ETH zürich

Recursive Formulae for Loop Integrands in Loop-Tree Duality

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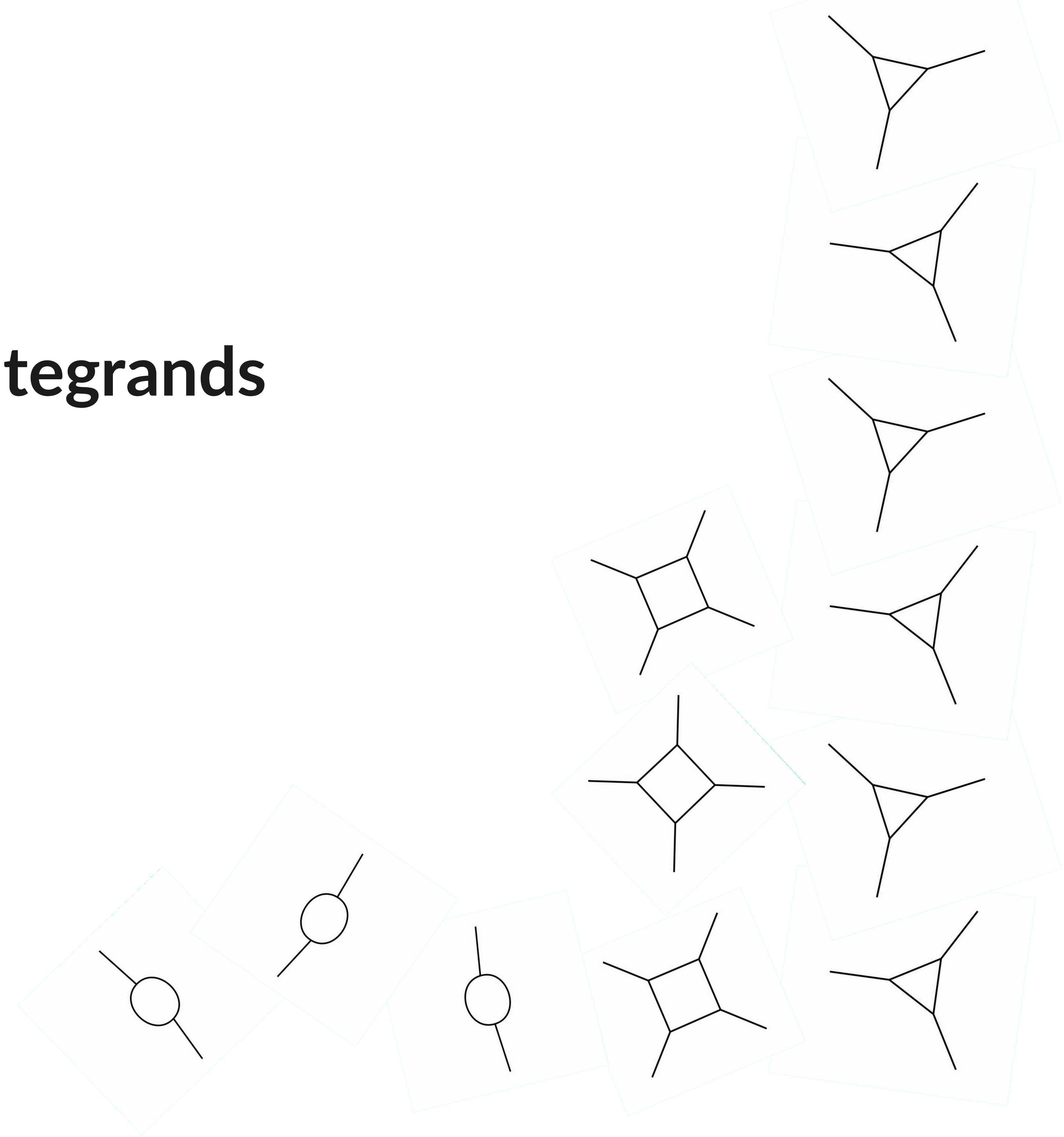


Numerical evaluation of Loop diagrams

Several analytical techniques to evaluate arbitrary loop integrals through a general strategy consisting in the reduction to Master Integrals using IBP and the solution of the PDEs relating them

[Tkachov 1981] [Chetyrkin, Tkachov 1981] [Kotikov 1991]
[Drummond, Henn, Trnka 2011] [Henn 2013]

Exploitation of different (3D) integral representations (LTD, cLTD, CFF) offers opportunities for direct numerical integration

[Soper 2000] [Catani, Gleisberg et al 2008]
[Capatti, Hirschi, Kermanschah 2019] [with Pelloni, Ruijl 2020]
[Capatti 2023]

$\longrightarrow$ Seemingly “pedestrian”, but not devoid of precious theoretical insights!

Loop-Tree duality

Exploits a different representation of Loop Integrals

$$I = \int \prod_{j=1}^{n} \frac{d^4 k_j}{(2\pi)^4} \frac{N}{\prod_{i\in\mathfrak{e}} D_i} \xrightarrow[\text{Residue Thm at the poles } D_i = 0]{\text{Energy integration}} I = (-i)^n \int \prod_{j=1}^{n} \frac{d^3 \vec{k}_j}{(2\pi)^3} \sum_{\mathfrak{b}\in\mathfrak{B}} \mathrm{Res}_{\mathfrak{b}}[\mathcal{I}].$$

Advantages:

- Simple singular structure in terms of (integrable) thresholds

  [Capatti, Hirschi, Kermanschah, Pelloni, Ruijl 2019]
  [Kermanschah 2021]

- Interpretation as phase space integrals allows for direct cancellation of real/virtual IR singular emissions [Capatti, Hirschi, Pelloni, Ruijl 2020] [Capatti 2021] [Capatti, Hirschi, Ruijl 2022]

- Allows for an interpretation of the integrand as a sum of tree-level diagrams

Loop-Tree duality

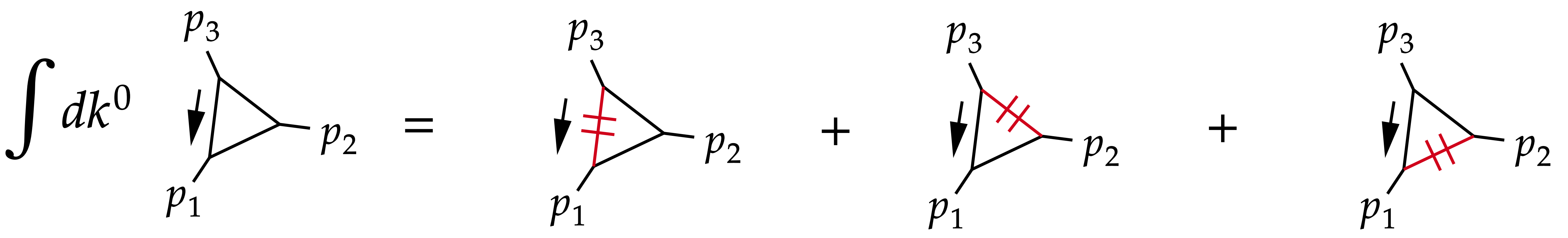
What is exactly the residue $\mathrm{Res}_{\mathfrak{b}}[\mathcal{I}]$?

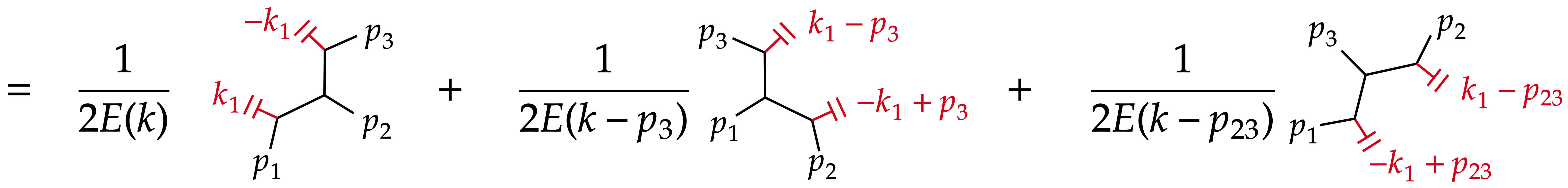
$$\mathrm{Res}_{\mathfrak{b}}[\mathcal{I}] = \frac{1}{\prod_{i\in\mathfrak{b}} 2E_i} \frac{N}{\prod_{i\in\mathfrak{e}/\mathfrak{b}} D_i}\Bigg|_{k^0 = k_{\mathfrak{b}}^{\sigma_{\mathfrak{b}}}}$$

$$D_i = q_i^2 - m_i^2 + i\varepsilon = (q_i^0 - E_i)(q_i^0 + E_i)$$

Each pole $D_i = 0$, corresponds to putting each time a different loop leg on-shell

$\longrightarrow$ The LTD integrand is a weighted sum of ***spanning trees***





Amplitudes?

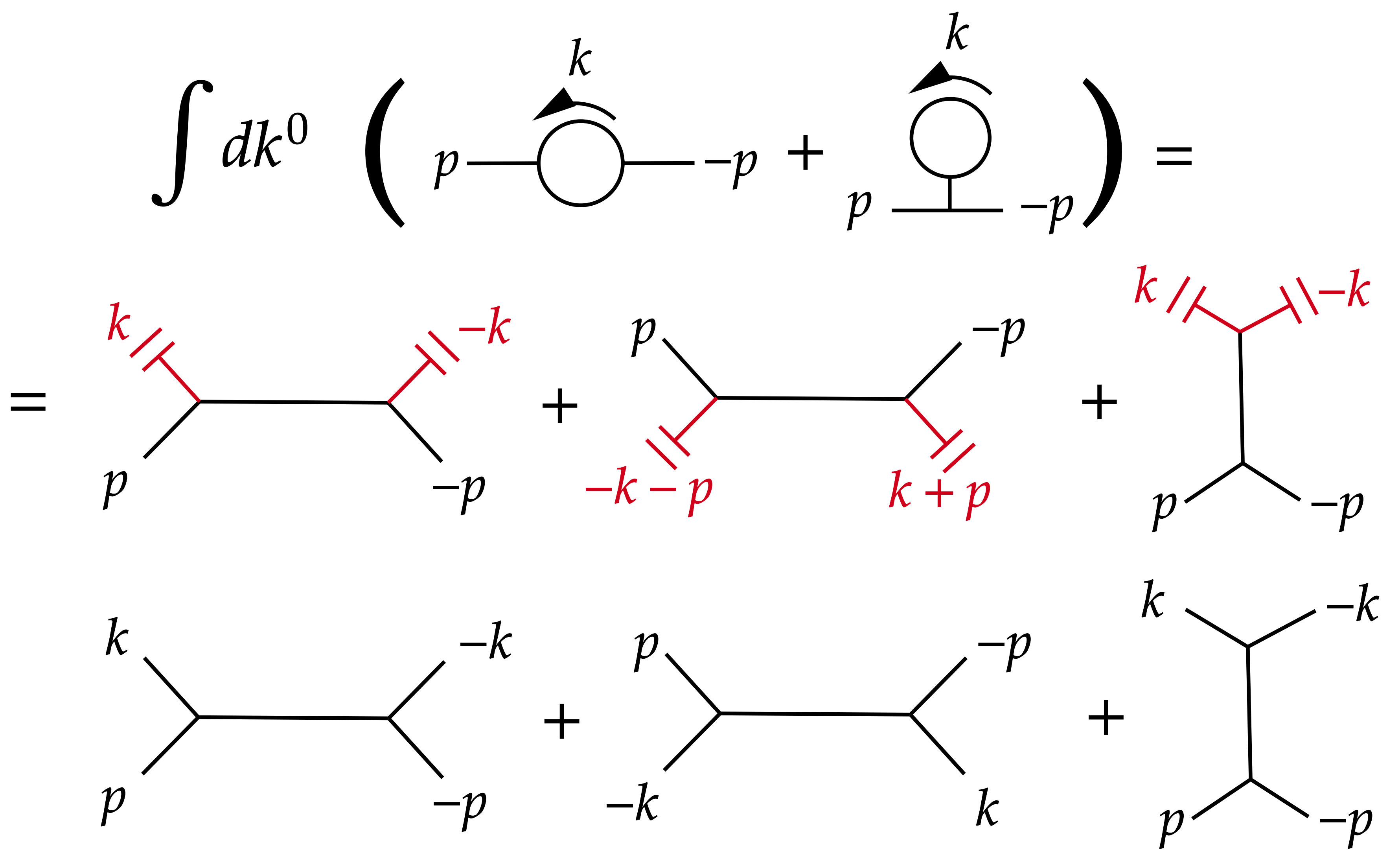
LTD provides an expression for integrands as sums of tree diagrams

However, when computing amplitudes, one is confronted with an explosion in the number of diagrams contributing

LTD integrands for amplitudes are even worse!
Instead of each diagram one should generate all the spanning trees

$\longrightarrow$ Totally unfeasible without simplifications

Integrands and tree amplitudes



$$\mathcal{I}_n^{\text{1-loop}} \sim \mathcal{M}_{n+2}^{\text{tree}}$$

Almost a 4-point tree amplitude!
If not for some kinematical shifts

[Catani, Gleisberg et al. 2008]
[Geyer, Mason et al. 2015]

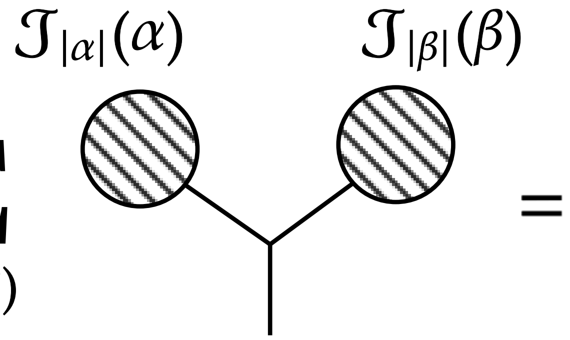
Can we exploit simplicity in the generation of tree amplitudes to obtain such kind of objects? $\longrightarrow$ YES! ***Modified Berends-Giele currents***

Berends-Giele recursion relations

Tree level amplitudes can be built recursively by combining at a vertex lower-point subamplitudes

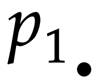
Off-shell currents:

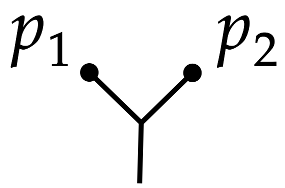
$$\mathcal{J}_n(\gamma) = \sum_{(\alpha,\beta)}$$

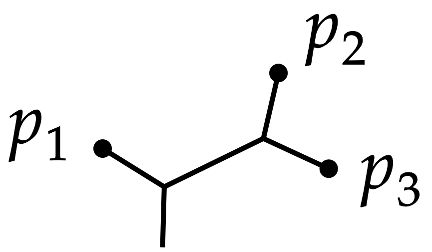


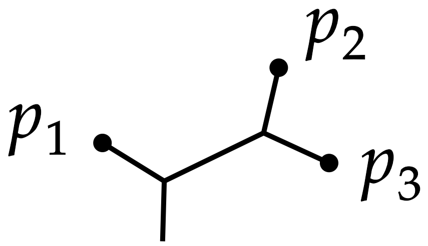
$$= \mathrm{Prop}(P_\gamma) \sum_{(\alpha,\beta)} V(P_\alpha, P_\beta, P_\gamma)\mathcal{J}_{|\alpha|}(\alpha)\mathcal{J}_{|\beta|}(\beta)$$

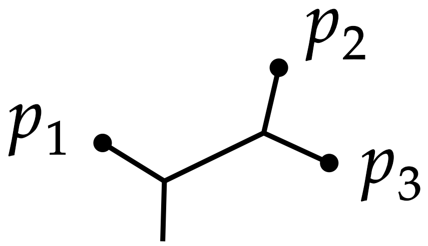
For every choice of subset pairs α, β

$\mathcal{J}_1(p_1) =$ 

$\mathcal{J}_2(p_1, p_2) =$ 

$\mathcal{J}_3(p_1, p_2, p_3) =$ 

$+$ 

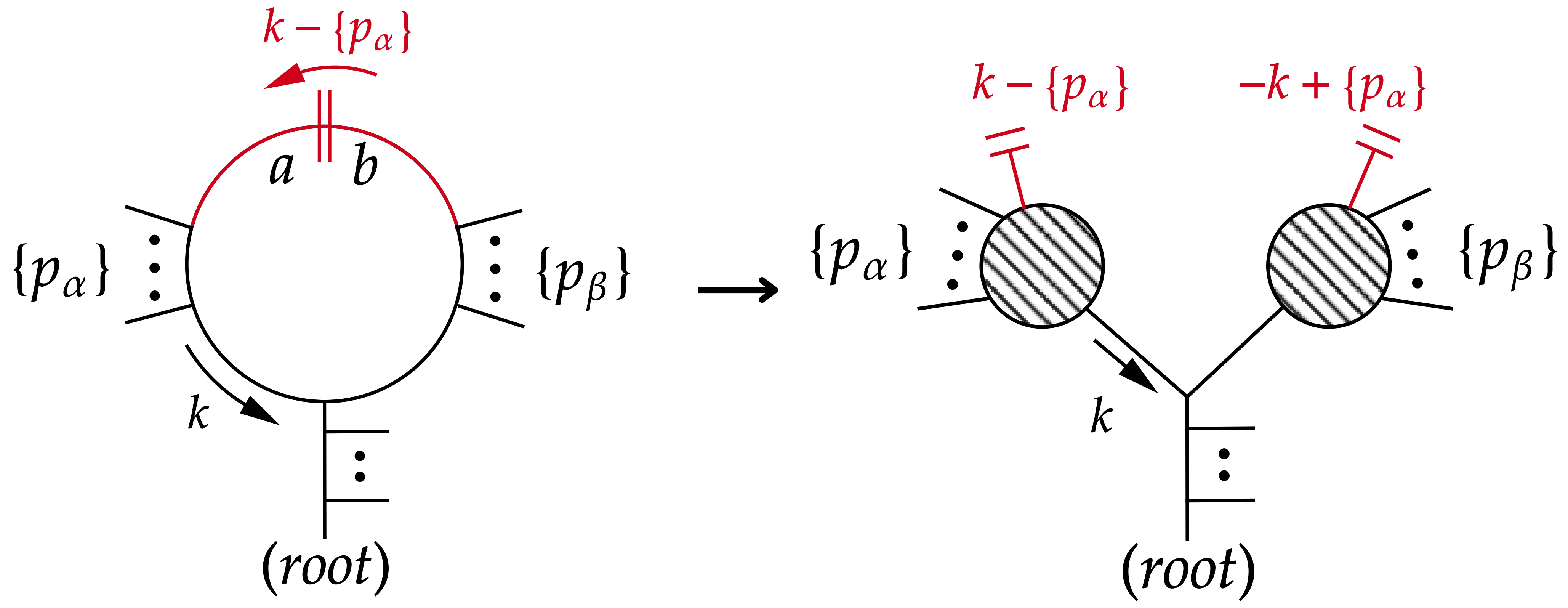
$+$ 

[Berends, Giele 1988]
[Duhr, Höche, Maltoni 2006]

Modified BG recursion relations

Can we build recursively the LTD integrands?

Fix a loop momentum routing and find all the trees respecting it



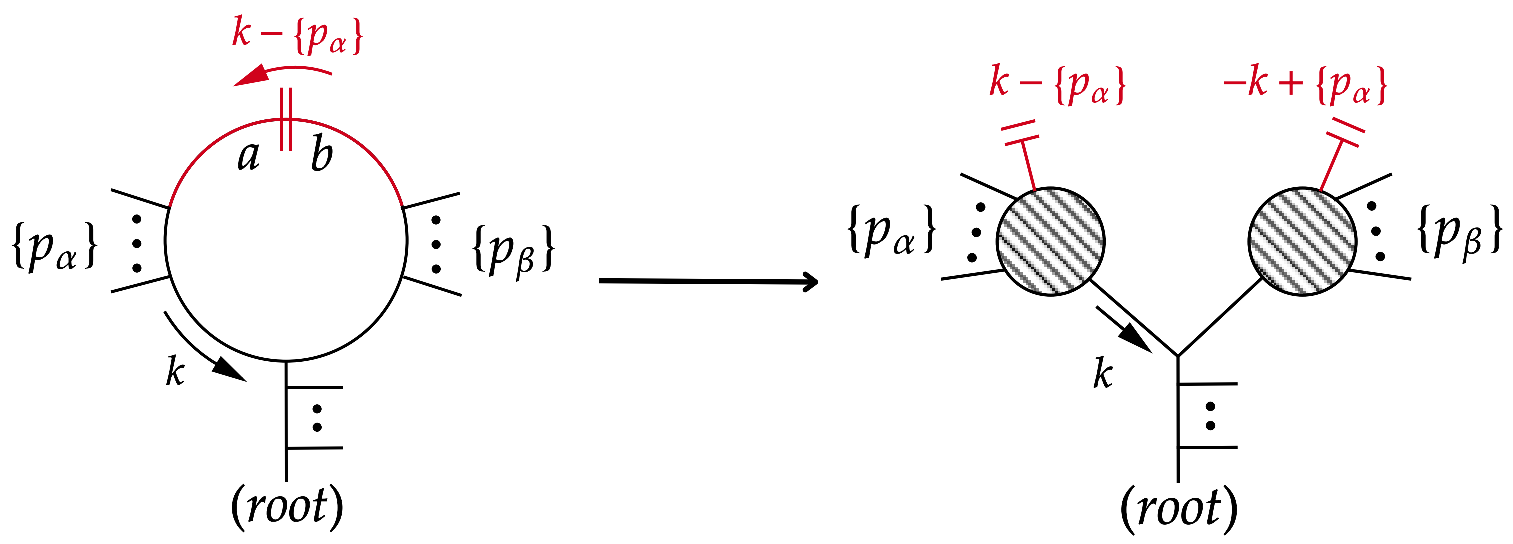
Ideally, it corresponds to taking all the trees with external momenta

$$\{p_2, p_3, \ldots, a, b\}$$

and assigning the values of a, b only when first appearing into two different branches

Modified BG recursion relations

Can we build recursively the LTD integrands?



We introduce a ***modified BG current*** that generates the desired trees

$$\mathcal{J}_n^{(1)}(k,p_2,\dots,p_{n-2}) =$$

$$\mathrm{Prop}(P_\gamma)\Bigg\{\sum_{\alpha\ni a,\ \beta\ni b} V(P_\alpha,P_\beta,-P_\alpha-P_\beta)\frac{1}{2E(k-P_\alpha)}\mathcal{J}_{|\alpha|}(k-P_\alpha,\{p_\alpha\})\mathcal{J}_{|\beta|}(-k+P_\alpha,\{p_\beta\})$$

Assigns the values of a, b
if in different subset

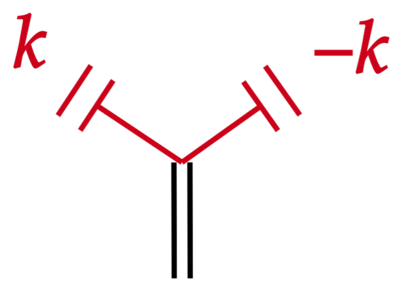
$$+\sum_{\alpha\ni a,b\ \beta\not\ni a,b} V(P_\alpha,P_\beta,-P_\alpha-P_\beta)\mathcal{J}_{|\alpha|}^{(1)}(k,\{p_\alpha\})\mathcal{J}_{|\beta|}(\{p_\beta\})$$

Postpones it to the
next iteration

$$+\sum_{\alpha\not\ni a,b\ \beta\ni a,b} V(P_\alpha,P_\beta,-P_\alpha-P_\beta)\mathcal{J}_{|\alpha|}(\{p_\alpha\})\mathcal{J}_{|\beta|}^{(1)}(k,\{p_\beta\})\Bigg\}$$

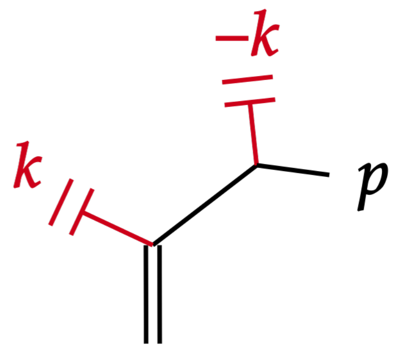
Modified BG recursion relations

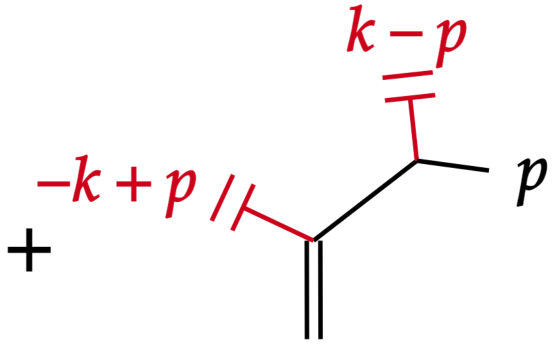
Enough for the amplitude integrand? NO, should remove 1/0 divergences

$\mathcal{J}_2^{(1)}(k) =$ 

$\stackrel{!}{=} 0$

Tadpoles

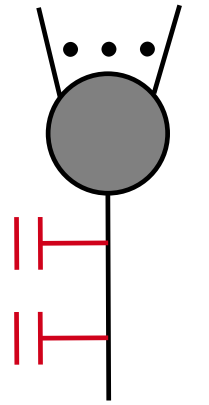
$\mathcal{J}_3^{(1)}(k,p) =$ 

$+$ 

$\stackrel{!}{=} 0$

Self-energy insertions
at the external legs

In addition, we should remove from the sum all the subset choices giving



$(root)$

Self-energy insertions
at the root

After that, we have a compact recursive expression for the integrand
Tested and verified numerically

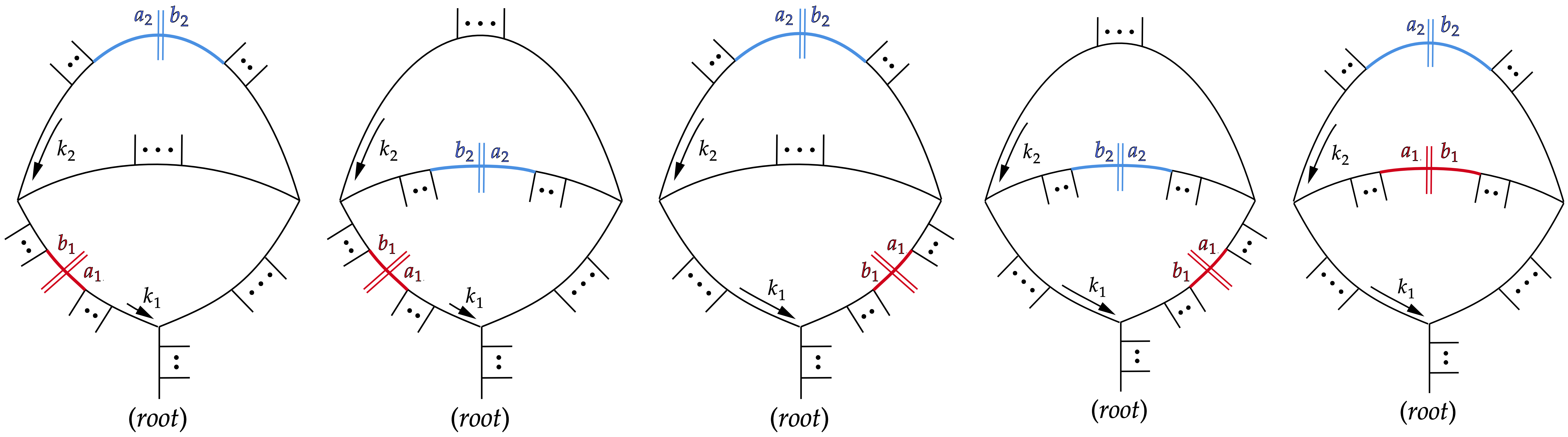
One-loop integrand formula

$$
\begin{aligned}
\mathcal{I}_n^{\text{1-loop}}(p_1,\dots,p_n) = \mathcal{J}_1(p_1)\sum_{\bar\alpha,\bar\beta}\Bigg\{&\bigg(\frac{1}{2E(k-P_{\bar\alpha})}\mathcal{J}_{|\bar\alpha|+1}\big(k-P_{\bar\alpha},\{p_{\bar\alpha}\}\big)\mathcal{J}_{|\bar\beta|+1}\big(-k+P_{\bar\alpha},\{p_{\bar\beta}\}\big)\\
&+\frac{1}{2E(k-P_{\bar\beta})}\mathcal{J}_{|\bar\alpha|+1}\big(-k+P_{\bar\beta},\{p_{\bar\alpha}\}\big)\mathcal{J}_{|\bar\beta|+1}\big(k-P_{\bar\beta},\{p_{\bar\beta}\}\big)\bigg)\\
&+\Big[\mathcal{J}^{(1)}_{|\bar\alpha|+2}\big(a,b,\{p_{\bar\alpha}\}\big)\mathcal{J}_{|\bar\beta|}\big(\{p_{\bar\beta}\}\big)+\mathcal{J}_{|\bar\alpha|}\big(\{p_{\bar\alpha}\}\big)\mathcal{J}^{(1)}_{|\bar\beta|+2}\big(a,b,\{p_{\bar\beta}\}\big)\Big]\\
+\frac{1}{2E(k)}\mathcal{J}_1(k)\mathrm{Prop}(-k+P_{\bar\alpha+\bar\beta})&\Big[\mathcal{J}_{|\bar\alpha|}\big(\{p_{\bar\alpha}\}\big)\mathcal{J}_{|\bar\beta|+1}\big(-k,\{p_{\bar\beta}\}\big)+\mathcal{J}_{|\bar\alpha|+1}\big(-k,\{p_{\bar\alpha}\}\big)\mathcal{J}_{|\bar\beta|}\big(\{p_{\bar\beta}\}\big)\Big]\\
&+\frac{1}{2E(k-P_{\bar\beta})}\mathcal{J}_1(-k+P_{\bar\beta})\mathrm{Prop}(k+P_{\bar\alpha})\mathcal{J}_{|\bar\alpha|}\big(\{p_{\bar\alpha}\}\big)\mathcal{J}_{|\bar\beta|+1}\big(k-P_{\bar\beta},\{p_{\bar\beta}\}\big)\\
&+\frac{1}{2E(k-P_{\bar\alpha})}\mathcal{J}_1(-k+P_{\bar\alpha})\mathrm{Prop}(k+P_{\bar\beta})\mathcal{J}_{|\bar\alpha|+1}\big(k-P_{\bar\alpha},\{p_{\bar\alpha}\}\big)\mathcal{J}_{|\bar\beta|}\big(\{p_{\bar\beta}\}\big)\Bigg\}
\end{aligned}
$$

Multiloop extension

At higher loops analogous strategy $\mathcal{I}_n^{l\text{-loop}} \sim \mathcal{M}_{n+2l}^{\text{tree}}$

Fix a consistent momentum routing and build currents that assign the extra parameters $a_1, b_1, a_2, b_2, \ldots, a_l, b_l$ to reproduce the relevant cuts of a general 1PI, *l*-loop diagram



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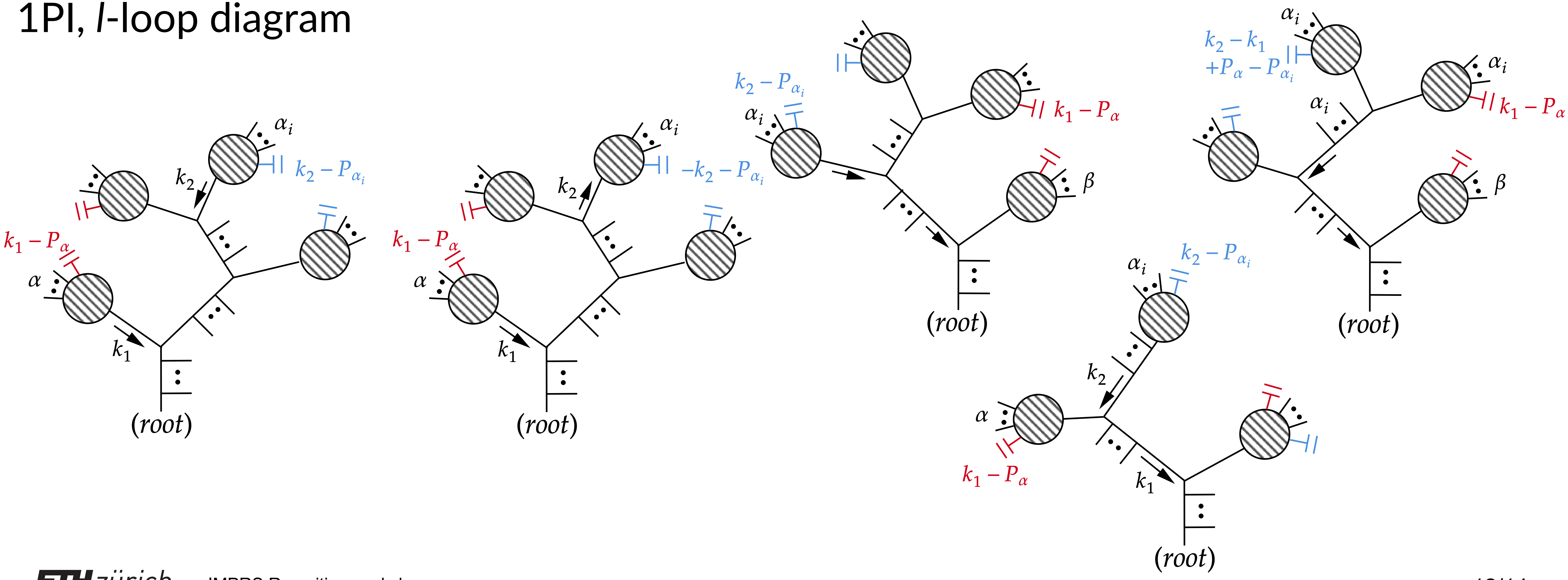
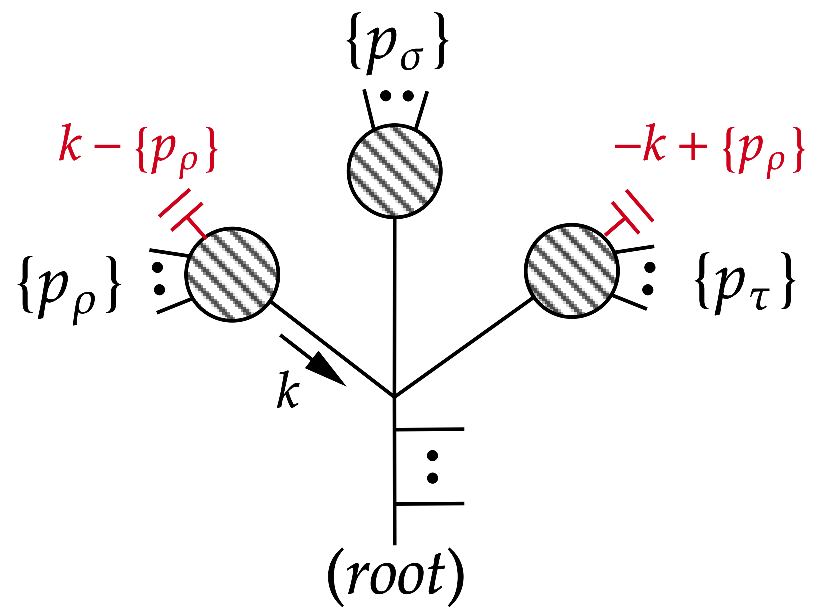


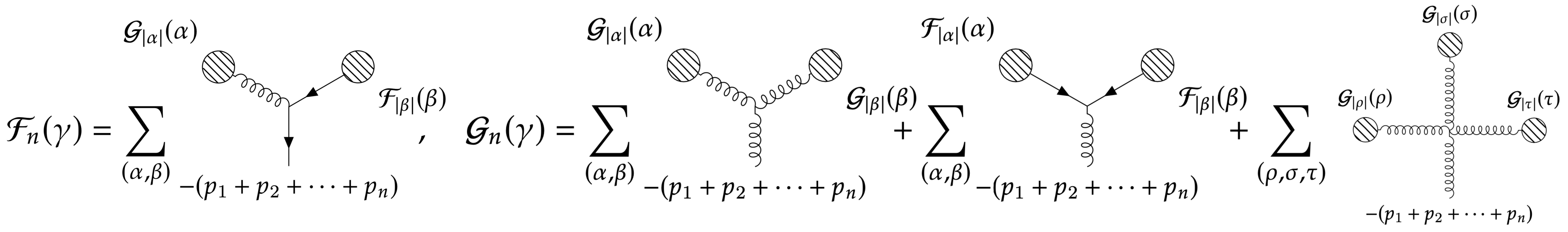
Conclusions and outlook

We have derived recursion relations to derive arbitrary *n*-point *l*-loop integrands in the LTD formalism through the construction of suitably modified BG currents



This formalism holds for any theory, just needs knowledge of the corresponding Feynman rules $\mathrm{Prop}(P_\gamma)$ and $V(P_\alpha, P_\beta, P_\gamma)$ (or $V(P_\rho, P_\sigma, P_\tau, P_\gamma)$)

In presence of amplitudes involving multiple particle species, one should really define different families of currents for each different species



Conclusions and outlook

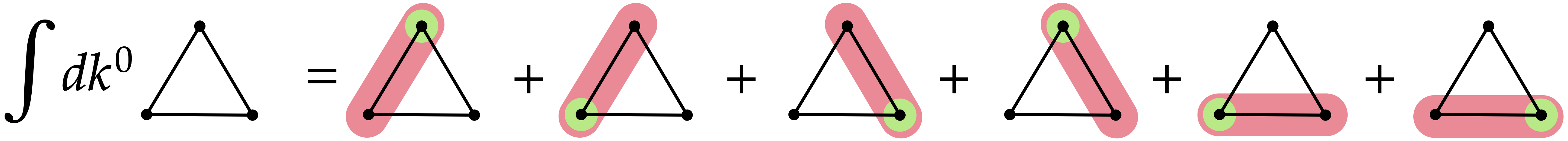
- Surfaceology methods derived differential equations relating loop integrands and tree amplitudes in planar gauge theories

$$\mathrm{Res}\,\mathcal{I}_n^{l-\text{loop}} = \hat{O}(\partial)\mathcal{M}_{n+2l}^{\text{tree}}$$

  Can we expect to derive analogous relations between pure and modified currents to extend these results to general theories?

[Arkani-Hamed, Cao, Dong, Figuereido, He 2024]
[Arkani-Hamed, Frost, Salvatori 2024]

- What about other 3D representations?



  Is there an Associahedron/Cosmohedron like correspondence between loop amplitudes and their CFF integrands?

[Capatti 2023]
[Arkani-Hamed, Benincasa, Postnikov 2017]
[Arkani-Hamed, Figuereido, Vazão 2024]

Thank you for your attention!