigg\{ f^{ba_k a_i}\boldsymbol{T}_q^b \boldsymbol{T}_k^{a_k} \boldsymbol{T}_i^{a_i} \times \Bigg[$$

$$\frac{1}{\epsilon^2}\left(\mathrm{i}\pi\log v_k^{1i} - \pi^2\right) - \frac{\mathrm{i}\pi^3}{6}\log v_k^{1i} + 4\mathrm{i}\pi\zeta_3 + 15\zeta_4 + \frac{8\pi}{3}\left(\arg(z_k^{1i})^3 - \pi^2\arg(z_k^{1i})\right)\Bigg] \rightarrow \Delta^{(2)}_{G,soft}$$

$$+ \left[(\boldsymbol{T}_q\cdot\boldsymbol{T}_i)(\boldsymbol{T}_q\cdot\boldsymbol{T}_k) + (\boldsymbol{T}_q\cdot\boldsymbol{T}_k)(\boldsymbol{T}_q\cdot\boldsymbol{T}_i)\right]\left(\frac{\pi^2}{\epsilon^2} - 15\zeta_4\right)\Bigg\}\mathbf{Sp}^{(0)},$$

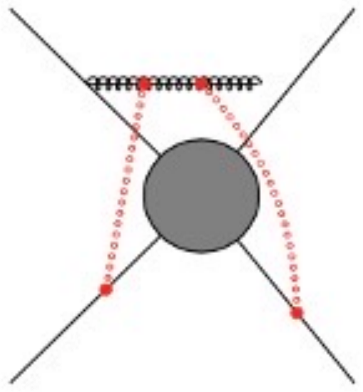
$$\subset \frac{1}{2}[\phi_G^{(1)}]\text{^}2$$

Discussions and outlook

$-\Delta_G^{(2)}$ and $\Delta_{G,soft}^{(2)}$ are identical.

Conjecture: the two-loop tripole function is universal

Dominated by pure soft/ Coulomb interactions (?)
-> region analysis for the two-loop nonplanar five-point integrals



$-\ \phi_G^{(1,2)}$ is a pure phase

does the exponentiation for dipole function also holds in QCD?

-- future investigations of more complicated factorization-breaking configurations

Thank you for your attention !