Asymmetric soft resummation of Semi-Inclusive Deep Inelastic scattering

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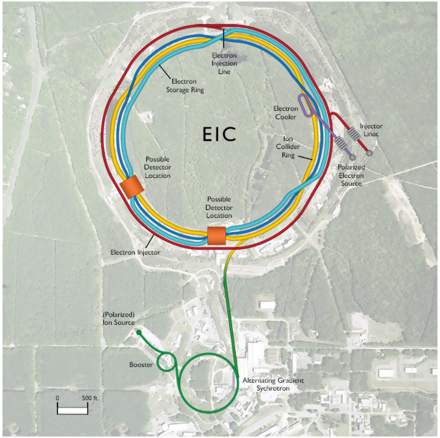
Why SIDIS precision calculations?

The new Electron-Ion Collider (EIC) at Brookhaven National Laboratory will be operational by 2032-2035.



SIDIS kinematic and large logs

SIDIS process at the hadronic level:

$$h_1(P_1) + l(k) \to h_2(P_2) + l(k') + X$$

$q = k' - k \Rightarrow Q^2 \equiv -q^2$ space-like

Scaling variables $x \equiv \dfrac{Q^2}{2P_1 \cdot q} \quad z \equiv \dfrac{P_1 \cdot P_2}{P_1 \cdot q}$

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at the LO $x \equiv \xi_1 \quad z \equiv \xi_2$

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Factorization $\quad \frac{d\sigma}{dxdzdy} = \sum_{a,b} f_a^{h_1} \otimes \hat{\sigma}_{ab} \otimes D_b^{h_2}$

qq channel $\Rightarrow \mathcal{O}(\alpha_s^k)$: $\delta(1-\hat{z})\left(\frac{\ln^m(1-\hat{x})}{1-\hat{x}}\right)_+, \quad \delta(1-\hat{x})\left(\frac{\ln^m(1-\hat{z})}{1-\hat{z}}\right)_+, \quad \left(\frac{\ln^n(1-\hat{x})}{1-\hat{x}}\right)_+ \times \left(\frac{\ln^m(1-\hat{z})}{1-\hat{z}}\right)_+$

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E.g. double-soft limit: $\lim_{N,M\to\infty} \hat{\sigma}(N, M, Q^2)$ only contains logs terms, constant terms and power suppressed terms $\Rightarrow$ Large logarithms can be resummed to all orders.

SIDIS vs DY resummation

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Process	Q^2	Momentum cons.	Frame	$\hat{\sigma}$ differential in	Radiation	Phase space
DY[1]	$q^2 > 0$ timelike	$p_1 + p_2 = q + k_1 + \cdots + k_n$	C.M.	$\hat{y}$	Collinear	$d\phi_{n+1} \propto (k^2_{\text{max}})^{p(\epsilon)} d\Omega^{n-1}(\epsilon) dv dy$
SIDIS	$q^2 < 0$ spacelike	$p_1 + q = p_2 + k_1 + \cdots + k_n$	Breit	$\hat{z}$	Collinear	$d\phi_{n+1} \propto (k^2_{\text{max}})^{p(\epsilon)} d\Omega^{n-1}(\epsilon) dv d\hat{z}$

with $d = 4 - 2\epsilon \quad k = k_1 + \cdots + k_n \quad p(\epsilon) = (n-1)(1-\epsilon) \quad \bar{N} = Ne^{\gamma}$ and $\bar{M} = Me^{\gamma}$.

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Process	k^2_{max}	Double-soft scale	single-soft scale
DY	$\frac{Q^2(1-x_1)(1-x_2)}{x_1 x_2}$	$Q^2(1-x_1)(1-x_2) \rightarrow \frac{Q^2}{\bar{N}\bar{M}}$	$Q^2(1-x_1) \vee Q^2(1-x_2) \rightarrow \frac{Q^2}{\bar{N}} \vee \frac{Q^2}{\bar{M}}$
SIDIS	$\frac{Q^2(1-\hat{x})(1-\hat{z})}{\hat{x}}$	$Q^2(1-\hat{x})(1-\hat{z}) \rightarrow \frac{Q^2}{\bar{N}\bar{M}}$	$Q^2(1-\hat{x}) \vee Q^2(1-\hat{z}) \rightarrow \frac{Q^2}{\bar{N}} \vee \frac{Q^2}{\bar{M}}$

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- E.g. $\hat{x}$ single-soft case $\Rightarrow$ it also resums all the M power suppressed terms (NLP corrections). The formula is

$$C_{qq}^{T,\mathrm{res}}\left(N,M,\alpha_s(Q^2)\right) = \left(1 + g_0^{(1)}(M)\alpha_s + g_0^{(2)}(M)\alpha_s^2 + \dots\right)$$
$$\exp\left\{\int_{\frac{Q^2}{\bar{N}}}^{Q^2}\frac{dk^2}{k^2}\left[\left(\alpha_s(k^2)A_1 + \alpha_s^2(k^2)A_2 + \dots\right)\ln\frac{\bar{N}k^2}{Q^2} - \left(\alpha_s(k^2)D_1(M) + \alpha_s^2(k^2)D_2(M) + \dots\right)\right]\right\}$$

with $\bar{N} = Ne^{\gamma}$

LL $\quad A_1 \Rightarrow \alpha_s \ln^2 N + \alpha_s^2 \ln^4 N + \dots$

NLL $\quad D_1, A_2 \Rightarrow \alpha_s \ln N + \alpha_s^2\left(\ln^3 N + \alpha_s \ln^2 N\right) + \dots$

NNLL $\quad D_2, g_0^{(1)}, A_3 \Rightarrow \alpha_s^2 \ln N + \dots$

$\dots$

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- dQCD: comparison with fixed order $\Rightarrow$ resummation coefficients.
- Single-soft limit for DY process up to NNLL obtained in dQCD by S. Forte, G. Ridolfi, L. De Ros, D. Tagliabue from a translation of a SCET result (Lustermans, Michel, and Tackmann 2019 and Mistlberger and Vita 2025).

Matching procedure

- NNLO result from Bonino, Gehrmann, and Stagnitto 2024.
- `Mathematica` software packages:
  - `MTMellin` Höschele et al. 2014: to compute Mellin transformation.
  - `Harmonic Sums` Ablinger 2013: to compute double- and single-soft limits.

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- Fixed order coefficient function in $\hat{x}$ single-soft limit:

$$C_{qq}(N,M) = 1 + \left(\frac{\alpha_s}{\pi}\right)\left(\ln^2 \bar{N}\, f_2^{(1)}(M) + \ln \bar{N}\, f_1^{(1)}(M) + f_0^{(1)}(M)\right)$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^2 \left(\ln^4 \bar{N}\, f_4^{(2)}(M) + \ln^3 \bar{N}\, f_3^{(2)}(M) + \ln^2 \bar{N}\, f_2^{(2)}(M) + \ln \bar{N}\, f_1^{(2)}(M) + f_0^{(2)}(M)\right) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}(\alpha_s^3)$$

- Resummation formula expanded

$$\tilde{C}_{qq}^{\mathrm{ss},T}\left(N,M,\alpha_s(Q^2)\right) = 1 + \frac{\alpha_s}{\pi}\left(\frac{1}{2}A_1 \ln^2 \bar{N} - D_1 \ln \bar{N} + g_0^{(1)}\right) + \frac{\alpha_s^2}{\pi^2}\left[g_0^{(2)} + \ln \bar{N}\left(-D_1 g_0^{(1)} - D_2\right)\right.$$
$$\left. + \ln^2 \bar{N}\left(\frac{A_1 g_0^{(1)}}{2} + \frac{A_2}{2} - \frac{b_0 \pi D_1}{2} + \frac{D_1^2}{2}\right) + \ln^3 \bar{N}\left(\frac{A_1 \pi b_0}{6} - \frac{A_1 D_1}{2}\right) + \frac{A_1^2 \ln^4 \bar{N}}{8}\right].$$

Matching procedure

$$f_2^{(1)}(M) = \frac{C_\mathrm{F}}{2}$$

$$f_1^{(1)}(M) = C_\mathrm{F}\left(S_1(M) - \frac{2}{4M^2+4M}\right)$$

$$f_0^{(1)}(M) = \frac{C_\mathrm{F}\left(2M^2-M-1\right)}{2M^2(M+1)^2} + \frac{1}{2}C_\mathrm{F}S_1(M)^2 - \frac{C_\mathrm{F}S_1(M)}{2M(M+1)} + \frac{3}{2}C_\mathrm{F}S_2(M) - \frac{C_\mathrm{F}\zeta(2)}{2} - 4C_\mathrm{F}$$

$$f_4^{(2)}(M) = \frac{1}{8}C_\mathrm{F}^2\,,$$

$$f_3^{(2)}(M) = C_\mathrm{F}^2\left(\frac{S_1(M)}{2} - \frac{1}{4M(M+1)}\right) + \frac{11C_\mathrm{F}C_\mathrm{A}}{72} - \frac{C_\mathrm{F}N_f}{36}\,,$$

$$f_2^{(2)}(M) = C_\mathrm{F}^2\left(\frac{4M^2-2M-1}{8M^2(M+1)^2} + \frac{3}{4}S_1(M)^2 - \frac{3S_1(M)}{4M(M+1)} + \frac{3S_2(M)}{4} - \frac{\zeta(2)}{4} - 2\right) + C_\mathrm{F}C_\mathrm{A}\left(\frac{11S_1(M)}{24} - \frac{11}{48M(M+1)} - \frac{\zeta(2)}{4} + \frac{67}{72}\right) + C_\mathrm{F}N_f\left(-\frac{1}{12}S_1(M) + \frac{1}{24M(M+1)} - \frac{5}{36}\right)\,,$$

$$\begin{aligned} f_1^{(2)}(M) = C_\mathrm{F}^2\Bigg(&-\frac{\left(3M^2+3M+5\right)S_2(M)}{4M(M+1)} + \frac{3\left(M^2+M+1\right)\zeta(2)}{4M(M+1)} \\ &- \frac{\left(16M^4+32M^3+12M^2+6M+3\right)S_1(M)}{4M^2(M+1)^2} + \frac{8M^4+19M^3+14M^2+8M+4}{4M^3(M+1)^3} \\ &- \frac{3}{2}\zeta(2)S_1(M) + \frac{1}{2}S_1(M)^3 - \frac{3S_1(M)^2}{4M(M+1)} + \frac{5}{2}S_2(M)S_1(M) - S_3(M) + \zeta(3)\Bigg) \\ &+ C_\mathrm{F}N_f\Bigg(-\frac{\left(10M^2+10M-3\right)S_1(M)}{36M(M+1)} - \frac{1}{12}S_1(M)^2 - \frac{S_2(M)}{12} \\ &+ \frac{5M+8}{36M(M+1)^2} + \frac{\zeta(2)}{12} - \frac{7}{27}\Bigg) \\ &+ C_\mathrm{F}C_\mathrm{A}\Bigg(\frac{\left(134M^2+134M-33\right)S_1(M)}{72M(M+1)} - \frac{85M^4+203M^3+154M^2+36M+18}{72M^3(M+1)^3} \\ &- \frac{1}{2}\zeta(2)S_1(M) + \frac{11}{24}S_1(M)^2 + \frac{11S_2(M)}{24} \\ &+ \frac{S_3(M)}{2} + \frac{\zeta(2)}{4M(M+1)} - \frac{11\zeta(2)}{24} - \frac{9\zeta(3)}{4} + \frac{101}{54}\Bigg) \end{aligned}$$

$$S_{a_1,\ldots,a_k}(N) = \sum_{N\geq i_1\geq i_2\geq\cdots\geq i_k\geq 1} \frac{\mathrm{sign}(a_1)^{i_1}}{i_1^{|a_1|}}\cdots\frac{\mathrm{sign}(a_k)^{i_k}}{i_k^{|a_k|}}$$

$$C_F = \frac{4}{3} \quad C_A = 3 \quad N_f = \text{active flavours}$$

Matching procedure

$$
\begin{aligned}
f_0^{(2)}(M) = C_\mathrm{F}^2 \Bigg(& \frac{1}{8}S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{4}\zeta(2)S_1(M)^2 \\
& + \frac{\left(-M^5+13M^4+41M^3+15M^2+2\right)S_1(M)}{8M^3(M+1)^3} + \frac{5\zeta(2)S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta(3)S_1(M) \\
& + \frac{S_1(M)^2\left(7M^2(M+1)^2S_2(M) - 2\left(4M^4+8M^3+3M^2+2M+1\right)\right)}{4M^2(M+1)^2} \\
& + \frac{\left(-7M(M+1)S_2(M)\right)S_1(M)}{4M^2(M+1)^2} + \frac{1}{2}\left(5S_3(M) - 2S_{2,1}(M)\right)S_1(M) - \frac{31\zeta(2)^2}{40} \\
& + \frac{-33M^6-50M^5+29M^4+81M^3+65M^2+48M+15}{8M^4(M+1)^4} \\
& + \frac{\left(4M^2-2M-1\right)\zeta(2)}{8M^2(M+1)^2} + \frac{\left(19M^4+38M^3+12M^2+M+1\right)\zeta(2)}{4M^2(M+1)^2} + \frac{13\zeta(2)}{8} \\
& - \frac{\zeta(3)}{2M(M+1)} - \frac{15\zeta(3)}{4} \\
& + \frac{3\left(7\zeta(3)M^2+7\zeta(3)M+2\zeta(3)\right)}{8M(M+1)} - \frac{\left(27M^4+54M^3+18M^2+7M+5\right)S_2(M)}{4M^2(M+1)^2} \\
& - \frac{5}{4}\zeta(2)S_2(M) - \frac{\left(9M^2+9M+10\right)S_3(M)}{8M(M+1)} - \frac{33S_4(M)}{8} - \frac{\left(3M^2+3M-2\right)S_{2,1}(M)}{4M(M+1)} \\
& + \frac{13}{4}S_{2,2}(M) - 2S_{3,1}(M) + \frac{3}{2}S_{2,1,1}(M) + \frac{511}{64}\Bigg) \\
& + \frac{1}{36}(11C_\mathrm{A} - 2N_f)\zeta(3)C_\mathrm{F} \\
& + N_f C_\mathrm{F}\Bigg(-\frac{1}{36}S_1(M)^3 - \frac{\left(10M^2+10M-3\right)S_1(M)^2}{72M(M+1)} \\
& - \frac{\left(28M^3+56M^2+13M-24\right)S_1(M)}{108M(M+1)^2} + \frac{1}{12}\zeta(2)S_1(M) - \frac{1}{12}S_2(M)S_1(M) \\
& - \frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta(2)}{36M(M+1)} \\
& \qquad + \frac{\zeta(2)}{24M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{\left(20M^2+20M-3\right)S_2(M)}{72M(M+1)} + \frac{S_3(M)}{36} + \frac{127}{96}\Bigg) \\
& + C_\mathrm{A}C_\mathrm{F}\Bigg(\frac{11}{72}S_1(M)^3 + \frac{\left(404M^5+1239M^4+1038M^3-70M^2-273M+54\right)S_1(M)}{216M^2(M+1)^3} \\
& - \frac{11}{24}\zeta(2)S_1(M) - \frac{13}{4}\zeta(3)S_1(M) \\
& - \frac{S_1(M)^2\left(-134M^2+36(M+1)S_2(M)M-134M+33\right)}{144M(M+1)} \\
& \frac{-6S_1(M)\left(11M^2+11M+6\right)S_2(M)}{144M(M+1)} - \frac{1}{2}\left(S_3(M) - 2S_{2,1}(M)\right)S_1(M) + \frac{21\zeta(2)^2}{20} + \\
& \frac{124M^6-1044M^5-2679M^4-2396M^3-1002M^2-765M-270}{432M^4(M+1)^4} \\
& - \frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^4+58\zeta(2)M^3+17\zeta(2)M^2-30\zeta(2)M-9\zeta(2)}{36M^2(M+1)^2} \\
& + \frac{\left(11M^2+11M+39\right)\zeta(3)}{24M(M+1)} + \frac{43\zeta(3)}{12} + \frac{\left(250M^3+250M^2-69M-36\right)S_2(M)}{144M^2(M+1)} \\
& - \frac{1}{2}\zeta(2)S_2(M) - \frac{\left(11M^2+11M-18\right)S_3(M)}{72M(M+1)} + S_4(M) - \frac{S_{2,1}(M)}{2M(M+1)} \\
& + S_{2,2}(M) + S_{3,1}(M) - \frac{3}{2}S_{2,1,1}(M) - \frac{1535}{192}\Bigg)
\end{aligned}
$$

Matching procedure

$$
\begin{aligned}
f_0^{(2)}(M) = C_\mathrm{F}^2 \Bigg(& \frac{1}{8}S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{4}\zeta(2)S_1(M)^2 \\
& + \frac{\left(-M^5+13M^4+41M^3+15M^2+2\right)S_1(M)}{8M^3(M+1)^3} + \frac{5\zeta(2)S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta(3)S_1(M) \\
& + \frac{S_1(M)^2\left(7M^2(M+1)^2S_2(M) - 2\left(4M^4+8M^3+3M^2+2M+1\right)\right)}{4M^2(M+1)^2} \\
& + \frac{\left(-7M(M+1)S_2(M)\right)S_1(M)}{4M^2(M+1)^2} + \frac{1}{2}\left(5S_3(M) - 2S_{2,1}(M)\right)S_1(M) - \frac{31\zeta(2)^2}{40} \\
& + \frac{-33M^6-50M^5+29M^4+81M^3+65M^2+48M+15}{8M^4(M+1)^4} \\
& + \frac{\left(4M^2-2M-1\right)\zeta(2)}{8M^2(M+1)^2} + \frac{\left(19M^4+38M^3+12M^2+M+1\right)\zeta(2)}{4M^2(M+1)^2} + \frac{13\zeta(2)}{8} \\
& - \frac{\zeta(3)}{2M(M+1)} - \frac{15\zeta(3)}{4} \\
& + \frac{3\left(7\zeta(3)M^2+7\zeta(3)M+2\zeta(3)\right)}{8M(M+1)} - \frac{\left(27M^4+54M^3+18M^2+7M+5\right)S_2(M)}{4M^2(M+1)^2} \\
& - \frac{5}{4}\zeta(2)S_2(M) - \frac{\left(9M^2+9M+10\right)S_3(M)}{8M(M+1)} - \frac{33S_4(M)}{8} - \frac{\left(3M^2+3M-2\right)S_{2,1}(M)}{4M(M+1)} \\
& + \frac{13}{4}S_{2,2}(M) - 2S_{3,1}(M) + \frac{3}{2}S_{2,1,1}(M) + \frac{511}{64} \Bigg) \\
& + \frac{1}{36}(11C_\mathrm{A} - 2N_f)\zeta(3)C_\mathrm{F} \\
& + N_fC_\mathrm{F} \Bigg(-\frac{1}{36}S_1(M)^3 - \frac{\left(10M^2+10M-3\right)S_1(M)^2}{72M(M+1)} \\
& - \frac{\left(28M^3+56M^2+13M-24\right)S_1(M)}{108M(M+1)^2} + \frac{1}{12}\zeta(2)S_1(M) - \frac{1}{12}S_2(M)S_1(M) \\
& - \frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta(2)}{36M(M+1)} \\
& \qquad + \frac{\zeta(2)}{24M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{\left(20M^2+20M-3\right)S_2(M)}{72M(M+1)} + \frac{S_3(M)}{36} + \frac{127}{96} \Bigg) \\
& + C_\mathrm{A}C_\mathrm{F} \Bigg(\frac{11}{72}S_1(M)^3 + \frac{\left(404M^5+1239M^4+1038M^3-70M^2-273M+54\right)S_1(M)}{216M^2(M+1)^3} \\
& - \frac{11}{24}\zeta(2)S_1(M) - \frac{13}{4}\zeta(3)S_1(M) \\
& - \frac{S_1(M)^2\left(-134M^2+36(M+1)S_2(M)M-134M+33\right)}{144M(M+1)} \\
& \frac{-6S_1(M)\left(11M^2+11M+6\right)S_2(M)}{144M(M+1)} - \frac{1}{2}\left(S_3(M)-2S_{2,1}(M)\right)S_1(M) + \frac{21\zeta(2)^2}{20} + \\
& \frac{124M^6-1044M^5-2679M^4-2396M^3-1002M^2-765M-270}{432M^4(M+1)^4} \\
& - \frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^4+58\zeta(2)M^3+17\zeta(2)M^2-30\zeta(2)M-9\zeta(2)}{36M^2(M+1)^2} \\
& + \frac{\left(11M^2+11M+39\right)\zeta(3)}{24M(M+1)} + \frac{43\zeta(3)}{12} + \frac{\left(250M^3+250M^2-69M-36\right)S_2(M)}{144M^2(M+1)} \\
& - \frac{1}{2}\zeta(2)S_2(M) - \frac{\left(11M^2+11M-18\right)S_3(M)}{72M(M+1)} + S_4(M) - \frac{S_{2,1}(M)}{2M(M+1)} \\
& + S_{2,2}(M) + S_{3,1}(M) - \frac{3}{2}S_{2,1,1}(M) - \frac{1535}{192} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
A^{(1)} &= C_F\,, \\
A^{(2)} &= \frac{1}{2}C_F\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right], \\
D^{(1)}(M) &= \tilde{P}_{qq}^{(0),T}(M) = \frac{C_F}{2}\left[\frac{1}{M(M+1)} - 2S_1(M)\right], \\
D^{(2)}(M) &= D^{(2)}_{\mathsf{ds},qq} - \pi\beta_0\tilde{F}(M) + \tilde{P}^{(1),T}_{qq,\mathsf{NS}}(M)\,, \\
g_0^{(1)}(M) &= g^{(1)}_{0,(\mathsf{ds})} + \tilde{F}(M) = f_0^{(1)}(M)\,, \\
g^{(2)}_{0,qq}(M) &= f_0^{(2)}(M)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}(M) &= \frac{1}{2}C_FS_1(M)^2 + \frac{C_F\left(2M^2-M-1\right)}{2M^2(M+1)^2} \\
&\quad + \frac{3}{2}C_F\left(S_2(M) - \zeta(2)\right) - \frac{C_FS_1(M)}{2M(M+1)}\,. \\
\tilde{P}^{(1)}_{qq}(M) &= C_F^2\Bigg(\frac{(2M+1)S_1(M)}{2M^2(M+1)^2} + \frac{\left(3M^2+3M+2\right)S_2(M)}{4M(M+1)} - \frac{\left(3M^2+3M+2\right)\zeta(2)}{4M(M+1)} \\
&\qquad - \frac{3M^3+8M^2+7M+3}{4M^3(M+1)^3} + \zeta(2)S_1(M) - S_1(M)S_2(M) + S_3(M) - \zeta(3)\Bigg) \\
&\qquad + C_FN_F\left(-\frac{11M^2+5M-3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6}\right) \\
&\qquad + C_FC_A\Bigg(-\frac{\left(11M^2+11M+3\right)\zeta(2)}{12M(M+1)} \\
&\qquad + \frac{151M^4+236M^3+88M^2+3M+18}{72M^3(M+1)^3} \\
&\qquad + \frac{1}{2}\zeta(2)S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_3(M)}{2} + \frac{\zeta(3)}{2}\Bigg)\,.
\end{aligned}
$$

$$
\begin{aligned}
g^{(1)}_{0,(\mathsf{ds},qq)} &= C_F\left(\zeta(2) - 4\right) \\
D^{(2)}_{\mathsf{ds},qq} &= \frac{1}{2}C_F\left[\left(-\frac{101}{27} + \frac{7}{2}\zeta(3)\right)C_A + \frac{14}{27}N_f\right], \\
g^{(2)}_{0,(\mathsf{ds},qq)} &= C_F^2\left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4}\zeta(3)\right] \\
&\quad + C_FC_A\left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36}\zeta(3)\right] \\
&\quad + C_FN_f\left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18}\right].
\end{aligned}
$$

Matching procedure

$$
\begin{aligned}
f_0^{(2)}(M) = C_{\mathrm{F}}^2 \Bigg(& \frac{1}{8}S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{4}\zeta(2)S_1(M)^2 \\
& + \frac{\left(-M^5+13M^4+41M^3+15M^2+2\right)S_1(M)}{8M^3(M+1)^3} + \frac{5\zeta(2)S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta(3)S_1(M) \\
& + \frac{S_1(M)^2\left(7M^2(M+1)^2S_2(M) - 2\left(4M^4+8M^3+3M^2+2M+1\right)\right)}{4M^2(M+1)^2} \\
& + \frac{\left(-7M(M+1)S_2(M)\right)S_1(M)}{4M^2(M+1)^2} + \frac{1}{2}\left(5S_3(M)-2S_{2,1}(M)\right)S_1(M) - \frac{31\zeta(2)^2}{40} \\
& + \frac{-33M^6-50M^5+29M^4+81M^3+65M^2+48M+15}{8M^4(M+1)^4} \\
& + \frac{\left(4M^2-2M-1\right)\zeta(2)}{8M^2(M+1)^2} + \frac{\left(19M^4+38M^3+12M^2+M+1\right)\zeta(2)}{4M^2(M+1)^2} + \frac{13\zeta(2)}{8} \\
& - \frac{\zeta(3)}{2M(M+1)} - \frac{15\zeta(3)}{4} \\
& + \frac{3\left(7\zeta(3)M^2+7\zeta(3)M+2\zeta(3)\right)}{8M(M+1)} - \frac{\left(27M^4+54M^3+18M^2+7M+5\right)S_2(M)}{4M^2(M+1)^2} \\
& - \frac{5}{4}\zeta(2)S_2(M) - \frac{\left(9M^2+9M+10\right)S_3(M)}{8M(M+1)} - \frac{33S_4(M)}{8} - \frac{\left(3M^2+3M-2\right)S_{2,1}(M)}{4M(M+1)} \\
& + \frac{13}{4}S_{2,2}(M) - 2S_{3,1}(M) + \frac{3}{2}S_{2,1,1}(M) + \frac{511}{64} \Bigg) \\
& + \frac{1}{36}(11C_{\mathrm{A}} - 2N_f)\zeta(3)C_{\mathrm{F}} \\
& + N_f C_{\mathrm{F}} \Bigg(-\frac{1}{36}S_1(M)^3 - \frac{\left(10M^2+10M-3\right)S_1(M)^2}{72M(M+1)} \\
& - \frac{\left(28M^3+56M^2+13M-24\right)S_1(M)}{108M(M+1)^2} + \frac{1}{12}\zeta(2)S_1(M) - \frac{1}{12}S_2(M)S_1(M) \\
& - \frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta(2)}{36M(M+1)} \\
& \qquad + \frac{\zeta(2)}{24M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{\left(20M^2+20M-3\right)S_2(M)}{72M(M+1)} + \frac{S_3(M)}{36} + \frac{127}{96} \Bigg) \\
& + C_{\mathrm{A}} C_{\mathrm{F}} \Bigg(\frac{11}{72}S_1(M)^3 + \frac{\left(404M^5+1239M^4+1038M^3-70M^2-273M+54\right)S_1(M)}{216M^2(M+1)^3} \\
& - \frac{11}{24}\zeta(2)S_1(M) - \frac{13}{4}\zeta(3)S_1(M) \\
& - \frac{S_1(M)^2\left(-134M^2+36(M+1)S_2(M)M-134M+33\right)}{144M(M+1)} \\
& \frac{-6S_1(M)\left(11M^2+11M+6\right)S_2(M)}{144M(M+1)} - \frac{1}{2}\left(S_3(M)-2S_{2,1}(M)\right)S_1(M) + \frac{21\zeta(2)^2}{20} + \\
& \frac{124M^6-1044M^5-2679M^4-2396M^3-1002M^2-765M-270}{432M^4(M+1)^4} \\
& - \frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^4+58\zeta(2)M^3+17\zeta(2)M^2-30\zeta(2)M-9\zeta(2)}{36M^2(M+1)^2} \\
& + \frac{\left(11M^2+11M+39\right)\zeta(3)}{24M(M+1)} + \frac{43\zeta(3)}{12} + \frac{\left(250M^3+250M^2-69M-36\right)S_2(M)}{144M^2(M+1)} \\
& - \frac{1}{2}\zeta(2)S_2(M) - \frac{\left(11M^2+11M-18\right)S_3(M)}{72M(M+1)} + S_4(M) - \frac{S_{2,1}(M)}{2M(M+1)} \\
& + S_{2,2}(M) + S_{3,1}(M) - \frac{3}{2}S_{2,1,1}(M) - \frac{1535}{192} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
& A^{(1)} = C_F\,, \\
& A^{(2)} = \frac{1}{2}C_F\left[\left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right], \\
& D^{(1)}(M) = \tilde{P}_{qq}^{(0),T}(M) = \frac{C_F}{2}\left[\frac{1}{M(M+1)} - 2S_1(M)\right], \\
& D^{(2)}(M) = D^{(2)}_{\mathsf{ds},qq} - \pi\beta_0\tilde{F}(M) + \tilde{P}^{(1),T}_{qq,\mathsf{NS}}(M)\,, \\
& g_0^{(1)}(M) = g^{(1)}_{0,(\mathsf{ds})} + \tilde{F}(M) = f_0^{(1)}(M)\,, \\
& g^{(2)}_{0,qq}(M) = f_0^{(2)}(M)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}(M) = {} & \frac{1}{2}C_F S_1(M)^2 + \frac{C_F\left(2M^2-M-1\right)}{2M^2(M+1)^2} \\
& + \frac{3}{2}C_F\left(S_2(M)-\zeta(2)\right) - \frac{C_F S_1(M)}{2M(M+1)}\,. \\
\tilde{P}^{(1)}_{qq}(M) = {} & C_F^2\Bigg(\frac{(2M+1)S_1(M)}{2M^2(M+1)^2} + \frac{\left(3M^2+3M+2\right)S_2(M)}{4M(M+1)} - \frac{\left(3M^2+3M+2\right)\zeta(2)}{4M(M+1)} \\
& \quad - \frac{3M^3+8M^2+7M+3}{4M^3(M+1)^3} + \zeta(2)S_1(M) - S_1(M)S_2(M) + S_3(M) - \zeta(3)\Bigg) \\
& + C_F N_F\left(-\frac{11M^2+5M-3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6}\right) \\
& + C_F C_A\Bigg(-\frac{\left(11M^2+11M+3\right)\zeta(2)}{12M(M+1)} \\
& + \frac{151M^4+236M^3+88M^2+3M+18}{72M^3(M+1)^3} \\
& + \frac{1}{2}\zeta(2)S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_3(M)}{2} + \frac{\zeta(3)}{2}\Bigg)\,.
\end{aligned}
$$

$$
\begin{aligned}
g^{(1)}_{0,(\mathsf{ds},qq)} &= C_F\left(\zeta(2)-4\right) \\
D^{(2)}_{\mathsf{ds},qq} &= \frac{1}{2}C_F\left[\left(-\frac{101}{27}+\frac{7}{2}\zeta(3)\right)C_A + \frac{14}{27}N_f\right], \\
g^{(2)}_{0,(\mathsf{ds},qq)} &= C_F^2\left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4}\zeta(3)\right] \\
&\quad + C_F C_A\left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36}\zeta(3)\right] \\
&\quad + C_F N_f\left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18}\right].
\end{aligned}
$$

- $\tilde{F}(M)$ process dependent function.

Matching procedure

$$
\begin{aligned}
f_0^{(2)}(M) = C_{\mathrm{F}}^2 \Bigg(& \frac{1}{8}S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{4}\zeta(2)S_1(M)^2 \\
& + \frac{\left(-M^5+13M^4+41M^3+15M^2+2\right)S_1(M)}{8M^3(M+1)^3} + \frac{5\zeta(2)S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta(3)S_1(M) \\
& + \frac{S_1(M)^2\left(7M^2(M+1)^2S_2(M) - 2\left(4M^4+8M^3+3M^2+2M+1\right)\right)}{4M^2(M+1)^2} \\
& + \frac{\left(-7M(M+1)S_2(M)\right)S_1(M)}{4M^2(M+1)^2} + \frac{1}{2}\left(5S_3(M) - 2S_{2,1}(M)\right)S_1(M) - \frac{31\zeta(2)^2}{40} \\
& + \frac{-33M^6-50M^5+29M^4+81M^3+65M^2+48M+15}{8M^4(M+1)^4} \\
& + \frac{\left(4M^2-2M-1\right)\zeta(2)}{8M^2(M+1)^2} + \frac{\left(19M^4+38M^3+12M^2+M+1\right)\zeta(2)}{4M^2(M+1)^2} + \frac{13\zeta(2)}{8} \\
& - \frac{\zeta(3)}{2M(M+1)} - \frac{15\zeta(3)}{4} \\
& + \frac{3\left(7\zeta(3)M^2+7\zeta(3)M+2\zeta(3)\right)}{8M(M+1)} - \frac{\left(27M^4+54M^3+18M^2+7M+5\right)S_2(M)}{4M^2(M+1)^2} \\
& - \frac{5}{4}\zeta(2)S_2(M) - \frac{\left(9M^2+9M+10\right)S_3(M)}{8M(M+1)} - \frac{33S_4(M)}{8} - \frac{\left(3M^2+3M-2\right)S_{2,1}(M)}{4M(M+1)} \\
& + \frac{13}{4}S_{2,2}(M) - 2S_{3,1}(M) + \frac{3}{2}S_{2,1,1}(M) + \frac{511}{64}\Bigg) \\
& + \frac{1}{36}(11C_{\mathrm{A}} - 2N_f)\zeta(3)C_{\mathrm{F}} \\
& + N_f C_{\mathrm{F}}\Bigg(-\frac{1}{36}S_1(M)^3 - \frac{\left(10M^2+10M-3\right)S_1(M)^2}{72M(M+1)} \\
& - \frac{\left(28M^3+56M^2+13M-24\right)S_1(M)}{108M(M+1)^2} + \frac{1}{12}\zeta(2)S_1(M) - \frac{1}{12}S_2(M)S_1(M) \\
& - \frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta(2)}{36M(M+1)} \\
& \qquad + \frac{\zeta(2)}{24M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{\left(20M^2+20M-3\right)S_2(M)}{72M(M+1)} + \frac{S_3(M)}{36} + \frac{127}{96}\Bigg) \\
& + C_{\mathrm{A}}C_{\mathrm{F}}\Bigg(\frac{11}{72}S_1(M)^3 + \frac{\left(404M^5+1239M^4+1038M^3-70M^2-273M+54\right)S_1(M)}{216M^2(M+1)^3} \\
& - \frac{11}{24}\zeta(2)S_1(M) - \frac{13}{4}\zeta(3)S_1(M) \\
& - \frac{S_1(M)^2\left(-134M^2+36(M+1)S_2(M)M-134M+33\right)}{144M(M+1)} \\
& \frac{-6S_1(M)\left(11M^2+11M+6\right)S_2(M)}{144M(M+1)} - \frac{1}{2}\left(S_3(M) - 2S_{2,1}(M)\right)S_1(M) + \frac{21\zeta(2)^2}{20} + \\
& \frac{124M^6-1044M^5-2679M^4-2396M^3-1002M^2-765M-270}{432M^4(M+1)^4} \\
& - \frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^4+58\zeta(2)M^3+17\zeta(2)M^2-30\zeta(2)M-9\zeta(2)}{36M^2(M+1)^2} \\
& + \frac{\left(11M^2+11M+39\right)\zeta(3)}{24M(M+1)} + \frac{43\zeta(3)}{12} + \frac{\left(250M^3+250M^2-69M-36\right)S_2(M)}{144M^2(M+1)} \\
& - \frac{1}{2}\zeta(2)S_2(M) - \frac{\left(11M^2+11M-18\right)S_3(M)}{72M(M+1)} + S_4(M) - \frac{S_{2,1}(M)}{2M(M+1)} \\
& + S_{2,2}(M) + S_{3,1}(M) - \frac{3}{2}S_{2,1,1}(M) - \frac{1535}{192}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
A^{(1)} &= C_F\,, \\
A^{(2)} &= \frac{1}{2}C_F\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right], \\
D^{(1)}(M) &= \tilde{P}_{qq}^{(0),T}(M) = \frac{C_F}{2}\left[\frac{1}{M(M+1)} - 2S_1(M)\right], \\
D^{(2)}(M) &= D^{(2)}_{\mathsf{ds},qq} - \pi\beta_0\tilde{F}(M) + \tilde{P}^{(1),T}_{qq,\mathsf{NS}}(M)\,, \\
g_0^{(1)}(M) &= g^{(1)}_{0,(\mathsf{ds})} + \tilde{F}(M) = f_0^{(1)}(M)\,, \\
g^{(2)}_{0,qq}(M) &= f_0^{(2)}(M)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}(M) &= \frac{1}{2}C_F S_1(M)^2 + \frac{C_F\left(2M^2-M-1\right)}{2M^2(M+1)^2} \\
&\quad + \frac{3}{2}C_F\left(S_2(M) - \zeta(2)\right) - \frac{C_F S_1(M)}{2M(M+1)}\,. \\
\tilde{P}^{(1)}_{qq}(M) &= C_F^2\Bigg(\frac{(2M+1)S_1(M)}{2M^2(M+1)^2} + \frac{\left(3M^2+3M+2\right)S_2(M)}{4M(M+1)} - \frac{\left(3M^2+3M+2\right)\zeta(2)}{4M(M+1)} \\
&\qquad - \frac{3M^3+8M^2+7M+3}{4M^3(M+1)^3} + \zeta(2)S_1(M) - S_1(M)S_2(M) + S_3(M) - \zeta(3)\Bigg) \\
&\qquad + C_F N_F\left(-\frac{11M^2+5M-3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6}\right) \\
&\qquad + C_F C_A\Bigg(-\frac{\left(11M^2+11M+3\right)\zeta(2)}{12M(M+1)} \\
&\qquad + \frac{151M^4+236M^3+88M^2+3M+18}{72M^3(M+1)^3} \\
&\qquad + \frac{1}{2}\zeta(2)S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_3(M)}{2} + \frac{\zeta(3)}{2}\Bigg)\,.
\end{aligned}
$$

$$
\begin{aligned}
g^{(1)}_{0,(\mathsf{ds},qq)} &= C_F\left(\zeta(2) - 4\right) \\
D^{(2)}_{\mathsf{ds},qq} &= \frac{1}{2}C_F\left[\left(-\frac{101}{27} + \frac{7}{2}\zeta(3)\right)C_A + \frac{14}{27}N_f\right], \\
g^{(2)}_{0,(\mathsf{ds},qq)} &= C_F^2\left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4}\zeta(3)\right] \\
&\quad + C_F C_A\left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36}\zeta(3)\right] \\
&\quad + C_F N_f\left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18}\right].
\end{aligned}
$$

- $\tilde{F}(M)$ process dependent function.
- $\tilde{P}^{(i)}_{qq}$s are the splitting functions without the constant terms.

Matching procedure

$$
\begin{aligned}
f_0^{(2)}(M) = C_{\mathrm{F}}^2 \Bigg(& \frac{1}{8} S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{4}\zeta(2) S_1(M)^2 \\
& + \frac{\left(-M^5+13M^4+41M^3+15M^2+2\right) S_1(M)}{8M^3(M+1)^3} + \frac{5\zeta(2)S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta(3)S_1(M) \\
& + \frac{S_1(M)^2\left(7M^2(M+1)^2 S_2(M) - 2\left(4M^4+8M^3+3M^2+2M+1\right)\right)}{4M^2(M+1)^2} \\
& + \frac{\left(-7M(M+1)S_2(M)\right) S_1(M)}{4M^2(M+1)^2} + \frac{1}{2}\left(5S_3(M) - 2S_{2,1}(M)\right) S_1(M) - \frac{31\zeta(2)^2}{40} \\
& + \frac{-33M^6-50M^5+29M^4+81M^3+65M^2+48M+15}{8M^4(M+1)^4} \\
& + \frac{\left(4M^2-2M-1\right)\zeta(2)}{8M^2(M+1)^2} + \frac{\left(19M^4+38M^3+12M^2+M+1\right)\zeta(2)}{4M^2(M+1)^2} + \frac{13\zeta(2)}{8} \\
& - \frac{\zeta(3)}{2M(M+1)} - \frac{15\zeta(3)}{4} \\
& + \frac{3\left(7\zeta(3)M^2+7\zeta(3)M+2\zeta(3)\right)}{8M(M+1)} - \frac{\left(27M^4+54M^3+18M^2+7M+5\right) S_2(M)}{4M^2(M+1)^2} \\
& - \frac{5}{4}\zeta(2)S_2(M) - \frac{\left(9M^2+9M+10\right) S_3(M)}{8M(M+1)} - \frac{33S_4(M)}{8} - \frac{\left(3M^2+3M-2\right) S_{2,1}(M)}{4M(M+1)} \\
& + \frac{13}{4}S_{2,2}(M) - 2S_{3,1}(M) + \frac{3}{2}S_{2,1,1}(M) + \frac{511}{64} \Bigg) \\
& + \frac{1}{36}(11C_{\mathrm{A}} - 2N_f)\zeta(3)C_{\mathrm{F}} \\
& + N_f C_{\mathrm{F}} \Bigg(-\frac{1}{36}S_1(M)^3 - \frac{\left(10M^2+10M-3\right) S_1(M)^2}{72M(M+1)} \\
& - \frac{\left(28M^3+56M^2+13M-24\right) S_1(M)}{108M(M+1)^2} + \frac{1}{12}\zeta(2)S_1(M) - \frac{1}{12}S_2(M)S_1(M) \\
& - \frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta(2)}{36M(M+1)} \\
& \qquad + \frac{\zeta(2)}{24M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{\left(20M^2+20M-3\right) S_2(M)}{72M(M+1)} + \frac{S_3(M)}{36} + \frac{127}{96} \Bigg) \\
& + C_{\mathrm{A}} C_{\mathrm{F}} \Bigg(\frac{11}{72}S_1(M)^3 + \frac{\left(404M^5+1239M^4+1038M^3-70M^2-273M+54\right) S_1(M)}{216M^2(M+1)^3} \\
& - \frac{11}{24}\zeta(2)S_1(M) - \frac{13}{4}\zeta(3)S_1(M) \\
& - \frac{S_1(M)^2\left(-134M^2+36(M+1)S_2(M)M-134M+33\right)}{144M(M+1)} \\
& \frac{-6S_1(M)\left(11M^2+11M+6\right) S_2(M)}{144M(M+1)} - \frac{1}{2}\left(S_3(M) - 2S_{2,1}(M)\right) S_1(M) + \frac{21\zeta(2)^2}{20} + \\
& \frac{124M^6-1044M^5-2679M^4-2396M^3-1002M^2-765M-270}{432M^4(M+1)^4} \\
& - \frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^4+58\zeta(2)M^3+17\zeta(2)M^2-30\zeta(2)M-9\zeta(2)}{36M^2(M+1)^2} \\
& + \frac{\left(11M^2+11M+39\right)\zeta(3)}{24M(M+1)} + \frac{43\zeta(3)}{12} + \frac{\left(250M^3+250M^2-69M-36\right) S_2(M)}{144M^2(M+1)} \\
& - \frac{1}{2}\zeta(2)S_2(M) - \frac{\left(11M^2+11M-18\right) S_3(M)}{72M(M+1)} + S_4(M) - \frac{S_{2,1}(M)}{2M(M+1)} \\
& + S_{2,2}(M) + S_{3,1}(M) - \frac{3}{2}S_{2,1,1}(M) - \frac{1535}{192} \Bigg)
\end{aligned}
$$

$$A^{(1)} = C_F \,,$$

$$A^{(2)} = \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right] ,$$

$$D^{(1)}(M) = \tilde{P}_{qq}^{(0),T}(M) = \frac{C_F}{2} \left[\frac{1}{M(M+1)} - 2S_1(M) \right] ,$$

$$D^{(2)}(M) = D^{(2)}_{\mathsf{ds},qq} - \pi\beta_0 \tilde{F}(M) + \tilde{P}^{(1),T}_{qq,\mathsf{NS}}(M) \,,$$

$$g_0^{(1)}(M) = g^{(1)}_{0,(\mathsf{ds})} + \tilde{F}(M) = f_0^{(1)}(M) \,,$$

$$g^{(2)}_{0,qq}(M) = f_0^{(2)}(M)$$

$$
\begin{aligned}
\tilde{F}(M) = & \frac{1}{2} C_F S_1(M)^2 + \frac{C_F\left(2M^2-M-1\right)}{2M^2(M+1)^2} \\
& + \frac{3}{2} C_F \left(S_2(M) - \zeta(2)\right) - \frac{C_F S_1(M)}{2M(M+1)} .
\end{aligned}
$$

$$
\begin{aligned}
\tilde{P}_{qq}^{(1)}(M) = C_F^2 \Bigg(& \frac{(2M+1)S_1(M)}{2M^2(M+1)^2} + \frac{\left(3M^2+3M+2\right) S_2(M)}{4M(M+1)} - \frac{\left(3M^2+3M+2\right)\zeta(2)}{4M(M+1)} \\
& - \frac{3M^3+8M^2+7M+3}{4M^3(M+1)^3} + \zeta(2)S_1(M) - S_1(M)S_2(M) + S_3(M) - \zeta(3) \Bigg) \\
& + C_F N_F \left(-\frac{11M^2+5M-3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6} \right) \\
& + C_F C_A \Bigg(-\frac{\left(11M^2+11M+3\right)\zeta(2)}{12M(M+1)} \\
& + \frac{151M^4+236M^3+88M^2+3M+18}{72M^3(M+1)^3} \\
& + \frac{1}{2}\zeta(2)S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_3(M)}{2} + \frac{\zeta(3)}{2} \Bigg) .
\end{aligned}
$$

$$g^{(1)}_{0,(\mathsf{ds},qq)} = C_F\left(\zeta(2) - 4\right)$$

$$D^{(2)}_{\mathsf{ds},qq} = \frac{1}{2} C_F \left[\left(-\frac{101}{27} + \frac{7}{2}\zeta(3) \right) C_A + \frac{14}{27} N_f \right] ,$$

$$
\begin{aligned}
g^{(2)}_{0,(\mathsf{ds},qq)} = & C_F^2 \left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4}\zeta(3) \right] \\
& + C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36}\zeta(3) \right] \\
& + C_F N_f \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18} \right] .
\end{aligned}
$$

- $\tilde{F}(M)$ process dependent function.
- $\tilde{P}_{qq}^{(i)}$s are the splitting functions without the constant terms.
- $\hat{z}-$single-soft limit same behavior aside for P_{qq} spacelike.

Matching procedure

$$
\begin{aligned}
f_0^{(2)}(M) = C_{\mathrm{F}}^2 \Bigg(& \frac{1}{8} S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{4}\zeta(2) S_1(M)^2 \\
& + \frac{\left(-M^5+13M^4+41M^3+15M^2+2\right) S_1(M)}{8M^3(M+1)^3} + \frac{5\zeta(2)S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta(3)S_1(M) \\
& + \frac{S_1(M)^2\left(7M^2(M+1)^2 S_2(M) - 2\left(4M^4+8M^3+3M^2+2M+1\right)\right)}{4M^2(M+1)^2} \\
& + \frac{\left(-7M(M+1)S_2(M)\right) S_1(M)}{4M^2(M+1)^2} + \frac{1}{2}\left(5S_3(M) - 2S_{2,1}(M)\right) S_1(M) - \frac{31\zeta(2)^2}{40} \\
& + \frac{-33M^6-50M^5+29M^4+81M^3+65M^2+48M+15}{8M^4(M+1)^4} \\
& + \frac{\left(4M^2-2M-1\right)\zeta(2)}{8M^2(M+1)^2} + \frac{\left(19M^4+38M^3+12M^2+M+1\right)\zeta(2)}{4M^2(M+1)^2} + \frac{13\zeta(2)}{8} \\
& - \frac{\zeta(3)}{2M(M+1)} - \frac{15\zeta(3)}{4} \\
& + \frac{3\left(7\zeta(3)M^2+7\zeta(3)M+2\zeta(3)\right)}{8M(M+1)} - \frac{\left(27M^4+54M^3+18M^2+7M+5\right) S_2(M)}{4M^2(M+1)^2} \\
& - \frac{5}{4}\zeta(2)S_2(M) - \frac{\left(9M^2+9M+10\right) S_3(M)}{8M(M+1)} - \frac{33S_4(M)}{8} - \frac{\left(3M^2+3M-2\right) S_{2,1}(M)}{4M(M+1)} \\
& + \frac{13}{4}S_{2,2}(M) - 2S_{3,1}(M) + \frac{3}{2}S_{2,1,1}(M) + \frac{511}{64} \Bigg) \\
& + \frac{1}{36}(11C_{\mathrm{A}} - 2N_f)\zeta(3)C_{\mathrm{F}} \\
& + N_f C_{\mathrm{F}} \Bigg(-\frac{1}{36}S_1(M)^3 - \frac{\left(10M^2+10M-3\right) S_1(M)^2}{72M(M+1)} \\
& - \frac{\left(28M^3+56M^2+13M-24\right) S_1(M)}{108M(M+1)^2} + \frac{1}{12}\zeta(2)S_1(M) - \frac{1}{12}S_2(M)S_1(M) \\
& - \frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta(2)}{36M(M+1)} \\
& \qquad + \frac{\zeta(2)}{24M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{\left(20M^2+20M-3\right) S_2(M)}{72M(M+1)} + \frac{S_3(M)}{36} + \frac{127}{96} \Bigg) \\
& + C_{\mathrm{A}} C_{\mathrm{F}} \Bigg(\frac{11}{72}S_1(M)^3 + \frac{\left(404M^5+1239M^4+1038M^3-70M^2-273M+54\right) S_1(M)}{216M^2(M+1)^3} \\
& - \frac{11}{24}\zeta(2)S_1(M) - \frac{13}{4}\zeta(3)S_1(M) \\
& - \frac{S_1(M)^2\left(-134M^2+36(M+1)S_2(M)M-134M+33\right)}{144M(M+1)} \\
& \frac{-6S_1(M)\left(11M^2+11M+6\right) S_2(M)}{144M(M+1)} - \frac{1}{2}\left(S_3(M) - 2S_{2,1}(M)\right) S_1(M) + \frac{21\zeta(2)^2}{20} + \\
& \frac{124M^6-1044M^5-2679M^4-2396M^3-1002M^2-765M-270}{432M^4(M+1)^4} \\
& - \frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^4+58\zeta(2)M^3+17\zeta(2)M^2-30\zeta(2)M-9\zeta(2)}{36M^2(M+1)^2} \\
& + \frac{\left(11M^2+11M+39\right)\zeta(3)}{24M(M+1)} + \frac{43\zeta(3)}{12} + \frac{\left(250M^3+250M^2-69M-36\right) S_2(M)}{144M^2(M+1)} \\
& - \frac{1}{2}\zeta(2)S_2(M) - \frac{\left(11M^2+11M-18\right) S_3(M)}{72M(M+1)} + S_4(M) - \frac{S_{2,1}(M)}{2M(M+1)} \\
& + S_{2,2}(M) + S_{3,1}(M) - \frac{3}{2}S_{2,1,1}(M) - \frac{1535}{192} \Bigg)
\end{aligned}
$$

$$A^{(1)} = C_F\,,$$

$$A^{(2)} = \frac{1}{2}C_F\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right],$$

$$D^{(1)}(M) = \tilde{P}_{qq}^{(0),T}(M) = \frac{C_F}{2}\left[\frac{1}{M(M+1)} - 2S_1(M)\right],$$

$$D^{(2)}(M) = D^{(2)}_{\mathsf{ds},qq} - \pi\beta_0 \tilde{F}(M) + \tilde{P}^{(1),T}_{qq,\mathsf{NS}}(M)\,,$$

$$g_0^{(1)}(M) = g^{(1)}_{0,(\mathsf{ds})} + \tilde{F}(M) = f_0^{(1)}(M)\,,$$

$$g^{(2)}_{0,qq}(M) = f_0^{(2)}(M)$$

$$
\begin{aligned}
\tilde{F}(M) = & \frac{1}{2}C_F S_1(M)^2 + \frac{C_F\left(2M^2-M-1\right)}{2M^2(M+1)^2} \\
& + \frac{3}{2}C_F\left(S_2(M) - \zeta(2)\right) - \frac{C_F S_1(M)}{2M(M+1)}\,.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{P}_{qq}^{(1)}(M) = C_F^2 \Bigg(& \frac{(2M+1)S_1(M)}{2M^2(M+1)^2} + \frac{\left(3M^2+3M+2\right)S_2(M)}{4M(M+1)} - \frac{\left(3M^2+3M+2\right)\zeta(2)}{4M(M+1)} \\
& - \frac{3M^3+8M^2+7M+3}{4M^3(M+1)^3} + \zeta(2)S_1(M) - S_1(M)S_2(M) + S_3(M) - \zeta(3)\Bigg) \\
& + C_F N_F\left(-\frac{11M^2+5M-3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6}\right) \\
& + C_F C_A\Bigg(-\frac{\left(11M^2+11M+3\right)\zeta(2)}{12M(M+1)} \\
& + \frac{151M^4+236M^3+88M^2+3M+18}{72M^3(M+1)^3} \\
& + \frac{1}{2}\zeta(2)S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_3(M)}{2} + \frac{\zeta(3)}{2}\Bigg)\,.
\end{aligned}
$$

$$g^{(1)}_{0,(\mathsf{ds},qq)} = C_F\left(\zeta(2) - 4\right)$$

$$D^{(2)}_{\mathsf{ds},qq} = \frac{1}{2}C_F\left[\left(-\frac{101}{27} + \frac{7}{2}\zeta(3)\right)C_A + \frac{14}{27}N_f\right],$$

$$
\begin{aligned}
g^{(2)}_{0,(\mathsf{ds},qq)} = & C_F^2\left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4}\zeta(3)\right] \\
& + C_F C_A\left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36}\zeta(3)\right] \\
& + C_F N_f\left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18}\right].
\end{aligned}
$$

- $\tilde{F}(M)$ process dependent function.
- $\tilde{P}^{(i)}_{qq}$s are the splitting functions without the constant terms.
- $\hat{z}-$single-soft limit same behavior aside for P_{qq} spacelike.
- Same structure of the results obtained for DY process. However in this case, in both single-soft limits the splitting functions are spacelike.

Checks

- $\hat{x}$ single-soft limit $\Rightarrow$ taking the large M limit $\Rightarrow$ I recovered the double-soft limit obtained by Abele, Florian, and Vogelsang 2021.
- Same for the $\hat{z}$ single-soft limit.

Checks

- $\hat{x}$ single-soft limit $\Rightarrow$ taking the large M limit $\Rightarrow$ I recovered the double-soft limit obtained by Abele, Florian, and Vogelsang 2021.
- Same for the $\hat{z}$ single-soft limit.
- $\hat{x}$ single-soft limit $\Rightarrow$ expand up to $\mathcal{O}(\frac{1}{M})$ $\Rightarrow$ I recovered the NLP corrections $\frac{\mathcal{L}^3}{M}$ predicted by Abele, Florian, and Vogelsang 2022.
- Same for the $\hat{z}$ single soft limit

Checks

- $\hat{x}$ single-soft limit $\Rightarrow$ taking the large M limit $\Rightarrow$ I recovered the double-soft limit obtained by Abele, Florian, and Vogelsang 2021.
- Same for the $\hat{z}$ single-soft limit.
- $\hat{x}$ single-soft limit $\Rightarrow$ expand up to $\mathcal{O}(\frac{1}{M})$ $\Rightarrow$ I recovered the NLP corrections $\frac{\mathcal{L}^3}{M}$ predicted by Abele, Florian, and Vogelsang 2022.
- Same for the $\hat{z}$ single soft limit
- **Final remark**: In agreement with my supervisor I am writing a paper on this new result ☺.

Thank you for your attention!

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