MAX PLANCK INSTITUTE
FOR PHYSICS

Positive Geometries in Amplitudes

Elia Mazzucchelli

IMPRS Colloquium

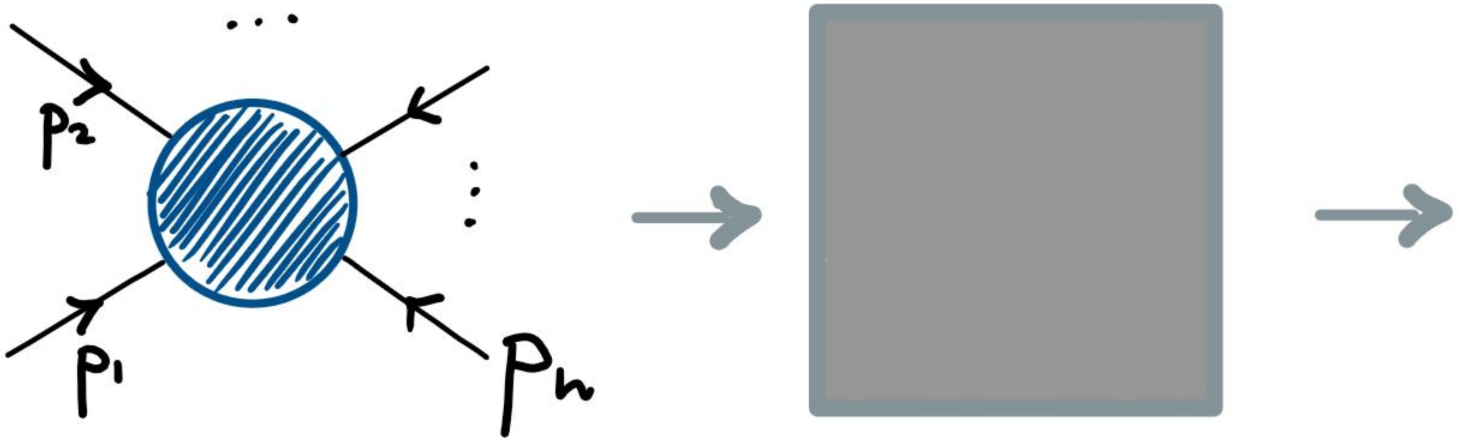
05.05.2025

To be answered

- What are (informally) positive geometries ?
- What are they useful for in QFT ?

Motivation

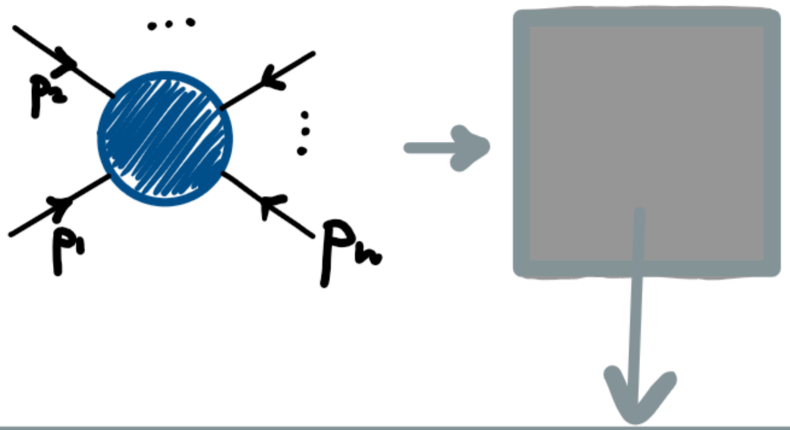
Description of a scattering process in a QFT = answering a question in kinematic space



amplitude

$A(p_1, \dots, p_n)$

Standard description



amplitude

$A(p_1,\dots,p_n)$

$$\boxed{\text{Lagrangian} \to \text{Path integral} \to \text{Perturbative expansion}}$$

$$A_n = \sum_{L\geq 0} g^L A_n^{(L)} \qquad g \text{ coupling constant}$$

$$A_n^{(L)} = \int \prod_{\ell=1}^{L} d^4 k_\ell \left(\sum \begin{array}{l}\text{Feynman diagrams} \\ \text{with } n \text{ legs and } L \text{ loops}\end{array} \right)$$

Geometric description

$$A_n^{(L)} = \int \prod_{\ell=1}^{L} d^4 k_\ell \ \mathrm{Vol}(P)$$

where P is a geom. in kinematic space

Universal Integrand

$$A_n^{(L)} = \int \prod_{\ell=1}^{L} d^4 k_\ell \; \boxed{\mathrm{Vol}(P)}$$

"The" L-loop n-point integrand: rational fct.
From the perspective of Feynmann diagrams, it requires an ordering on the external momenta $\longrightarrow$ colored $U(N)$ theories at large N, planar limit

When is such description available?

- colored scalar theories : $\mathrm{Tr}\, \phi^p$
- $\mathcal{N}=4$ super Yang-Mills

- recent progress towards pure Yang-Mills and non-linear sigma model (pions)

A concrete example : $\mathrm{Tr}\,\phi^3$

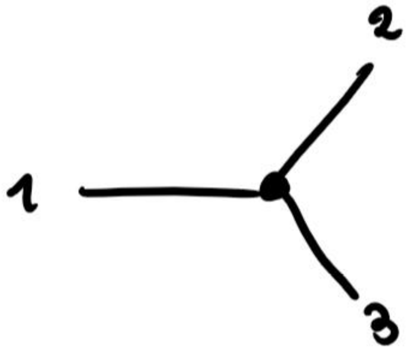
Lagrangian : $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{g}{3}\mathrm{Tr}\,\phi^3$

ϕ scalar field in the adj. repr. of $SU(N)$.

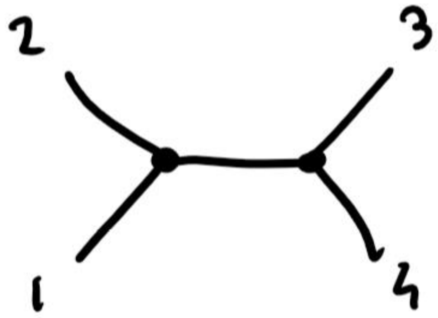
At large N, only planar diagrams contribute, i.e. those with a cyclic ordering on the external particles.

Tree diagrams

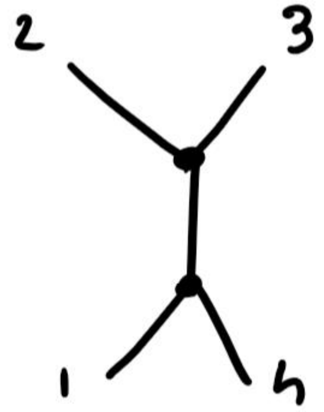
$n=3$:



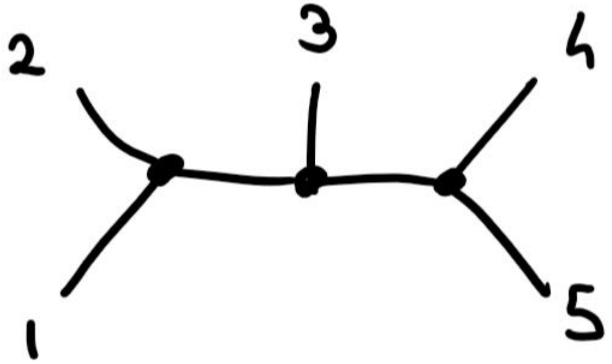
$n=4$:



$+$



$n=5$:

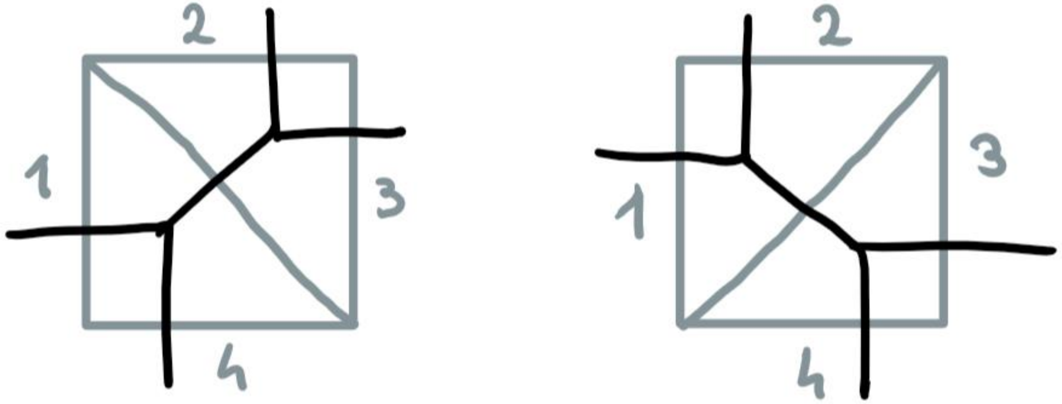


$+$ cycl.

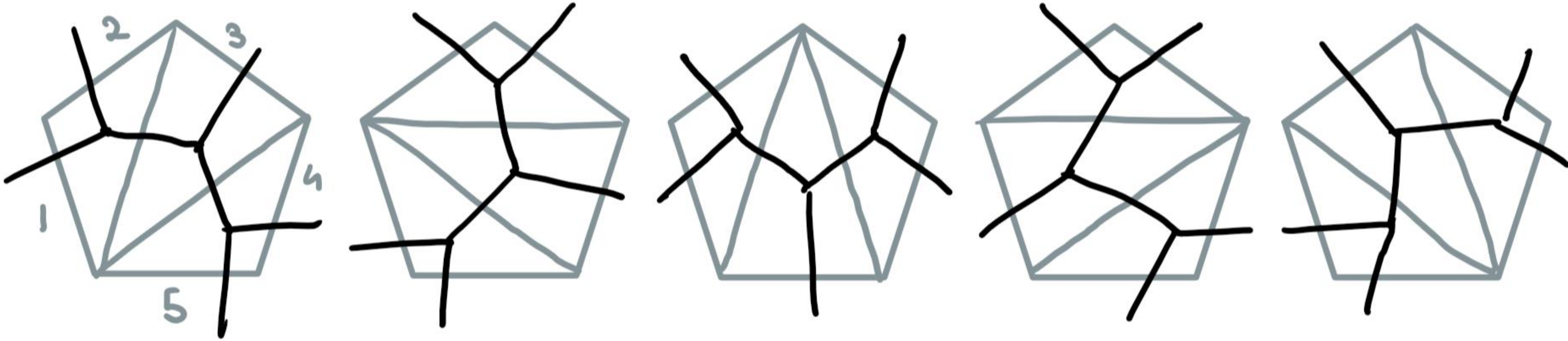
5 diagrams in total

Diagrams $\simeq$ Triangulations

$n=4$:



$n=5$:

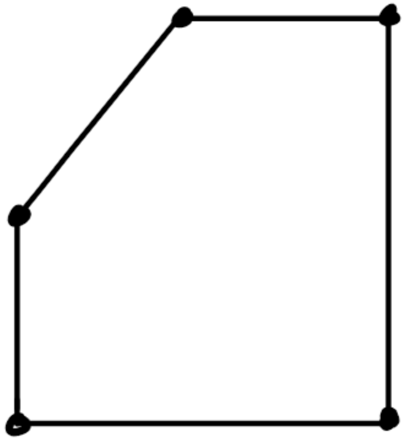


Geometry: Associahedron

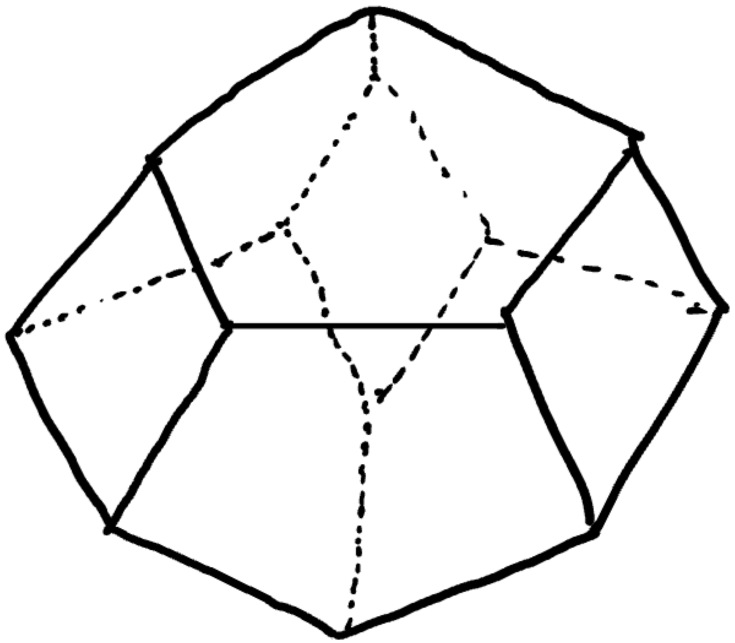
The positive geometry for this theory (at tree level) is a polytope:

the Associahedron

$n=4$



$n=5$

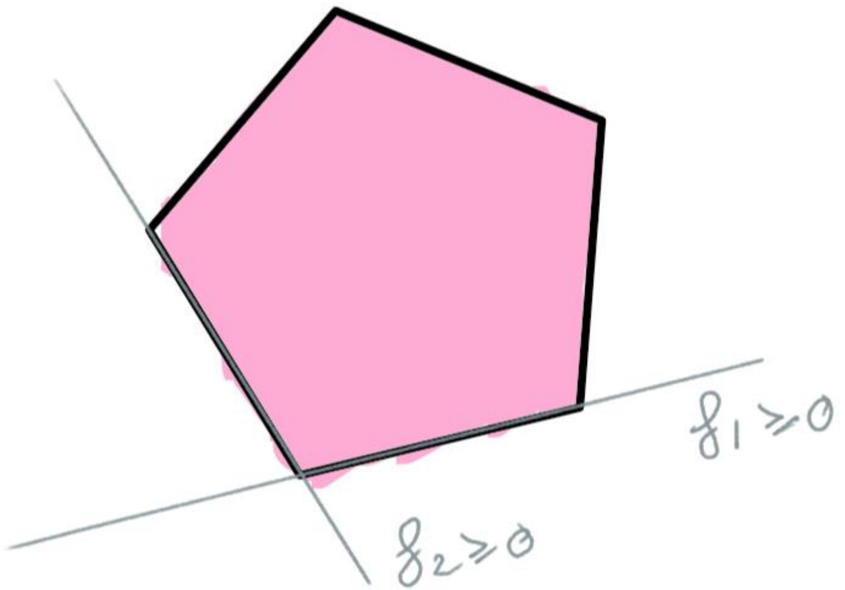


$n=6$

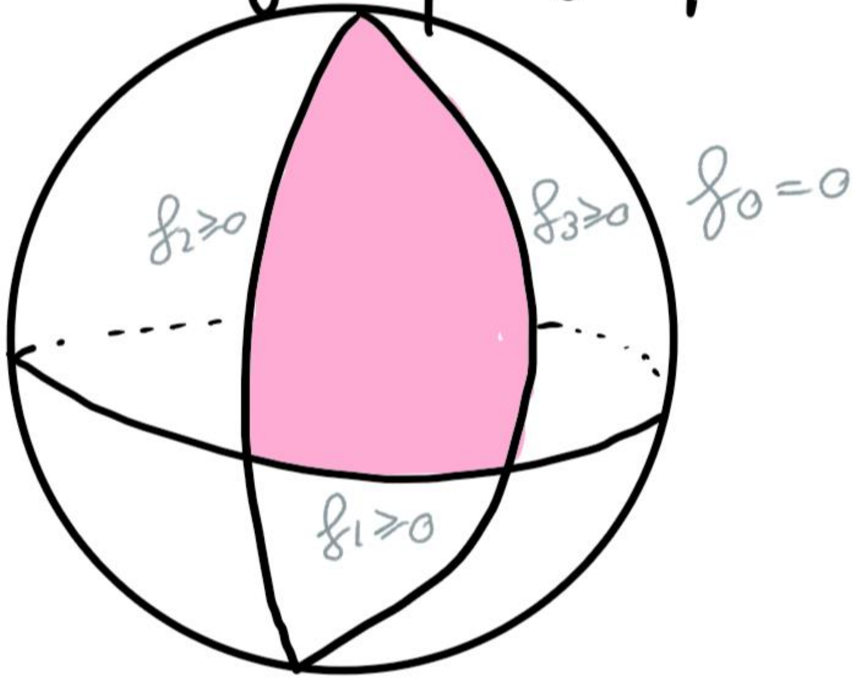
What is a positive geometry

It's a semialgebraic set P (e.g. in $\mathbb{R}^n$) cut out by polynomials $= 0$ and ≥ 0.

Exp polytopes



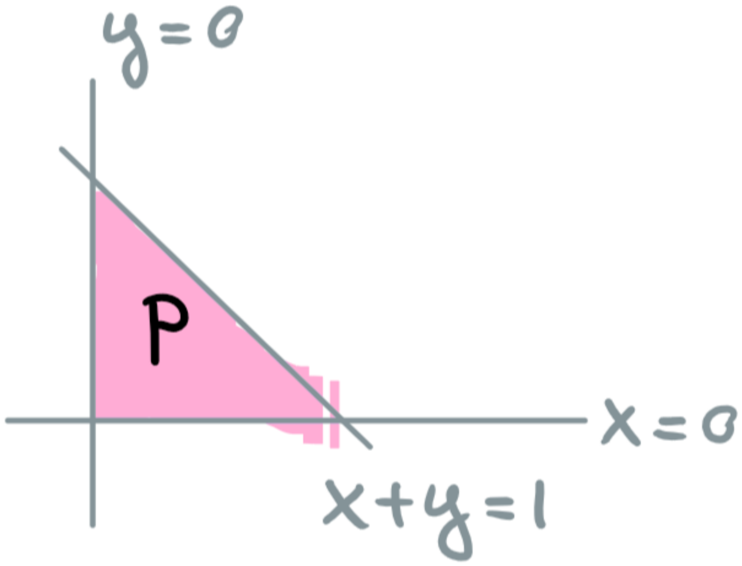
"curvy" polytopes



The canonical form

P is equipped with a unique top-form $\Omega(P)$ with logarithmic singularities on ∂P and nowhere else

Exp



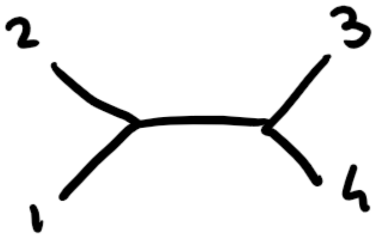
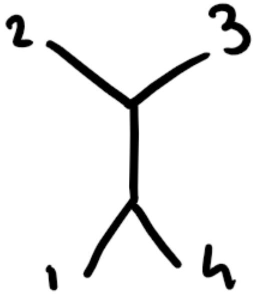
$$\Omega(P) = \frac{dx \wedge dy}{x \cdot y \cdot (1-x-y)}$$

Tr $(\phi^3) \simeq$ Associahedron

$$A_n^{(L=0)} = \Omega(P_n)$$

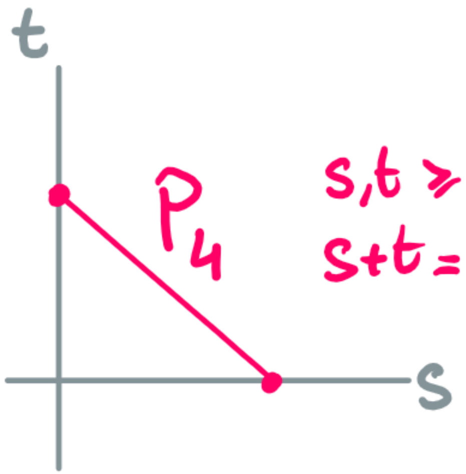
(ABHY) associahedron of dim. $n-3$
embedded in kinematic space

$n = 4$

$A_4 =$  $+$  $= \dfrac{1}{s} + \dfrac{1}{t}$

$$s = (p_1 + p_2)^2$$
$$t = (p_2 + p_3)^2$$
$$s + t = -u > 0$$

$$= \Omega(P_4)$$



$s, t \geqslant 0$

$s + t = -u > 0$

$\mathcal{N}=4$ SYM $\simeq$ Amplituhedron

There exists a positive geometry $\mathcal{A}^{(0)}_{n,k}$ in the Grassmannian $Gr(k,k+4)$ of k-planes in $\mathbb{C}^{k+4}$ s.t. in $\mathcal{N}=4$ SYM

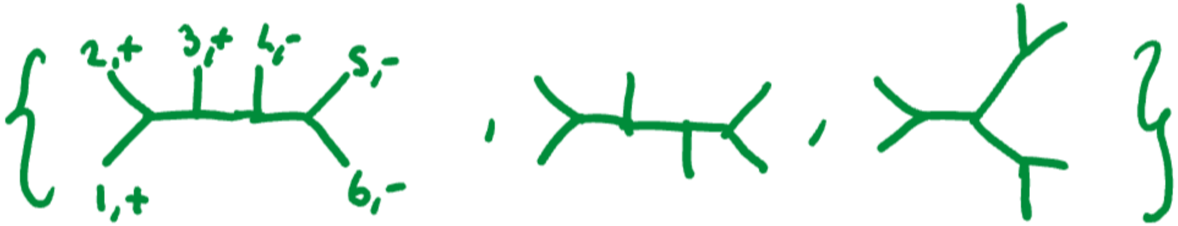
$$A^{(L=0)}_{n,k} = \Omega\left(\mathcal{A}^{(0)}_{n,k}\right) \quad \text{and more gener.}$$

$$\text{integrand of } A^{(L)}_{n,k} = \Omega\left(\mathcal{A}^{(L)}_{n,k}\right)$$

What does a positive geometry description bring us ?

Triangulations vs Feynm. diagr.

PG yield an efficient way of computing amplitudes/integrands via triangulating the geometry rather than summing over FD.

Exp $A^{(0)}_{6,1} = \sum_{a=1}^{220}$ 

($\mathcal{N}=4$ SYM)

also $= \Omega(\mathcal{A}^{(0)}_{6,1}) = [23456]+[45612]+[61234]$

where $[\cdot\cdot\cdot\cdot\cdot]$ are can. forms of simplices in $\mathbb{R}^4$.

Manifest locality

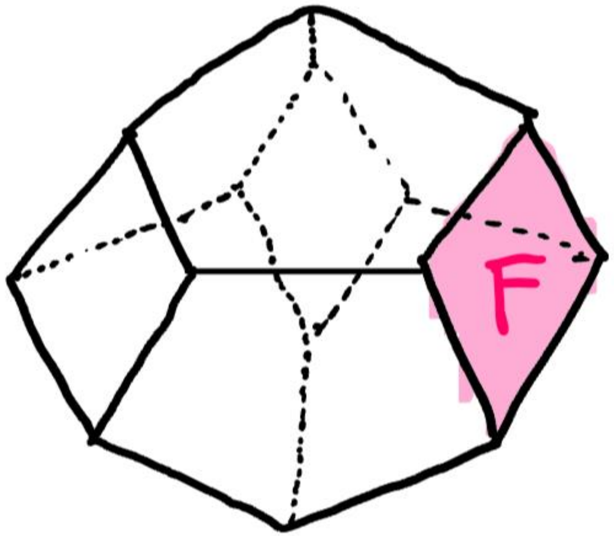
FD are full of spurious poles that cancel when taking the sum over diagrams.

The PG tells you exactly what the physical singularities are, i.e. they are the boundary components of the PG.

Factorization at poles

Singularities arise by collinear / soft conf. of momenta. In QFT there are universal factorization thms, reflected by geometric factorization of a boundary.

Exp



$$\mathrm{Res}_F\, A_6 = \Omega\left(\boxed{F}\right)$$

$$= \Omega(\;) \wedge \Omega(\;)$$

$$= A_4 \cdot A_4 \qquad (\mathrm{Tr}\,\phi^3)$$

Computing integrals

We said (at loop level $L > 0$) :

$$\mathrm{PG} \quad P \rightsquigarrow \text{integrand } \Omega(P)$$

but what about $A^{(L)} = \int_{\mathbb{R}^{4L}} \Omega(P)$?

Geometries help !

Polylogrithms

Determining the function space of an amplitude at fixed n and L is a great problem (state of the art $n+L \leq 8$).

For $\mathcal{N}=4$ SYM, many ampl. are general. polylogs

$$A_n^{(L)} = \sum_i \overset{\mathbb{Q}}{\overset{\cup}{c_i}} \overbrace{\int dt_1 \, d\log(a_1) \int dt_2 \, d\log(a_2) \ldots}^{2L\text{-fold iterated integral}}$$

letters

Geometric Bootstrap

Idea: compute $A_n^{(L)}$ by:

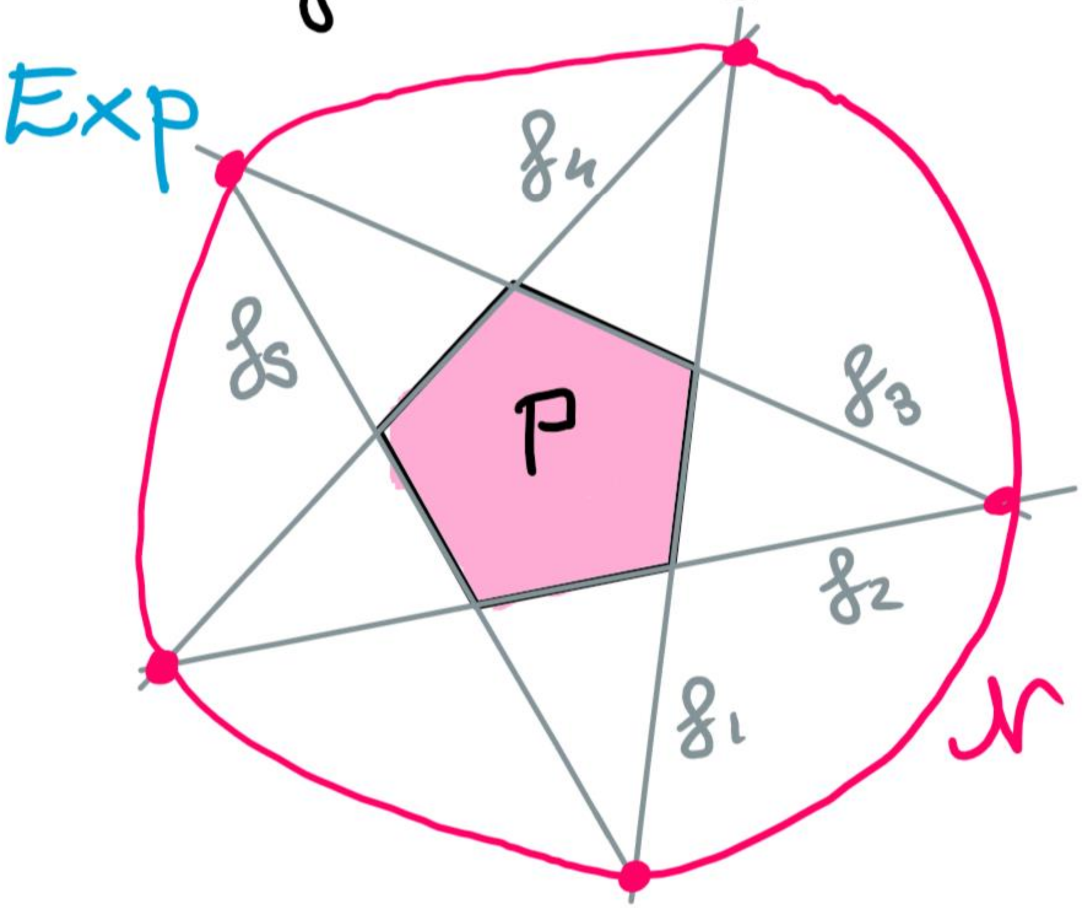
- $\boxed{\text{"guessing" the relevant alphabet } \{a_1, \ldots, a_N\},\ a_i = \text{rat. fct of kin. var.}}$

PG provide an efficient algorithm for this step

- Make an Ansatz $A_n^{(L)} = \sum_I c_I \, a_{I_1} \otimes \ldots \otimes a_{I_{2L}}$ and fix all $c_I \in \mathbb{Q}$ by physical consid: right physical limits, ...

Positivity ...

Positivity properties of amplitudes have a long history. PG often certify positivity.



$$\Omega(P) = \frac{\overset{\geq 0}{\widetilde{\mathcal{N}}}}{\underset{\geq 0}{\underset{\sim}{f_1}}\,\underset{\geq 0}{\underset{\sim}{f_2}}\, f_3\, f_4\, f_5}$$

≥ 0 inside P

... and more !

Positivity is the tip of the iceberg:

a fct. $F: C \to \mathbb{R}$, $C \subset \mathbb{R}^n$ convex cone,

is completely monotone if

$$(-1)^r D_{v_1} \cdots D_{v_r} F(x) \geq 0$$

$$\forall\, v_1, \ldots, v_r, x \in C.$$

Recently, many observables in QFTs have been found to be CM!

CM $\simeq$ volumes

Thm $F : C \to \mathbb{R}$ is CM $\iff$

$$F = \mathrm{Vol}_\mu (C^*) \quad \text{where}$$

μ is a non-negative measure on C^*.

Example : polytopes

Fact: P polytope $\Rightarrow$ $\Omega(P) = \mathrm{Vol}_1(P^*)$.

Exp All tree-level amplitedes in $\mathrm{Tr}\phi^3$ are volumes, hence CM.

The dual Amplituhedron??

For $\mathcal{N}=4$ SYM, can we find

$$\Omega(\mathcal{A}^{(L)}_{n,k}) = \mathrm{Vol}_{\underbrace{\mu^{(L)}_{n,k}}_{?}}\Big(\underbrace{(\mathcal{A}^{(L)}_{n,k})^*}_{?}\Big).$$

Are amplitudes in this theory volumes?

Summary

- What are (informally) positive geometries ?

A PG is a semialgebraic set P with a canonical form $\Omega(P)$:

- amplitude's local poles $\simeq$ ∂P
- factorization $\simeq$ geometric fact. of ∂P.

Summary

- What are they useful for in QFT ?

→ computing integrands by triangulations

→ bootstrapping actual integrals

→ they provide positivity certificates (and more: complete monotonicity),

providing a volume picture for amplitudes.