

Asymmetric soft resummation of Semi-Inclusive Deep Inelastic scattering

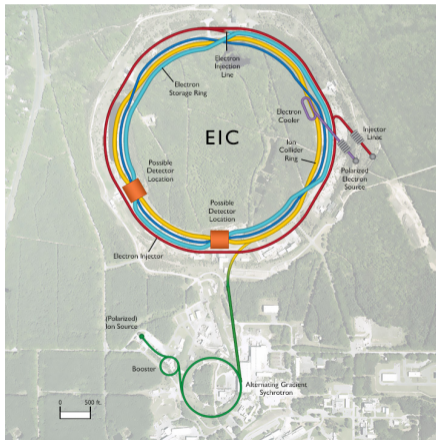
Francesco Ventola¹ Supervisor: Prof. Stefano Forte²

^{1,2}Università degli studi di Milano

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Why SIDIS precision calculations?

The new Electron-Ion Collider (EIC) at Brookhaven National Laboratory will be operational by 2032-2035.



SIDIS kinematic and large logs

SIDIS process at the hadronic level:

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$$q = k' - k \Rightarrow Q^2 \equiv -q^2 \text{ space-like}$$

$$\text{Scaling variables } x \equiv \frac{Q^2}{2P_1 \cdot q} \quad z \equiv \frac{P_1 \cdot P_2}{P_1 \cdot q}$$

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$$\text{Factorization } \frac{d\sigma}{dx dz dy} = \sum_{a,b} f_a^{h_1} \otimes \hat{\sigma}_{ab} \otimes D_b^{h_2}$$

$$qq \text{ channel} \Rightarrow \mathcal{O}(\alpha_s^k): \delta(1 - \hat{x}) \left(\frac{\ln^m(1 - \hat{x})}{1 - \hat{x}} \right)_+, \quad \delta(1 - \hat{x}) \left(\frac{\ln^m(1 - \hat{z})}{1 - \hat{z}} \right)_+, \quad \left(\frac{\ln^n(1 - \hat{x})}{1 - \hat{x}} \right)_+ \times \left(\frac{\ln^m(1 - \hat{z})}{1 - \hat{z}} \right)_+$$

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E.g. double-soft limit: $\lim_{N, M \rightarrow \infty} \hat{\sigma}(N, M, Q^2)$ only contains logs terms, constant terms and power suppressed terms \Rightarrow Large logarithms can be resummed to all orders.

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Process	Q^2	Momentum cons.	Frame	$\hat{\sigma}$ differential in	Radiation	Phase space
DY ¹	$q^2 > 0$ timelike	$p_1 + p_2 = q + k_1 + \dots + k_n$	C.M.	\hat{y}	Collinear	$d\phi_{n+1} \propto (k_{\max}^2)^{p(\epsilon)} d\Omega^{n-1}(\epsilon) dv dy$
SIDIS	$q^2 < 0$ spacelike	$p_1 + q = p_2 + k_1 + \dots + k_n$	Breit	\hat{z}	Collinear	$d\phi_{n+1} \propto (k_{\max}^2)^{p(\epsilon)} d\Omega^{n-1}(\epsilon) dv d\hat{z}$

with $d = 4 - 2\epsilon$ $k = k_1 + \dots + k_n$ $p(\epsilon) = (n - 1)(1 - \epsilon)$ $\bar{N} = Ne^\gamma$ and $\bar{M} = Me^\gamma$.

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Process	k_{\max}^2	Double-soft scale	single-soft scale
DY	$\frac{Q^2(1-x_1)(1-x_2)}{x_1 x_2}$	$Q^2(1-x_1)(1-x_2) \rightarrow \frac{Q^2}{\bar{N}\bar{M}}$	$Q^2(1-x_1) \vee Q^2(1-x_2) \rightarrow \frac{Q^2}{\bar{N}} \vee \frac{Q^2}{\bar{M}}$
SIDIS	$\frac{Q^2(1-\hat{x})(1-\hat{z})}{\hat{x}}$	$Q^2(1-\hat{x})(1-\hat{z}) \rightarrow \frac{Q^2}{\bar{N}\bar{M}}$	$Q^2(1-\hat{x}) \vee Q^2(1-\hat{z}) \rightarrow \frac{Q^2}{\bar{N}} \vee \frac{Q^2}{\bar{M}}$

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- E.g. \hat{x} single-soft case \Rightarrow it also resums all the M power suppressed terms (NLP corrections). The formula is

$$C_{qq}^{T,\text{res}}(N, M, \alpha_s(Q^2)) = \left(1 + g_0^{(1)}(M)\alpha_s + g_0^{(2)}(M)\alpha_s^2 + \dots\right) \exp \left\{ \int_{\frac{Q^2}{\bar{N}}}^{Q^2} \frac{dk^2}{k^2} \left[\left(\alpha_s(k^2)A_1 + \alpha_s^2(k^2)A_2 + \dots \right) \ln \frac{\bar{N}k^2}{Q^2} - \left(\alpha_s(k^2)D_1(M) + \alpha_s^2(k^2)D_2(M) + \dots \right) \right] \right\}$$

with $\bar{N} = Ne^\gamma$

$$\text{LL} \quad A_1 \Rightarrow \alpha_s \ln^2 N + \alpha_s^2 \ln^4 N + \dots$$

$$\text{NLL} \quad D_1, A_2 \Rightarrow \alpha_s \ln N + \alpha_s^2 \left(\ln^3 N + \alpha_s \ln^2 N \right) + \dots$$

$$\text{NNLL} \quad D_2, g_0^{(1)}, A_3 \Rightarrow \alpha_s^2 \ln N + \dots$$

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- dQCD: comparison with fixed order \Rightarrow resummation coefficients.

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- dQCD: comparison with fixed order \Rightarrow resummation coefficients.
- Single-soft limit for DY process up to NNLL obtained in dQCD by S. Forte, G. Ridolfi, L. De Ros, D. Tagliabue from a translation of a SCET result (Lustermans, Michel, and Tackmann 2019 and Mistlberger and Vita 2025).

Matching procedure

- NNLO result from Bonino, Gehrmann, and Stagnitto 2024.
- Mathematica software packages:
 - `MTMellin` Höschele et al. 2014: to compute Mellin transformation.
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- Fixed order coefficient function in \hat{x} single-soft limit:

$$C_{qq}(N, M) = 1 + \left(\frac{\alpha_s}{\pi}\right) \left(\ln^2 \bar{N} f_2^{(1)}(M) + \ln \bar{N} f_1^{(1)}(M) + f_0^{(1)}(M) \right) \\ + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\ln^4 \bar{N} f_4^{(2)}(M) + \ln^3 \bar{N} f_3^{(2)}(M) + \ln^2 \bar{N} f_2^{(2)}(M) + \ln \bar{N} f_1^{(2)}(M) + f_0^{(2)}(M) \right) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}(\alpha_s^3)$$

- Resummation formula expanded

$$\tilde{C}_{qq}^{ss,T}(N, M, \alpha_s(Q^2)) = 1 + \frac{\alpha_s}{\pi} \left(\frac{1}{2} A_1 \ln^2 \bar{N} - D_1 \ln \bar{N} + g_0^{(1)} \right) + \frac{\alpha_s^2}{\pi^2} \left[g_0^{(2)} + \ln \bar{N} \left(-D_1 g_0^{(1)} - D_2 \right) \right. \\ \left. + \ln^2 \bar{N} \left(\frac{A_1 g_0^{(1)}}{2} + \frac{A_2}{2} - \frac{b_0 \pi D_1}{2} + \frac{D_1^2}{2} \right) + \ln^3 \bar{N} \left(\frac{A_1 \pi b_0}{6} - \frac{A_1 D_1}{2} \right) + \frac{A_1^2 \ln^4 \bar{N}}{8} \right].$$

Matching procedure

$$f_2^{(1)}(M) = \frac{C_F}{2}$$

$$f_1^{(1)}(M) = C_F \left(S_1(M) - \frac{2}{4M^2 + 4M} \right)$$

$$f_0^{(1)}(M) = \frac{C_F(2M^2 - M - 1)}{2M^2(M+1)^2} + \frac{1}{2}C_F S_1(M)^2 - \frac{C_F S_1(M)}{2M(M+1)} + \frac{3}{2}C_F S_2(M) - \frac{C_F \zeta(2)}{2} - 4C_F$$

$$f_4^{(2)}(M) = \frac{1}{8}C_F^2,$$

$$f_3^{(2)}(M) = C_F^2 \left(\frac{S_1(M)}{2} - \frac{1}{4M(M+1)} \right) + \frac{11C_F C_A}{72} - \frac{C_F N_f}{36},$$

$$f_2^{(2)}(M) = C_F^2 \left(\frac{4M^2 - 2M - 1}{8M^2(M+1)^2} + \frac{3}{4}S_1(M)^2 - \frac{3S_1(M)}{4M(M+1)} + \frac{3S_2(M)}{4} - \frac{\zeta(2)}{4} - 2 \right) + C_F C_A \left(\frac{11S_1(M)}{24} - \frac{11}{48M(M+1)} - \frac{\zeta(2)}{4} + \frac{67}{72} \right) + C_F N_f \left(-\frac{1}{12}S_1(M) + \frac{1}{24M(M+1)} - \frac{5}{36} \right),$$

$$f_1^{(2)}(M) = C_F^2 \left(-\frac{(3M^2 + 3M + 5)S_2(M)}{4M(M+1)} + \frac{3(M^2 + M + 1)\zeta(2)}{4M(M+1)} - \frac{(16M^4 + 32M^3 + 12M^2 + 6M + 3)S_1(M)}{4M^2(M+1)^2} + \frac{8M^4 + 19M^3 + 14M^2 + 8M + 4}{4M^3(M+1)^3} - \frac{3}{2}\zeta(2)S_1(M) + \frac{1}{2}S_1(M)^3 - \frac{3S_1(M)^2}{4M(M+1)} + \frac{5}{2}S_2(M)S_1(M) - S_3(M) + \zeta(3) \right) + C_F N_f \left(-\frac{(10M^2 + 10M - 3)S_1(M)}{36M(M+1)} - \frac{1}{12}S_1(M)^2 - \frac{S_2(M)}{12} + \frac{5M + 8}{36M(M+1)^2} + \frac{\zeta(2)}{12} - \frac{7}{27} \right) + C_F C_A \left(\frac{(134M^2 + 134M - 33)S_1(M)}{72M(M+1)} - \frac{85M^4 + 203M^3 + 154M^2 + 36M + 18}{72M^3(M+1)^3} - \frac{1}{2}\zeta(2)S_1(M) + \frac{11}{24}S_1(M)^2 + \frac{11S_2(M)}{24} + \frac{S_3(M)}{2} + \frac{\zeta(2)}{4M(M+1)} - \frac{11\zeta(2)}{24} - \frac{9\zeta(3)}{4} + \frac{101}{54} \right)$$

$$S_{a_1, \dots, a_k}(N) = \sum_{N \geq i_1 \geq i_2 \geq \dots \geq i_k \geq 1} \frac{\text{sign}(a_1)^{i_1}}{i_1^{|a_1|}} \dots \frac{\text{sign}(a_k)^{i_k}}{i_k^{|a_k|}}$$

$$C_F = \frac{4}{3} \quad C_A = 3 \quad N_f = \text{active flavours}$$

Matching procedure

$$\begin{aligned}
 f^{(6)}(M) = & C_1^2 \left(\frac{1}{2} S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{1}{2} (2) S_1(M)^2 \right. \\
 & \left. (-M^2 + 13M^4 + 41M^3 + 15M^2 + 2) S_1(M) + \frac{S_1(2) S_1(M)}{8M^2(M+1)^2} - \frac{1}{2} (3) S_1(M) \right. \\
 & \left. \frac{S_1(M)^2 (7M^2(M+1)^2 S_1(M) - 2(43M^2 + 81M^2 + 35M^2 + 25M + 1))}{4M^2(M+1)^2} \right. \\
 & \left. + \frac{(-7M(M+1) S_1(M) S_1(M))}{4M^2(M+1)^2} + \frac{1}{2} (5) S_1(M) - 2S_{1,1}(M) S_1(M) - \frac{33(2)^2}{40} \right. \\
 & \left. + \frac{-33M^4 - 30M^3 + 203M^2 + 81M^2 + 65M^2 + 48M + 15}{8M^2(M+1)^2} \right. \\
 & \left. \frac{(4M^2 - 2M - 1)(2)}{8M^2(M+1)^2} + \frac{(10M^2 + 30M^2 + 12M^2 + M + 1)(2)}{4M^2(M+1)^2} \left(\frac{13(2)}{8} \right) \right. \\
 & \left. - \frac{(2)}{2M(M+1)} - \frac{15(2)}{4} \right. \\
 & \left. \frac{3(7(2)M^2 + 7(2)M + 2(2))}{8M(M+1)} - \frac{(27M^4 + 54M^3 + 18M^2 + 7M + 5) S_1(M)}{4M^2(M+1)^2} \right. \\
 & \left. \frac{5}{4} (2) S_1(M) - \frac{(9M^2 + 9M + 10) S_1(M)}{8M(M+1)} - \frac{33S_1(M)}{8} - \frac{(3M^2 + 3M - 2) S_{1,1}(M)}{4M(M+1)} \right. \\
 & \left. + \frac{15}{4} S_{1,1}(M) - 2S_{1,1}(M) + \frac{2}{5} S_{1,1,1}(M) + \frac{31}{64} \right) \\
 & + \frac{1}{360} (13C_A - 2N_f)(3)C_F \\
 & + N_f C_F \left(-\frac{1}{36} S_1(M)^3 - \frac{(10M^2 + 10M - 3) S_1(M)^2}{72M(M+1)} \right. \\
 & \left. \frac{(28M^2 + 56M^2 + 13M - 24) S_1(M)}{108M(M+1)^2} + \frac{1}{22} (2) S_1(M) - \frac{1}{12} S_1(M) S_1(M) \right. \\
 & \left. \frac{11M^4 - 74M^3 - 109M^2 - 63M + 9}{216M^2(M+1)^2} + \frac{(5M^2 + 5M - 3)(2)}{36M(M+1)} \right. \\
 & \left. + \frac{(2)}{24M(M+1)} + \frac{7(2)}{18} + \frac{(3)}{12} - \frac{230M^2 + 20M - 3) S_1(M)}{72M(M+1)} + \frac{S_1(M)}{36} + \frac{17}{90} \right) \\
 & + C_A C_F \left(\frac{1}{72} S_1(M)^3 + \frac{489M^3 + 1220M^2 + 1038M^2 - 703M^2 - 273M + 54) S_1(M)}{216M^2(M+1)^2} \right. \\
 & \left. \frac{11}{20} (2) S_1(M) - \frac{13}{8} (2) S_1(M) \right. \\
 & \left. \frac{S_1(M)^2 (-13M^2 + 36M + 1) S_1(M) S_1(M) - 134M + 30}{144M(M+1)} \right. \\
 & \left. - \frac{65(M)(11M^2 + 11M + 6) S_1(M)}{144M(M+1)} + \frac{1}{2} (5) S_1(M) - 2S_{1,1}(M) S_1(M) + \frac{21(2)^2}{20} \right. \\
 & \left. \frac{124M^6 - 1044M^5 - 2078M^4 - 2296M^3 - 1602M^2 - 703M - 270}{432M^2(M+1)^2} \right. \\
 & \left. \frac{14(2)}{48M(M+1)} - \frac{101(2)}{36} - \frac{29(2)M^2 + 56(2)M^2 + 17(2)M^2 - 38(2)M - 9(2)}{36M^2(M+1)^2} \right. \\
 & \left. \frac{(11M^2 + 11M + 39)(2)}{36M(M+1)} - \frac{43(2)}{18} - \frac{259M^2 + 259M^2 - 69M - 36) S_1(M)}{144M^2(M+1)} \right. \\
 & \left. + \frac{1}{2} (2) S_1(M) - \frac{(11M^2 + 11M - 16) S_1(M)}{72M(M+1)} + S_1(M) - \frac{S_1(M)}{2M(M+1)} \right. \\
 & \left. + S_{1,1}(M) + S_{1,1}(M) - \frac{3}{2} S_{1,1,1}(M) - \frac{1535}{192} \right)
 \end{aligned}$$

Matching procedure

$$\begin{aligned}
 f_0^{(1)}(M) &= C_F^2 \left(\frac{1}{2} S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{1}{2} \zeta(2) S_1(M)^2 \right. \\
 &\quad \left. - (M^2 + 13M^3 + 41M^4 + 15M^5 + 2) S_1(M) + \frac{5 \zeta(2) S_1(M)}{63M^2(M+1)^2} - \frac{1}{2} \zeta(3) S_1(M) \right. \\
 &\quad \left. + \frac{S_1(M)^2 (7M^2(M+1) + 7) S_2(M) - 2(43M^2 + 81M^3 + 35M^4 + 25M + 1)}{4M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(-7M(M+1) + 15) S_1(M) S_2(M)}{4M^2(M+1)^2} + \frac{1}{2} (5S_1(M) - 2S_{1,1}(M)) S_2(M) - \frac{3 \zeta(2)^2}{40} \right. \\
 &\quad \left. + \frac{-333M^4 - 303M^5 + 205M^6 + 81M^7 + 65M^8 + 48M + 15}{882M^2(M+1)^4} \right. \\
 &\quad \left. + \frac{(4M^2 - 2M - 1) \zeta(2)}{84M^2(M+1)^2} + \frac{(10M^2 + 36M^3 + 12M^4 + M + 3) \zeta(2)}{432M^2(M+1)^2} - \frac{13 \zeta(2)}{8} \right. \\
 &\quad \left. - \frac{\zeta(2)}{25M(M+1)} - \frac{15 \zeta(2)}{4} \right. \\
 &\quad \left. + \frac{3(7 \zeta(2) M^2 + 7 \zeta(2) M + 2) \zeta(3)}{84M(M+1)} - \frac{(27M^4 + 54M^5 + 183M^6 + 7M + 5) S_1(M)}{432M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{5 \zeta(2) S_1(M)}{4} - \frac{(9M^2 + 9M + 10) S_1(M)}{84M(M+1)} - \frac{33S_1(M)}{8} - \frac{(3M^2 + 3M - 2) S_{1,1}(M)}{40M(M+1)} \right. \\
 &\quad \left. + \frac{15}{4} S_{1,1}(M) - 2S_{1,1}(M) + \frac{3}{2} S_{1,1,1}(M) + \frac{311}{64} \right) \\
 &\quad + \frac{1}{36} (13C_A - 2N_f) \zeta(3) C_F \\
 &\quad + N_f C_F \left(-\frac{1}{36} S_1(M)^3 - \frac{(10M^2 + 10M - 3) S_1(M)^2}{72M(M+1)} \right. \\
 &\quad \left. + \frac{(28M^4 + 56M^5 + 13M - 24) S_1(M)}{108M(M+1)^2} + \frac{1}{12} \zeta(2) S_1(M) - \frac{1}{12} S_1(M) S_2(M) \right. \\
 &\quad \left. + \frac{11M^2 - 74M^3 - 109M^4 - 63M + 9}{216M^2(M+1)^2} + \frac{(5M^2 + 5M - 3) \zeta(2)}{36M(M+1)} \right. \\
 &\quad \left. + \frac{\zeta(2)}{24M(M+1)} + \frac{7 \zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{20M^2 + 20M - 3}{72M^2(M+1)} + \frac{S_2(M)}{36} + \frac{177}{96} \right) \\
 &\quad + C_F C_A \left(\frac{1}{12} S_1(M)^3 + \frac{499M^2 + 1209M^4 + 693M^5 - 703M^6 - 273M + 54) S_1(M)}{216M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{11}{20} \zeta(2) S_1(M) - \frac{13}{4} \zeta(2) S_1(M) \right. \\
 &\quad \left. + \frac{S_1(M)^2 (-13M^2 + 36M + 1) S_2(M) M - 134M + 30}{144M(M+1)} \right. \\
 &\quad \left. - \frac{65(M)(11M^2 + 11M + 6) S_2(M)}{144M(M+1)} + \frac{1}{2} (5S_1(M) - 2S_{1,1}(M)) S_2(M) + \frac{21 \zeta(2)^2}{20} \right. \\
 &\quad \left. + \frac{124M^6 - 1044M^7 - 2076M^8 - 2296M^9 - 1662M^{10} - 705M - 270}{432M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{11 \zeta(2)}{48M(M+1)} - \frac{101 \zeta(2)}{36} - \frac{29 \zeta(2) M^2 + 58 \zeta(2) M^3 + 17 \zeta(2) M^4 - 38 \zeta(2) M - 9 \zeta(2)}{36M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(11M^2 + 11M + 39) \zeta(2)}{36M(M+1)} - \frac{43 \zeta(3)}{72} + \frac{259M^2 + 260M^3 - 69M - 36) S_1(M)}{144M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{(11M^2 + 11M + 18) S_1(M)}{72M(M+1)} + S_2(M) - \frac{S_2(M)}{20M(M+1)} \right. \\
 &\quad \left. + S_{1,1}(M) + S_{1,1}(M) - \frac{3}{2} S_{1,1,1}(M) - \frac{1535}{192} \right)
 \end{aligned}$$

$$\begin{aligned}
 A^{(1)} &= C_F, \\
 A^{(2)} &= \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right], \\
 D^{(1)}(M) &= \tilde{F}_{qq}^{(0),T}(M) = \frac{C_F}{2} \left[\frac{1}{M(M+1)} - 2S_1(M) \right], \\
 D^{(2)}(M) &= D_{\text{ds},qq}^{(2)} - \pi \beta_0 \tilde{F}(M) + \tilde{F}_{qq,NS}^{(1),T}(M), \\
 9_0^{(1)}(M) &= 9_0^{(1),\text{ds}} + \tilde{F}(M) = f_0^{(1)}(M), \\
 9_0^{(2)}(M) &= f_0^{(2)}(M)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{F}(M) &= \frac{1}{2} C_F S_1(M)^2 + \frac{C_F (2M^2 - M - 1)}{2M^2(M+1)^2} \\
 &\quad + \frac{3}{2} C_F (S_2(M) - \zeta(2)) - \frac{C_F S_1(M)}{2M(M+1)}, \\
 \tilde{F}_{99}^{(1)}(M) &= C_F^2 \left(\frac{(2M+1) S_1(M)}{2M^2(M+1)^2} + \frac{(3M^2 + 3M + 2) S_2(M)}{4M(M+1)} - \frac{(3M^2 + 3M + 2) \zeta(2)}{4M(M+1)} \right. \\
 &\quad \left. - \frac{3M^2 + 8M^2 + 7M + 3}{4M^2(M+1)^2} + \zeta(2) S_1(M) - S_1(M) S_2(M) + S_2(M) - \zeta(3) \right) \\
 &\quad + C_F N_f \left(-\frac{11M^2 + 5M - 3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6} \right) \\
 &\quad + C_F C_A \left(-\frac{(11M^2 + 11M + 3) \zeta(2)}{12M(M+1)} \right. \\
 &\quad \left. + \frac{151M^4 + 236M^5 + 88M^6 + 3M + 18}{72M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_2(M)}{2} + \frac{\zeta(3)}{2} \right).
 \end{aligned}$$

$$\begin{aligned}
 9_{0,\text{ds},qq}^{(1)} &= C_F (\zeta(2) - 4) \\
 D_{\text{ds},qq}^{(2)} &= \frac{1}{2} C_F \left[\left(-\frac{101}{27} + \frac{7}{2} \zeta(3) \right) C_A + \frac{14}{27} N_f \right], \\
 9_{0,\text{ds},qq}^{(2)} &= C_F^2 \left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4} \zeta(3) \right] \\
 &\quad + C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36} \zeta(3) \right] \\
 &\quad + C_F N_f \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18} \right].
 \end{aligned}$$

Matching procedure

$$\begin{aligned}
 f_0^{(1)}(M) &= C_F^2 \left(\frac{1}{2} S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{1}{2} \zeta(2) S_1(M)^2 \right. \\
 &\quad \left. - (-M^2 + 13M^3 + 4M^4 + 15M^2 + 2) S_1(M) + \frac{S_1(2)S_1(M)}{63M^2(M+1)^2} - \frac{1}{12M^2 + 4M^2} - \frac{1}{2} \zeta(3) S_1(M) \right. \\
 &\quad \left. + \frac{S_1(M)^2 (7M^2 M^2 + 1) S_1(M) - 2(43M^2 + 81M^2 + 35M^2 + 25M + 1)}{48M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(-7M(M+1)S_1(M)S_1(M))}{48M^2(M+1)^2} + \frac{1}{2} [S_1(M) - 2S_{1,1}(M)] S_1(M) - \frac{3\zeta(2)^2}{40} \right. \\
 &\quad \left. + \frac{-333M^4 - 303M^2 + 203M^2 + 81M^2 + 65M^2 + 48M + 15}{882M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(4M^2 - 2M - 1)\zeta(2)}{882M^2(M+1)^2} + \frac{(103M^2 + 36M^2 + 12M^2 + M + 3)\zeta(2)}{432M^2(M+1)^2} - \frac{13\zeta(2)}{8} \right. \\
 &\quad \left. - \frac{\zeta(3)}{252(M+1)} - \frac{15\zeta(3)}{4} \right. \\
 &\quad \left. + \frac{3(7\zeta(3)M^2 + 7\zeta(3)M + 2\zeta(3))}{48M(M+1)} - \frac{(27M^4 + 54M^2 + 183M^2 + 7M + 5)S_1(M)}{432M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{5}{4} \zeta(2) S_1(M) - \frac{(9M^2 + 9M + 10)S_1(M)}{432M(M+1)} - \frac{33S_1(M)}{8} - \frac{(3M^2 + 3M - 2)S_{1,1}(M)}{48(M+1)} \right. \\
 &\quad \left. + \frac{15}{4} S_{1,1}(M) - 2S_{1,1}(M) + \frac{2}{3} S_{1,1}(M) + \frac{311}{64} \right) \\
 &\quad + \frac{1}{36} (13C_A - 2N_f) \zeta(3) C_F \\
 &\quad + N_f C_F \left(-\frac{1}{36} S_1(M)^2 - \frac{(10M^2 + 10M - 3)S_1(M)^2}{72M(M+1)} \right. \\
 &\quad \left. + \frac{(28M^2 + 56M^2 + 13M - 24)S_1(M)}{108M(M+1)^2} + \frac{1}{12} \zeta(2) S_1(M) - \frac{1}{12} S_1(M) S_1(M) \right. \\
 &\quad \left. + \frac{11M^2 - 74M^2 - 109M^2 - 63M + 9}{216M^2(M+1)^2} + \frac{(53M^2 + 5M - 3)\zeta(2)}{36M(M+1)} \right. \\
 &\quad \left. + \frac{\zeta(2)}{243M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{203M^2 + 20M - 3}{72M(M+1)} - \frac{S_1(M)}{36} + \frac{127}{96} \right) \\
 &\quad + C_F C_A \left(\frac{1}{12} S_1(M)^3 + \frac{489M^2 + 1209M^2 + 639M^2 - 703M^2 - 273M + 54}{216M^2(M+1)^2} S_1(M) \right. \\
 &\quad \left. + \frac{11}{20} \zeta(2) S_1(M) - \frac{13}{4} \zeta(3) S_1(M) \right. \\
 &\quad \left. + \frac{S_1(M)^2 (-13M^2 + 36M + 1) S_1(M) M - 134M + 30}{144M(M+1)} \right. \\
 &\quad \left. - \frac{65(M)(11M^2 + 11M + 6)S_1(M)}{144M(M+1)} + \frac{1}{2} [S_1(M) - 2S_{1,1}(M)] S_1(M) + \frac{2\zeta(2)^2}{20} \right. \\
 &\quad \left. + \frac{124M^6 - 1044M^2 - 2076M^2 - 2296M^2 - 1662M^2 - 705M - 270}{432M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^2 + 56\zeta(2)M^2 + 17\zeta(2)M^2 - 38\zeta(2)M - 9\zeta(2)}{36M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(11M^2 + 11M - 39)\zeta(3)}{36M(M+1)} - \frac{43\zeta(3)}{72} + \frac{2594M^2 + 2604M^2 - 69M - 365}{144M(M+1)} \right. \\
 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{(11M^2 + 11M - 16)S_1(M)}{72M(M+1)} + S_1(M) - \frac{S_1(M)}{20M(M+1)} \right. \\
 &\quad \left. + S_{1,1}(M) + S_{1,1}(M) - \frac{2}{3} S_{1,1}(M) - \frac{1535}{108} \right)
 \end{aligned}$$

$$\begin{aligned}
 A^{(1)} &= C_F, \\
 A^{(2)} &= \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right], \\
 D^{(1)}(M) &= \tilde{F}_{qq}^{(0),T}(M) = \frac{C_F}{2} \left[\frac{1}{M(M+1)} - 2S_1(M) \right], \\
 D^{(2)}(M) &= D_{\text{ds},qq}^{(2)} - \pi\beta_0 \tilde{F}(M) + \tilde{F}_{qq,NS}^{(1),T}(M), \\
 9_0^{(1)}(M) &= 9_0^{(1),\text{ds}} + \tilde{F}(M) = f_0^{(1)}(M), \\
 9_0^{(2)}(M) &= f_0^{(2)}(M)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{F}(M) &= \frac{1}{2} C_F S_1(M)^2 + \frac{C_F (2M^2 - M - 1)}{2M^2(M+1)^2} \\
 &\quad + \frac{3}{2} C_F (S_2(M) - \zeta(2)) - \frac{C_F S_1(M)}{2M(M+1)}, \\
 \tilde{F}_{99}^{(1)}(M) &= C_F^2 \left(\frac{(2M+1)S_1(M)}{2M^2(M+1)^2} + \frac{(3M^2 + 3M + 2)S_2(M)}{4M(M+1)} - \frac{(3M^2 + 3M + 2)\zeta(2)}{4M(M+1)} \right. \\
 &\quad \left. - \frac{3M^2 + 8M^2 + 7M + 3}{4M^2(M+1)^2} + \zeta(2) S_1(M) - S_1(M) S_2(M) + S_3(M) - \zeta(3) \right) \\
 &\quad + C_F N_F \left(-\frac{11M^2 + 5M - 3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6} \right) \\
 &\quad + C_F C_A \left(-\frac{(11M^2 + 11M + 3)\zeta(2)}{12M(M+1)} \right. \\
 &\quad \left. + \frac{151M^4 + 236M^2 + 88M^2 + 3M + 18}{72M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_3(M)}{2} + \frac{\zeta(3)}{2} \right).
 \end{aligned}$$

$$\begin{aligned}
 9_{0,\text{ds},qq}^{(1)} &= C_F (\zeta(2) - 4) \\
 D_{\text{ds},qq}^{(2)} &= \frac{1}{2} C_F \left[\left(-\frac{101}{27} + \frac{7}{2} \zeta(3) \right) C_A + \frac{14}{27} N_f \right], \\
 9_{0,\text{ds},qq}^{(2)} &= C_F^2 \left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4} \zeta(3) \right] \\
 &\quad + C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36} \zeta(3) \right] \\
 &\quad + C_F N_F \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18} \right].
 \end{aligned}$$

- $\tilde{F}(M)$ process dependent function.

Matching procedure

$$\begin{aligned}
 f_0^{(1)}(M) &= C_F^2 \left(\frac{1}{2} S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{1}{2} \zeta(2) S_1(M)^2 \right. \\
 &\quad \left. - (-M^2 + 13M^3 + 4M^4 + 15M^2 + 2) S_1(M) + \frac{S_1(M) S_2(M)}{63M^2(M+1)^2} - \frac{1}{12M^2 + 4M} - \frac{1}{2} \zeta(3) S_1(M) \right. \\
 &\quad \left. + \frac{S_1(M)^2 (7M^2 M + 1) S_2(M) - 2(43M^2 + 81M^2 + 35M^2 + 2M + 1)}{4M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(-7M(M+1) S_2(M) S_3(M))}{4M^2(M+1)^2} + \frac{1}{2} (5S_1(M) - 2S_{1,1}(M)) S_3(M) - \frac{3\zeta(2)^2}{40} \right. \\
 &\quad \left. + \frac{-333M^4 - 303M^2 + 203M^2 + 81M^2 + 65M^2 + 48M + 15}{882M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(4M^2 - 2M - 1) \zeta(2)}{882M^2(M+1)^2} + \frac{(10M^2 + 36M^2 + 12M^2 + M + 3) \zeta(2)}{432M^2(M+1)^2} - \frac{13\zeta(2)}{8} \right. \\
 &\quad \left. - \frac{\zeta(3)}{252(M+1)} - \frac{15\zeta(3)}{4} \right. \\
 &\quad \left. + \frac{3(7\zeta(3)M^2 + 7\zeta(3)M + 2\zeta(3))}{43M^2(M+1)} - \frac{(27M^4 + 54M^2 + 18M^2 + 7M + 5) S_1(M)}{43M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{5}{4} \zeta(2) S_1(M) - \frac{(9M^2 + 9M + 10) S_1(M)}{43M^2(M+1)} - \frac{33S_1(M)}{43M^2(M+1)} - \frac{(3M^2 + 3M - 2) S_{1,1}(M)}{43M^2(M+1)} \right. \\
 &\quad \left. + \frac{15}{4} S_{1,1}(M) - 2S_{1,1}(M) + \frac{2}{5} S_{1,1}(M) + \frac{211}{64} \right) \\
 &\quad + \frac{1}{36} (13C_A - 2N_f) \zeta(3) C_F \\
 &\quad + N_f C_F \left(-\frac{1}{36} S_1(M)^2 - \frac{(10M^2 + 10M - 3) S_1(M)^2}{72M^2(M+1)} \right. \\
 &\quad \left. + \frac{(28M^2 + 56M^2 + 13M - 24) S_1(M)}{108M^2(M+1)^2} + \frac{1}{12} \zeta(2) S_1(M) - \frac{1}{12} S_1(M) S_2(M) \right. \\
 &\quad \left. + \frac{11M^2 - 74M^2 - 109M^2 - 63M + 9}{2163M^2(M+1)^2} + \frac{(5M^2 + 5M - 3) \zeta(2)}{363M(M+1)} \right. \\
 &\quad \left. + \frac{\zeta(2)}{243M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{20M^2 + 20M - 3) S_1(M)}{72M^2(M+1)} - \frac{S_1(M)}{36} + \frac{177}{96} \right) \\
 &\quad + C_s C_F \left(\frac{1}{12} S_1(M)^2 + \frac{489M^2 + 1209M^2 + 639M^2 - 703M^2 - 273M + 54) S_1(M)}{216M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{11}{20} \zeta(2) S_1(M) - \frac{13}{4} \zeta(3) S_1(M) \right. \\
 &\quad \left. + \frac{S_1(M)^2 (-13M^2 + 36M + 1) S_2(M) M - 134M + 30}{144M^2(M+1)} \right. \\
 &\quad \left. - \frac{65(M) (11M^2 + 11M + 6) S_1(M)}{144M^2(M+1)} + \frac{1}{2} (5S_1(M) - 2S_{1,1}(M)) S_3(M) + \frac{21\zeta(2)^2}{20} \right. \\
 &\quad \left. + \frac{124M^6 - 1044M^2 - 2079M^2 - 2296M^2 - 1680M^2 - 703M - 270}{432M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^2 + 58\zeta(2)M^2 + 17\zeta(2)M^2 - 38\zeta(2)M - 9\zeta(2)}{36M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(11M^2 + 11M - 39) \zeta(2)}{363M(M+1)} - \frac{43\zeta(3)}{72} + \frac{2594M^2 + 2604M^2 - 60M - 36) S_1(M)}{1443M^2(M+1)} \right. \\
 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{(11M^2 + 11M - 16) S_1(M)}{216M(M+1)} + S_1(M) - \frac{S_1(M)}{216M(M+1)} \right. \\
 &\quad \left. + S_{1,1}(M) + S_{1,1}(M) - \frac{3}{2} S_{1,1}(M) - \frac{1535}{192} \right)
 \end{aligned}$$

$$\begin{aligned}
 A^{(1)} &= C_F, \\
 A^{(2)} &= \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right], \\
 D^{(1)}(M) &= \tilde{F}_{qq}^{(0),T}(M) = \frac{C_F}{2} \left[\frac{1}{M(M+1)} - 2S_1(M) \right], \\
 D^{(2)}(M) &= D_{ds,qq}^{(2)} - \pi \beta_0 \tilde{F}(M) + \tilde{F}_{qq,NS}^{(1),T}(M), \\
 9_0^{(1)}(M) &= 9_{0,(ds)}^{(1)} + \tilde{F}(M) = f_0^{(1)}(M), \\
 9_{0,qq}^{(2)}(M) &= f_0^{(2)}(M)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{F}(M) &= \frac{1}{2} C_F S_1(M)^2 + \frac{C_F (2M^2 - M - 1)}{2M^2(M+1)^2} \\
 &\quad + \frac{3}{2} C_F (S_2(M) - \zeta(2)) - \frac{C_F S_1(M)}{2M(M+1)}, \\
 \tilde{F}_{99}^{(1)}(M) &= C_F^2 \left(\frac{(2M+1) S_1(M)}{2M^2(M+1)^2} + \frac{(3M^2 + 3M + 2) S_2(M)}{4M(M+1)} - \frac{(3M^2 + 3M + 2) \zeta(2)}{4M(M+1)} \right. \\
 &\quad \left. - \frac{3M^2 + 8M^2 + 7M + 3}{4M^2(M+1)^2} + \zeta(2) S_1(M) - S_1(M) S_2(M) + S_3(M) - \zeta(3) \right) \\
 &\quad + C_F N_F \left(-\frac{11M^2 + 5M - 3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6} \right) \\
 &\quad + C_F C_A \left(-\frac{(11M^2 + 11M + 3) \zeta(2)}{12M(M+1)} \right. \\
 &\quad \left. + \frac{151M^4 + 236M^2 + 88M^2 + 3M + 18}{72M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_3(M)}{2} + \frac{\zeta(3)}{2} \right).
 \end{aligned}$$

$$\begin{aligned}
 9_{0,(ds,qq)}^{(1)} &= C_F (\zeta(2) - 4) \\
 D_{ds,qq}^{(2)} &= \frac{1}{2} C_F \left[\left(-\frac{101}{27} + \frac{7}{2} \zeta(3) \right) C_A + \frac{14}{27} N_f \right], \\
 9_{0,(ds,qq)}^{(2)} &= C_F^2 \left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4} \zeta(3) \right] \\
 &\quad + C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36} \zeta(3) \right] \\
 &\quad + C_F N_F \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18} \right].
 \end{aligned}$$

- $\tilde{F}(M)$ process dependent function.
- $\tilde{P}_{qq}^{(i)}$ s are the splitting functions without the constant terms.

Matching procedure

$$\begin{aligned}
 f_0^{(1)}(M) &= C_F^2 \left(\frac{1}{2} S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{2} \zeta(2) S_1(M)^2 \right. \\
 &\quad \left. - (-M^2 + 13M^2 + 4M^2 + 15M^2 + 2) S_1(M) + \frac{5}{2} \zeta(2) S_1(M) \right. \\
 &\quad \left. - \frac{63M^2(M+1)^2}{4M^2(M+1)^2} - \frac{1}{2M^2} + 2M^2 + 1 \right) \\
 &\quad + \frac{S_1(M)^2 (7M^2 M + 1) S_1(M) - 2(43M^2 + 81M^2 + 35M^2 + 2M + 1)}{4M^2(M+1)^2} \\
 &\quad + \frac{(-7M(M+1) S_1(M) S_1(M))}{4M^2(M+1)^2} + \frac{1}{2} (5S_1(M) - 2S_{1,1}(M)) S_1(M) - \frac{33(2)^2}{40} \\
 &\quad + \frac{-33M^2 - 30M^2 + 20M^2 + 81M^2 + 65M^2 + 48M + 15}{88M^2(M+1)^2} \\
 &\quad + \frac{(4M^2 - 2M - 1) \zeta(2)}{88M^2(M+1)^2} + \frac{(10M^2 + 36M^2 + 12M^2 + M + 3) \zeta(2)}{43M^2(M+1)^2} - \frac{13 \zeta(2)}{8} \\
 &\quad - \frac{\zeta(3)}{2M(M+1)} - \frac{15 \zeta(3)}{4} \\
 &\quad + \frac{3(7 \zeta(3) M^2 + 7 \zeta(3) M + 2 \zeta(3))}{43M^2(M+1)} - \frac{(27M^2 + 54M^2 + 18M^2 + 7M + 5) S_1(M)}{43M^2(M+1)^2} \\
 &\quad + \frac{5}{4} \zeta(2) S_1(M) - \frac{(9M^2 + 9M + 10) S_1(M)}{43M^2(M+1)} - \frac{33S_1(M)}{43M^2(M+1)} - \frac{(3M^2 + 3M - 2) S_{1,1}(M)}{43M^2(M+1)} \\
 &\quad + \frac{15}{4} S_{1,1}(M) - 2S_{1,1}(M) + \frac{5}{2} S_{1,1}(M) + \frac{311}{64} \\
 &\quad + \frac{1}{36} (13C_A - 2N_f) \zeta(3) C_F \\
 &\quad + N_f C_F \left(-\frac{1}{36} S_1(M)^3 - \frac{(10M^2 + 10M - 3) S_1(M)^2}{72M(M+1)} \right. \\
 &\quad \left. + \frac{(28M^2 + 56M^2 + 13M - 24) S_1(M)}{108M(M+1)^2} + \frac{1}{12} \zeta(2) S_1(M) - \frac{1}{12} S_1(M) S_1(M) \right. \\
 &\quad \left. + \frac{11M^2 - 74M^2 - 109M^2 - 63M + 9}{2163M^2(M+1)^2} + \frac{(5M^2 + 5M - 3) \zeta(2)}{363M(M+1)} \right. \\
 &\quad \left. + \frac{\zeta(2)}{243M(M+1)} + \frac{7 \zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{20M^2 + 20M - 3) S_1(M)}{72M(M+1)} - \frac{S_1(M)}{36} + \frac{127}{96} \right) \\
 &\quad + C_A C_F \left(\frac{1}{12} S_1(M)^3 + \frac{49M^2 + 120M^2 + 103M^2 - 70M^2 - 273M + 54) S_1(M)}{216M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{11}{20} \zeta(2) S_1(M) - \frac{13}{4} \zeta(2) S_1(M) \right. \\
 &\quad \left. + \frac{S_1(M)^2 (-13M^2 + 36M + 1) S_1(M) M - 134M + 30}{144M(M+1)} \right. \\
 &\quad \left. - \frac{65(M)(11M^2 + 11M + 6) S_1(M)}{144M(M+1)} + \frac{1}{2} (5S_1(M) - 2S_{1,1}(M)) S_1(M) + \frac{21 \zeta(2)^2}{20} \right. \\
 &\quad \left. + \frac{124M^2 - 1044M^2 - 2079M^2 - 2296M^2 - 1680M^2 - 705M - 270}{432M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{11 \zeta(2)}{48M(M+1)} - \frac{101 \zeta(2)}{36} - \frac{29 \zeta(2) M^2 + 56 \zeta(2) M + 17 \zeta(2) M^2 - 38 \zeta(2) M - 9 \zeta(2)}{36M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(11M^2 + 11M + 39) \zeta(3)}{363M(M+1)} - \frac{43 \zeta(3)}{72} + \frac{2594M^2 + 2604M^2 - 60M - 36) S_1(M)}{1443M^2(M+1)} \right. \\
 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{(11M^2 + 11M - 18) S_1(M)}{72M(M+1)} + S_1(M) - \frac{S_1(M)}{2M(M+1)} \right. \\
 &\quad \left. + S_{1,1}(M) + S_{1,1}(M) - \frac{3}{2} S_{1,1}(M) - \frac{1535}{192} \right)
 \end{aligned}$$

$$\begin{aligned}
 A^{(1)} &= C_F, \\
 A^{(2)} &= \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right], \\
 D^{(1)}(M) &= \tilde{F}_{qq}^{(0),T}(M) = \frac{C_F}{2} \left[\frac{1}{M(M+1)} - 2S_1(M) \right], \\
 D^{(2)}(M) &= D_{ds,qq}^{(2)} - \pi \beta_0 \tilde{F}(M) + \tilde{F}_{qq,NS}^{(1),T}(M), \\
 9_0^{(1)}(M) &= 9_0^{(1),ds} + \tilde{F}(M) = f_0^{(1)}(M), \\
 9_0^{(2)}(M) &= f_0^{(2)}(M)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{F}(M) &= \frac{1}{2} C_F S_1(M)^2 + \frac{C_F (2M^2 - M - 1)}{2M^2(M+1)^2} \\
 &\quad + \frac{3}{2} C_F (S_2(M) - \zeta(2)) - \frac{C_F S_1(M)}{2M(M+1)}, \\
 \tilde{F}_{qq}^{(1)}(M) &= C_F^2 \left(\frac{(2M+1) S_1(M)}{2M^2(M+1)^2} + \frac{(3M^2 + 3M + 2) S_2(M)}{4M(M+1)} - \frac{(3M^2 + 3M + 2) \zeta(2)}{4M(M+1)} \right. \\
 &\quad \left. - \frac{3M^2 + 8M^2 + 7M + 3}{4M^2(M+1)^2} + \zeta(2) S_1(M) - S_1(M) S_2(M) + S_3(M) - \zeta(3) \right) \\
 &\quad + C_F N_f \left(-\frac{11M^2 + 5M - 3}{36M^2(M+1)^2} + \frac{5S_1(M)}{18} - \frac{S_2(M)}{6} + \frac{\zeta(2)}{6} \right) \\
 &\quad + C_F C_A \left(-\frac{(11M^2 + 11M + 3) \zeta(2)}{12M(M+1)} \right. \\
 &\quad \left. + \frac{151M^4 + 236M^2 + 88M^2 + 3M + 18}{72M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{67S_1(M)}{36} + \frac{11S_2(M)}{12} - \frac{S_3(M)}{2} + \frac{\zeta(3)}{2} \right).
 \end{aligned}$$

$$\begin{aligned}
 9_0^{(1)}(ds,qq) &= C_F (\zeta(2) - 4) \\
 D_{ds,qq}^{(2)} &= \frac{1}{2} C_F \left[\left(-\frac{101}{27} + \frac{7}{2} \zeta(3) \right) C_A + \frac{14}{27} N_f \right], \\
 9_0^{(2)}(ds,qq) &= C_F^2 \left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4} \zeta(3) \right] \\
 &\quad + C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36} \zeta(3) \right] \\
 &\quad + C_F N_f \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18} \right].
 \end{aligned}$$

- $\tilde{F}(M)$ process dependent function.
- $\tilde{P}_{qq}^{(i)}$ s are the splitting functions without the constant terms.
- \hat{z} —single-soft limit same behavior aside for P_{qq} spacelike.

Matching procedure

$$\begin{aligned}
 f_0^{(1)}(M) &= C_F^2 \left(\frac{1}{2} S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{2} \zeta(2) S_1(M)^2 \right. \\
 &\quad \left. - (M^2 + 13M^2 + 41M^2 + 15M^2 + 2) S_1(M) + \frac{S_1(M) S_2(M)}{63M^2(M+1)^2} - \frac{1}{12M^2} + 2\zeta(3) S_1(M) \right. \\
 &\quad \left. - \frac{S_1(M)^2 (7M^2 M + 1) S_2(M) - 2(43M^2 + 81M^2 + 35M^2 + 42M^2 + 1)}{4M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(-7M(M+1) S_1(M) S_2(M))}{4M^2(M+1)^2} + \frac{1}{2} (5S_1(M) - 2S_1(M) S_2(M)) S_3(M) - \frac{33(2)^2}{40} \right. \\
 &\quad \left. + \frac{-333M^4 - 303M^2 + 203M^2 + 81M^2 + 65M^2 + 48M + 15}{882M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{(4M^2 - 2M - 1) \zeta(2)}{84M^2(M+1)^2} + \frac{(10M^2 + 36M^2 + 12M^2 + M + 3) \zeta(2)}{432M^2(M+1)^2} - \frac{13 \zeta(2)}{8} \right. \\
 &\quad \left. - \frac{\zeta(3)}{252(M+1)} - \frac{15 \zeta(3)}{4} \right. \\
 &\quad \left. + \frac{3(7 \zeta(3) M^2 + 7 \zeta(3) M + 2 \zeta(3))}{43M(M+1)} - \frac{(27M^4 + 54M^2 + 183M^2 + 7M + 5) S_1(M)}{43M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{5}{4} \zeta(2) S_1(M) - \frac{(9M^2 + 9M + 10) S_1(M)}{84M(M+1)} - \frac{33S_1(M)}{43M(M+1)} - \frac{(3M^2 + 3M - 2) S_1(M)}{40(M+1)} \right. \\
 &\quad \left. + \frac{15}{4} S_2(M) - 2S_1(M) + \frac{5}{2} S_{1,1}(M) + \frac{311}{64} \right) \\
 &\quad + \frac{1}{36} (13C_A - 2N_f) \zeta(3) C_F \\
 &\quad + N_f C_F \left(-\frac{1}{36} S_1(M)^2 - \frac{(10M^2 + 10M - 3) S_1(M)^2}{72M(M+1)} \right. \\
 &\quad \left. + \frac{(28M^2 + 56M^2 + 13M - 24) S_1(M)}{108M(M+1)^2} + \frac{1}{12} \zeta(2) S_1(M) - \frac{1}{12} S_1(M) S_2(M) \right. \\
 &\quad \left. + \frac{11M^2 - 74M^2 - 109M^2 - 63M + 9}{2163M^2(M+1)^2} + \frac{(5M^2 + 5M - 3) \zeta(2)}{363M(M+1)} \right. \\
 &\quad \left. + \frac{\zeta(2)}{243M(M+1)} + \frac{7 \zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{20M^2 + 20M - 3) S_1(M)}{72M^2(M+1)} - \frac{S_1(M)}{36} + \frac{127}{96} \right) \\
 &\quad + C_s C_A \left(\frac{1}{12} S_1(M)^2 + \frac{498M^2 + 120M^2 + 1038M^2 - 704M^2 - 273M - 273M + 54) S_1(M)}{216M^2(M+1)^2} \right. \\
 &\quad \left. + \frac{11}{20} \zeta(2) S_1(M) - \frac{11}{4} \zeta(3) S_1(M) \right. \\
 &\quad \left. + \frac{S_1(M)^2 (-134M^2 + 36M + 1) S_2(M) M - 134M + 30}{144M(M+1)} \right. \\
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 &\quad \left. + \frac{1}{2} \zeta(2) S_1(M) - \frac{(11M^2 + 11M - 16) S_1(M)}{72M(M+1)} + S_1(M) - \frac{S_1(M)}{20M(M+1)} \right. \\
 &\quad \left. + S_{1,1}(M) + S_{1,1}(M) - \frac{3}{2} S_{1,1}(M) - \frac{1535}{108} \right)
 \end{aligned}$$

$$\begin{aligned}
 A^{(1)} &= C_F, \\
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 D^{(1)}(M) &= \tilde{F}_{qq}^{(0)} T(M) = \frac{C_F}{2} \left[\frac{1}{M(M+1)} - 2S_1(M) \right], \\
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 \tilde{F}_{qq}^{(1)}(M) &= C_F^2 \left(\frac{(2M+1) S_1(M)}{2M^2(M+1)^2} + \frac{(3M^2 + 3M + 2) S_2(M)}{4M(M+1)} - \frac{(3M^2 + 3M + 2) \zeta(2)}{4M(M+1)} \right. \\
 &\quad \left. - \frac{3M^2 + 8M^2 + 7M + 3}{4M^2(M+1)^2} + \zeta(2) S_1(M) - S_1(M) S_2(M) + S_3(M) - \zeta(3) \right) \\
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 &\quad + C_F C_A \left(-\frac{(11M^2 + 11M + 3) \zeta(2)}{12M(M+1)} \right. \\
 &\quad \left. + \frac{151M^4 + 236M^2 + 88M^2 + 3M + 18}{72M^2(M+1)^2} \right. \\
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 9_0^{(2)}(ds,qq) &= C_F^2 \left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4} \zeta(3) \right] \\
 &\quad + C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36} \zeta(3) \right] \\
 &\quad + C_F N_f \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta(3)}{18} \right].
 \end{aligned}$$

- $\tilde{F}(M)$ process dependent function.
- $\tilde{P}_{qq}^{(i)}$ s are the splitting functions without the constant terms.
- \hat{z} —single-soft limit same behavior aside for P_{qq} spacelike.
- Same structure of the results obtained for DY process. However in this case, in both single-soft limits the splitting functions are spacelike.

Checks

- \hat{x} single-soft limit \Rightarrow taking the large M limit \Rightarrow I recovered the double-soft limit obtained by Abele, Florian, and Vogelsang 2021.
- Same for the \hat{z} single-soft limit.

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- \hat{x} single-soft limit \Rightarrow taking the large M limit \Rightarrow I recovered the double-soft limit obtained by Abele, Florian, and Vogelsang 2021.
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- \hat{x} single-soft limit \Rightarrow expand up to $\mathcal{O}(\frac{1}{M}) \Rightarrow$ I recovered the NLP corrections $\frac{\mathcal{L}^3}{M}$ predicted by Abele, Florian, and Vogelsang 2022.
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- Same for the \hat{z} single soft limit
- **Final remark:** In agreement with my supervisor I am writing a paper on this new result ☺.

Thank you for your attention!

francesco.ventola@studenti.unimi.it