Max Planck Institute for Physics, LMU and TU Munich

Hamiltonian Cobordism:

Einstein's Equations meet the Cobordism Conjecture

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Motivation

Do we truly need a theory of Quantum Gravity (QG)?

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A consistent theory of QG promises not only conceptual completeness but also the potential to resolve these deep empirical mysteries.

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- ✷ The network of string **dualities** (S-duality, T-duality, mirror symmetry) reveals deep non-perturbative equivalences between apparently different theories.
- ✷ **Rich mathematical structure**: the study of string theory has driven profound advances in mathematics.

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QG effects can manifest at low energies, imposing constraints on compatible theories.

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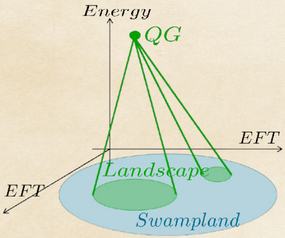

Figure: Swampland and Landscape

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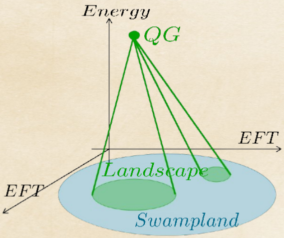

Figure: Swampland and Landscape

- Top-Down & Bottom-Up Approaches
- Predictive Power
- Guiding Model Building

[Vafa 2005; Palti 2019; van Beest, Calderon-Infante, Mirfendereski, Valenzuela 2022]

Background

No Global Symmetries Conjecture

There are no global symmetries in QG, i.e. any symmetry is either broken or gauged

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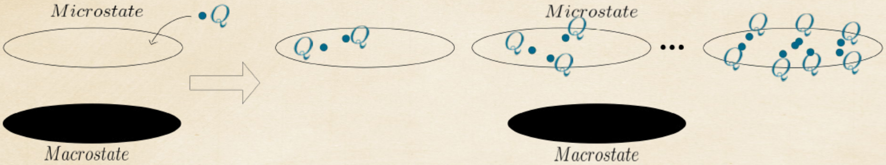

Figure: Gedankenexperiment - Black Hole entropy

Topology and QG

In QG, spacetime can have **non-trivial topology**!

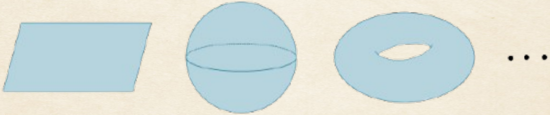

Figure: Different topologies

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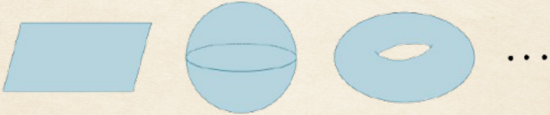

Figure: Different topologies

... BUT non-trivial topologies lead to **(topological) global symmetry**, which are detected by **bordism groups**.

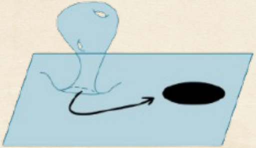

Figure: Gendankenexperiment - Non-trivial topology

Cobordism Conjecture

M_d and N_d are said to be **cobordant** if there exists a $(d+1)$-dimensional manifold W_{d+1} interpolating between them.

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All cobordism classes must vanish $\Omega_d^{QG} = 0$.

[McNamara, Vafa 2019]

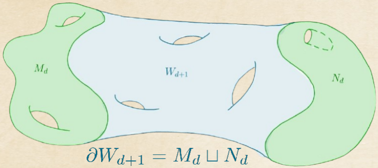

Figure: Cobordism

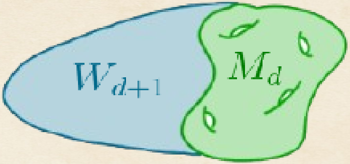

Figure: Cobordism to nothing

Predictive power: **existence of additional defects**!

Hamiltonian Cobordism

Hamiltonian Cobordism: A gap in the conjecture

Cobordism Conjecture has **no connection to dynamics**!

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Topologically, the answer is yes, but once gravity is in the picture, one needs to define a metric that satisfies Einstein's equations.

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Dynamical refinement of the Cobordism Conjecture.

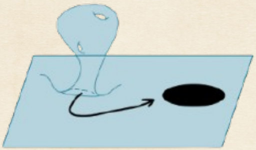

Figure: Non-trivial topology

Hamiltonian Cobordism: ADM decomposition

Hamiltonian description of GR. $\rightarrow$ Spatial hypersurfaces ("*slices*") evolve in time.

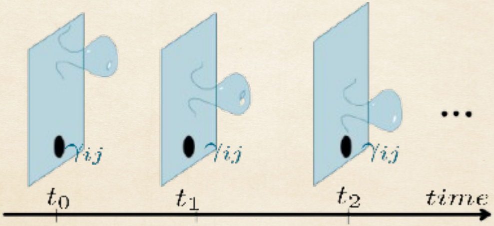

Figure: Foliation of spacetime

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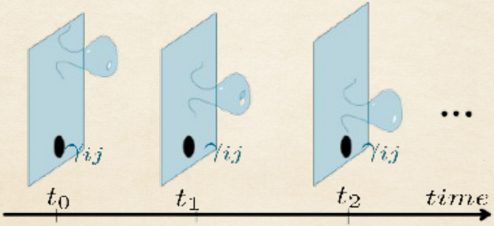

Figure: Foliation of spacetime

$$g_{00} = -\alpha^2 + \gamma^{ij}\beta_i\beta_j \ , \ g_{0i} = g_{i0} = \beta_i \ , \ g_{ij} = \gamma_{ij} \tag{1}$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (-\alpha^2 + \gamma_{ij}\beta^i\beta^j)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j, \tag{2}$$

[Arnowitt, Deser, Misner 1959; Bertschinger 2005; Lao 2021]

Hamiltonian Cobordism: Initial & Energy Conditions

Hamiltonian and momentum constraints for GR as initial conditions:

$$-\mathcal{H} = \alpha^2 T^{00} \quad \text{with} \quad \mathcal{H} = K_{ij}K^{ij} - K^2 - {}^{(d)}R, \tag{3}$$

$$\alpha\mathcal{H}^i + \beta^i\mathcal{H} = \alpha^2 T^{0i} \quad \text{with} \quad \mathcal{H}^i = 2\nabla_j(K^{ij} - K\gamma^{ij})\,. \tag{4}$$

For asymtotically flat manifold $|x| \to \infty$: $\alpha(x_i, t) \to 1$, $\beta_i(x_i, t) \to 0$, and $\gamma_{ij}(x_i, t) \to \delta_{ij}$.

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Weak Energy Condition (WEC):

Given any future-directed timelike vector field X^μ, $T_{\mu\nu}X^\mu X^\nu \geq 0$.

$$\implies T^{00} \geq 0 \tag{5}$$

Hamiltonian Cobordism: Assumptions & Results

Assume there exists a metric for the gluing such that:

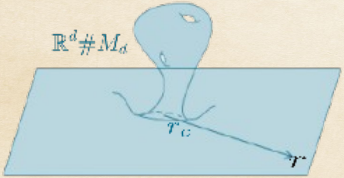

Figure: Initial Configuration

- $\gamma_{ij}(x_i, t) = \begin{cases} \gamma_{ij}^{M_d} \text{ for } r < r_c \\ \delta_{ij} + \mathcal{O}(r^{-(d-2)}) \text{ for } r \geq r_c \end{cases}$
- Classical matter $\to$ WEC
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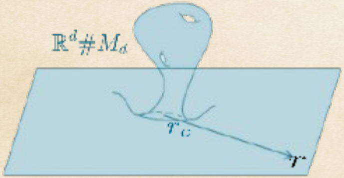

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$\Downarrow$

M_d must admit a metric of positive scalar curvature (PSC).

Hamiltonian Cobordism: Importance of PSC

STEP 1: Conformal transformation of γ to obtain another metric $\hat{\gamma}$ which is also asymptotically flat and whose scalar curvature is everywhere non-negative.

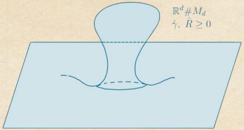

Figure: Step 1

$$\hat{R} = e^{-2\varphi}\left[R - \frac{4(d-1)}{(d-2)} e^{-(d-2)\varphi/2} \Delta\left(e^{(d-2)\varphi/2}\right)\right] \tag{6}$$

$$e^{2\varphi}\hat{R} + 6e^{-\varphi}\Delta\left(e^{\varphi}\right) + K^2 = \alpha^2 T^{00} + K_{ij}K^{ij} \tag{7}$$

$$\Delta f + \frac{K^2}{6} f \leq 0\,. \tag{8}$$

Hamiltonian Cobordism: Importance of PSC

STEP 2: Conformal transformation on $\hat{\gamma}$ to a manifold $S^d \# M_d$ is *compact* with metric $\hat{\gamma}_c$ and $\hat{R}_c \geq 0$.

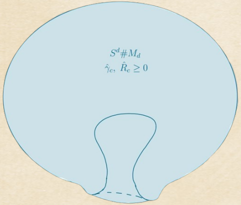

Figure: Step 2

$$u(r) = \begin{cases} 1 \quad \text{for} \quad r < r_0\,, \\ u_s(r) \quad \text{for} \quad r \to \infty\,, \end{cases} \tag{9}$$

with smooth boundary conditions at $r = r_0$:

$$u(r_0) = 1\,,\ u'(r_0) = u''(r_0) = 0 \ \text{ and } \ \hat{R}_c\big|_{\substack{r \to r_0 \\ r > r_0}} = 0\,, \tag{10}$$

Hamiltonian Cobordism: Importance of PSC

STEP 3: Conformal transformation on $\hat{\gamma}_p$ to a manifold with positive Yamabe invariant, which implies that with another conformal rescaling one obtains a PSC metric.

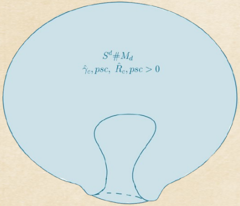

Figure: Step 3

Yamabe functional:
$$\mathcal{E}(\hat{\gamma}_c, \nu) = \frac{\int_{M_d \# R^d} \hat{R}_{c,psc} dV_{\hat{\gamma}_{c,psc}}}{\left(\int_{M_d \# R^d} dV_{\hat{\gamma}_{c,psc}}\right)^{\frac{d-2}{d}}}, \quad (11)$$

Yamabe invariant:
$$Y(M_d \# R^d, \hat{\gamma}_c) = \inf_{\nu \neq 0} \mathcal{E}(\hat{\gamma}_c, \nu). \quad (12)$$

Hamiltonian Cobordism: Results

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Cobordism Conjecture requires refinement $\Omega \to \Omega_{PSC}$.

Possible implications:

- New bordism classes might appear, some known ones might be trivialized;
- Defects/singularities beyond classical GR (e.g., Casimir potentials, orbifolds, ...)

Outlook

Hamiltonian Cobordism: Outlook

- How do these ideas interact with other known Swampland conjectures?
- Can we use this to make phenomenological predictions about what kinds of low-energy EFTs are allowed?
- How does this affect the defects?

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- How does this affect the defects?

Important first and innovative step in connecting the Cobordism Conjecture to dynamics.

Thank you!

Backup slides

AdS/CFT Proof

Assumption: splittability (you can "split" the boundary in subsets)

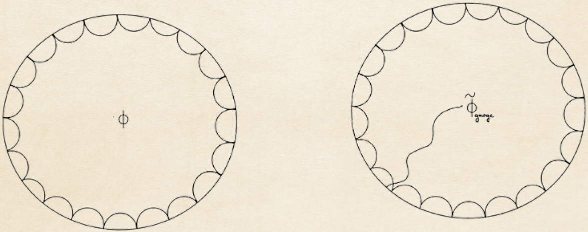

Figure: AdS/CFT proof

No Global Symmetries Conjecture

The No Global Symmetries conjecture is believed to hold when the notion of symmetry is extended, i.e., not only for ordinary 0-form symmetries but also for generalized symmetries.

Definition: p-form global symmetry

A p-form global symmetry acts on p-dimensional ‘defects’ or operators. Its conserved charge lives on a $(d - p - 1)$-dimensional sphere surrounding the defect in a d-dimensional spacetime.

Cobordism and String Theory

Cobordism classes form abelian groups depending on the dimension of the manifold and the conserved structure.

Definition: Cobordism Group of Quantum Gravity Ω^{QG}

- $\Omega^{QG,D}$: This group classifies all *D*-dimensional theories of quantum gravity, where the *D*-dimensions can be non-compact.
- Ω_k^{QG}: This group classifies all theories compactified on a k-dimensional manifold.

PSC Bordism?

Should we consider PSC bordism?

A priori, the above result just means that we should drop all manifolds that do not admit a PSC metric; it doesn't necessarily mean that we should consider the bordism of PSC manifolds.

To do so, we should be able to argue that the interpolating $(d + 1)$-dimensional manifold should also have PSC.

Yamabe Invariant and PSC

$\hat{\gamma}_{c,psc} = \nu^{\frac{4}{d-2}}\hat{\gamma}_c$ with a function ν that satisfies $\nu(r) > 0\ \forall r$.

Yamabe equation: $L_{\hat{\gamma}_c}\nu = \hat{R}_{c,psc}\nu^{\frac{d+2}{d-2}}$ where $L_{\hat{\gamma}_c} = -\frac{4(d-1)}{d-2}\Delta_{\hat{\gamma}_c} + \hat{R}_c$ (13)

Yamabe functional:
$$\mathcal{E}(\hat{\gamma}_c, \nu) = \frac{\int_M \hat{R}_{c,psc} dV_{\hat{\gamma}_{c,psc}}}{\left(\int_M dV_{\hat{\gamma}_{c,psc}}\right)^{\frac{d-2}{d}}} = \frac{\int_M \left(\frac{4(d-1)}{d-2}|\nabla_{\hat{\gamma}_c}\nu|^2 + \hat{R}_c\nu^2\right) dV_{\hat{\gamma}_c}}{\left(\int_M \nu^{\frac{2d}{d-2}} dV_{\hat{\gamma}_c}\right)^{\frac{d-2}{d}}}, \tag{14}$$

where we have used the fact that $dV_{\hat{\gamma}_{c,psc}} = \nu^{\frac{2d}{d-2}} dV_{\hat{\gamma}_c}$ and since M has no boundary, we have
$$\int_M \nu\Delta_{\hat{\gamma}_c}\nu dV_{\hat{\gamma}_c} = -\int_M |\nabla_{\hat{\gamma}_c}\nu|^2 dV_{\hat{\gamma}_c}\,. \tag{15}$$

Yamabe Invariant and PSC

$$\textbf{Yamabe invariant: } Y(M, \hat{\gamma}_c) = \inf_{\nu \not\equiv 0} \mathcal{E}(\hat{\gamma}_c, \nu), \tag{16}$$

where the infimum is taken over the smooth real-valued positive functions ν.

By construction, the metric $\hat{\gamma}_c$ is such that $\hat{R}_c \geq 0$ everywhere and in particular $\hat{R}_c > 0$ in the sphere. We can calculate that $Y(M_d \# R^d, \hat{\gamma}_c) > 0$.

Therefore, it follows that: $\sigma(M_d \# R^d) = sup_{[g] \in C_M} Y_{[g]} > 0$ and one can conclude that the manifold admits a metric of positive scalar curvature (**Yamabe problem**).