Reconstructing Landau Singularities over Finite Fields

Giulio Crisanti

Munich, 03/02/26

Based on

[Chestnov, Crisanti, Giroux, 2026 — TBA]

[Chestnov, Crisanti, 2025]

Introduction and Motivation (1/3)

Why Amplitudes?

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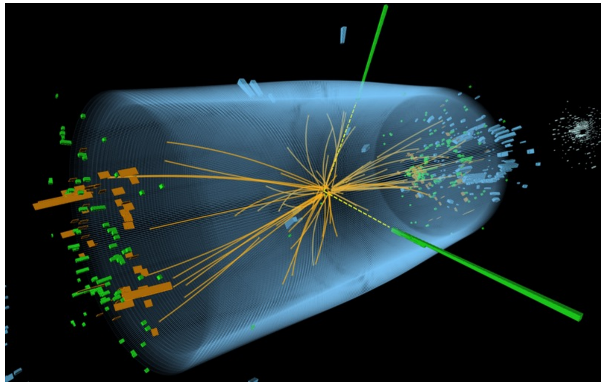
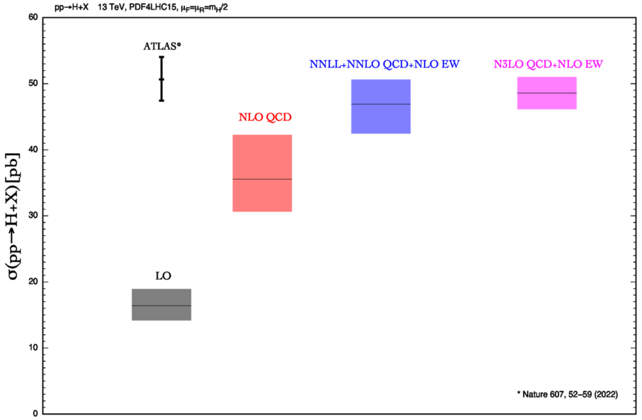
Collider Phenomenology





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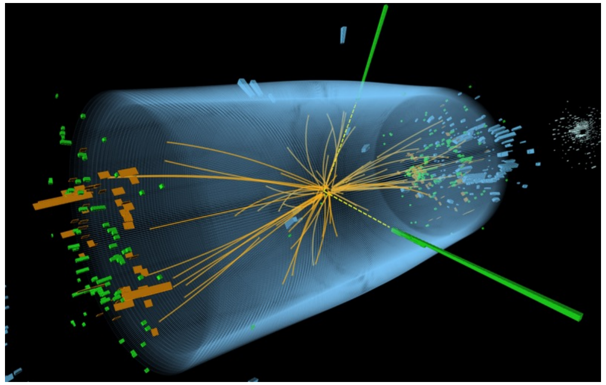
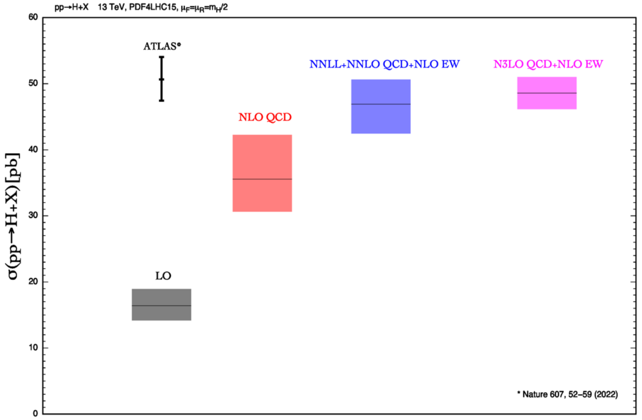
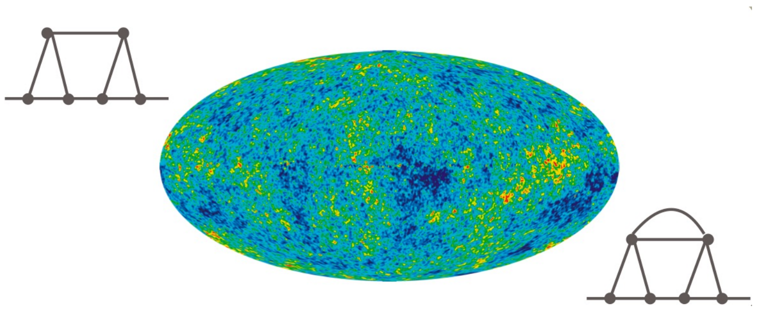




Cosmological Correlators



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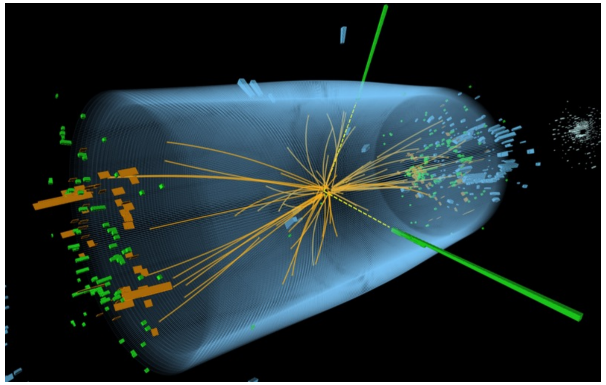

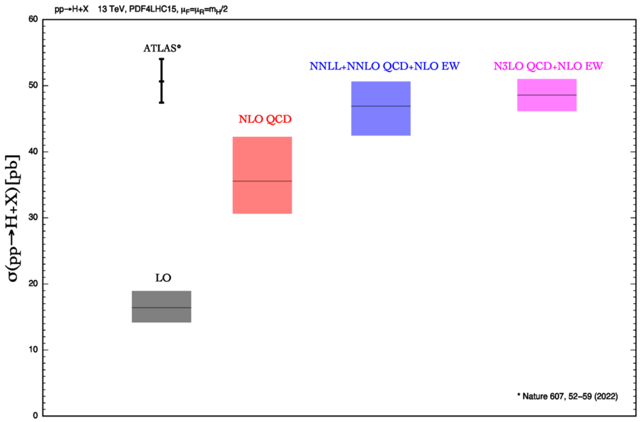

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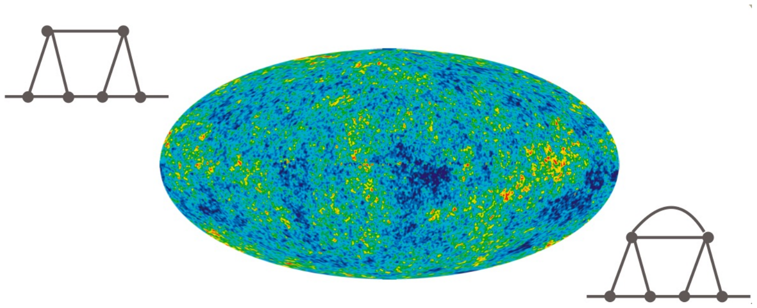
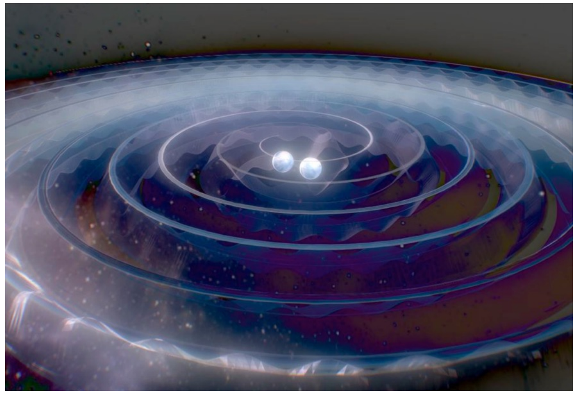
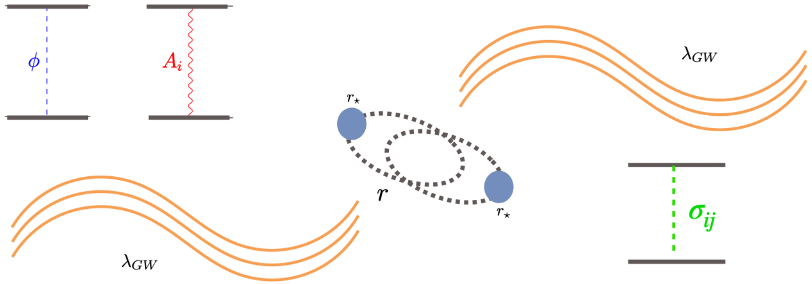


Classical Scattering





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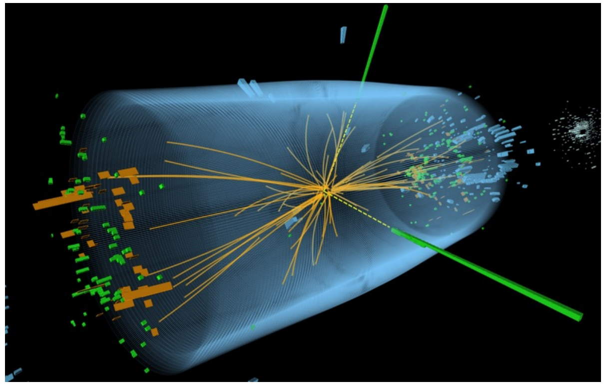

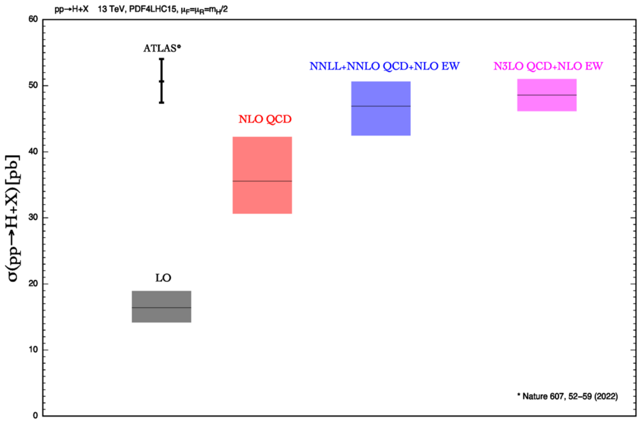

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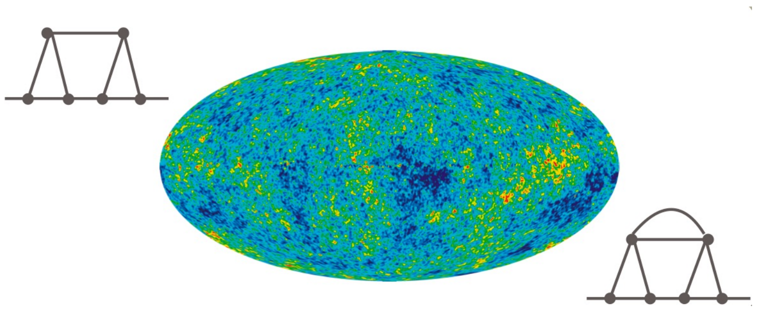

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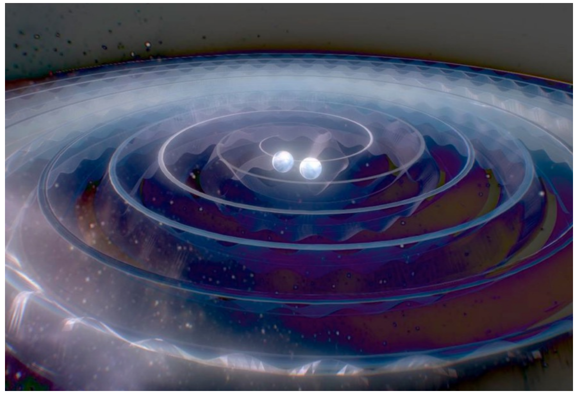
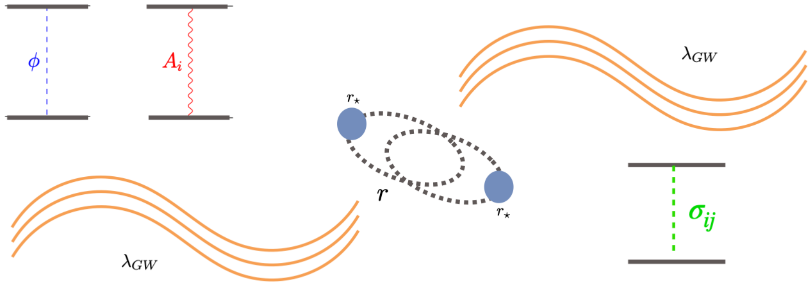
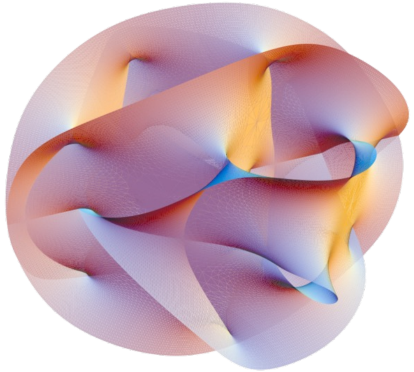




Algebraic/Differential Geometry



Introduction and Motivation (2/3)

How are amplitudues computed?

Most practical method: Feynman integrals (for now at least)

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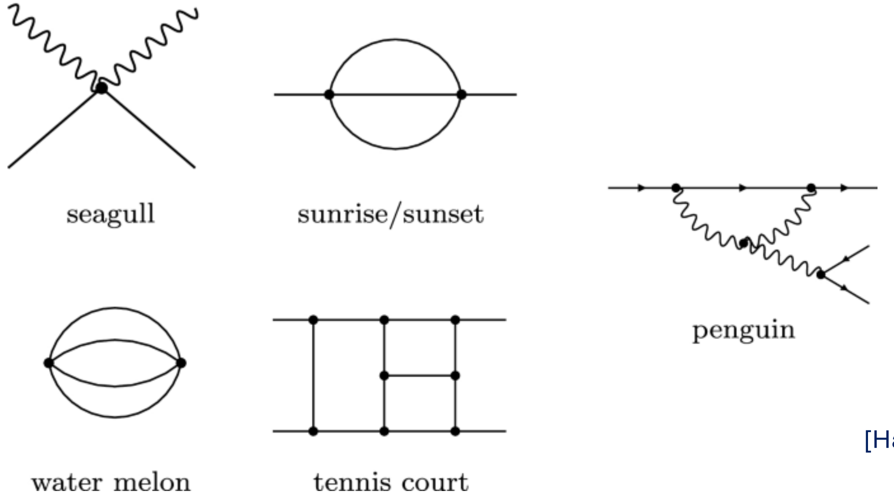
number of *loops*

regulated spacetime dimension

$$I_{a_1 \cdots a_n}(s_{ij}) = \int \prod_{i=1}^{l} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \frac{1}{D_1^{a_1} \cdots D_n^{a_n}}$$

propagators/denominators

Mandelstam invariants $s_{ij} = p_i \cdot p_j$



[Harlander, 2021]

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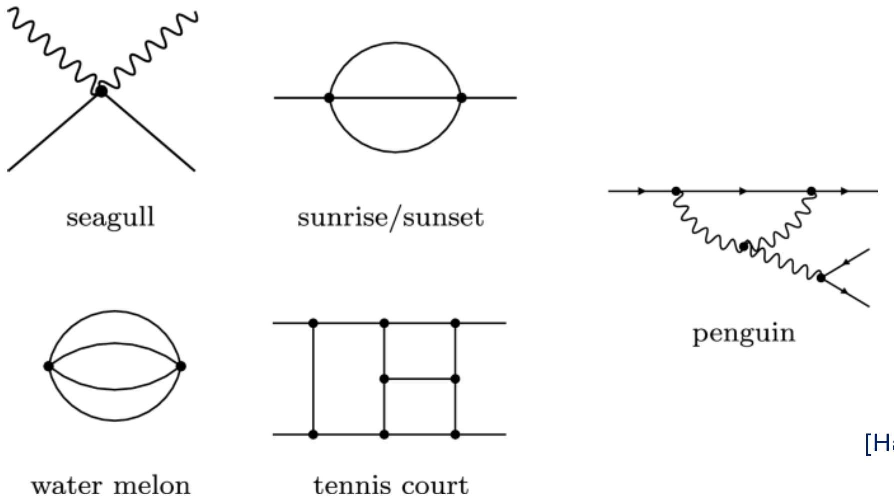

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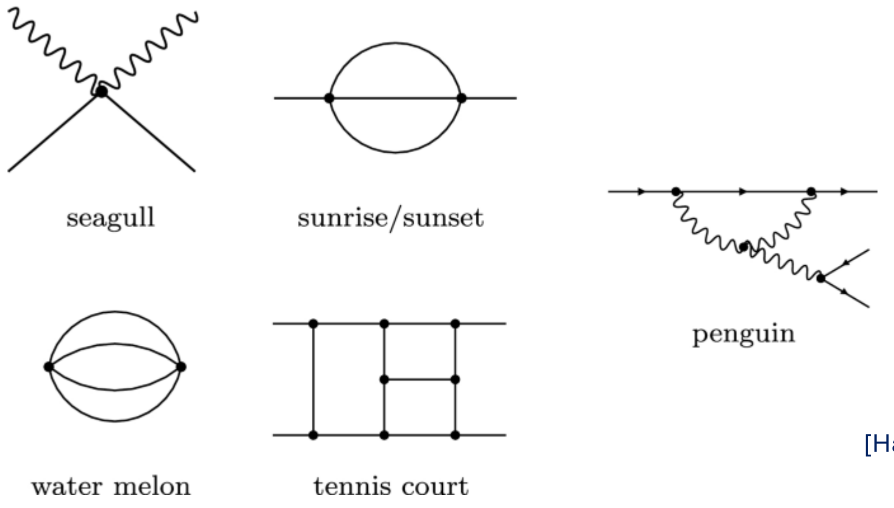

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- the theory: particles/fields, masses and spin (via its Feynman rules)
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Terrible scaling!

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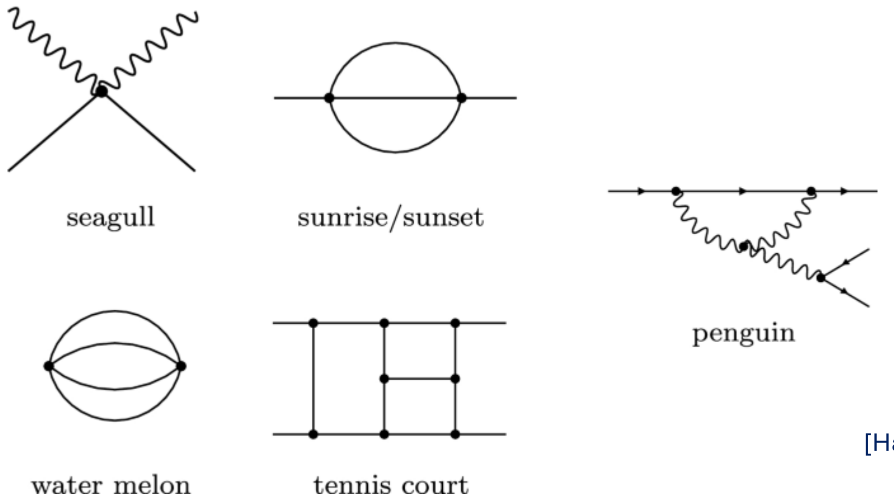

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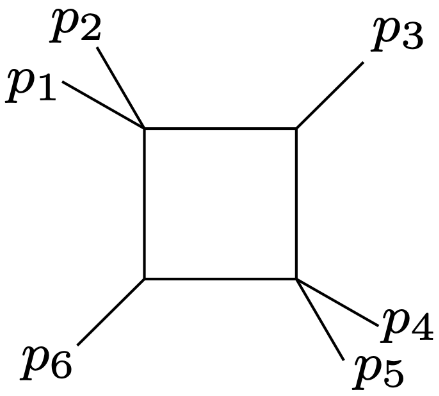
Characterising and understanding the general as well as specific properties of Feynman integrals is crucial to continue making progress towards computing higher precision amplitudes

Introduction and Motivation (3/3)

What is a Feynman integral made of?



$$
\begin{aligned}
= \quad & \frac{2}{\epsilon^2}\left((-s_{345})^{-\epsilon} + (-s_{123})^{-\epsilon} + (-s_{12})^{-\epsilon} + (-s_{45})^{-\epsilon}\right) \\
& -2\mathrm{Li}_2\left(1 - \frac{s_{12}}{s_{345}}\right) - 2\mathrm{Li}_2\left(1 - \frac{s_{12}}{s_{123}}\right) - 2\mathrm{Li}_2\left(1 - \frac{s_{45}}{s_{345}}\right) \\
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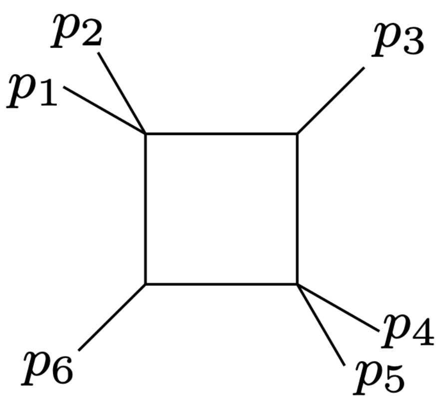

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Function space

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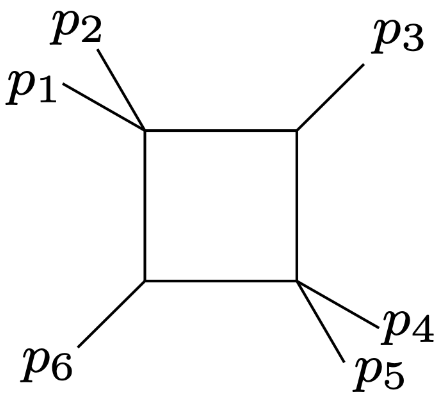

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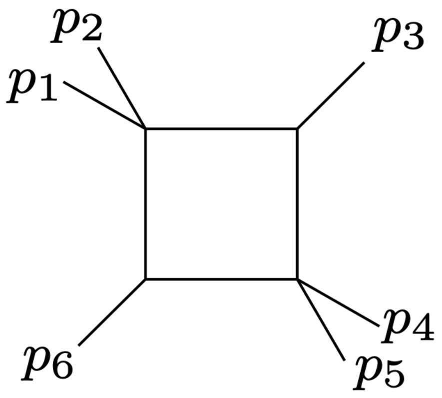

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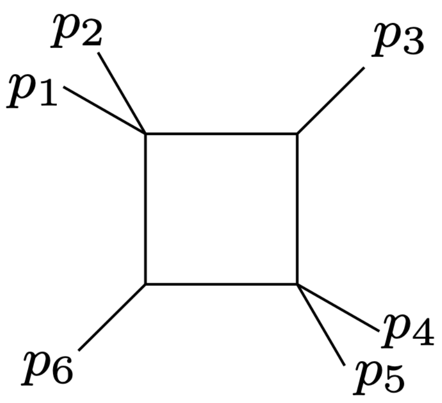

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Can we predict all the branch points of a Feynman integral efficiently?

Problem known as Landau analysis — location of branches is given by polynomials in the Mandelstam invariants

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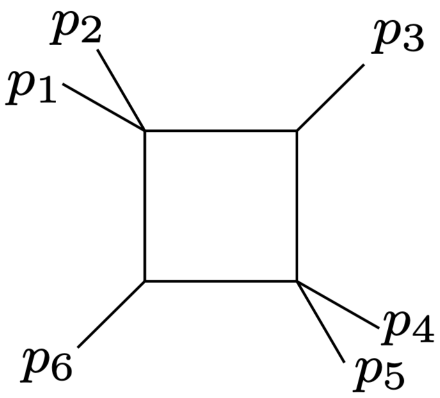

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A lot of work has gone into this recently! [Landau, 1960] [Cutkosky, 1960]

[Abreu, Berghoff, Bourjaily, Britto, Chestnov, Crisanti, Correia, Duhr, Fevola, Gardi, Giroux, Hannesdottir, Helmer, McLeod, Mizera, Panzer, Papathanasiou, Schwartz, Tellander, Telen, Vergu, 2017-2026]

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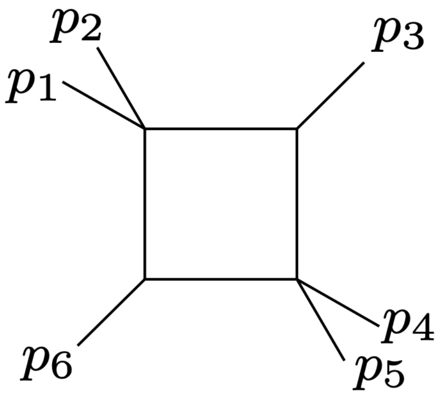

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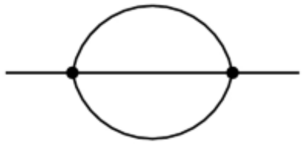
Knowing the locations of the branch points is crucial information when trying to compute Feynman integrals and amplitudes

Goal and Outline (1/2)

Goal

Work towards the function:



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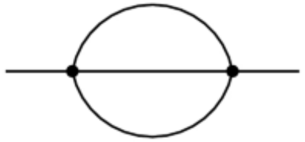
$\text{Singularities}\Big[$  $\Big] = \{s, m^2, s - m^2, s - 9m^2\}$

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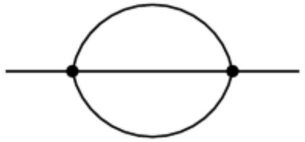
$\text{Singularities}\Big[$  $\Big] = \{s, m^2, s - m^2, s - 9m^2\}$

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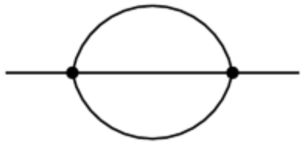
Singularities [] $= \{s, m^2, s - m^2, s - 9m^2\}$

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Needs to be automated practical and universal

[Fevola, Mizera, Telen, 2023]
[Helmer, Papathanasiou, Tellander, 2024]
[Correia, Giroux, Mizera, 2025]

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Outline

Present a new approach that satisfies these criteria

Algorithm

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1) Landau Singularities and Euler Characteristics

2) Identifying discontinuous changes in systems of polynomial equations

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Practical implementation

1) Elimination Theory over Finite Fields

2) Implementation inside SPQR

Parametric Representations of Feynman Integrals

From momentum space to parameter space

$$I(s_{ij}) = \int \prod_{i=1}^{l} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \frac{1}{D_1 \cdots D_n} \longrightarrow I(s_{ij}) = \int_0^\infty G(x, s_{ij}, m)^{-d/2} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

[Lee, Pomeransky, 2013]

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Parametric integrals are convenient as they avoid the mathematical ambiguities of non integer dimension vectors

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Integration variables

[Lee, Pomeransky, 2013]

Integration over loop momenta is exchanged with integration over (scalar) Schwinger parameters x

$G(x, s_{ij}, m)$ is a polynomial in both x and

Due to dimensional regularisation $G(x, s_{ij}, m)^{-d/2}$ is a multivalued function with branch points at $G = 0$

Parametric integrals are convenient as they avoid the mathematical ambiguities of non integer dimension vectors

Feynman integrals in parametric representation belong to the broad class of Euler/Twisted period integrals

Euler Characteristics of Feynman Integrals (1/3)

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$$\omega = 0 \longrightarrow \begin{cases} \dfrac{\partial_1 G}{G} = 0 \\ \vdots \\ \dfrac{\partial_n G}{G} = 0 \end{cases}$$

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system of polynomial equations!

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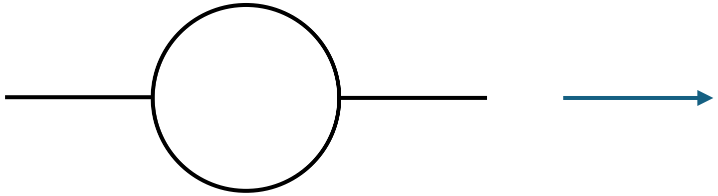
system of polynomial equations!

Linking the Euler characteristic to Landau Singularities

[Chestnov, Matsubara-Heo, Munch, Takayama, 2023]
[Mizera, Fevola, Telen, 2023/24]

Euler Characteristics of Feynman Integrals (2/3)

Example — One loop massive bubble



$$G = -m_1^2 x_1^2 - m_1^2 x_2^2 - 2m_1^2 x_1 x_2 + s x_1 x_2 + x_1 + x_2$$

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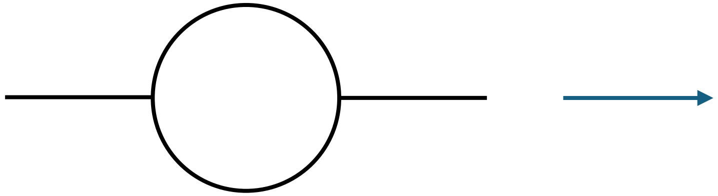

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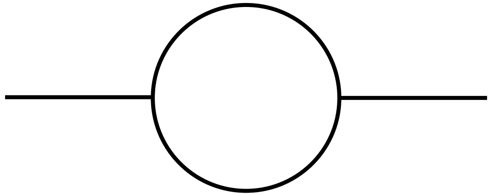

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Find one critical point: $\left\{x_0 = 4m_1^2 - s, x_1 = \frac{1}{4m_1^2 - s}, x_2 = \frac{1}{4m_1^2 - s}\right\} \longrightarrow \chi = 1$

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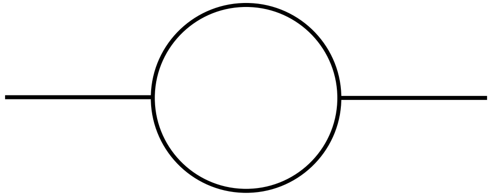

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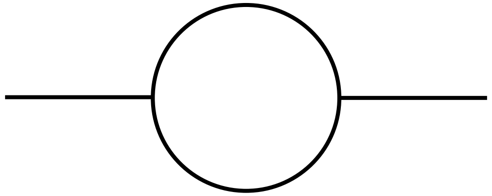

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Sum first two equations: $\frac{1}{2}\left(-s x_1 + s x_2 + 2\right) + \frac{1}{2}\left(s x_1 - s x_2 + 2\right) = 2 \neq 0 \longrightarrow \chi_{s-4m_1^2} = 0 < \chi$

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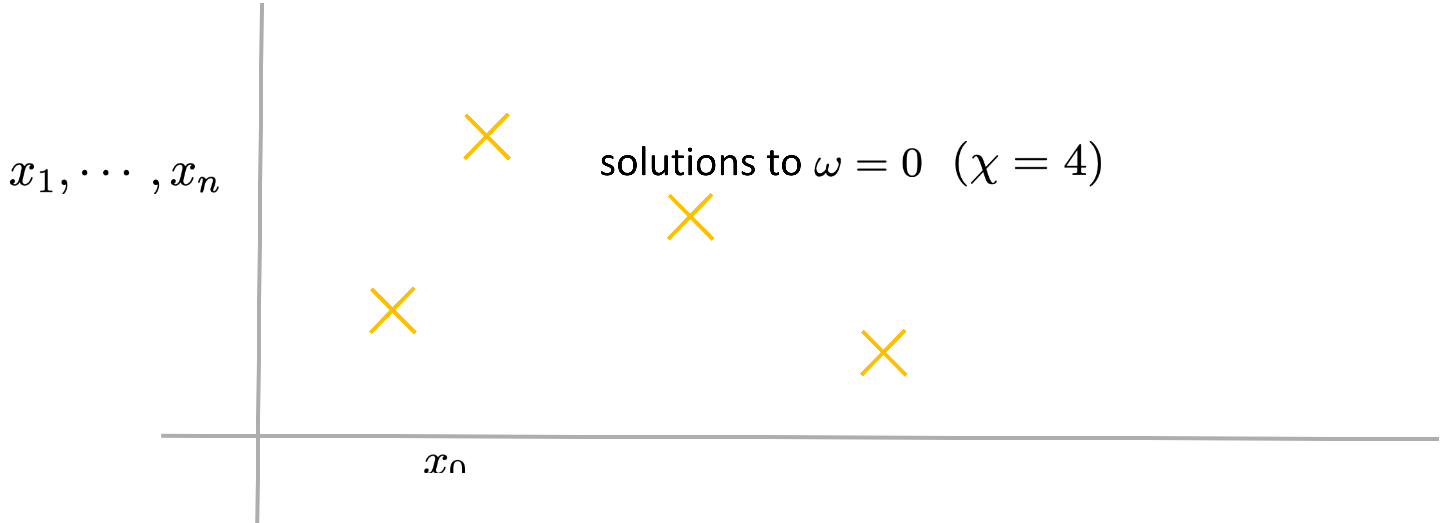
Key idea: as the limit $l(s) \to 0$ is approached, a subset of the critical points must behave badly



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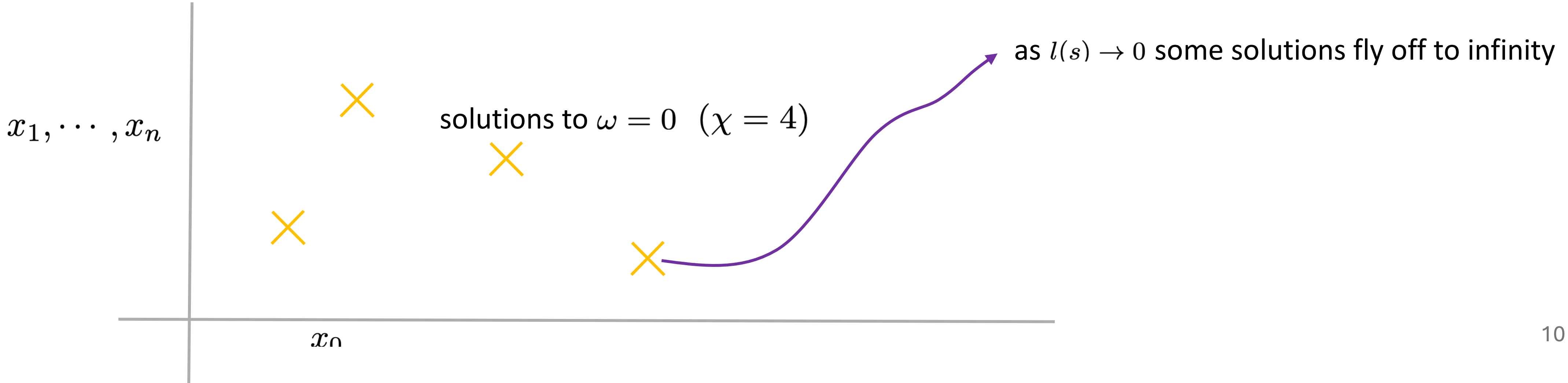
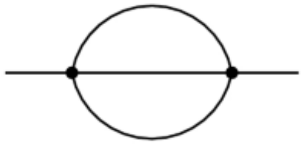


Demo

$$\text{Singularities}\Bigg[$$  $$\Bigg] = \{s, m^2, s - m^2, s - 9m^2\}$$

Detecting when roots go to infinity (1/2)

Univariate case

How do we detect for which values of parameters do roots move to infinity?

$$0 = a\,x^2 + b\,x + c$$

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Generalises straightforwardly to higher degree polynomials:

$$0 = c_n\,x^n + c_{n-1}\,x^{n-1} + \cdots + c_1 x + c + 0$$

As $c_n \to 0$ at least one solution diverges

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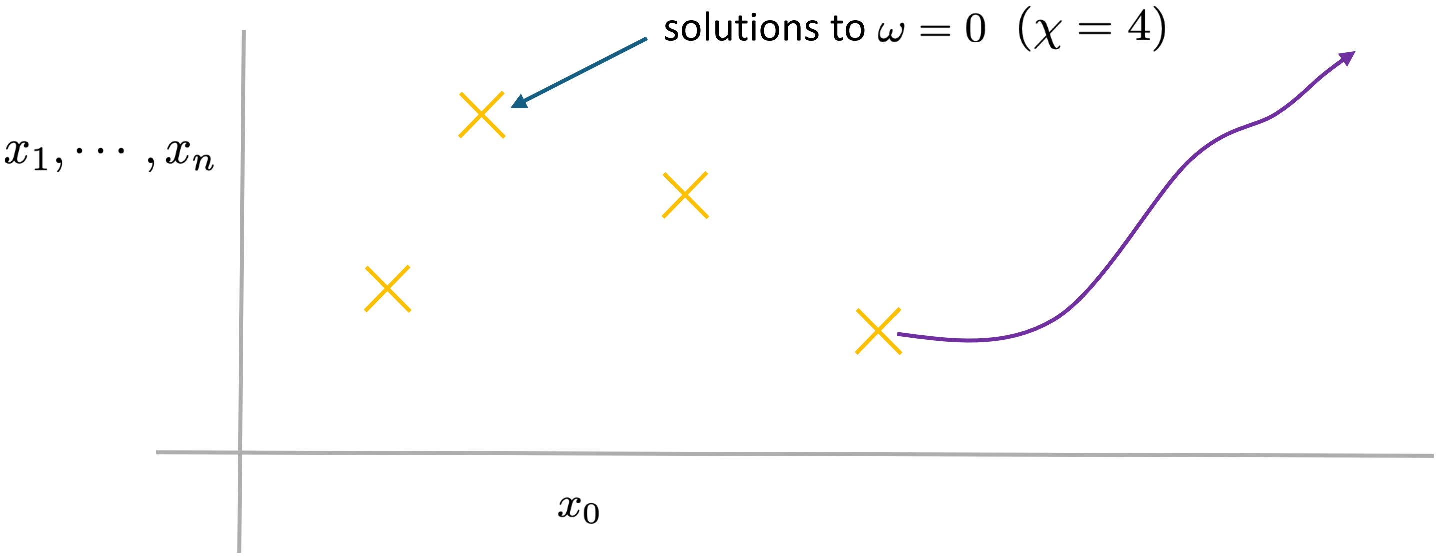

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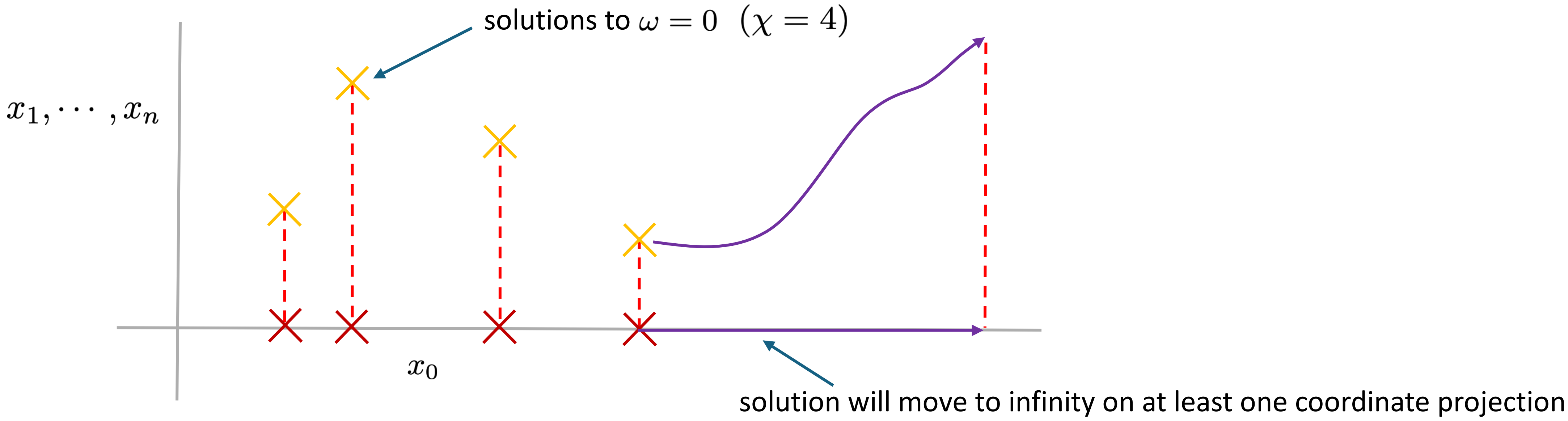

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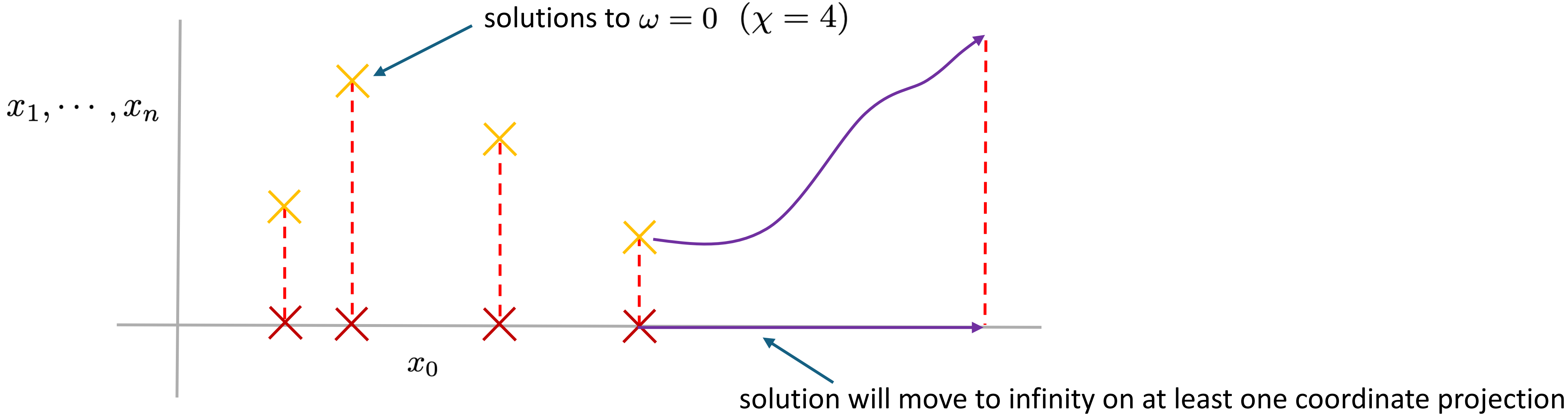

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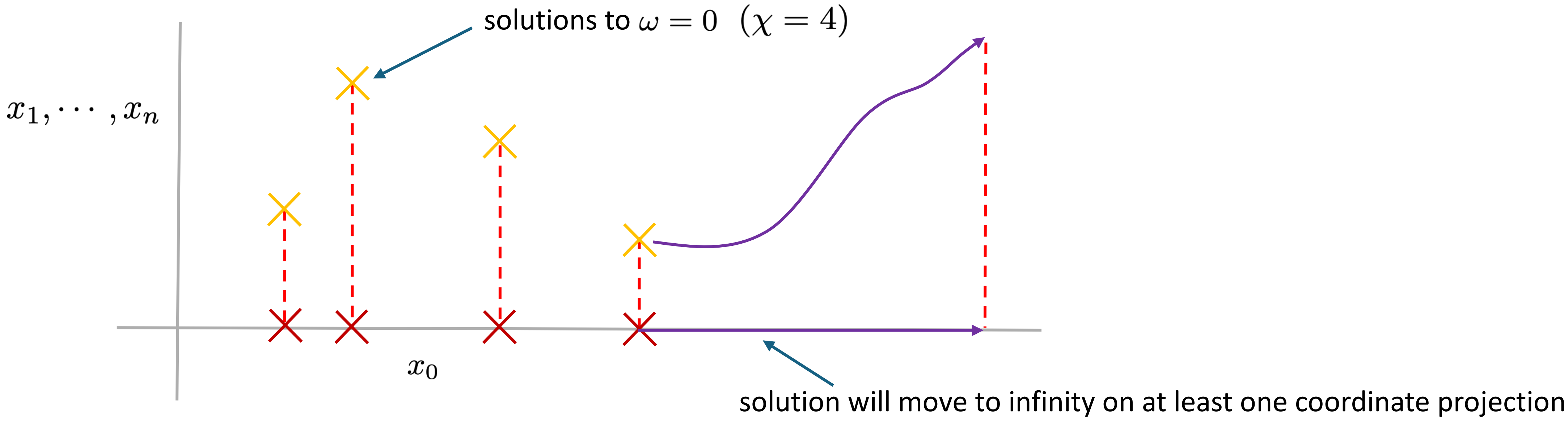

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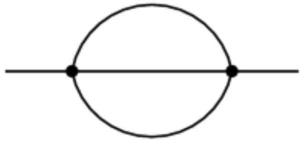
Avoids having to solve the system explicitly → Never have to deal with complicated roots (only rationals)

Demo

Singularities[] $= \{s, m^2, s - m^2, s - 9m^2\}$

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Solution: Build Gröbner bases and eliminate variables over Finite Fields — New SPQR package

[Chestnov, Crisanti, 2025]

Finite Field Reconstruction (1/2)

Operations on Matrices

$$\mathbb{M}(a_1, \cdots, a_n)$$

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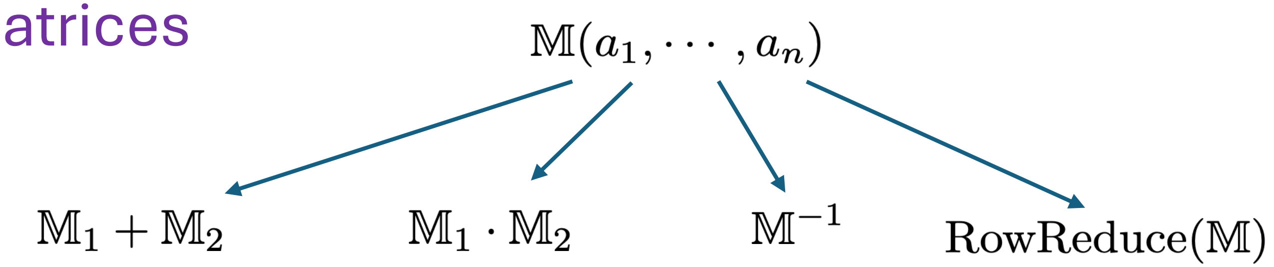

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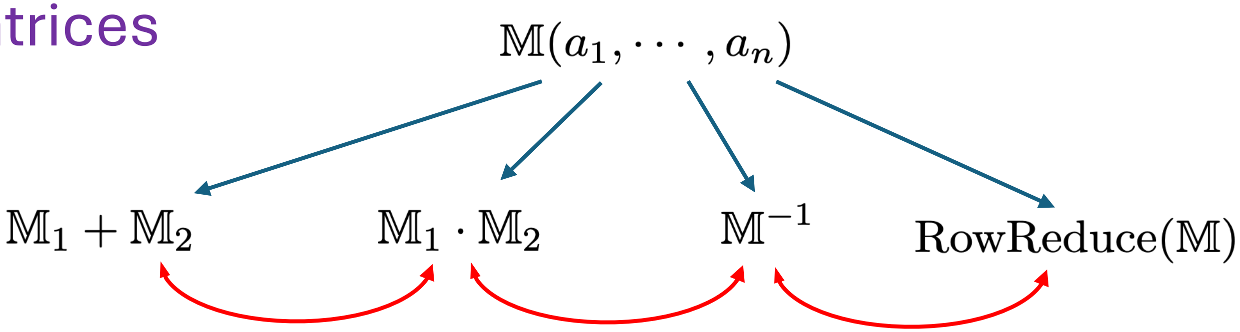

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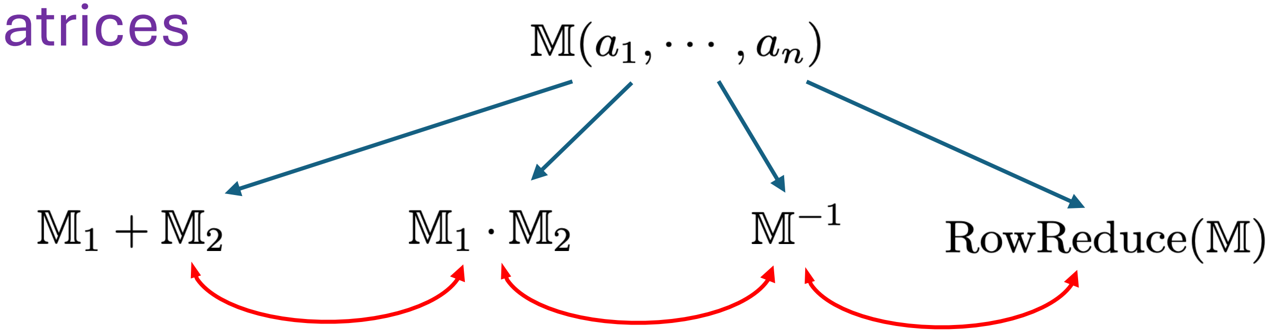

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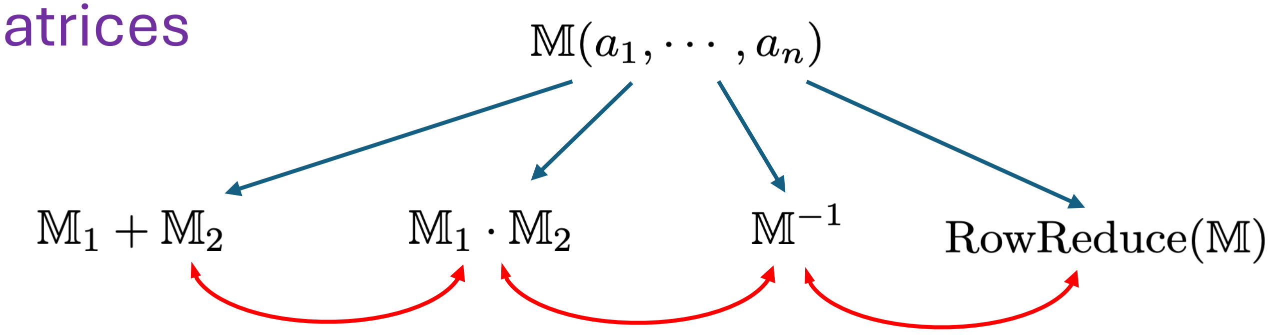

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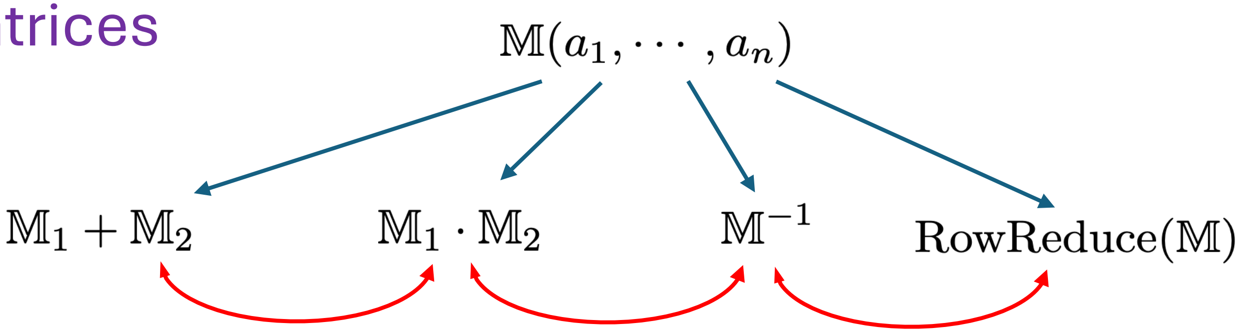

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It is possible to recast building Gröbner bases as operations on matrices (over finite fields)

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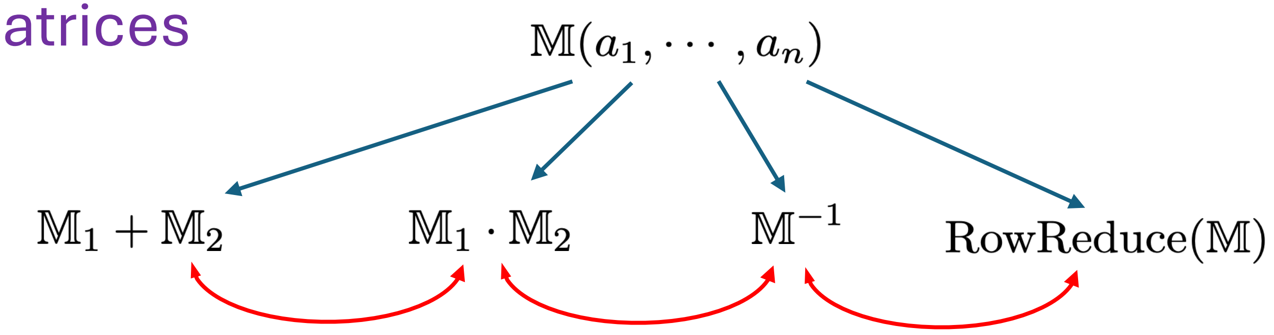

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Find 6+ order of magnitude improvement for finding Landau singularities

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Finite Field Reconstruction (2/2)

Example

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$$
M = \begin{pmatrix}
1+\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)} & \frac{(a^2+b^2)(b-c)}{(1+c)(1+bc)} & \frac{(a^2+b^2)(-a+c)}{(1+c)(1+ac)} & \frac{(a+b)(a^2+b^2)}{(1+c)^2} & \frac{(a^2+b^2)(b+c)}{(1+a)(1+c)} & \frac{(a^2+b^2)(a+c)}{(1+b)(1+c)} & \frac{a^2+b^2}{1+c} & \frac{(a^2+b^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+c)} \\
\frac{(a-b)(b^2+c^2)}{(1+a)(1+ab)} & 1+\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)} & \frac{(-a+c)(b^2+c^2)}{(1+a)(1+ac)} & \frac{(a+b)(b^2+c^2)}{(1+a)(1+c)} & \frac{(b+c)(b^2+c^2)}{(1+a)^2} & \frac{(a+c)(b^2+c^2)}{(1+a)(1+b)} & \frac{b^2+c^2}{1+a} & \frac{(b^2+c^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a)} \\
\frac{(a-b)(a^2+c^2)}{(1+b)(1+ab)} & \frac{(b-c)(a^2+c^2)}{(1+b)(1+bc)} & 1+\frac{(-a+c)(a^2+c^2)}{(1+b)(1+ac)} & \frac{(a+b)(a^2+c^2)}{(1+b)(1+c)} & \frac{(b+c)(a^2+c^2)}{(1+a)(1+b)} & \frac{(a+c)(a^2+c^2)}{(1+b)^2} & \frac{a^2+c^2}{1+b} & \frac{(a^2+c^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+b)} \\
\frac{(a-b)(ab-c)}{(1+ab)(1+abc)} & \frac{(b-c)(ab-c)}{(1+bc)(1+abc)} & \frac{(ab-c)(-a+c)}{(1+ac)(1+abc)} & 1+\frac{(a+b)(ab-c)}{(1+c)(1+abc)} & \frac{(ab-c)(b+c)}{(1+a)(1+abc)} & \frac{(ab-c)(a+c)}{(1+b)(1+abc)} & \frac{ab-c}{1+abc} & \frac{(ab-c)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+abc)} \\
\frac{(a-b)(-a+bc)}{(1+ab)(1+a+c)} & \frac{(b-c)(-a+bc)}{(1+a+c)(1+bc)} & \frac{(a-c)(a-bc)}{(1+a+c)(1+ac)} & \frac{(a+b)(-a+bc)}{(1+c)(1+a+c)} & 1+\frac{(b+c)(-a+bc)}{(1+a)(1+a+c)} & \frac{(a+c)(-a+bc)}{(1+b)(1+a+c)} & \frac{-a+bc}{1+a+c} & \frac{(-a+bc)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a+c)} \\
\frac{(a-b)(-b+ac)}{(1+a+b)(1+ab)} & \frac{(b-c)(-b+ac)}{(1+a+b)(1+bc)} & \frac{(-a+c)(-b+ac)}{(1+a+b)(1+ac)} & \frac{(a+b)(-b+ac)}{(1+a+b)(1+c)} & \frac{(b+c)(-b+ac)}{(1+a)(1+a+b)} & 1+\frac{(a+c)(-b+ac)}{(1+b)(1+a+b)} & \frac{-b+ac}{1+a+b} & \frac{(-b+ac)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a+b)} \\
\frac{(a-b)(a+b+c)}{(1+ab)(bc+a(b+c))} & \frac{(b-c)(a+b+c)}{(1+bc)(bc+a(b+c))} & \frac{(-a+c)(a+b+c)}{(1+ac)(bc+a(b+c))} & \frac{(a+b)(a+b+c)}{(1+c)(bc+a(b+c))} & \frac{(b+c)(a+b+c)}{(1+a)(bc+a(b+c))} & \frac{(a+c)(a+b+c)}{(1+b)(bc+a(b+c))} & 1+\frac{a+b+c}{bc+a(b+c)} & \frac{(a+b+c)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(bc+a(b+c))} \\
\frac{2(a-b)}{1+ab} & \frac{2(b-c)}{1+bc} & \frac{2(-a+c)}{1+ac} & \frac{2(a+b)}{1+c} & \frac{2(b+c)}{1+a} & \frac{2(a+c)}{1+b} & 2 & 1-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}
\end{pmatrix}
$$

Finite Field Reconstruction (2/2)

Example

$$M = \begin{pmatrix}
1+\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)} & \frac{(a^2+b^2)(b-c)}{(1+c)(1+bc)} & \frac{(a^2+b^2)(-a+c)}{(1+c)(1+ac)} & \frac{(a+b)(a^2+b^2)}{(1+c)^2} & \frac{(a^2+b^2)(b+c)}{(1+a)(1+c)} & \frac{(a^2+b^2)(a+c)}{(1+b)(1+c)} & \frac{a^2+b^2}{1+c} & \frac{(a^2+b^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+c)} \\
\frac{(a-b)(b^2+c^2)}{(1+a)(1+ab)} & 1+\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)} & \frac{(-a+c)(b^2+c^2)}{(1+a)(1+ac)} & \frac{(a+b)(b^2+c^2)}{(1+a)(1+c)} & \frac{(b+c)(b^2+c^2)}{(1+a)^2} & \frac{(a+c)(b^2+c^2)}{(1+a)(1+b)} & \frac{b^2+c^2}{1+a} & \frac{(b^2+c^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a)} \\
\frac{(a-b)(a^2+c^2)}{(1+b)(1+ab)} & \frac{(b-c)(a^2+c^2)}{(1+b)(1+bc)} & 1+\frac{(-a+c)(a^2+c^2)}{(1+b)(1+ac)} & \frac{(a+b)(a^2+c^2)}{(1+b)(1+c)} & \frac{(b+c)(a^2+c^2)}{(1+a)(1+b)} & \frac{(a+c)(a^2+c^2)}{(1+b)^2} & \frac{a^2+c^2}{1+b} & \frac{(a^2+c^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+b)} \\
\frac{(a-b)(ab-c)}{(1+ab)(1+abc)} & \frac{(b-c)(ab-c)}{(1+bc)(1+abc)} & \frac{(ab-c)(-a+c)}{(1+ac)(1+abc)} & 1+\frac{(a+b)(ab-c)}{(1+c)(1+abc)} & \frac{(ab-c)(b+c)}{(1+a)(1+abc)} & \frac{(ab-c)(a+c)}{(1+b)(1+abc)} & \frac{ab-c}{1+abc} & \frac{(ab-c)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+abc)} \\
\frac{(a-b)(-a+bc)}{(1+ab)(1+a+c)} & \frac{(b-c)(-a+bc)}{(1+a+c)(1+bc)} & \frac{(a-c)(a-bc)}{(1+a+c)(1+ac)} & \frac{(a+b)(-a+bc)}{(1+c)(1+a+c)} & 1+\frac{(b+c)(-a+bc)}{(1+a)(1+a+c)} & \frac{(a+c)(-a+bc)}{(1+b)(1+a+c)} & \frac{-a+bc}{1+a+c} & \frac{(-a+bc)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a+c)} \\
\frac{(a-b)(-b+ac)}{(1+a+b)(1+ab)} & \frac{(b-c)(-b+ac)}{(1+a+b)(1+bc)} & \frac{(-a+c)(-b+ac)}{(1+a+b)(1+ac)} & \frac{(a+b)(-b+ac)}{(1+a+b)(1+c)} & \frac{(b+c)(-b+ac)}{(1+a)(1+a+b)} & 1+\frac{(a+c)(-b+ac)}{(1+b)(1+a+b)} & \frac{-b+ac}{1+a+b} & \frac{(-b+ac)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a+b)} \\
\frac{(a-b)(a+b+c)}{(1+ab)(bc+a(b+c))} & \frac{(b-c)(a+b+c)}{(1+bc)(bc+a(b+c))} & \frac{(-a+c)(a+b+c)}{(1+ac)(bc+a(b+c))} & \frac{(a+b)(a+b+c)}{(1+c)(bc+a(b+c))} & \frac{(b+c)(a+b+c)}{(1+a)(bc+a(b+c))} & \frac{(a+c)(a+b+c)}{(1+b)(bc+a(b+c))} & 1+\frac{a+b+c}{bc+a(b+c)} & \frac{(a+b+c)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(bc+a(b+c))} \\
\frac{2(a-b)}{1+ab} & \frac{2(b-c)}{1+bc} & \frac{2(-a+c)}{1+ac} & \frac{2(a+b)}{1+c} & \frac{2(b+c)}{1+a} & \frac{2(a+c)}{1+b} & 2 & 1-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}
\end{pmatrix}$$

$$\det(M) = -b\,a^2 + a^2c + ab^2 - ac^2 - b^2c + bc^2 + 1$$

super simple!

Finite Field Reconstruction (2/2)

Example

$$M = \begin{pmatrix}
1+\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)} & \frac{(a^2+b^2)(b-c)}{(1+c)(1+bc)} & \frac{(a^2+b^2)(-a+c)}{(1+c)(1+ac)} & \frac{(a+b)(a^2+b^2)}{(1+c)^2} & \frac{(a^2+b^2)(b+c)}{(1+a)(1+c)} & \frac{(a^2+b^2)(a+c)}{(1+b)(1+c)} & \frac{a^2+b^2}{1+c} & \frac{(a^2+b^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+c)} \\
\frac{(a-b)(b^2+c^2)}{(1+a)(1+ab)} & 1+\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)} & \frac{(-a+c)(b^2+c^2)}{(1+a)(1+ac)} & \frac{(a+b)(b^2+c^2)}{(1+a)(1+c)} & \frac{(b+c)(b^2+c^2)}{(1+a)^2} & \frac{(a+c)(b^2+c^2)}{(1+a)(1+b)} & \frac{b^2+c^2}{1+a} & \frac{(b^2+c^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a)} \\
\frac{(a-b)(a^2+c^2)}{(1+b)(1+ab)} & \frac{(b-c)(a^2+c^2)}{(1+b)(1+bc)} & 1+\frac{(-a+c)(a^2+c^2)}{(1+b)(1+ac)} & \frac{(a+b)(a^2+c^2)}{(1+b)(1+c)} & \frac{(b+c)(a^2+c^2)}{(1+a)(1+b)} & \frac{(a+c)(a^2+c^2)}{(1+b)^2} & \frac{a^2+c^2}{1+b} & \frac{(a^2+c^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+b)} \\
\frac{(a-b)(ab-c)}{(1+ab)(1+abc)} & \frac{(b-c)(ab-c)}{(1+bc)(1+abc)} & \frac{(ab-c)(-a+c)}{(1+ac)(1+abc)} & 1+\frac{(a+b)(ab-c)}{(1+c)(1+abc)} & \frac{(ab-c)(b+c)}{(1+a)(1+abc)} & \frac{(ab-c)(a+c)}{(1+b)(1+abc)} & \frac{ab-c}{1+abc} & \frac{(ab-c)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+abc)} \\
\frac{(a-b)(-a+bc)}{(1+ab)(1+a+c)} & \frac{(b-c)(-a+bc)}{(1+a+c)(1+bc)} & \frac{(a-c)(a-bc)}{(1+a+c)(1+ac)} & \frac{(a+b)(-a+bc)}{(1+c)(1+a+c)} & 1+\frac{(b+c)(-a+bc)}{(1+a)(1+a+c)} & \frac{(a+c)(-a+bc)}{(1+b)(1+a+c)} & \frac{-a+bc}{1+a+c} & \frac{(-a+bc)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a+c)} \\
\frac{(a-b)(-b+ac)}{(1+a+b)(1+ab)} & \frac{(b-c)(-b+ac)}{(1+a+b)(1+bc)} & \frac{(-a+c)(-b+ac)}{(1+a+b)(1+ac)} & \frac{(a+b)(-b+ac)}{(1+a+b)(1+c)} & \frac{(b+c)(-b+ac)}{(1+a)(1+a+b)} & 1+\frac{(a+c)(-b+ac)}{(1+b)(1+a+b)} & \frac{-b+ac}{1+a+b} & \frac{(-b+ac)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a+b)} \\
\frac{(a-b)(a+b+c)}{(1+ab)(bc+a(b+c))} & \frac{(b-c)(a+b+c)}{(1+bc)(bc+a(b+c))} & \frac{(-a+c)(a+b+c)}{(1+ac)(bc+a(b+c))} & \frac{(a+b)(a+b+c)}{(1+c)(bc+a(b+c))} & \frac{(b+c)(a+b+c)}{(1+a)(bc+a(b+c))} & \frac{(a+c)(a+b+c)}{(1+b)(bc+a(b+c))} & 1+\frac{a+b+c}{bc+a(b+c)} & \frac{(a+b+c)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(bc+a(b+c))} \\
\frac{2(a-b)}{1+ab} & \frac{2(b-c)}{1+bc} & \frac{2(-a+c)}{1+ac} & \frac{2(a+b)}{1+c} & \frac{2(b+c)}{1+a} & \frac{2(a+c)}{1+b} & 2 & 1-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}
\end{pmatrix}$$

$$\det(M) = -b\,a^2 + a^2c + ab^2 - ac^2 - b^2c + bc^2 + 1$$

super simple!

CAS

```
M // Det
((… 836 … + a2 b5 c8 + 4 a3 b5 c8 + 3 a4 b5 c8 - a6 b5 c8 + a2 b6 c8 + 3 a3 b6 c8 + 3 a4 b6 c8 + a5 b6 c8 + a3 b7 c8 + a
  (-2 (1 + a) (1 + b) (1 + a + b) … 5 … (1 + a c) (a b + a c + b c) (1 + a b c) (… 7 … + … 1 …) + … 8 … +
  … 10 …) / (128 (1 + a)9 (1 + b)9 (1 + a + b)9 (1 + a b)8 (1 + c)9 (1 + a + c)9 (1 + a c)8 (1 + b c)8 (a b + a c + b
Size in memory: 0.8 GB   + Show more   Show all   Iconize   Store full expression in notebook
```

Finite Field Reconstruction (2/2)

Example

$$M = \begin{pmatrix}
1+\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)} & \frac{(a^2+b^2)(b-c)}{(1+c)(1+bc)} & \frac{(a^2+b^2)(-a+c)}{(1+c)(1+ac)} & \frac{(a+b)(a^2+b^2)}{(1+c)^2} & \frac{(a^2+b^2)(b+c)}{(1+a)(1+c)} & \frac{(a^2+b^2)(a+c)}{(1+b)(1+c)} & \frac{a^2+b^2}{1+c} & \frac{(a^2+b^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+c)} \\
\frac{(a-b)(b^2+c^2)}{(1+a)(1+ab)} & 1+\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)} & \frac{(-a+c)(b^2+c^2)}{(1+a)(1+ac)} & \frac{(a+b)(b^2+c^2)}{(1+a)(1+c)} & \frac{(b+c)(b^2+c^2)}{(1+a)^2} & \frac{(a+c)(b^2+c^2)}{(1+a)(1+b)} & \frac{b^2+c^2}{1+a} & \frac{(b^2+c^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a)} \\
\frac{(a-b)(a^2+c^2)}{(1+b)(1+ab)} & \frac{(b-c)(a^2+c^2)}{(1+b)(1+bc)} & 1+\frac{(-a+c)(a^2+c^2)}{(1+b)(1+ac)} & \frac{(a+b)(a^2+c^2)}{(1+b)(1+c)} & \frac{(b+c)(a^2+c^2)}{(1+a)(1+b)} & \frac{(a+c)(a^2+c^2)}{(1+b)^2} & \frac{a^2+c^2}{1+b} & \frac{(a^2+c^2)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+b)} \\
\frac{(a-b)(ab-c)}{(1+ab)(1+abc)} & \frac{(b-c)(ab-c)}{(1+bc)(1+abc)} & \frac{(ab-c)(-a+c)}{(1+ac)(1+abc)} & 1+\frac{(a+b)(ab-c)}{(1+c)(1+abc)} & \frac{(ab-c)(b+c)}{(1+a)(1+abc)} & \frac{(ab-c)(a+c)}{(1+b)(1+abc)} & \frac{ab-c}{1+abc} & \frac{(ab-c)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+abc)} \\
\frac{(a-b)(-a+bc)}{(1+ab)(1+a+c)} & \frac{(b-c)(-a+bc)}{(1+a+c)(1+bc)} & \frac{(a-c)(a-bc)}{(1+a+c)(1+ac)} & \frac{(a+b)(-a+bc)}{(1+c)(1+a+c)} & 1+\frac{(b+c)(-a+bc)}{(1+a)(1+a+c)} & \frac{(a+c)(-a+bc)}{(1+b)(1+a+c)} & \frac{-a+bc}{1+a+c} & \frac{(-a+bc)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a+c)} \\
\frac{(a-b)(-b+ac)}{(1+a+b)(1+ab)} & \frac{(b-c)(-b+ac)}{(1+a+b)(1+bc)} & \frac{(-a+c)(-b+ac)}{(1+a+b)(1+ac)} & \frac{(a+b)(-b+ac)}{(1+a+b)(1+c)} & \frac{(b+c)(-b+ac)}{(1+a)(1+a+b)} & 1+\frac{(a+c)(-b+ac)}{(1+b)(1+a+b)} & \frac{-b+ac}{1+a+b} & \frac{(-b+ac)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(1+a+b)} \\
\frac{(a-b)(a+b+c)}{(1+ab)(bc+a(b+c))} & \frac{(b-c)(a+b+c)}{(1+bc)(bc+a(b+c))} & \frac{(-a+c)(a+b+c)}{(1+ac)(bc+a(b+c))} & \frac{(a+b)(a+b+c)}{(1+c)(bc+a(b+c))} & \frac{(b+c)(a+b+c)}{(1+a)(bc+a(b+c))} & \frac{(a+c)(a+b+c)}{(1+b)(bc+a(b+c))} & 1+\frac{a+b+c}{bc+a(b+c)} & \frac{(a+b+c)\left(-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}\right)}{2(bc+a(b+c))} \\
\frac{2(a-b)}{1+ab} & \frac{2(b-c)}{1+bc} & \frac{2(-a+c)}{1+ac} & \frac{2(a+b)}{1+c} & \frac{2(b+c)}{1+a} & \frac{2(a+c)}{1+b} & 2 & 1-\frac{(a-b)(a^2+b^2)}{(1+ab)(1+c)}+(a-b)(b-c)(-a+c)+\frac{(a+c)(b-ac)}{(1+b)(1+a+b)}+\frac{(b+c)(a-bc)}{(1+a)(1+a+c)}-\frac{(a+b)(ab-c)}{(1+c)(1+abc)}+\frac{(a-c)(a^2+c^2)}{(1+b)(1+ac)}-\frac{(b-c)(b^2+c^2)}{(1+a)(1+bc)}-\frac{a+b+c}{bc+a(b+c)}
\end{pmatrix}$$

$$\det(M) = -b\,a^2 + a^2c + ab^2 - ac^2 - b^2c + bc^2 + 1$$

super simple!

SPQR

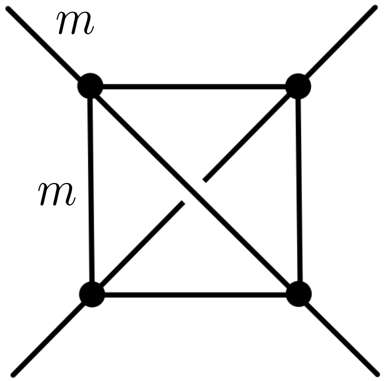
```
M // FFDet // AbsoluteTiming
Generated 20 sample points
Approximate time per evaluation (single core): 0.000146 sec.
{0.023711, 1 - a^2 b + a b^2 + a^2 c - b^2 c - a c^2 + b c^2}
```

CAS

```
M // Det
(( ... 836 ... + a^2 b^5 c^8 + 4 a^3 b^5 c^8 + 3 a^4 b^5 c^8 - a^6 b^5 c^8 + a^2 b^6 c^8 + 3 a^3 b^6 c^8 + 3 a^4 b^6 c^8 + a^5 b^6 c^8 + a^3 b^7 c^8 + a
  (-2 (1 + a) (1 + b) (1 + a + b) ... 5 ... (1 + a c) (a b + a c + b c) (1 + a b c) ( ... 7 ... + ... 1 ... ) + ... 8 ... +
  ... 10 ... ) / (128 (1 + a)^9 (1 + b)^9 (1 + a + b)^9 (1 + a b)^8 (1 + c)^9 (1 + a + c)^9 (1 + a c)^8 (1 + b c)^8 (a b + a c + b
Size in memory: 0.8 GB   + Show more   Show all   Iconize ▾   Store full expression in notebook
```

State of the Art Examples (1/2)

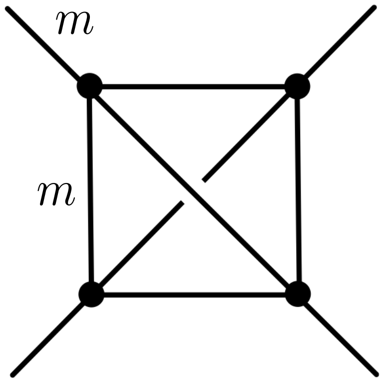
Three loop envelope



[Correia, Sever, Zhibodeov, 2021]

State of the Art Examples (1/2)

Three loop envelope

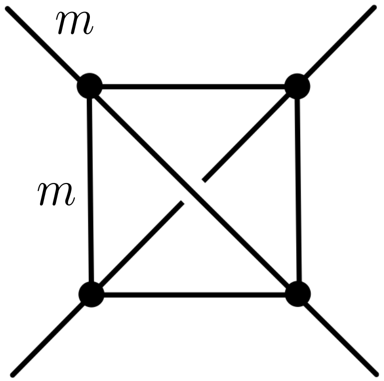


[Correia, Sever, Zhibodeev, 2021]

Horrendous integral: $\chi = 60(!)$ in the top (max cut) sector alone

State of the Art Examples (1/2)

Three loop envelope



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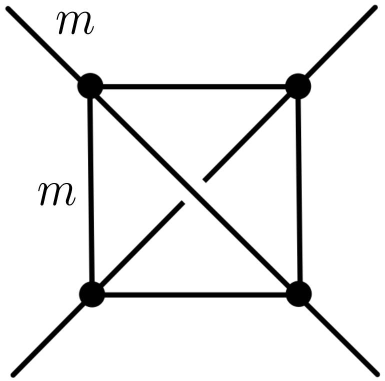
PLD/SOFIA most complicated letter found: $27(m^2)^3 + 4s^2t + 4st^2$

[Fevola, Mizera, Telen, 2023]
[Correia, Giroux, Mizera, 2025]

State of the Art Examples (1/2)

Three loop envelope



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[Fevola, Mizera, Telen, 2023]
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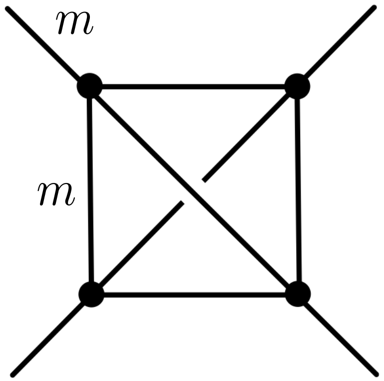
Singularities

Two new simple letters: $\{s^2 + st + t^2, m^2s^2 + m^2st + s^2t + m^2t^2 + st^2\}$

State of the Art Examples (1/2)

Three loop envelope



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Singularities

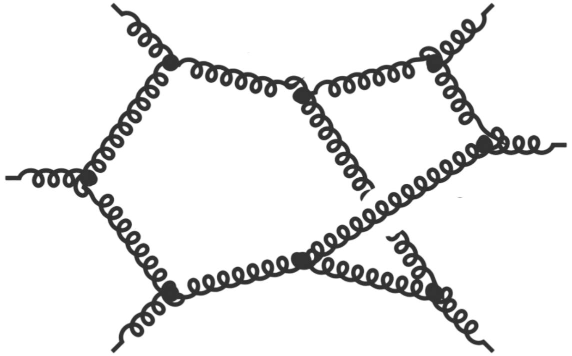
Two new simple letters: $\{s^2 + st + t^2, m^2s^2 + m^2st + s^2t + m^2t^2 + st^2\}$

Four new complicated letters:

$$\{27\,\mathrm{m2}^4 s^2 + 108\,\mathrm{m2}^4 s\, t + 162\,\mathrm{m2}^3 s^2 t + 54\,\mathrm{m2}^2 s^3 t + 4\,\mathrm{m2}\, s^4 t + 108\,\mathrm{m2}^4 t^2 + 162\,\mathrm{m2}^3 s\, t^2 + 45\,\mathrm{m2}^2 s^2 t^2 -$$
$$6\,\mathrm{m2}\, s^3 t^2 - s^4 t^2 - 18\,\mathrm{m2}^2 s\, t^3 - 20\,\mathrm{m2}\, s^2 t^3 - 2 s^3 t^3 - 9\,\mathrm{m2}^2 t^4 - 10\,\mathrm{m2}\, s\, t^4 - s^2 t^4,\ 108\,\mathrm{m2}^4 s^2 - 9\,\mathrm{m2}^2 s^4 + 108\,\mathrm{m2}^4 s\, t + 162\,\mathrm{m2}^3 s^2 t -$$
$$18\,\mathrm{m2}^2 s^3 t - 10\,\mathrm{m2}\, s^4 t + 27\,\mathrm{m2}^4 t^2 + 162\,\mathrm{m2}^3 s\, t^2 + 45\,\mathrm{m2}^2 s^2 t^2 - 20\,\mathrm{m2}\, s^3 t^2 - s^4 t^2 + 54\,\mathrm{m2}^2 s\, t^3 - 6\,\mathrm{m2}\, s^2 t^3 - 2 s^3 t^3 + 4\,\mathrm{m2}\, s\, t^4 - s^2 t^4,$$
$$27\,\mathrm{m2}^4 s^2 - 54\,\mathrm{m2}^4 s\, t + 162\,\mathrm{m2}^3 s^2 t - 54\,\mathrm{m2}^2 s^3 t + 4\,\mathrm{m2}\, s^4 t + 27\,\mathrm{m2}^4 t^2 + 162\,\mathrm{m2}^3 s\, t^2 - 117\,\mathrm{m2}^2 s^2 t^2 + 22\,\mathrm{m2}\, s^3 t^2 - s^4 t^2 - 54\,\mathrm{m2}^2 s\, t^3 + 22\,\mathrm{m2}\, s^2 t^3 - 2 s^3 t^3 + 4\,\mathrm{m2}\, s\, t^4 - s^2 t^4,$$
$$65\,536\,\mathrm{m2}^{12} + 270\,336\,\mathrm{m2}^{10} s^2 + 33\,024\,\mathrm{m2}^8 s^4 + 1024\,\mathrm{m2}^6 s^6 + 270\,336\,\mathrm{m2}^{10} s\, t - 458\,752\,\mathrm{m2}^9 s^2 t + 66\,048\,\mathrm{m2}^8 s^3 t - 1\,276\,416\,\mathrm{m2}^7 s^4 t + 3072\,\mathrm{m2}^6 s^5 t - 137\,472\,\mathrm{m2}^5 s^6 t -$$
$$4096\,\mathrm{m2}^3 s^8 t + 270\,336\,\mathrm{m2}^{10} t^2 - 458\,752\,\mathrm{m2}^9 s\, t^2 + 99\,072\,\mathrm{m2}^8 s^2 t^2 - 2\,552\,832\,\mathrm{m2}^7 s^3 t^2 - 3\,427\,584\,\mathrm{m2}^6 s^4 t^2 - 412\,416\,\mathrm{m2}^5 s^5 t^2 + 149\,472\,\mathrm{m2}^4 s^6 t^2 - 16\,384\,\mathrm{m2}^3 s^7 t^2 +$$
$$768\,\mathrm{m2}^2 s^8 t^2 + 66\,048\,\mathrm{m2}^8 s\, t^3 - 2\,552\,832\,\mathrm{m2}^7 s^2 t^3 - 6\,860\,288\,\mathrm{m2}^6 s^3 t^3 - 687\,360\,\mathrm{m2}^5 s^4 t^3 + 448\,416\,\mathrm{m2}^4 s^5 t^3 - 49\,888\,\mathrm{m2}^3 s^6 t^3 + 3072\,\mathrm{m2}^2 s^7 t^3 - 48\,\mathrm{m2}\, s^8 t^3 +$$
$$33\,024\,\mathrm{m2}^8 t^4 - 1\,276\,416\,\mathrm{m2}^7 s\, t^4 - 3\,427\,584\,\mathrm{m2}^6 s^2 t^4 - 687\,360\,\mathrm{m2}^5 s^3 t^4 + 597\,888\,\mathrm{m2}^4 s^4 t^4 - 92\,320\,\mathrm{m2}^3 s^5 t^4 + 6144\,\mathrm{m2}^2 s^6 t^4 - 192\,\mathrm{m2}\, s^7 t^4 + s^8 t^4 +$$
$$3072\,\mathrm{m2}^6 s\, t^5 - 412\,416\,\mathrm{m2}^5 s^2 t^5 + 448\,416\,\mathrm{m2}^4 s^3 t^5 - 92\,320\,\mathrm{m2}^3 s^4 t^5 + 7680\,\mathrm{m2}^2 s^5 t^5 - 336\,\mathrm{m2}\, s^6 t^5 + 4 s^7 t^5 + 1024\,\mathrm{m2}^6 t^6 - 137\,472\,\mathrm{m2}^5 s\, t^6 + 149\,472\,\mathrm{m2}^4 s^2 t^6 -$$
$$49\,888\,\mathrm{m2}^3 s^3 t^6 + 6144\,\mathrm{m2}^2 s^4 t^6 - 336\,\mathrm{m2}\, s^5 t^6 + 6 s^6 t^6 - 16\,384\,\mathrm{m2}^3 s^2 t^7 + 3072\,\mathrm{m2}^2 s^3 t^7 - 192\,\mathrm{m2}\, s^4 t^7 + 4 s^5 t^7 - 4096\,\mathrm{m2}^3 s\, t^8 + 768\,\mathrm{m2}^2 s^2 t^8 - 48\,\mathrm{m2}\, s^3 t^8 + s^4 t^8\}$$

State of the Art Examples (2/2)

Non Planar Double Pentagon 2

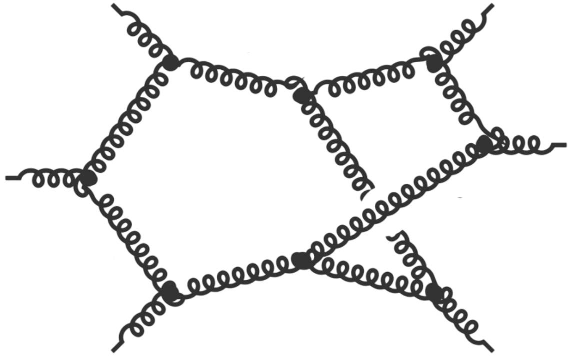


Also a horrendous integral: $\chi = 13$ in the top (max cut) sector

9 Mandelstam variables

State of the Art Examples (2/2)

Non Planar Double Pentagon 2



Also a horrendous integral: $\chi = 13$ in the top (max cut) sector

9 Mandelstam variables

Singularities in the top sector

One new very complicated letter:

$$s12^4\, s16^4\, s34^4 - 4\, s12^3\, s123\, s16^4\, s34^4 + 6\, s12^2\, s123^2\, s16^4\, s34^4 - 4\, s12\, s123^3\, s16^4\, s34^4 + s123^4\, s16^4\, s34^4 - 4\, s12^4\, s16^3\, s23\, s34^4 + 12\, s12^3\, s123\, s16^3\, s23\, s34^4 -$$
$$12\, s12^2\, s123^2\, s16^3\, s23\, s34^4 + 4\, s12\, s123^3\, s16^3\, s23\, s34^4 + 6\, s12^4\, s16^2\, s23^2\, s34^4 - 12\, s12^3\, s123\, s16^2\, s23^2\, s34^4 + 6\, s12^2\, s123^2\, s16^2\, s23^2\, s34^4 -$$
$$4\, s12^4\, s16\, s23^3\, s34^4 + 4\, s12^3\, s123\, s16\, s23^3\, s34^4 + s12^4\, s23^4\, s34^4 + 4\, s12^3\, s123\, s16^4\, s34^3\, s345 - 12\, s12^2\, s123^2\, s16^4\, s34^3\, s345 +$$
$$12\, s12\, s123^3\, s16^4\, s34^3\, s345 - 4\, s123^4\, s16^4\, s34^3\, s345 + 4\, s12^4\, s16^3\, s23\, s34^3\, s345 - 24\, s12^3\, s123\, s16^3\, s23\, s34^3\, s345 + 36\, s12^2\, s123^2\, s16^3\, s23\, s34^3\, s345 -$$
$$16\, s12\, s123^3\, s16^3\, s23\, s34^3\, s345 - 12\, s12^4\, s16^2\, s23^2\, s34^3\, s345 + 36\, s12^3\, s123\, s16^2\, s23^2\, s34^3\, s345 - 24\, s12^2\, s123^2\, s16^2\, s23^2\, s34^3\, s345 + \cdots 8173 \cdots +$$
$$s16^4\, s345^4\, s56^4 - 4\, s16^3\, s345^5\, s56^4 + 6\, s16^2\, s345^6\, s56^4 - 4\, s16\, s345^7\, s56^4 + s345^8\, s56^4 - 4\, s16^4\, s345^3\, s45\, s56^4 + 16\, s16^3\, s345^4\, s45\, s56^4 -$$
$$24\, s16^2\, s345^5\, s45\, s56^4 + 16\, s16\, s345^6\, s45\, s56^4 - 4\, s345^7\, s45\, s56^4 + 6\, s16^4\, s345^2\, s45^2\, s56^4 - 24\, s16^3\, s345^3\, s45^2\, s56^4 + 36\, s16^2\, s345^4\, s45^2\, s56^4 -$$
$$24\, s16\, s345^5\, s45^2\, s56^4 + 6\, s345^6\, s45^2\, s56^4 - 4\, s16^4\, s345\, s45^3\, s56^4 + 16\, s16^3\, s345^2\, s45^3\, s56^4 - 24\, s16^2\, s345^3\, s45^3\, s56^4 + 16\, s16\, s345^4\, s45^3\, s56^4 -$$
$$4\, s345^5\, s45^3\, s56^4 + s16^4\, s45^4\, s56^4 - 4\, s16^3\, s345\, s45^4\, s56^4 + 6\, s16^2\, s345^2\, s45^4\, s56^4 - 4\, s16\, s345^3\, s45^4\, s56^4 + s345^4\, s45^4\, s56^4$$

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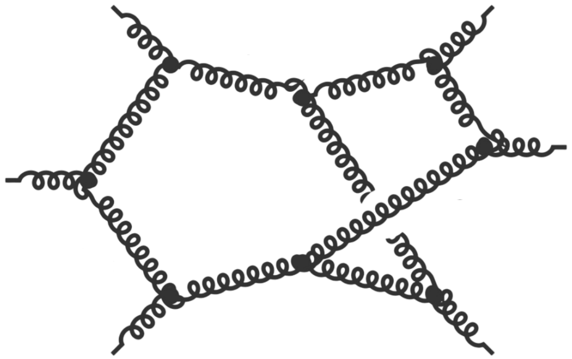
State of the Art Examples (2/2)

Non Planar Double Pentagon 2



Also a horrendous integral: $\chi = 13$ in the top (max cut) sector

9 Mandelstam variables

Singularities in the top sector

One new very complicated letter:

$$s12^4\, s16^4\, s34^4 - 4\, s12^3\, s123\, s16^4\, s34^4 + 6\, s12^2\, s123^2\, s16^4\, s34^4 - 4\, s12\, s123^3\, s16^4\, s34^4 + s123^4\, s16^4\, s34^4 - 4\, s12^4\, s16^3\, s23\, s34^4 + 12\, s12^3\, s123\, s16^3\, s23\, s34^4 -$$
$$12\, s12^2\, s123^2\, s16^3\, s23\, s34^4 + 4\, s12\, s123^3\, s16^3\, s23\, s34^4 + 6\, s12^4\, s16^2\, s23^2\, s34^4 - 12\, s12^3\, s123\, s16^2\, s23^2\, s34^4 + 6\, s12^2\, s123^2\, s16^2\, s23^2\, s34^4 -$$
$$4\, s12^4\, s16\, s23^3\, s34^4 + 4\, s12^3\, s123\, s16\, s23^3\, s34^4 + s12^4\, s23^4\, s34^4 + 4\, s12^3\, s123\, s16^4\, s34^3\, s345 - 12\, s12^2\, s123^2\, s16^4\, s34^3\, s345 +$$
$$12\, s12\, s123^3\, s16^4\, s34^3\, s345 - 4\, s123^4\, s16^4\, s34^3\, s345 + 4\, s12^4\, s16^3\, s23\, s34^3\, s345 - 24\, s12^3\, s123\, s16^3\, s23\, s34^3\, s345 + 36\, s12^2\, s123^2\, s16^3\, s23\, s34^3\, s345 -$$
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5543 terms

degree 12

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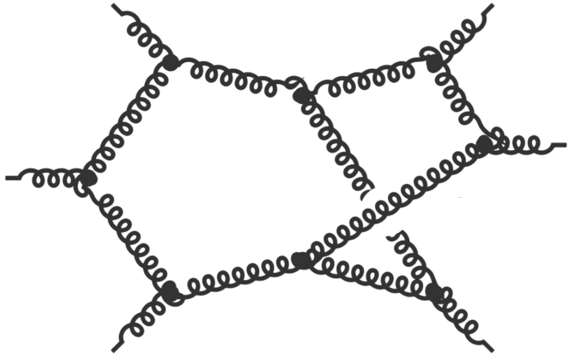

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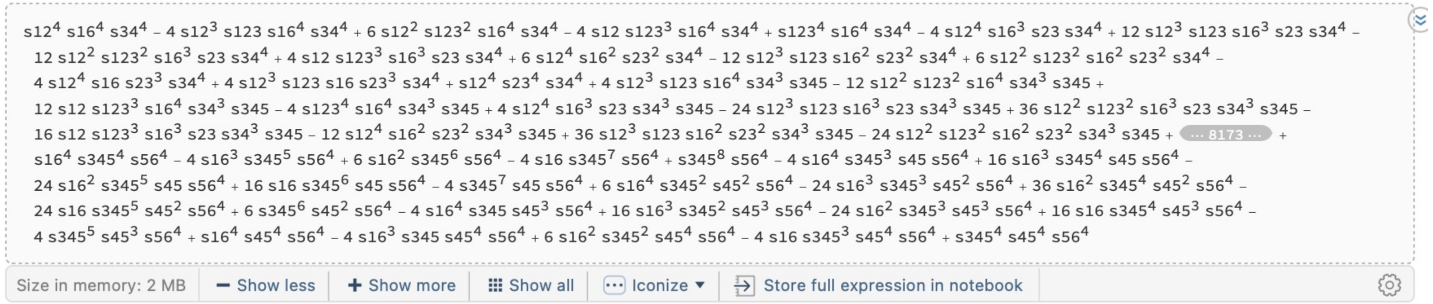

5543 terms

degree 12

Also find one (smaller) letter that PLD found but fit incorrectly

Many other new singularities in the subsectors

Does this Method Find all the Singularities?

Short answer

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[Lee, Pomeransky, 2013]

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For certain with this work: a proof-of-concept set of functions that can already push the state of the art in multiple examples (such as the non-planar envelope)

Thank you for listening!