MAX-PLANCK-INSTITUT
FÜR PHYSIK

Cosmohedra

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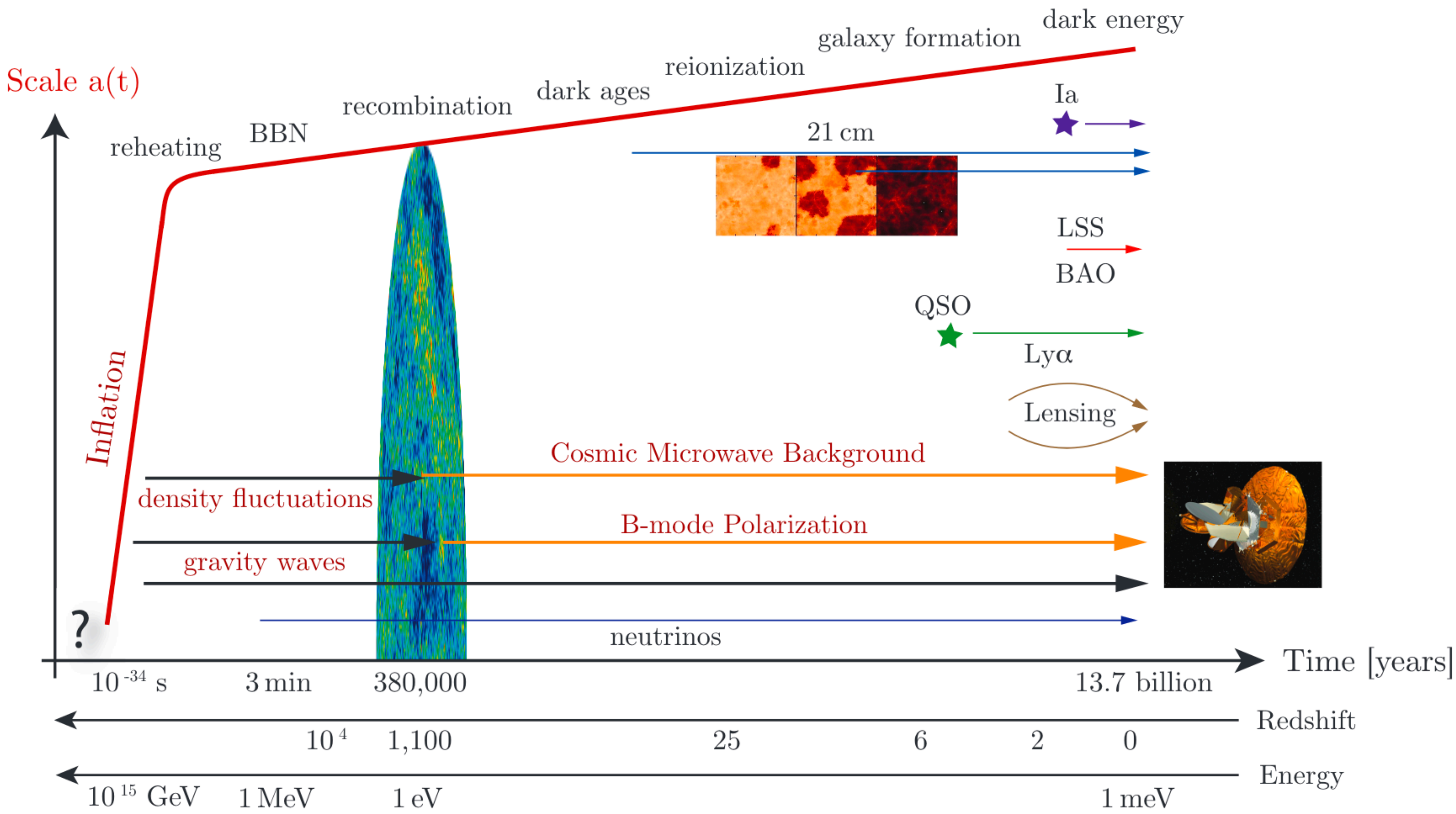
Outline

- The Wavefunction.
- The ABHY Associahedron.
- Cosmohedra.

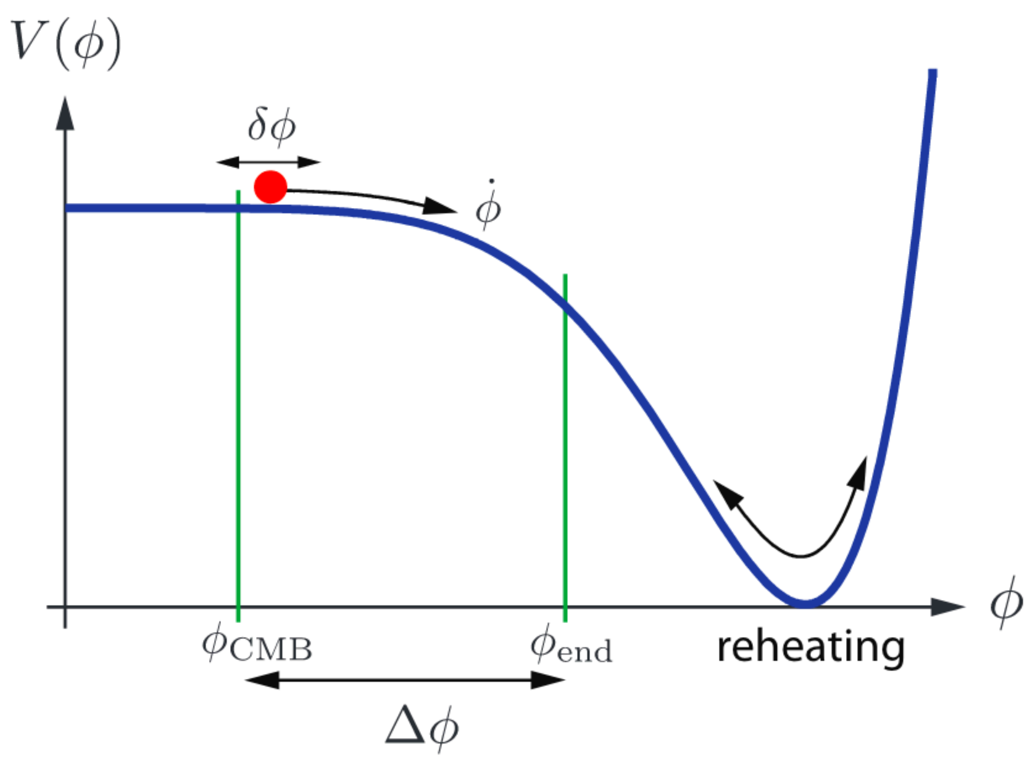
The Wavefunction

Inflation



Scale a(t)
reheating
BBN
recombination
dark ages
reionization
galaxy formation
dark energy
Inflation
Ia
21 cm
LSS
BAO
QSO
Lyα
Lensing
Cosmic Microwave Background
density fluctuations
B-mode Polarization
gravity waves
neutrinos
?
Time [years]
10^{-34} s
3 min
380,000
13.7 billion
Redshift
10^4
1,100
25
6
2
0
Energy
10^{15} GeV
1 MeV
1 eV
1 meV

Inflationary models



Scalar field sitting in a false vacuum

Scalar Potential such that the SEC is violated and the scalar field generates negative pressure

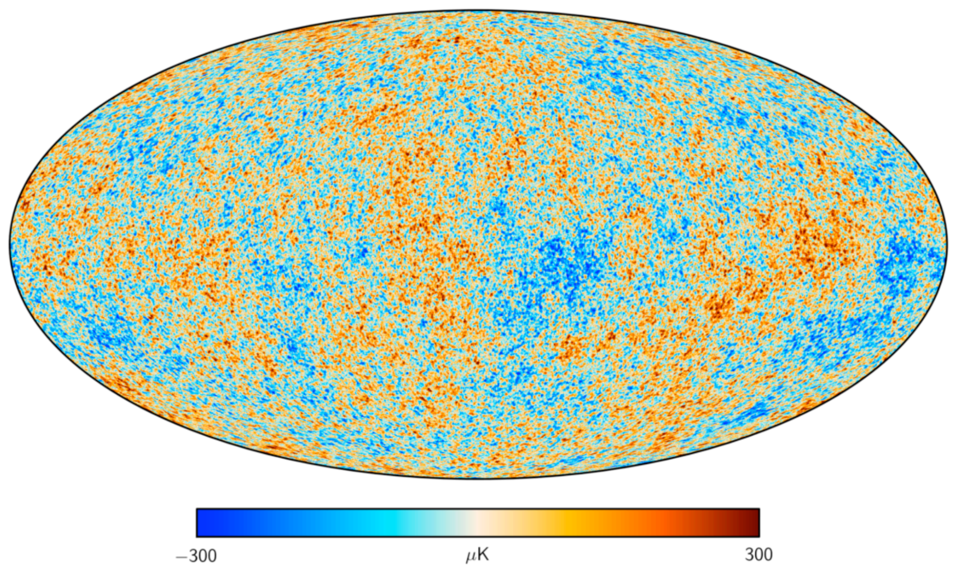
↓

Accelerated expansion while field slow rolls down the potential

↓

Field approaches the true vacuum, inflation stops and reheating starts

Cosmological Observables

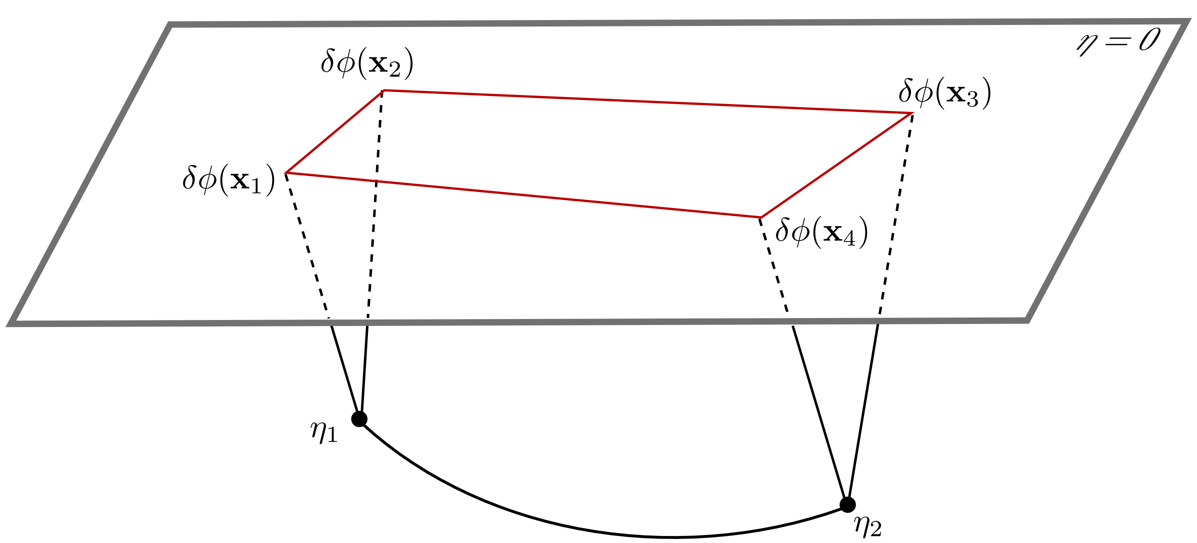


We expect that correlations of the temperature at different points in the Cosmic Microwave Background (CMB), $\langle \delta T(\mathbf{x}_1)\delta T(\mathbf{x}_2)...\delta T(\mathbf{x}_n)\rangle$ are related to quantum fluctuations of a scalar field during inflation, $\langle \delta\phi(\mathbf{x}_1)\delta\phi(\mathbf{x}_2)...\delta\phi(\mathbf{x}_n)\rangle$

The correlation functions of the fields at the end of inflation, can be defined as:

$$\langle \delta\phi(\mathbf{x}_1)\delta\phi(\mathbf{x}_2)...\delta\phi(\mathbf{x}_n)\rangle = \int_{\delta\phi(-\infty)}^{\delta\phi(0)} [\mathcal{D}\delta\phi]^n \delta\phi(\mathbf{x}_1)\delta\phi(\mathbf{x}_2)...\delta\phi(\mathbf{x}_n)|\Psi[\delta\phi]|^2$$

$\Psi[\delta\phi]$ is the vacuum wavefunction.



Wavefunction

The wavefunction can be defined as:

Colored scalars with cubic interactions

$$\Psi[\delta\phi] = \int_{\delta\phi(-\infty(1-i\epsilon))}^{\delta\phi(0)} \mathcal{D}[\varphi] e^{i\mathcal{S}[\varphi]} \qquad \mathcal{S}[\phi] = \int d^d x d\eta \, \frac{1}{2} \mathrm{Tr}\,(\partial\phi)^2 - \frac{\lambda_3(\eta)}{3} \mathrm{Tr}\,\phi^3$$

Bunch-Davies condition

$$\lambda_3(\eta) \equiv \lambda_3\, a(\eta) \qquad a(\eta) = \eta^{-(1+\varepsilon)}$$

$\varepsilon = -1$ Flat-space

$\varepsilon = 0$ dS

And it can be computed perturbatively, as follows:

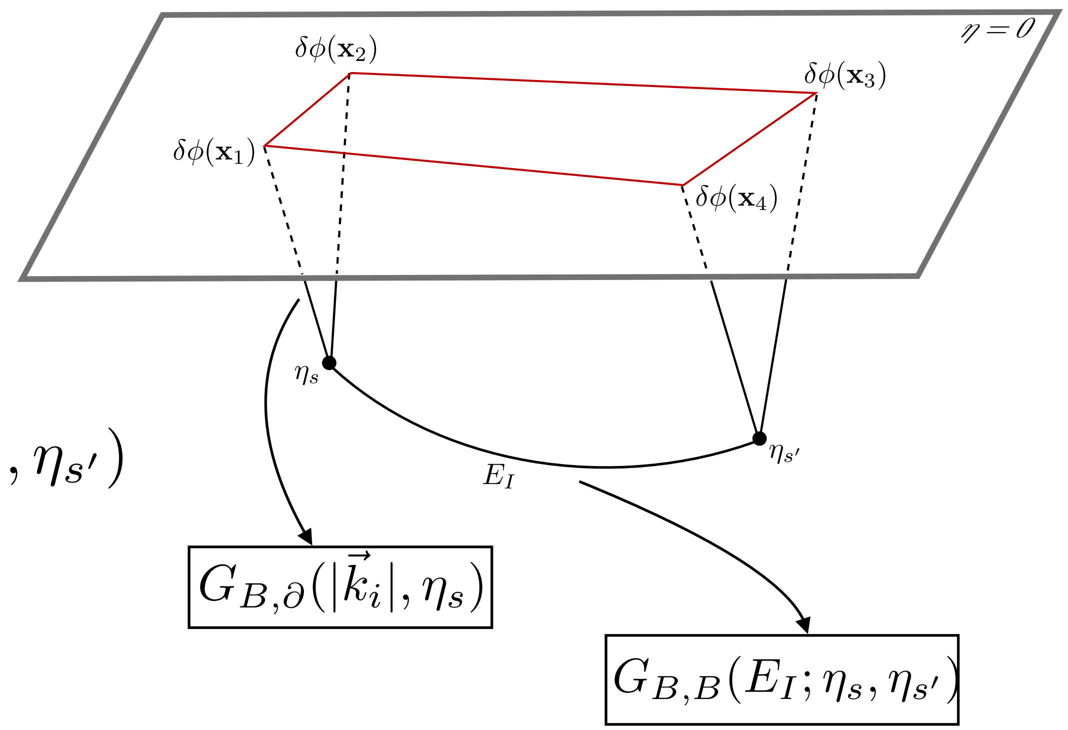
$$\Psi[\delta\phi] = \exp\left\{ \sum_{n \geq 2} \frac{1}{n!} \int \prod_{i=1}^{n} d^d k_i \, \delta\phi_i(k_i) \boxed{\psi^{(n)}[\vec{k}_i]} \delta^d \left(\textstyle\sum_i \vec{k}_i \right) \right\}$$

Wavefunction coefficient

$$\psi^{(n)} = \sum_{\mathcal{G}} \psi_{\mathcal{G}}$$

Wavefunction

Each wavefunction coefficient as an integral form of the type:



$$\psi_{\mathcal{G}} = \int_{-\infty}^{0} \prod_{s \in \mathcal{V}_s} \left[d\eta_s \lambda(\eta_s) \prod_{\vec{k}_i} G_{B,\partial}(|\vec{k}_i|, \eta_s) \right] \prod_{e \in \mathcal{E}} G_{B,B}(E_I; \eta_s, \eta_{s'})$$

$$G_{B,\partial}(E_k, \eta) = e^{iE_k \eta}$$

Bulk-to-Boundary propagator

$$G_{B,B}(E_k; \eta_1, \eta_2) = \frac{1}{2E_k} \left(e^{iE_k(\eta_1 - \eta_2)} \Theta(\eta_1 - \eta_2) + e^{iE_k(\eta_2 - \eta_1)} \Theta(\eta_2 - \eta_1) - e^{iE_k(\eta_1 + \eta_2)} \right)$$

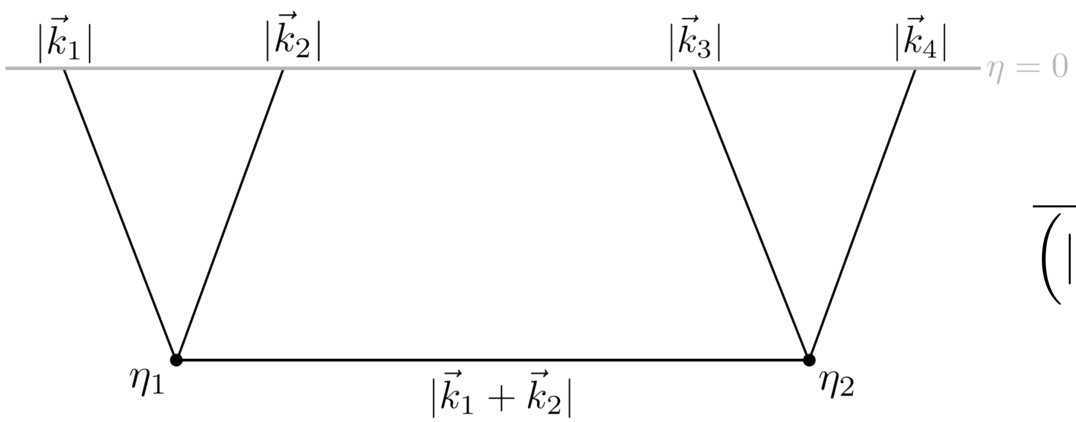
Bulk-to-Bulk propagator

For FLRW backgrounds, it is possible to perform the time integration by Fourier transforming the couplings, trading the time integration by a integration over the total external energy entering each site:

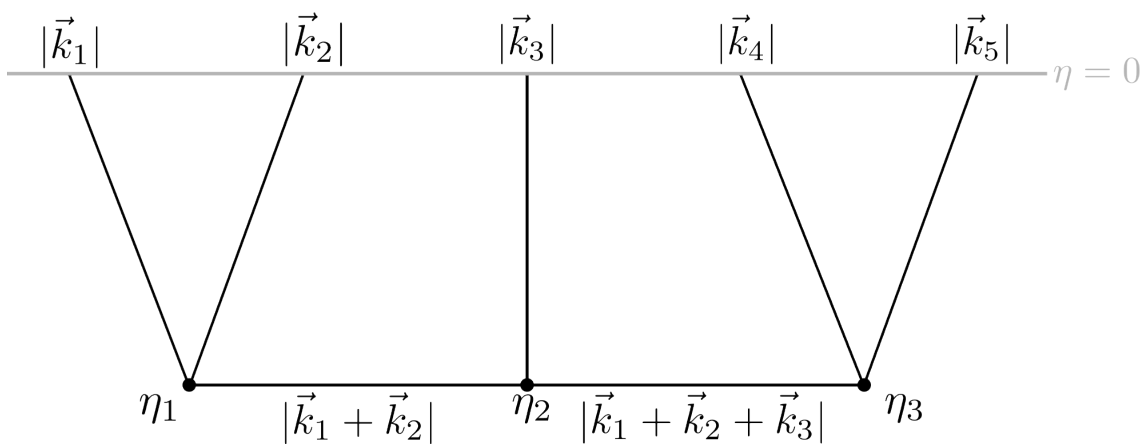
In FLRW: $$\lambda_3(\eta_s) = \int_{-\infty}^{+\infty} d\omega_s e^{i\omega_s \eta_s} \lambda_3(\omega_s)$$

Wavefunction: Diagrammatic Expansion

After performing the time integration, at tree-level, $\psi_{\mathcal{G}}$ is a rational function of the norm of the momenta:



$$\frac{1}{\left(|\vec{k}_1| + |\vec{k}_2| + |\vec{k}_3| + |\vec{k}_4|\right)\left(|\vec{k}_1| + |\vec{k}_2| + |\vec{k}_1 + \vec{k}_2|\right)\left(|\vec{k}_3| + |\vec{k}_4| + |\vec{k}_1 + \vec{k}_2|\right)}$$

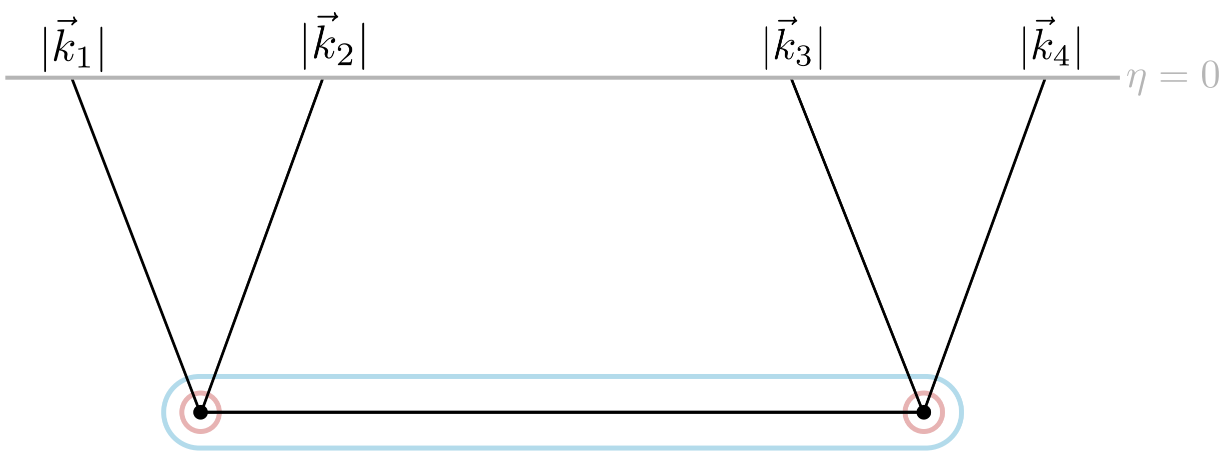


$$\frac{1}{\left(|\vec{k}_1| + |\vec{k}_2| + |\vec{k}_3| + |\vec{k}_4| + |\vec{k}_5|\right)\left(|\vec{k}_1| + |\vec{k}_2| + |\vec{k}_1 + \vec{k}_2|\right)\left(|\vec{k}_3| + |\vec{k}_1 + \vec{k}_2| + |\vec{k}_1 + \vec{k}_2 + \vec{k}_3|\right)} \times$$

$$\times \frac{1}{|\vec{k}_4| + |\vec{k}_5| + |\vec{k}_1 + \vec{k}_2 + \vec{k}_3|}\left(\frac{1}{|\vec{k}_3| + |\vec{k}_4| + |\vec{k}_5| + |\vec{k}_1 + \vec{k}_2|} + \frac{1}{|\vec{k}_1| + |\vec{k}_2| + |\vec{k}_3| + |\vec{k}_1 + \vec{k}_2 + \vec{k}_3|}\right)$$

Singularities of the Wavefunction

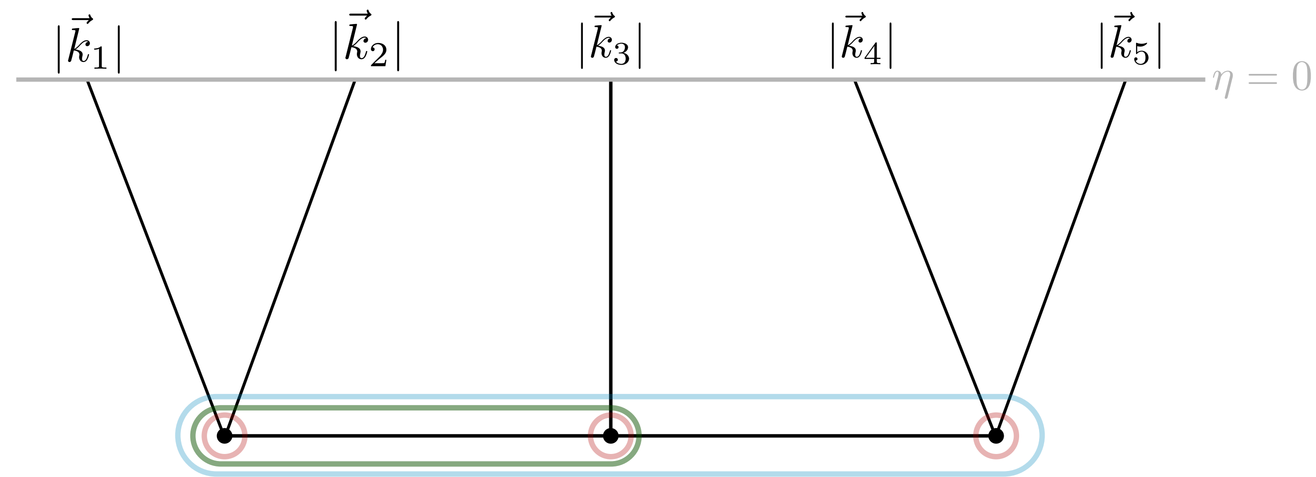
Every singularity of the integrand of the wavefunction, is directly associated with a subprocess. Diagrammatically, these subprocesses are represented as tubes in the graph:



$$\frac{1}{\left(|\vec{k}_1|+|\vec{k}_2|+|\vec{k}_3|+|\vec{k}_4|\right)\left(|\vec{k}_1|+|\vec{k}_2|+|\vec{k}_1+\vec{k}_2|\right)\left(|\vec{k}_3|+|\vec{k}_4|+|\vec{k}_1+\vec{k}_2|\right)}$$

Total energy pole, E_t

$\psi_4^{(0)}$



Total energy pole, E_t

$$\frac{1}{\left(|\vec{k}_1|+|\vec{k}_2|+|\vec{k}_3|+|\vec{k}_4|+|\vec{k}_5|\right)\left(|\vec{k}_1|+|\vec{k}_2|+|\vec{k}_1+\vec{k}_2|\right)\left(|\vec{k}_3|+|\vec{k}_1+\vec{k}_2|+|\vec{k}_1+\vec{k}_2+\vec{k}_3|\right)}\times$$
$$\times\frac{1}{|\vec{k}_4|+|\vec{k}_5|+|\vec{k}_1+\vec{k}_2+\vec{k}_3|}\left(\frac{1}{|\vec{k}_3|+|\vec{k}_4|+|\vec{k}_5|+|\vec{k}_1+\vec{k}_2|}+\frac{1}{|\vec{k}_1|+|\vec{k}_2|+|\vec{k}_3|+|\vec{k}_1+\vec{k}_2+\vec{k}_3|}\right)$$

$\psi_5^{(0)}$

The residue of the wavefunction on the total energy pole is the corresponding scattering amplitude!

$$\mathrm{Res}_{E_t=0}\Psi_n = \mathcal{A}_n$$

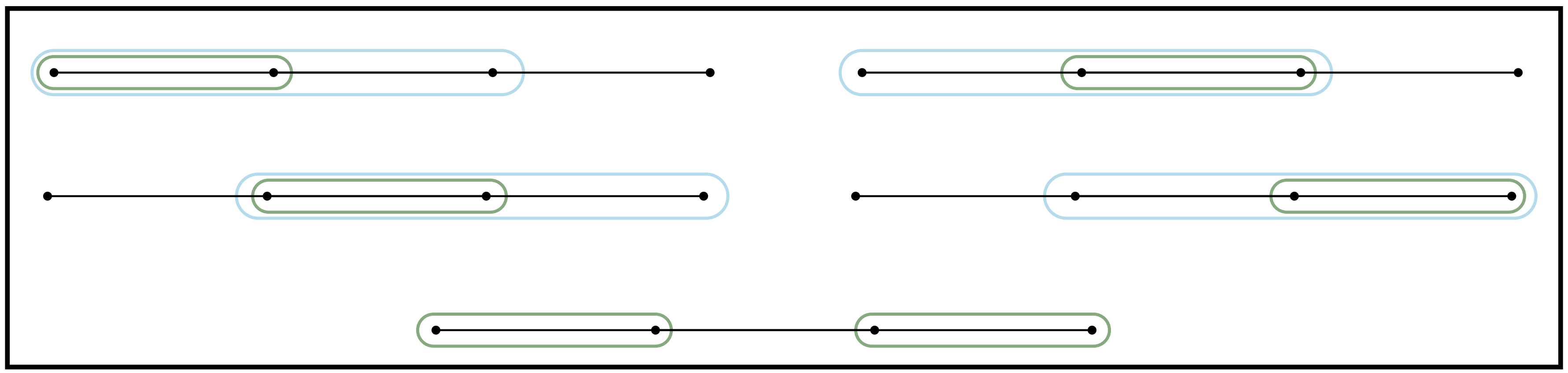
Example:

$$\underset{E_t=0}{\mathrm{Res}}\ \psi_4^{(0)} = \frac{1}{-(|\vec{k}_1|+|\vec{k}_2|)^2+(\vec{k}_1+\vec{k}_2)^2}$$

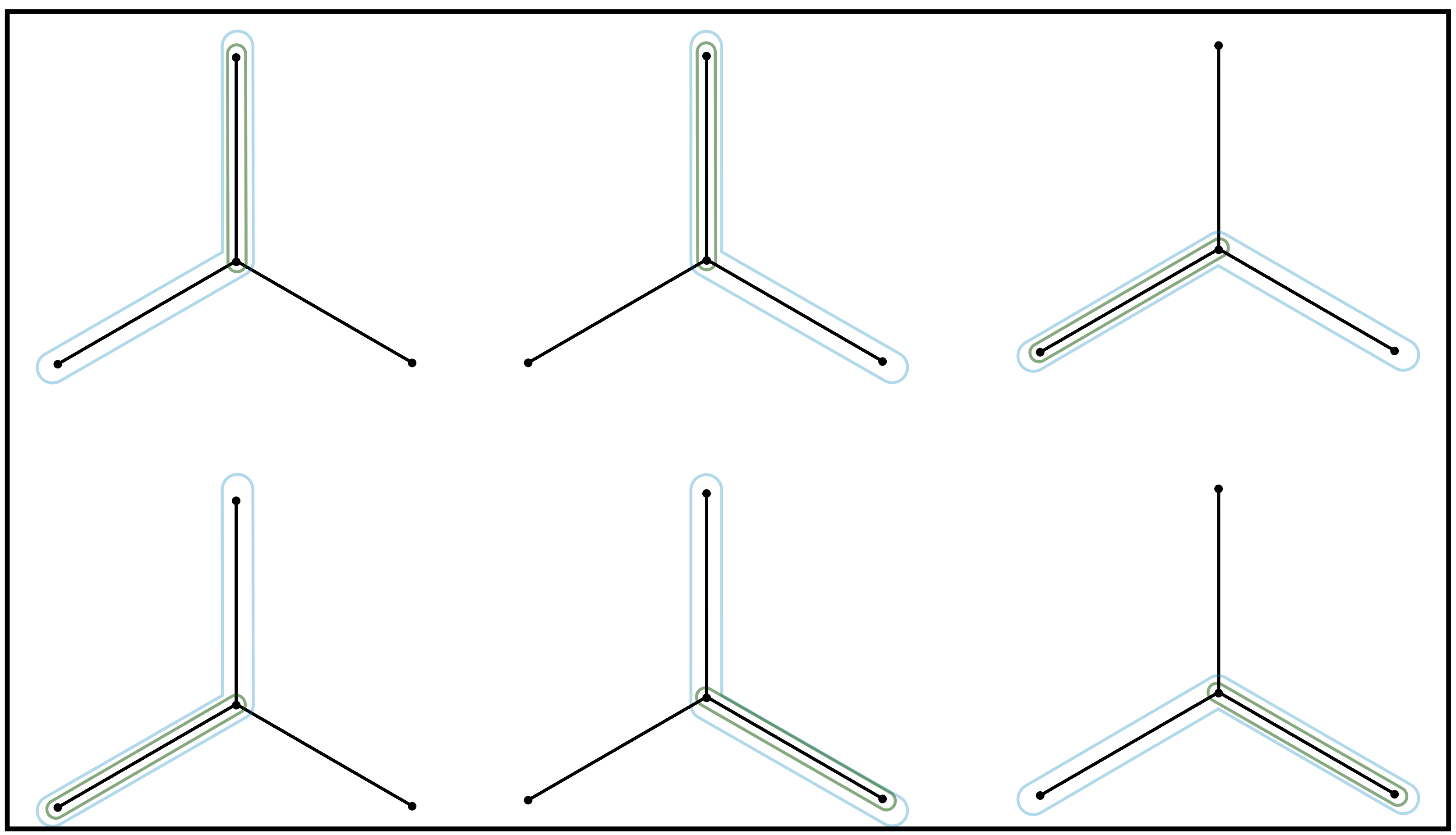
Integrand from Russian Dolls

A Russian-Doll / Tubing is a maximal collection of non-overlapping tubes. The integrand of a given diagram is the sum over all possible tubings one can draw:

Five-point chain topology

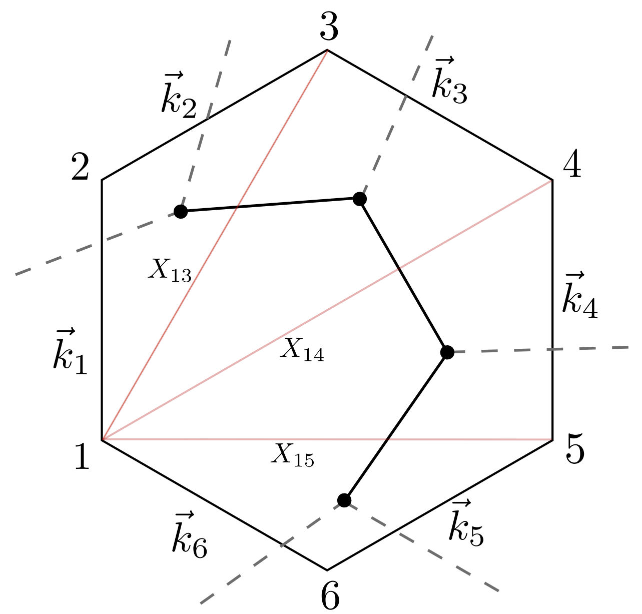


Six-point chain topology

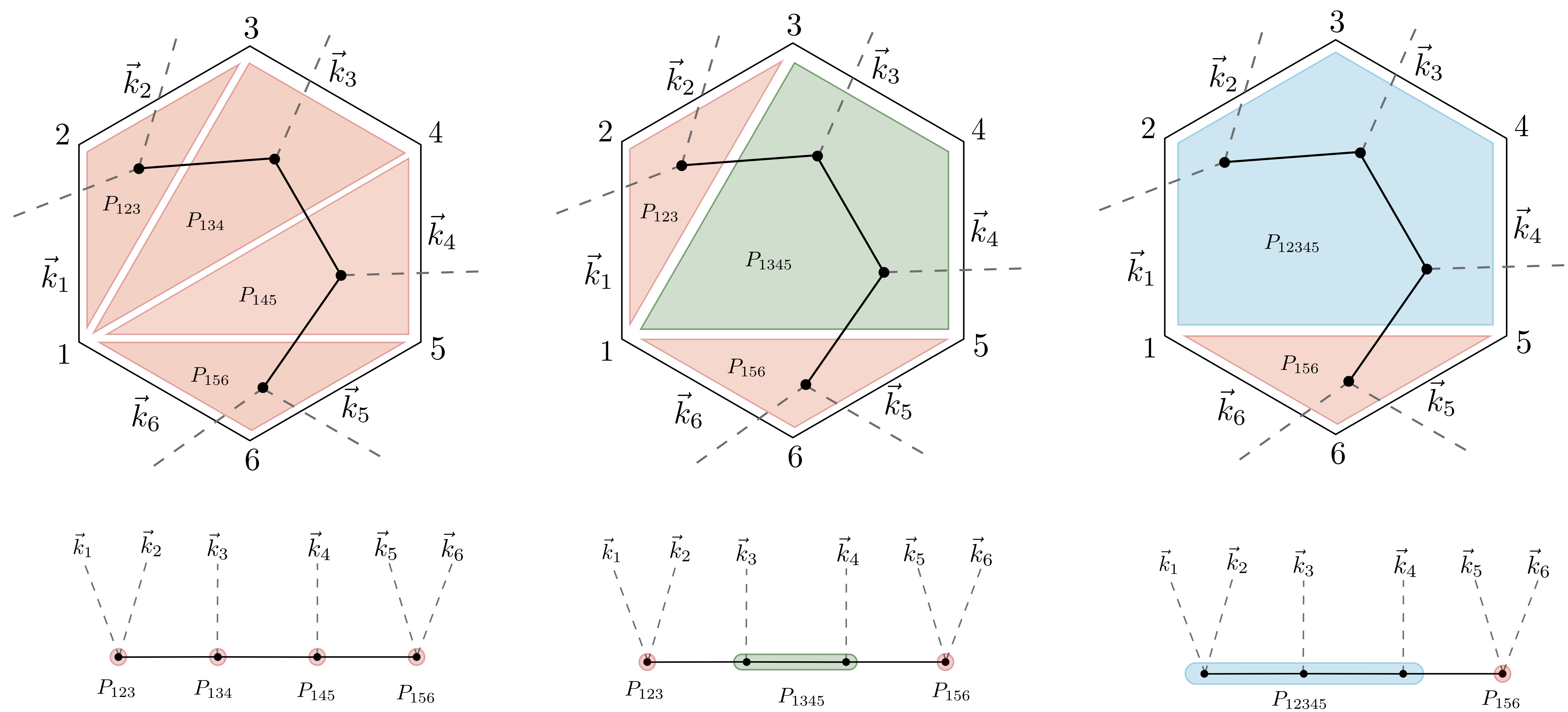


Six-point star topology

Kinematics From Sub-polygons



Every graph is dual to a triangulation of the momentum polygon. For the scattering amplitude, the dual variables are $X_{i,j} = (\vec{k}_i + \vec{k}_{i+1} + \ldots + \vec{k}_{j-1})^2$. For the wavefunction, the dual variables will be the perimeters of the sub-polygons of the momentum polygon.



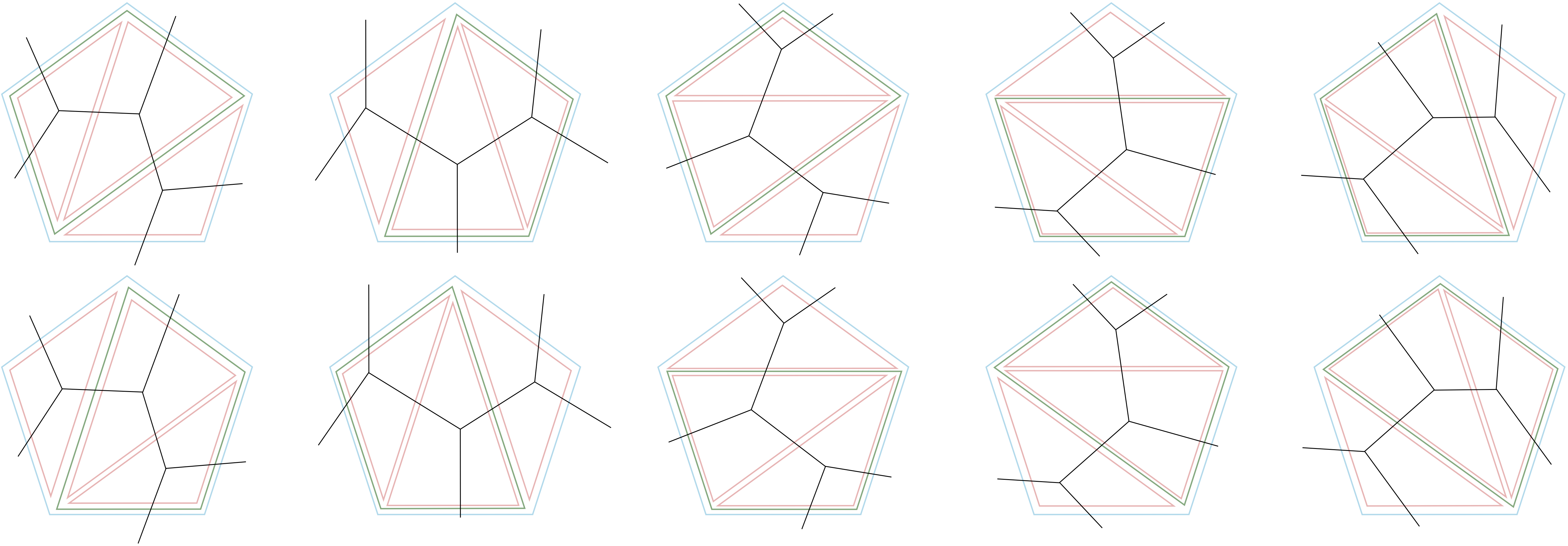
$P_{123} = |\vec{k}_1| + |\vec{k}_2| + |\vec{k}_1 + \vec{k}_2|$

$P_{1345} = |\vec{k}_3| + |\vec{k}_4| + |\vec{k}_1 + \vec{k}_2| + |\vec{k}_5 + \vec{k}_6|$

$P_{12345} = |\vec{k}_1| + |\vec{k}_2| + |\vec{k}_3| + |\vec{k}_4| + |\vec{k}_5 + \vec{k}_6|$

Wavefunction From Sub-polygons

The wavefunction can be written as a sum over all maximal sets of non-overlapping sub-polygons.



This allows us to write the following form for the wavefunction:

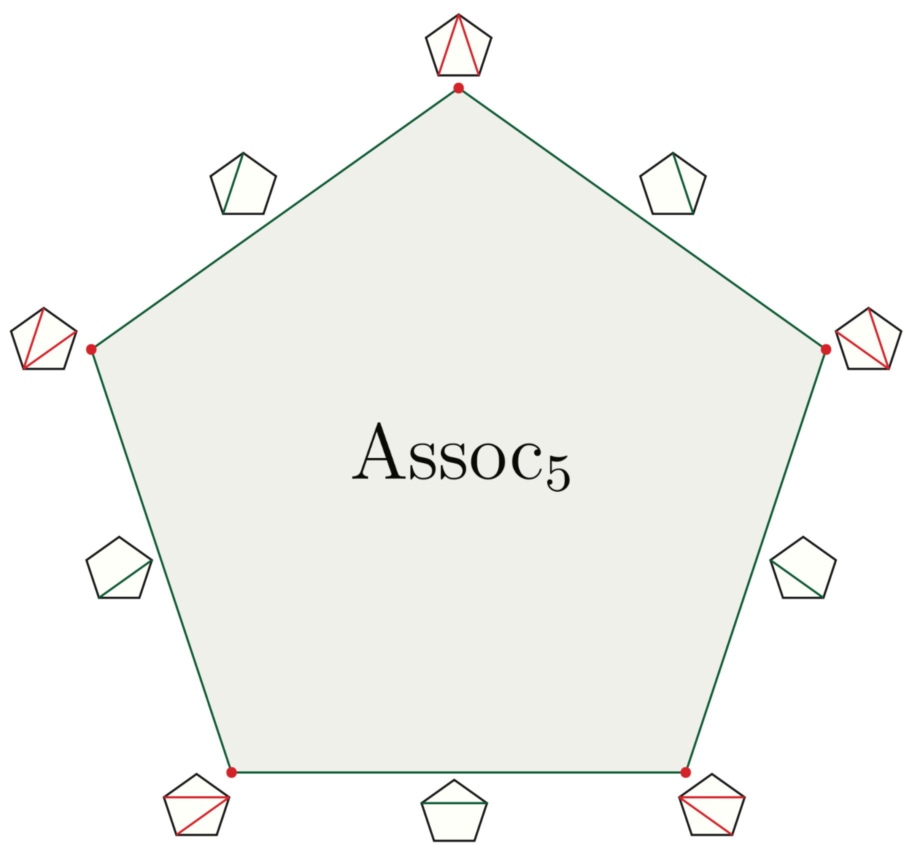
$$\Psi = \sum_{\mathbf{P}} \prod_{P \subset \mathbf{P}} \frac{1}{\mathcal{P}_P}$$

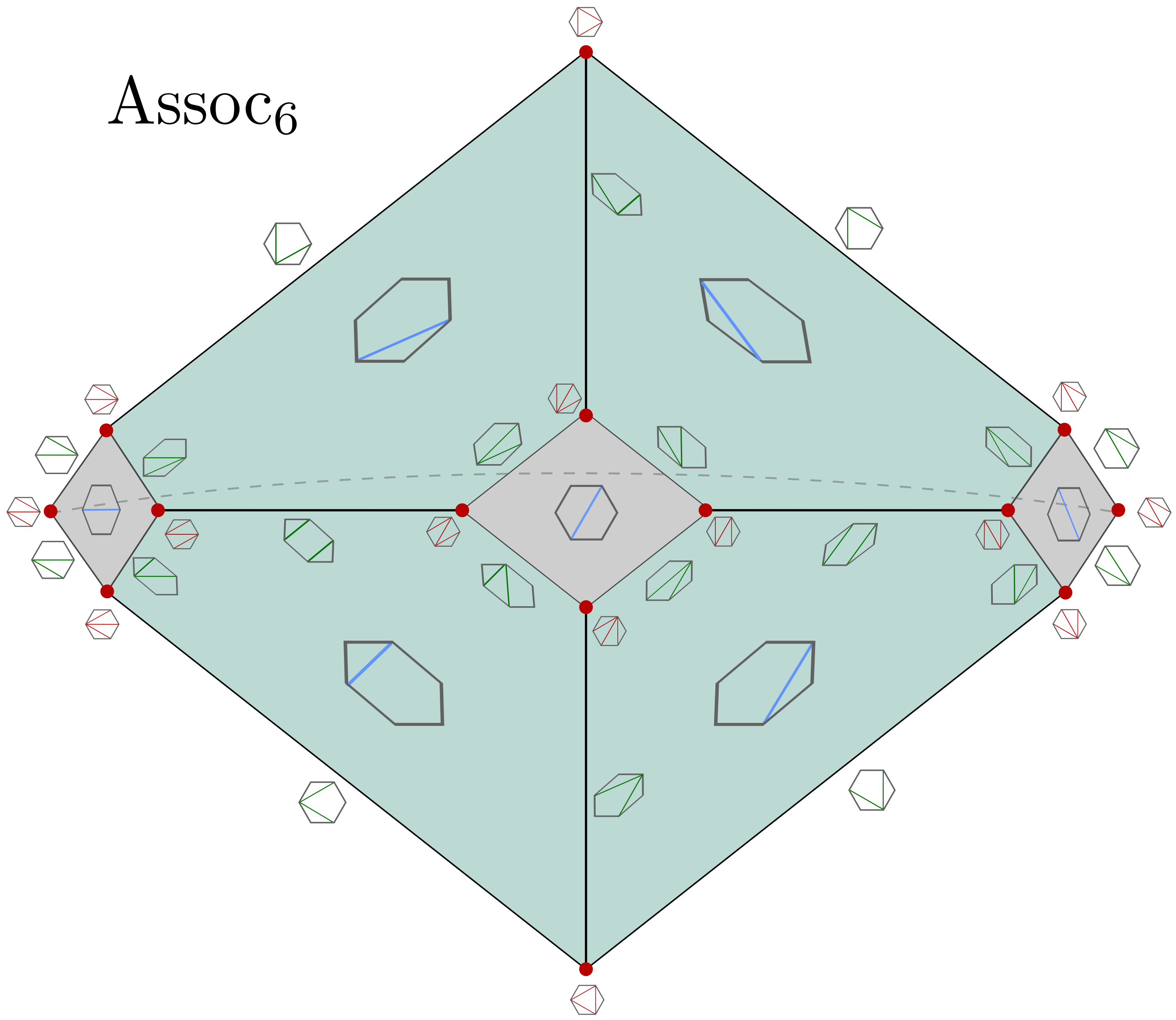
Maximal sets of non-overlapping sub-polygons.

The ABHY Associahedron

Associahedron

The associahedron encodes the combinatorial information of triangulations of polygons:





Consider a set of cords, C. We say that C' is a refinement of C, if $C \subset C'$. Then:

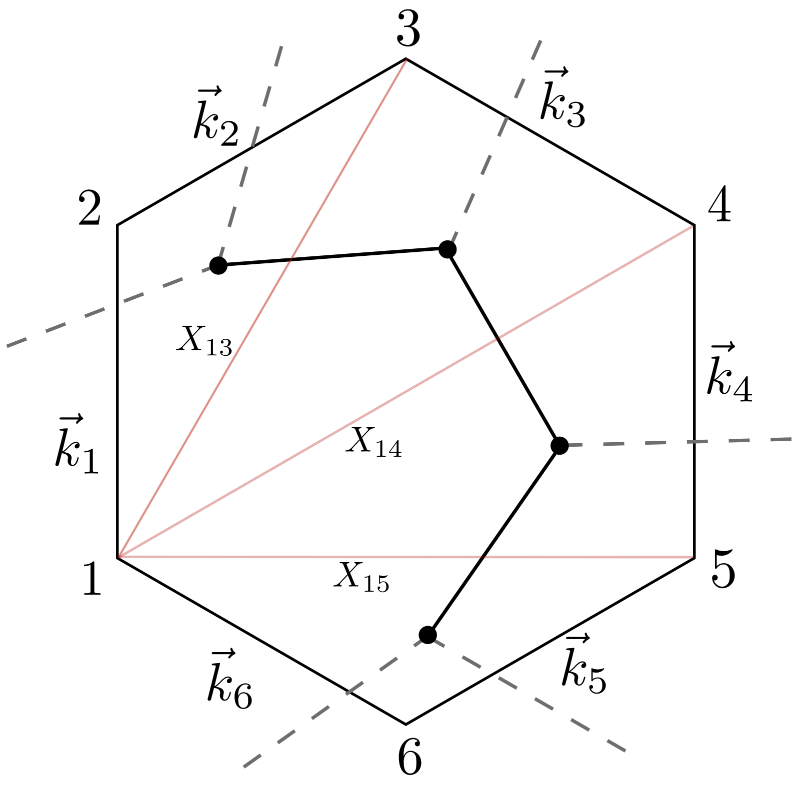
$$C' \text{ is a face of } C \text{ if } C \subset C'$$

Amplitudes from the Associahedron

The ABHY associahedron is carved out by the inequalities $X_{i,j} \geq 0$. One can embed it in $(n-3)$-dimensional space, by picking a basis triangulation, *e.g.* $\{X_{1,3}, X_{1,4}, X_{1,5}\}$, and write the remaining $X_{i,j}$-variables in terms of the base X's, and the non-planar variables:

$$c_{i,j} = X_{i,j} + X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1}$$

By fixing the c's to be positive, one obtains the $(n-3)$-dimensional embedding of the associahedron.



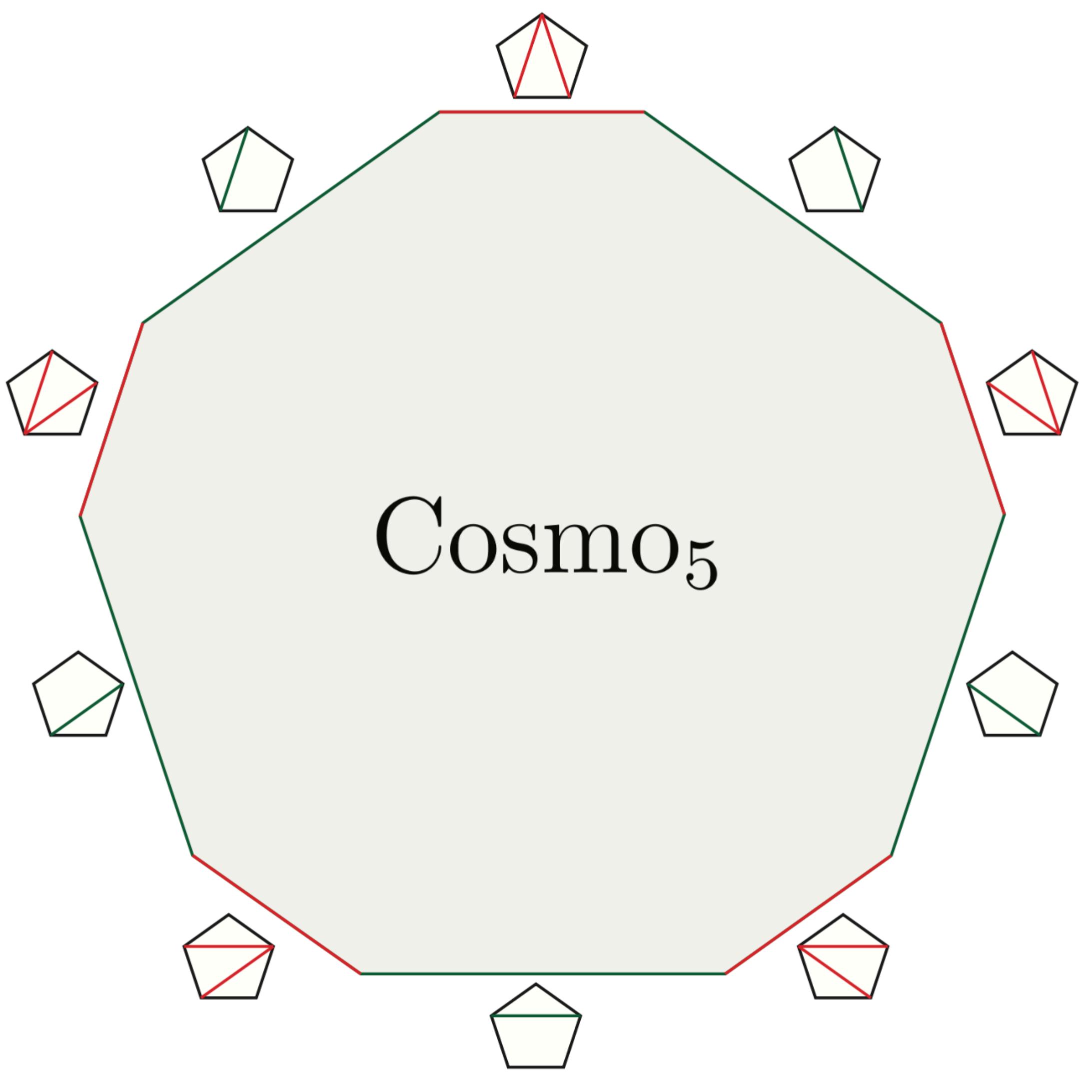
The amplitude is given by the canonical form of the polytope.

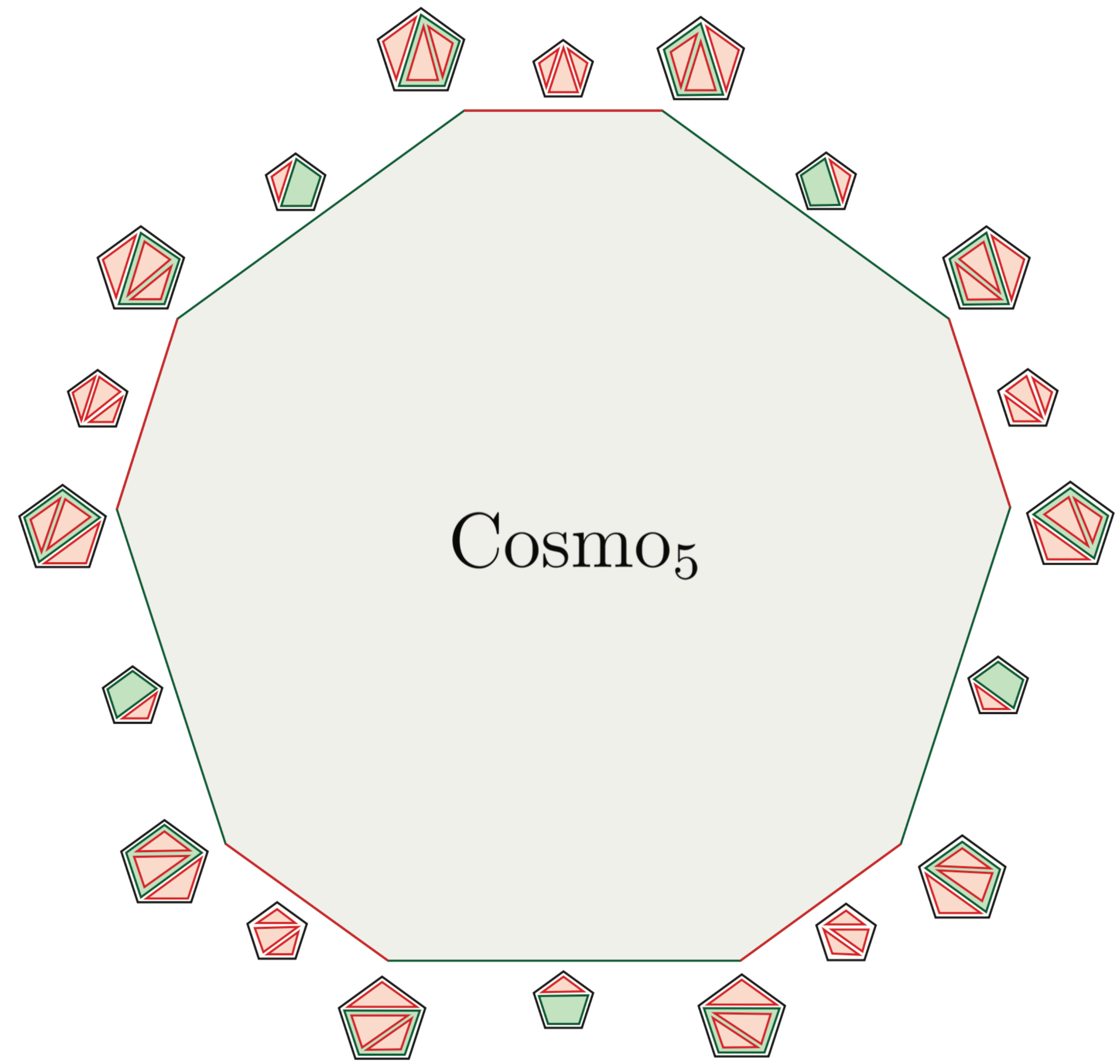
$$\mathcal{A}_n(X_{i,j}) = \sum_{\text{triang. } \mathcal{T}} \prod_{X_{i,j} \in \mathcal{T}} \frac{1}{X_{i,j}}$$

Cosmohedra

Cosmohedra

The cosmohedron captures the combinatorics of the full wavefunction!

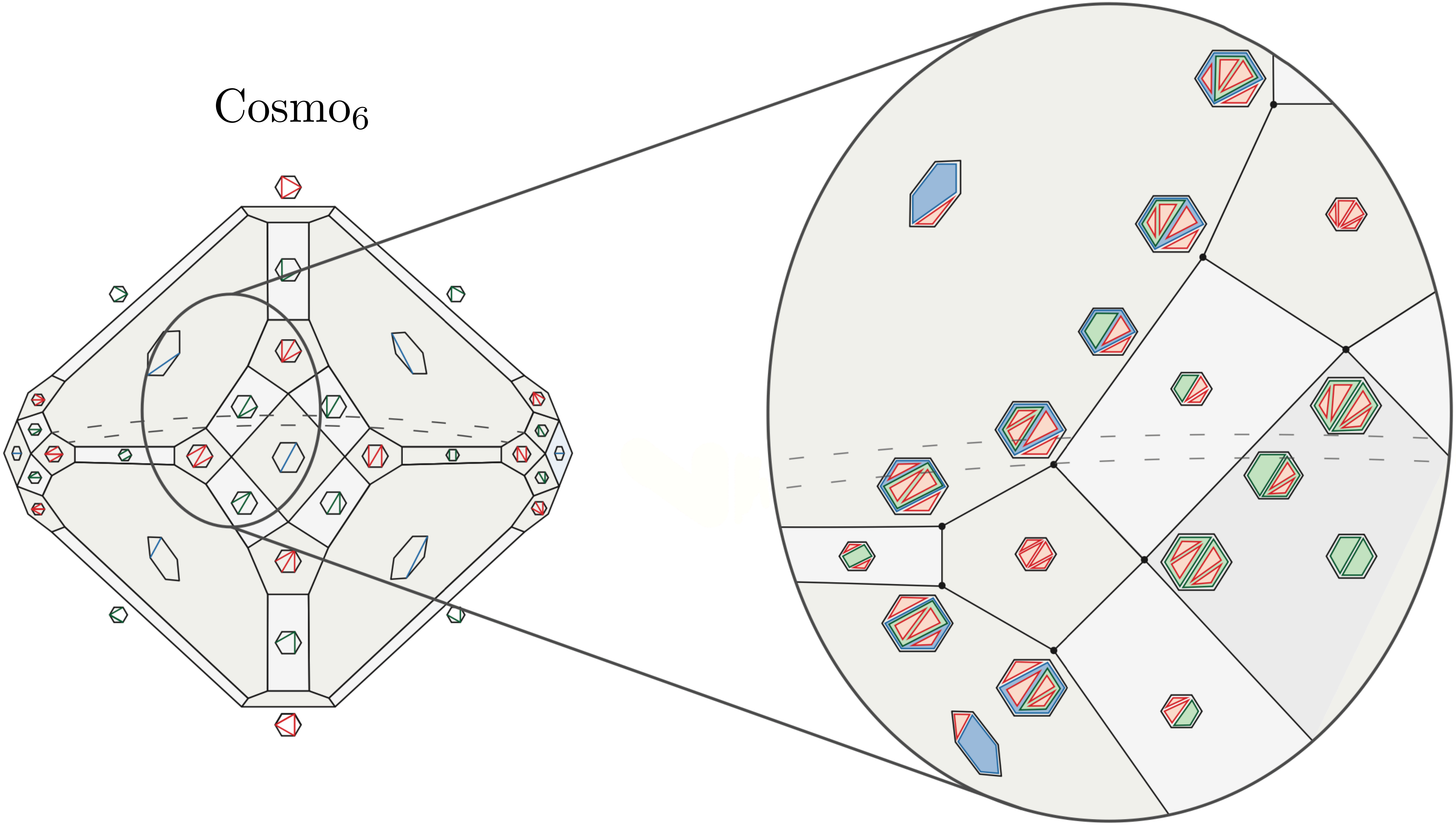




Considering that P is a collection of non-overlapping sub-polygons, then the cosmohedron has faces such that:

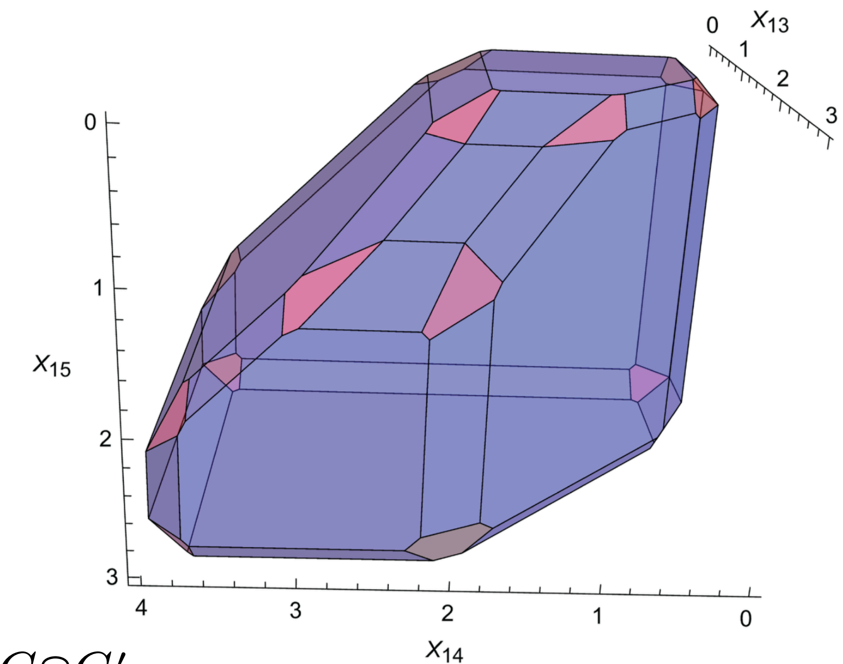
$$P' \text{ is a face of } P \text{ if } P \subset P'$$

Cosmohedra



Cosmohedra: Embedding

The cosmohedron will be carved out by the inequalities:



Cosmo$_6$

$$\sum_{c \in C} X_c \geq \epsilon_C$$

Set of non-overlapping cords, partial triangulation.

Where the constants need to satisfy inequalities: $\epsilon_C + \epsilon_{C'} < \epsilon_{C \cup C'} + \epsilon_{C \cap C'}$

If $C \cap C'$ is entirely to left, or right of C, C', then: $\epsilon_C + \epsilon_{C'} = \epsilon_{C \cup C'} + \epsilon_{C \cap C'}$

The above equalities are automatically satisfied if we impose $\epsilon_C = \sum_{P \text{ of } C} \delta_P$

P are sub-polygons in C.

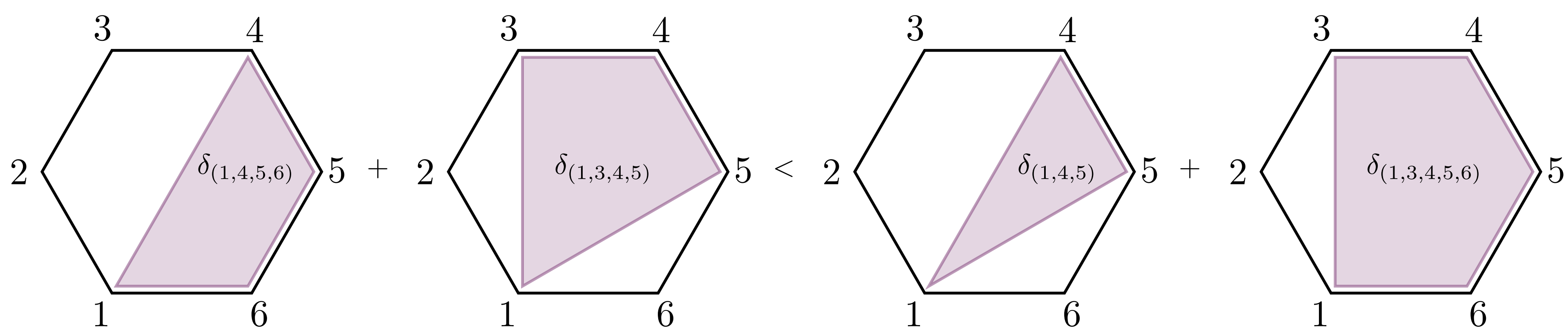
And the above inequalities are satisfied if we impose the following:

$$\delta_P + \delta_{P'} < \delta_{P \cap P'} + \delta_{P \cup P'}$$

True if imposing the parameterization

$$\delta_P = \delta(n - \#P)^2$$

Example:



Cosmohedra: Counting

Associahedron

$$\#\text{Facets}(\text{Assoc})_n \to \frac{n^2}{2}$$

$$\#\text{Vertices}(\text{Assoc})_n \to \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

	codim-1	codim-2	codim-3	codim-4	codim-5	codim-6
4-points	2	—	—	—	—	—
5-points	5	5	—	—	—	—
6-points	9	21	14	—	—	—
7-points	14	56	84	42	—	—
8-points	20	120	300	330	132	—
9-points	27	225	825	1485	1287	429

Cosmohedron

$$\#\text{Facets}(\text{Cosmo})_n \to \frac{c}{n^{3/2}}(3+\sqrt{8})^n$$

$$\#\text{Vertices}(\text{Cosmo})_n \to c n^4 n!$$

	codim-1	codim-2	codim-3	codim-4	codim-5	codim-6
4-points	2	—	—	—	—	—
5-points	10	10	—	—	—	—
6-points	44	114	72	—	—	—
7-points	196	952	1400	644	—	—
8-points	902	7116	18040	18528	6704	—
9-points	4278	50550	194616	332664	262728	78408

Outlook

Associahedron ⟷ **Particle/String Amplitudes**

[N. Arkani-Hamed, S. He, T. Lam, 19']

Scalars ⟶ **Pions and Gluons**

[N. Arkani-Hamed, Q. Cao, J. Dong, C. Figueiredo, S. He, 23']

What about Cosmology?

Thank You!