On Non-Invertible Duality Symmetries in Maxwell Theory

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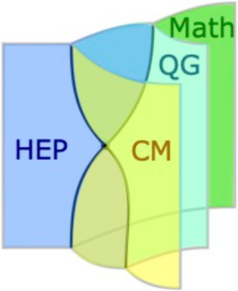
The aim of this thesis consists to explore the modular symmetry, extension of duality symmetry, in the context of Maxwell theory and study its Non-Invertible counterpart, exploiting the modern advancement in the context of **Generalized Symmetries**.

What is the importance of Generalized Symmetries?

- IR Constraints
- Symmetries in String theory/Holography
- Quantum Gravity Constraints
- Relation to Condensed Matter



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Generalized symmetries

Ordinary self-duality symmetry → Non invertible self-duality symmetry

Anomalies in Maxwell theory

Topological constraints

Anomalies in Maxwell theory

Topological constraints

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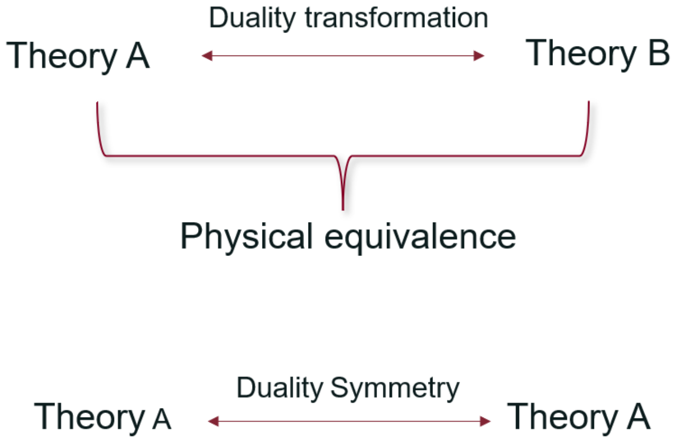
Duality $\neq$ Symmetry



Setup

[Witten, 1995]

We are in a 4d compact, torsion-free, manifold.

Free Maxwell theory with topological θ term

$$S = \frac{1}{2e^2}\int_X F\wedge *F + i\frac{\theta}{8\pi^2}\int_X F\wedge F =$$
$$= \int_X d^4x\left[\frac{1}{4e^2}\sqrt{g}F_{\mu\nu}F^{\mu\nu} + i\frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]$$

Here

$$\theta \in \begin{cases}[0,2\pi] & \text{if } X \text{ Spin Manifold}\\ [0,4\pi] & \text{if } X \text{ non-Spin Manifold}\end{cases}$$

given

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \in \mathbb{H}$$

The Maxwell action can be written as

$$
\begin{aligned}
S = \frac{i}{4\pi}\int_X \left[\overline{\tau} F^+ \wedge F^+ + \tau F^- \wedge F^-\right] = \\
= \frac{i}{8\pi}\int_X d^4x\sqrt{g}\left[\overline{\tau}(F^+)_{\mu\nu}(F^+)^{\mu\nu} - \tau(F^-)_{\mu\nu}(F^-)^{\mu\nu}\right]
\end{aligned}
$$

We want to analyze transformations of the kind

$$
\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \qquad ad - bc = 1 \quad a, b, c, d \in \mathbb{Z}
$$

$$
\mathbb{T} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \longleftrightarrow T : \tau \to \tau + 1,
$$

$$
\mathbb{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longleftrightarrow S : \tau \to -\frac{1}{\tau}
$$

$$
PSL(2,\mathbb{Z}) = \frac{SL(2,\mathbb{Z})}{\pm\mathbb{I}}
$$

$$
\mathbb{S}^2 = \mathbb{I} \quad (\mathbb{S}\mathbb{T})^3 = \mathbb{I}
$$

Since $\tau + 1 \longleftrightarrow \theta + 2\pi$, for a not Spin manifold we should consider the subgroup $\langle \mathbb{S}, \mathbb{T}^2 \rangle \subset PSL(2,\mathbb{Z})$.

Anomaly result

$$Z\left[\frac{a\tau+b}{c\tau+d}\right] = (i)^{\sigma/2}(c\tau+d)^{(\chi-\sigma)/2}(c\bar{\tau}+d)^{(\chi+\sigma)/2}Z[\tau]$$

σ Hirzebruch signature

χ Euler characteristic

$\tau = i$ fixed under S, not anomalous

$\tau = e^{2\pi i/3}$ fixed under ST, anomalous

Validity conditions for σ and χ

Spin Manifold

Symmetry group generated by $\mathbb{S}$ and $\mathbb{T}$

$$\sigma = 16n \quad \text{with } n \in \mathbb{Z}$$
$$Z[-\frac{1}{\tau}] = (\tau)^{\frac{\chi-\sigma}{4}}(\bar{\tau})^{\frac{\chi+\sigma}{4}} Z[\tau]$$
$$Z[\tau+1] = Z[\tau]$$

charge conjugation: $\mathbb{S}^2 = -\mathbb{I} \implies (-1)^{\chi/2} = 1 \implies \chi = 4n \quad$ with $n \in \mathbb{Z}$.
It implies that under $\mathbb{S}^4$ the anomaly disappears.
Therefore

$$\chi \in 4\mathbb{Z}$$

non Spin Manifold

Symmetry group generated by $\mathbb{S}$ and $\mathbb{T}^2$

$$Z[-\frac{1}{\tau}] = (i)^{\sigma/2}(\tau)^{\frac{\chi-\sigma}{4}}(\bar{\tau})^{\frac{\chi+\sigma}{4}} Z[\tau]$$
$$Z[\tau+2] = Z[\tau]$$

- Anomaly disappearing under $\mathbb{S}^2$:
  $(i^{\sigma/2})^2(-1)^{\chi/2} = e^{\frac{i\pi}{2}(\sigma+\chi)} \implies \sigma + \chi \in 4\mathbb{Z}$
- Anomaly disappearing under $\mathbb{S}^4$:
  $(i^{\sigma})^2 = 1 \implies \sigma \in 2\mathbb{Z}$

Therefore

$$\frac{\sigma}{2} = \frac{\chi}{2} mod2 \quad \text{with } \chi, \sigma \in 2\mathbb{Z}$$

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3 Generalized symmetries

Ordinary symmetries

- Parameter of the transformation: **0-form**
- Charged objects: fields
- **Group-like** composition rules

Higher-form symmetries

- Parameter of the transformation: **q-form**
- Charged objects: operators supported on q-dimensional submanifolds
- **Group-like** composition rules
- **Abelian** group structure

Non-Invertible symmetries

Not group-like fusion rules

Symmetry $\longleftrightarrow$ Topological operator

Non-invertible symmetries lack a group-like composition law. The topological operators instead obey the so-called **fusion rules**:

$$U^a(\Sigma_{d-q-1}) \otimes U^b(\Sigma_{d-q-1}) = \bigoplus_c (N^q)^{ab}_c U^c(\Sigma_{d-q-1})$$

with $q =, .., d-1$

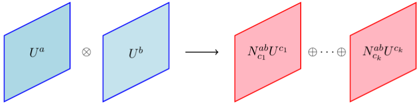

Figure 2.5. Representation of the fusion product for generic $(d-q-1)$-dimensional non invertible defects.

How do we construct defects with non-invertible fusion rules?

General construction: **Stacking with a TQFT** and **gauging of a global symmetry**.

Example:

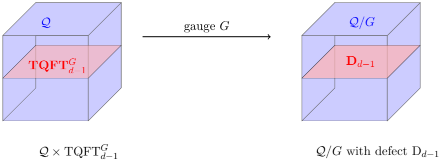

Figure 2.6. d-dimensional theory $\mathcal{Q}$ with global symmetry G twisted with the $(d-1)$ topological field theory having the same global symmetry. After gauging the TQFT has become a defect.

Construction of the Non-Invertible symmetry

We make a composition of the S-duality with the following two operation:

- Stacking with the topological term $\frac{ipN_e}{4\pi}\int B_e \wedge B_e$ [Choi, Cordova, Hsin, Lam, Chao 2022]
- Gauging of $\mathbb{Z}^{(1)}_{N_e} \times \mathbb{Z}^{(1)}_{N_m}$ with $gcd(N_e, N_m) = 1$ [Cordova, Ohomori 2023]

We find

$$Z_{\mathcal{G}\mathcal{T}^p(\tau)} = (N_m)^{\chi} \cdot Z_{\frac{N_m^2}{N_e^2}(\tau+pN_e)}$$

i.e. the p-twisted theory is isomorphic to the original one, with a rescaling of the coupling

$$\tau \longrightarrow \frac{N_m^2}{N_e^2}(\tau + pN_e)$$

The goal now consists to find the fixed points of $\mathbb{S} \circ \mathcal{G}[(\mathcal{Z}^{(1)}_{N_e})_p \times \mathcal{Z}^{(1)}_{N_m}]$

The non-invertible transformation consists of:

$$\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \in SL(2,\mathbb{Q}) \qquad \tau \longrightarrow \frac{q_1\tau + q_2}{q_3\tau + q_4}$$

$$q_1q_4 - q_2q_3 = 1 \qquad q_1, q_2, q_3, q_4 \in \mathbb{Q}$$

For $p = 0$: $\quad \tau_{\text{fixed}} = i\frac{N_e}{N_m}$

For $p = N_m$: $\quad \tau_{\text{fixed}} = \frac{N_e}{N_m}e^{2i\pi/3}$

$$Z_{\mathcal{G}(\tau_{\text{fixed}})} = \mathcal{A} \cdot Z_{\tau_{\text{fixed}}} \quad \neq \quad Z_{\tau_{\text{fixed}}} = \mathcal{A} \cdot Z_{\tau_{\text{fixed}}}$$

How to find the anomalies?

First step in this direction could be to analyze the zeros of the partition function. In fact, under certain conditions

$$Z[\tau] = 0 \iff \text{Presence of an anomaly}$$

However we should have:

- A manifold for which $\exists G$ such that $Z[\overline{G}, \tau, \overline{\tau}] = Z[\overline{G}, \tau]Z[\overline{G}, \overline{\tau}]$
- An intersection form for which the corresponding generalize theta functions becomes 0 in correspondence of "interesting τ values" (example: rational multiples of i)
- An indefinite metric G (for Donaldson's theorem)

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5 Conclusions

The main results of this thesis are:

- An extension of Witten's calculation of the $SL(2,\mathbb{Z})$-duality anomaly to non-Spin manifolds.
- Identification of topological constraints on four-manifolds that can support Maxwell theory.
- A generalization of $SL(2,\mathbb{Z})$ symmetry to its non-invertible counterpart, $SL(2,\mathbb{Q})$.
- A conceptual idea for studying anomalies associated with non-invertible duality.

Developing the proposed procedure for identifying anomalous fixed points could bring us closer to the goal of finding a functional expression for the anomaly, potentially allowing for the imposition of new constraints.