Of Schottky and Jacobi

Based on arXiv:2602.04715 with Johannes Broedel
my Master's thesis at ETH Zürich

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Max Planck Institute for Physics

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Motivation: Higher-genus endeavors

Special functions on higher genus Riemann surfaces are useful to describe observables in QFT and string theory.

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- Riemann theta function, Weierstrass $\wp$, ...

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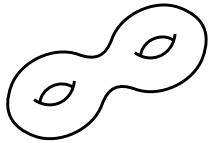
Understand, compare and translate the auxiliary symmetries in the different descriptions of higher-genus surfaces.

Review: Jacobi and Schottky languages

Compact genus-g Riemann surface $\Sigma \cong$  yields a period matrix τ.

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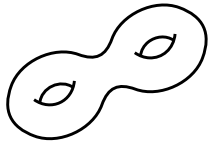
Compact genus-g Riemann surface $\Sigma \cong$  yields a period matrix τ.

Jacobi

Lattice $\Lambda_\tau = \mathbb{Z}^g + \tau\mathbb{Z}^g$,

$$\Sigma \hookrightarrow \mathrm{Jac}(\Sigma) := \mathbb{C}^g/\Lambda_\tau.$$

Freedom: $\mathrm{Sp}(2g, \mathbb{Z})$ action $\tau \mapsto M \cdot \tau$, sympletic transformations.

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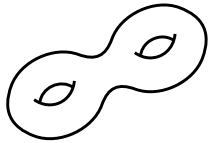
Compact genus-g Riemann surface $\Sigma \cong$  yields a period matrix τ.

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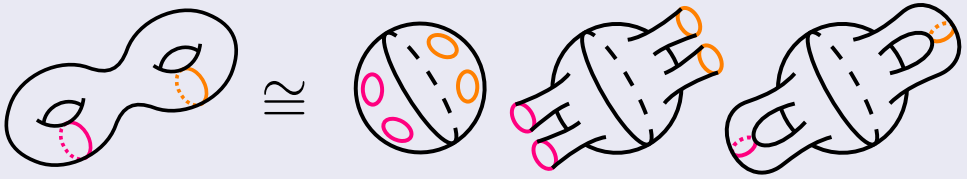
Schottky

Schottky group $G \cong F_g$, generated by Möbius transformations $\gamma_1, \ldots, \gamma_g$.

$$\Sigma \cong \Omega(G)/G,$$



Freedom: $\mathrm{Aut}(F_g)$ action $\vec{\gamma} \mapsto \alpha(\vec{\gamma})$, changes of marking.

Translating the languages

Decorated Schottky

Additional parameter, "decoration" $S \in \mathbb{Z}_{\mathrm{sym}}^{g\times g}$, fixes B-cycles up to reference (given by log branch choice).

$$\text{Decorated Schottky group:} \quad (S, \vec{\gamma}).$$

Freedom: $R_g := \mathbb{Z}_{\mathrm{sym}}^{g\times g} \rtimes \mathrm{Aut}(F_g)$ action $(S, \vec{\gamma}) \mapsto r \diamond (S, \gamma)$.

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Translating the languages and their freedom

Period matrix from Schottky data:

$$\text{Jacobi} \quad \tau = \tau(S, \vec{\gamma}) \quad \text{Schottky.}$$

Translation of the freedoms via homomorphism

$$\Phi_g \colon R_g \to \mathrm{Sp}(2g, \mathbb{Z})$$

such that

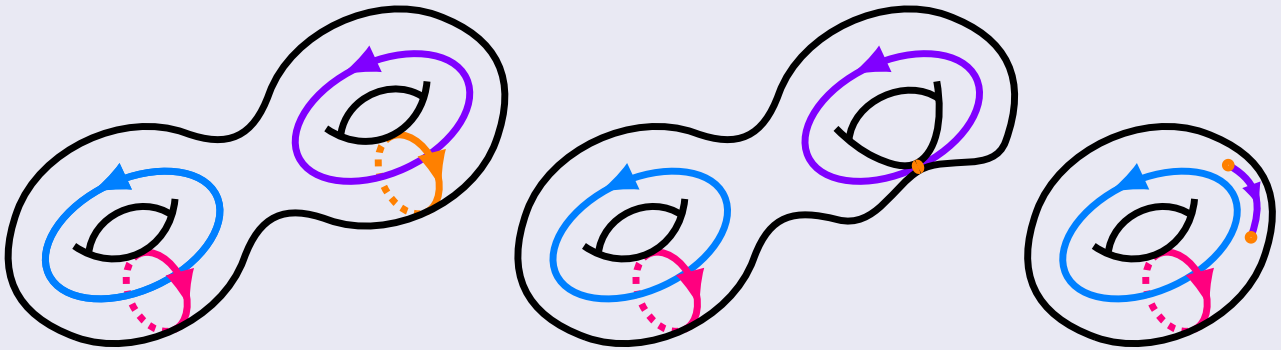
$$\tau(r \diamond (S, \vec{\gamma})) = \Phi_g(r) \cdot \tau(S, \vec{\gamma}) + B_{\mathrm{def}}.$$

"Defect" $B_{\mathrm{def}}(r, \vec{\gamma}) \in \mathbb{Z}_{\mathrm{sym}}^{g\times g}$ takes care of log branch choice.

A-cycle degeneration

A-cycle degeneration

Easy to describe in Schottky language:



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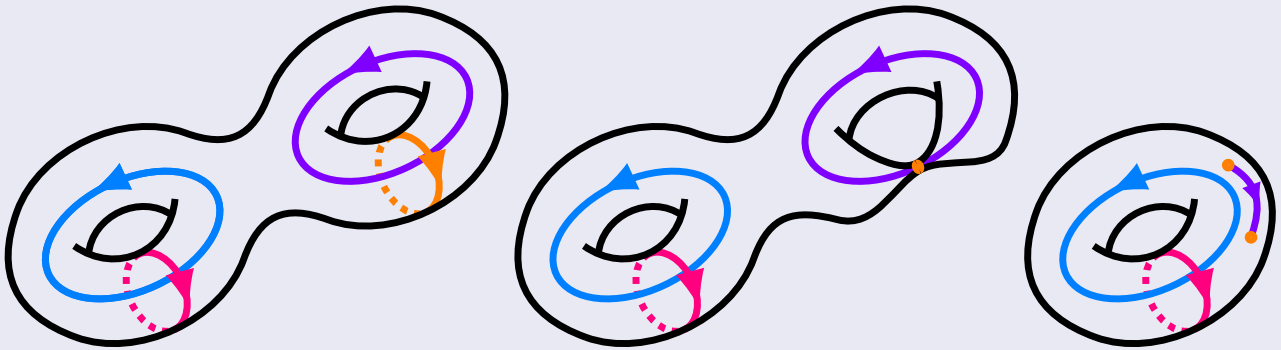

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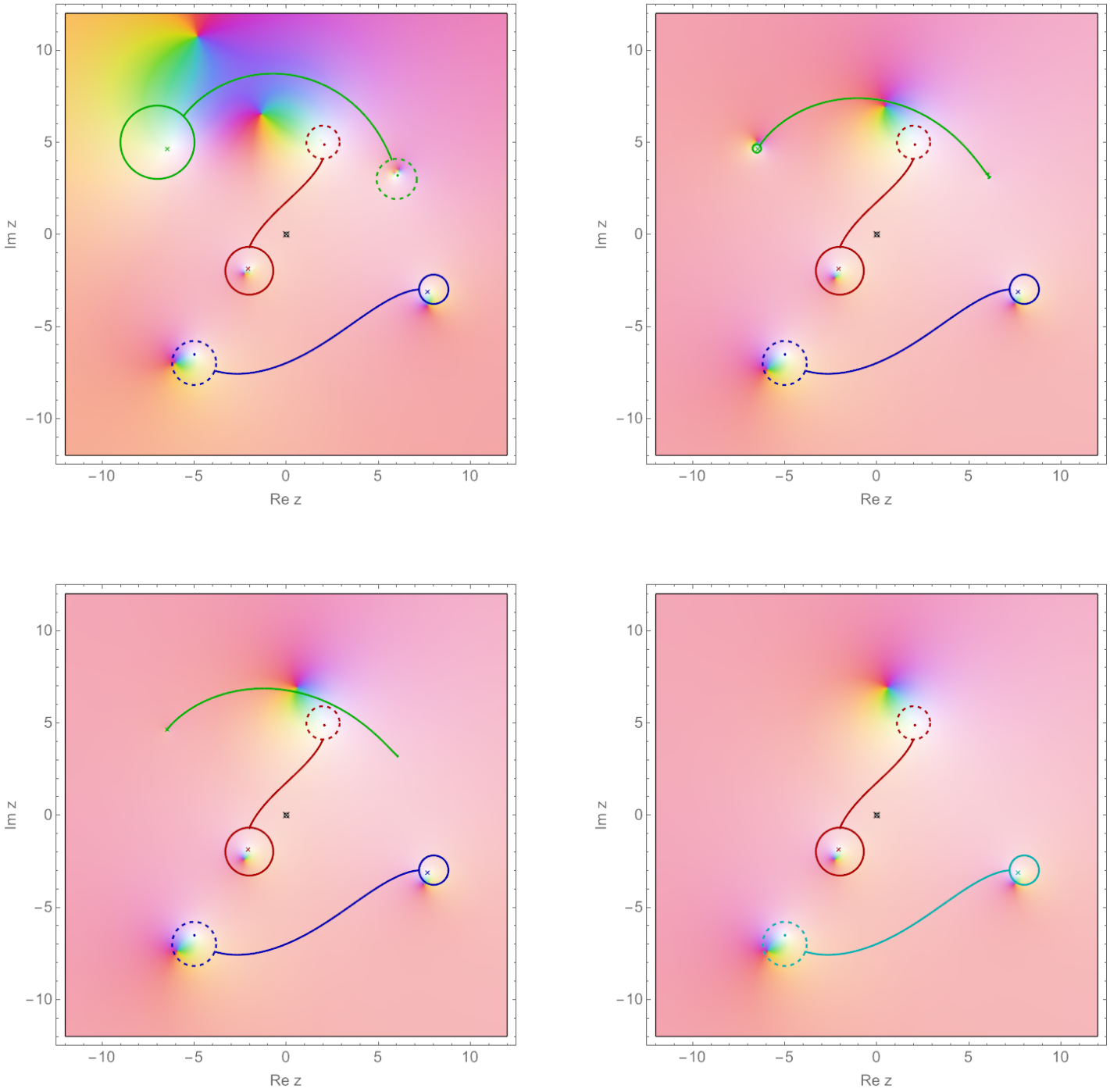
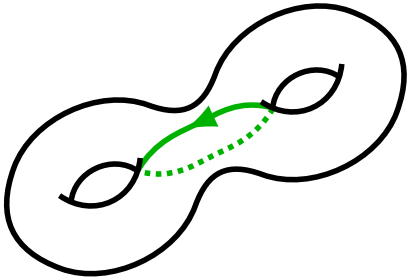
For the period matrix:

$$\tau(\vec{\gamma})^{\hat{k}} \longrightarrow \tau\left(\vec{\gamma}^{\hat{k}}\right) = \tau_{\text{degen}}, \quad \tau(\vec{\gamma})_{kk} \longrightarrow \mathrm{i}\infty.$$


Im z
Re z
10
5
0
−5
−10

Application: More general degenerations

What about other cycles?



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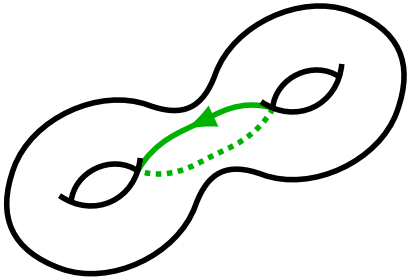

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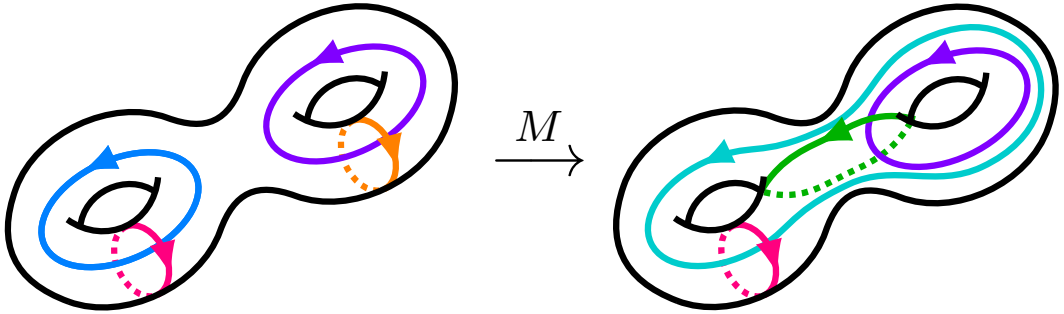

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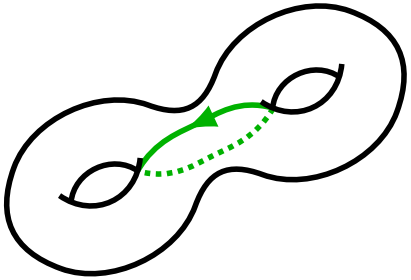

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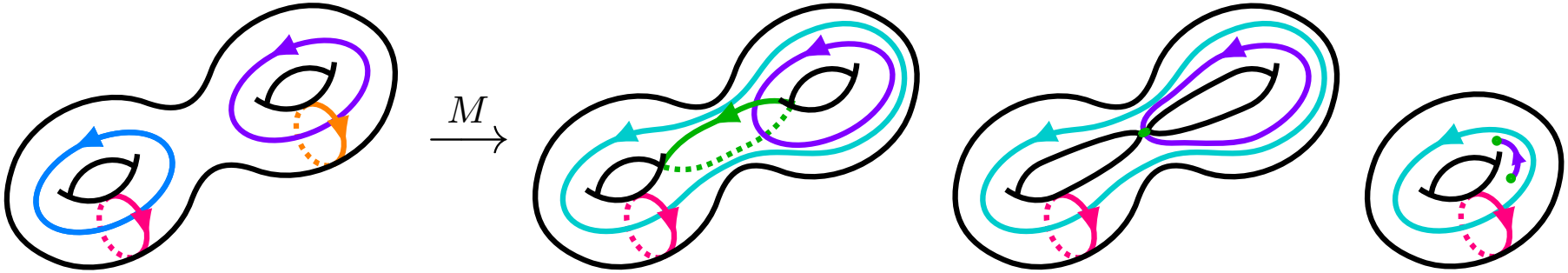

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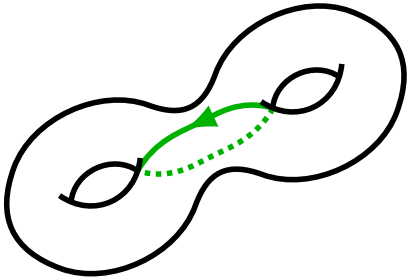

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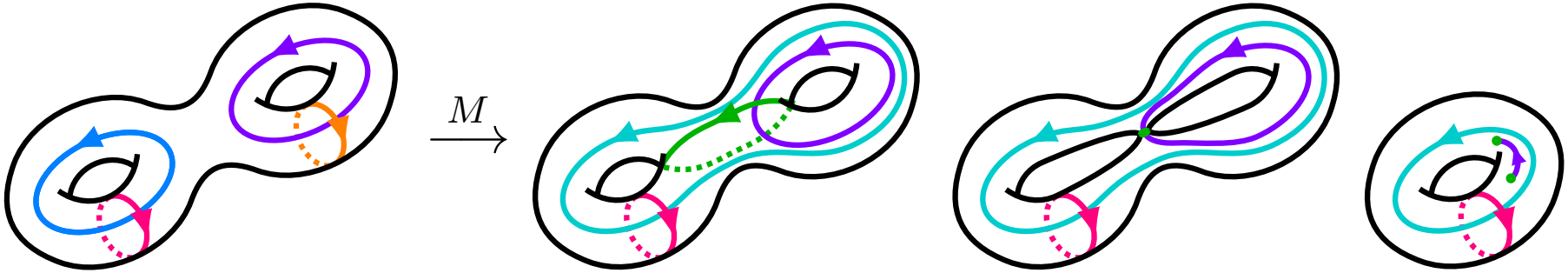

If $\exists r \in R_g \colon M = \Phi_g(r)$, then

$$\tau_{\text{degen}} = \tau\left(\left(r \diamond (S, \vec{\gamma})\right)^{\hat{k}}\right).$$

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Outlook and open questions

- Proof of main statement requires numerical checks.
- Φ_g is not surjective; i.e. the translation is not yet complete.
- Other languages for Riemann surfaces such as Fuchsian models and algebraic curves.
- Possible application: Tune Schottky covers to speed up numerical computations.

Appendix: Additional details

- Symplectic group action. For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2g, \mathbb{Z})$:

$$M \cdot \tau := (A\tau + B)(C\tau + D)^{-1}. \tag{1}$$

- Schottky period matrix formula & decorated version:

$$\tau(\vec{\gamma})_{ii} := \frac{1}{2\pi \mathrm{i}} \ln \lambda_i + \frac{1}{2\pi \mathrm{i}} \sum_{\substack{\gamma \in G_i \backslash G / G_i \\ \gamma \neq \mathrm{id}}} \ln\left\{P_i', \gamma P_i', P_i, \gamma P_i\right\}, \tag{2a}$$

$$\tau(\vec{\gamma})_{ij} := \frac{1}{2\pi \mathrm{i}} \sum_{\gamma \in G_j \backslash G / G_i} \ln\left\{P_j', \gamma P_i', P_j, \gamma P_i\right\}, \quad i \neq j, \tag{2b}$$

$$\tau(S, \vec{\gamma}) := S + \tau(\vec{\gamma}), \tag{2c}$$

with λ_i the multiplier, and P_i, P_i' the attractive and repulsive fixed points of γ_i.

- Construction of Φ_g. For $r = [B, \alpha] \in R_g$:

$$\Phi_g(r) := \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \Psi_g(\alpha), \tag{3a}$$

with Ψ_g constructed via

$$\begin{tikzcd}
F_g \arrow[d, two heads, "\mathrm{ab}"'] & \mathrm{Aut}(F_g) \arrow[r, hook] \arrow[d, two heads, "\psi_g"] \arrow[dr, "\Psi_g"] & R_g \arrow[dl, "\Phi_g"] \\
\mathbb{Z}^g & \mathrm{Aut}(\mathbb{Z}^g) \arrow[ur, hook, "A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{-\mathsf{T}} \end{pmatrix}"'] & \mathrm{Sp}(2g, \mathbb{Z})
\end{tikzcd} \tag{3b}$$

- R_g group action:

$$[B, \alpha] \diamond (S, \vec{\gamma}) := ({}^{\alpha}S + B, \alpha(\vec{\gamma})), \tag{4a}$$

$${}^{\alpha}S := \psi_g(\alpha) S \psi_g(\alpha)^{\mathsf{T}}. \tag{4b}$$

- Defect (for standard log branch):

$$B_{\mathrm{def}}(\alpha, \vec{\gamma}) := \tau(\alpha(\vec{\gamma})) - \Psi_g(\alpha) \cdot \tau(\vec{\gamma}) \tag{5a}$$

$$= -\operatorname{rhd} \operatorname{Re}(\Psi_g(\alpha) \cdot \tau(\vec{\gamma})). \tag{5b}$$