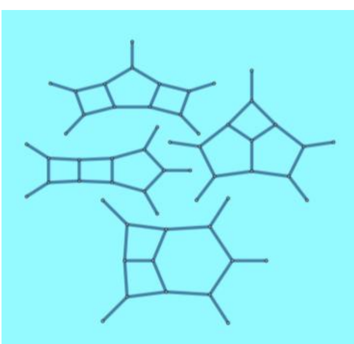
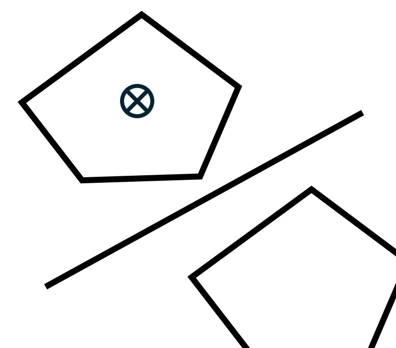
Pentagonal Feynman Integrals and Wilson Loop with Lagrangian Insertion at Three-Loop

Base on:
Dmitry Chicherin, Johannes Henn, Yongqun Xu, Shun-Qing Zhang, Yang Zhang [2512.17881]
Dmitry Chicherin, Yu Wu, Zihao Wu, Yongqun Xu, Shun-Qing Zhang, Yang Zhang [2512.17330]
and
Yuanche Liu, Antonela Matijašić, Julian Miczajka, Yingxuan Xu, Yongqun Xu, Yang Zhang [2411.18697]
(Phys. Rev. D 112, 016021)

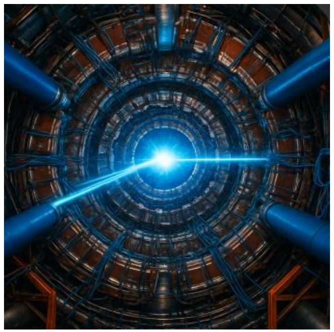


Yongqun Xu
University of Science and Technology of China

IMPRS Recruitment Workshop
April 13, 2026



Background: Feynman Integrals as the basic building block of perturbative Quantum Field Theory

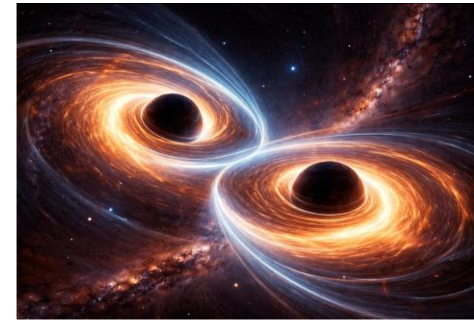


Collider Phenomenology

Scattering Amplitudes
Precision Physics

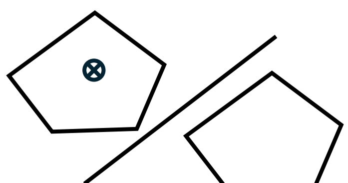
Future Collider:
CEPC, FCC,
HL-LHC…

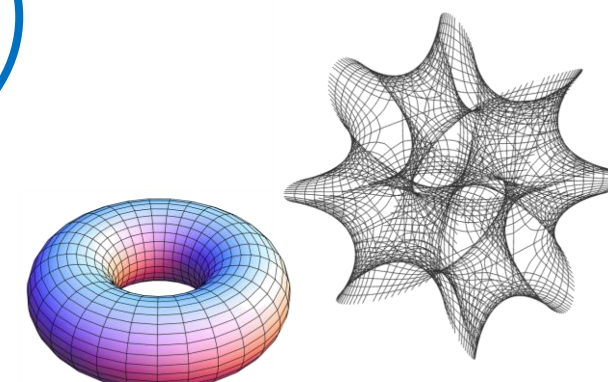
Feynman Integrals



Gravitational Wave Physics

Understanding Structure of QFT

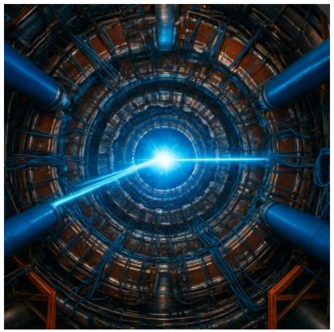




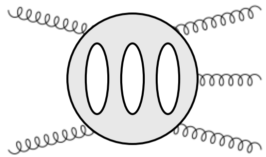
Mathematical Interest

[Henn'13] [Arkani-Hamed, Henn, Trnka '21] [Bree, Gasparotto, Matijašić, Mazloumi, Melnichenko, Pögel, Teschke, Wang, Weinzierl, Wu, Xu '25] [Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch '24] [Gehrmann, Henn, Lo Presti '18] [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18] [Abreu, Page, Zeng '18] and many more..

Motivation: Planar massless Three-Loop Five-Point Feynman Integrals In dimensional regularization



Collider Phenomenology



$\mathcal{A}_5^{(3)}(1,2,3,4,5)$

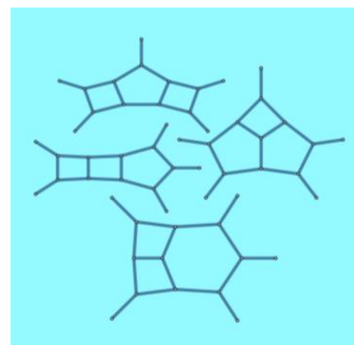
Future Collider:
CEPC, FCC,
HL-LHC…

Ingredients for Scattering Amplitudes with three massless final states

$pp \to \gamma\gamma\gamma \qquad pp \to \gamma jj$

$pp \to \gamma\gamma j \qquad pp \to jjj$

@NNLO

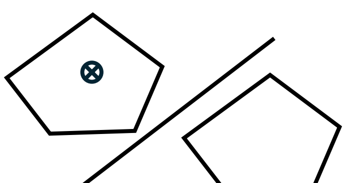


[Liu, Matijašić, Miczajka, Xu, YX, Zhang '24]
[Chicherin, Wu, Wu, YX, Zhang, Zhang '25]

Prelude of various masses configuration e.g.

$pp \to t\bar{b}j \qquad pp \to t\bar{t}j$

Understanding Structure of QFT



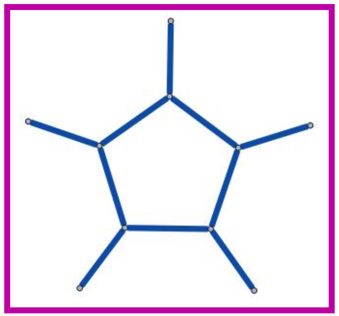
null Wilson loop with a Lagrangian insertion
(closely related to scattering amplitude)

[Chicherin, Henn, YX, Zhang, Zhang '25]

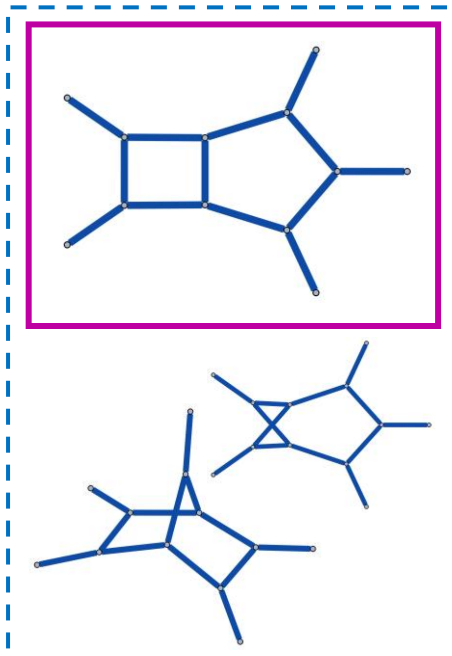
History: Pentagon Feynman Integrals

Massless five-point integrals in dimensional regularization

Planar Part



[Bern, Dixon, Kosower ' 92]
[A. Kniehl, V. Tarasov '10]
[Gehrmann, Henn, Lo Presti '18]
[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]
[Abreu, Page, Zeng '18]
and so on…



Method of
Canonical Differential Equation
[Henn'13]

Function Space

[Gehrmann, Henn, Lo Presti '18] [Chicherin, Sotnikov '20]

Scattering Amplitudes

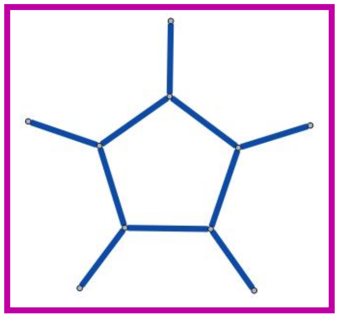
[Abreu, Page, Pascual, Sotnikov '20]
[Chawdhry, Czakon, Mitov, Poncelet '21]
[Abreu, Febres-Cordero, Ita, Page, Sotnikov '21]
[Agarwal, Buccioni, von Manteuffel, Tancredi '21] $\times 2$
[A. Chawdhry, Czakon, Mitov, Poncelet '21]
[Badger, Brønnum-Hansen, Chicherin, Gehrmann, Bayu Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

Cross Sections

[Kallweit, Sotnikov, Wiesemann '20]
[Chawdhry, Czakon, Mitov, Poncelet '20]
[Chawdhry, Czakon, Mitov, Poncelet '21]
[Badger, Gehrmann, Marcoli, Moodie '21]
[Czakon, Mitov, Poncelet '21]
[Badger, Czakon, Bayu Hartanto, Moodie, Peraro, Poncelet, Zoia '21]
[Chen, Gehrmann, Glover, Huss, Marcoli '22]

Pentagon Feynman Integrals at Three-Loop

Planar Part



[Bern, Dixon, Kosower ' 92]
[A. Kniehl, V. Tarasov '10]
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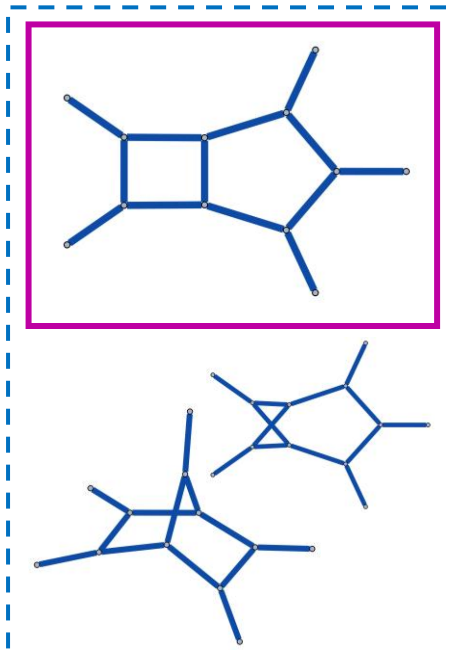
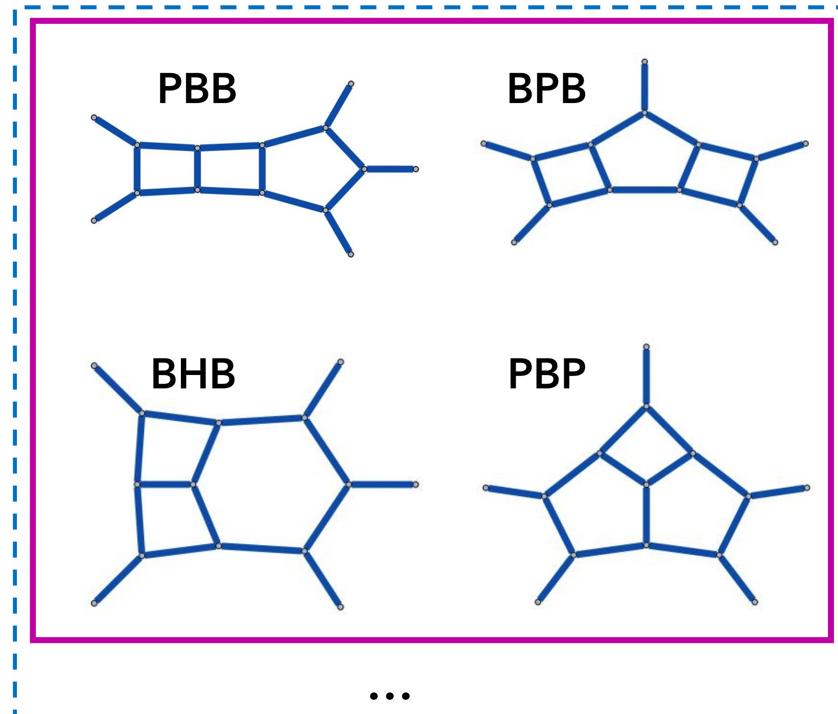
[Abreu, Page, Zeng '18]
and so on…



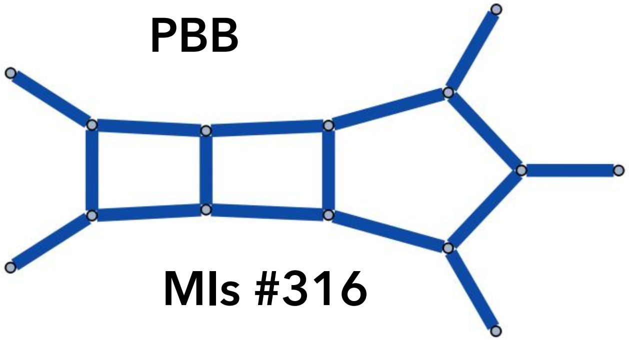
Method of
Canonical Differential Equation
[Henn'13]



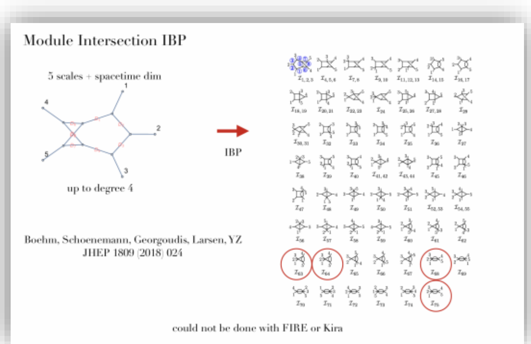
[Liu, Matijašić, Miczajka, Xu, YX, Zhang '24]
[Chicherin, Wu, Wu, YX, Zhang, Zhang '25]

Integration-By-Parts (IBP) Reduction

The current bottleneck

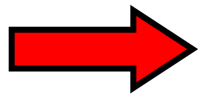


Picture created by Azurite 1.1.0

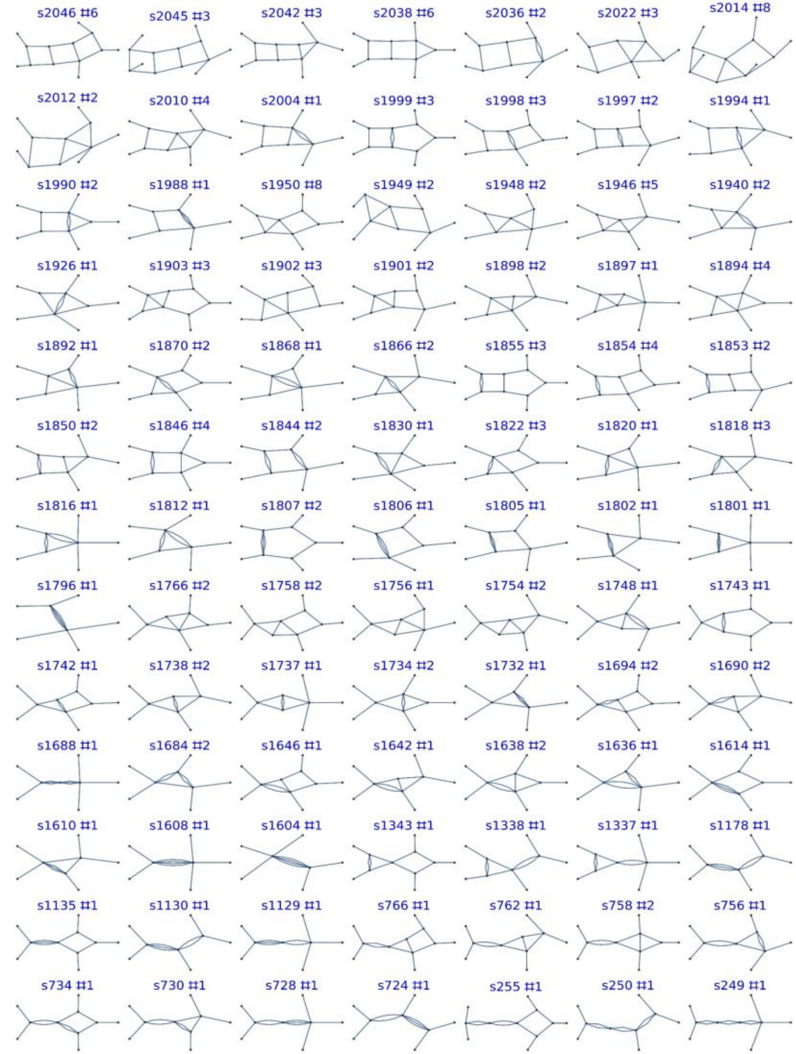


Similar Stories at Two Loop
Picture from Yang Zhang

NeatIBP

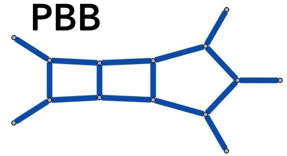
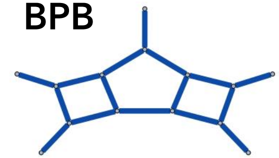
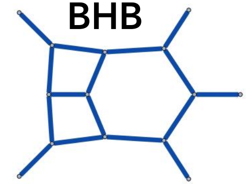
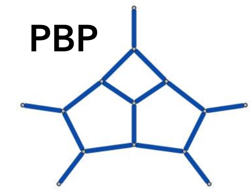


~85,000 IBPs



Could not be done with FIRE or Kira

Algebraic Geometry Accelerating IBP

Family	PBB 	BPB 	BHB 	PBP 
Master Integrals	316	367	431	734
Five Point Integrals	120	164	169	342
# IBPs	~85,000	~183,000	Spanning Cuts	

PRD Editor's Suggestion
[Liu, Matijašić, Miczajka, Xu, Xu, Zhang '24]

NEATIBP

Powered by 1.1.0.3

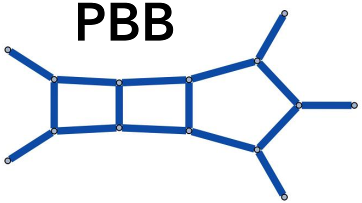
[Wu, Böhm, Ma, Xu, Zhang '23]
[Wu, Böhm, Ma, Usovitsch, Xu, Zhang '25]

Loop ++
Leg ++
Scale ++

Pure Integrals: Leading Singularities Analysis

Sample expressions..

PBB

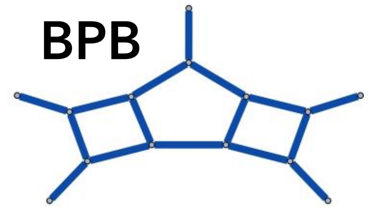


$$\mathcal{N}_1 = s_{12}s_{23}s_{45}^2(\ell_1+k_5)^2$$

$$\mathcal{N}_3 = \frac{s_{45}^2}{\epsilon_5}G(\ell_1,k_1,k_2,k_3,k_4)$$

$$\mathcal{N}_4 = \frac{s_{45}^2}{\epsilon_5}G\begin{pmatrix}\ell_1,k_1,k_2,k_3,k_4\\ \ell_2,k_1,k_2,k_3,k_4\end{pmatrix}$$

BPB

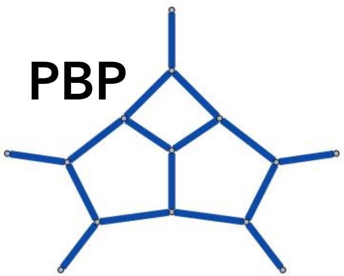


$$\mathcal{N}_1 = s_{12}s_{34}s_{45}^2(\ell_2-k_1)^2$$

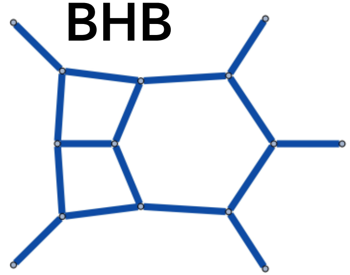
$$\mathcal{N}_5 = \frac{s_{45}s_{12}}{\epsilon_5}G(\ell_2,k_1,k_2,k_3,k_4)$$

$$\mathcal{N}_6 = \frac{s_{45}s_{12}}{\epsilon_5}G\begin{pmatrix}\ell_1,k_1,k_2,k_3,k_4\\ \ell_2,k_1,k_2,k_3,k_4\end{pmatrix}$$

PBP



BHB



$$\begin{aligned}\mathcal{N}_3 = &-2\frac{s_{12}s_{15}s_{23}s_{34}s_{45}}{\epsilon_5}s_{15}\ell_2^2\\ &+\frac{s_{23}s_{34}\left(s_{12}s_{15}+s_{45}s_{15}-s_{12}s_{23}+s_{23}s_{34}-s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^4\\ &+\frac{s_{34}s_{45}\left(s_{12}s_{15}-s_{45}s_{15}+s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2-k_1\right)^2\\ &-\frac{s_{15}s_{45}\left(s_{12}s_{15}-s_{45}s_{15}-s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2-k_1-k_2\right)^2\\ &+\frac{s_{12}s_{15}\left(s_{12}s_{15}-s_{45}s_{15}-s_{12}s_{23}+s_{23}s_{34}+s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2-k_1-k_2-k_3\right)^2\\ &-\frac{s_{12}s_{23}\left(s_{12}s_{15}-s_{45}s_{15}-s_{12}s_{23}+s_{23}s_{34}-s_{34}s_{45}\right)}{\epsilon_5}s_{15}\ell_2^2\left(\ell_2-k_1-k_2-k_3-k_4\right)^2\end{aligned}$$

$$\mathcal{N}_6 = \frac{\varepsilon}{1+2\varepsilon}\frac{s_{12}-s_{45}}{\epsilon_5}G\begin{pmatrix}\ell_1,\ell_2,k_1,k_2,k_3,k_4\\ \ell_1,\ell_2,k_1,k_2,k_3,k_4\end{pmatrix}$$

← By leading singularities analysis in d-dimensional Baikov Rep.

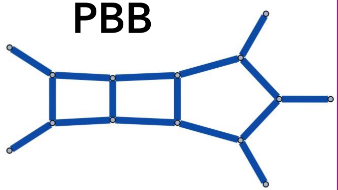
3L4P+1 off-shell leg from
[Di Vita, and Mastrolia, and Schubert, and Yundin '14]
[Henn, Lim, Bobadila'23]
DlogBasis [Henn, Mistlberger, Smirnov, Wasser '20]

Many thanks to my incredible collaborators!

Canonical Differential Equation & Alphabet

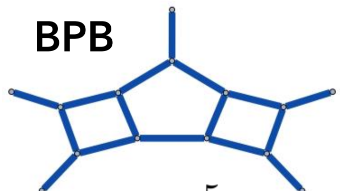
$$\mathrm{d}\mathbf{J} = \varepsilon \sum_{i=1}^{56} \mathbf{a}_i \,\mathrm{dlog} W_i \cdot \mathbf{J}$$

PBB



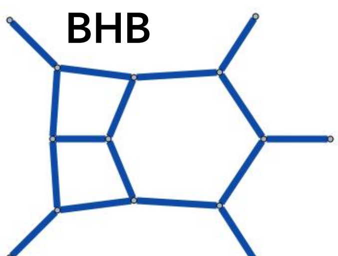
$\mathbb{A}_{\text{pl}}^{\text{2-loop}}$

BPB

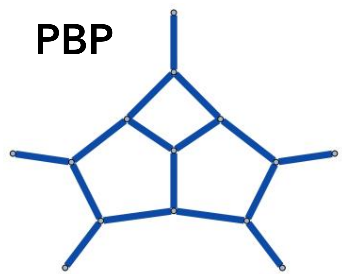


$\left\{\widehat{W}_i\right\}_{i=1}^{5}$

BHB



PBP



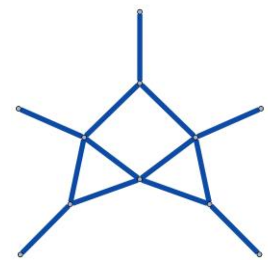
$\left\{\widetilde{W}_i\right\}_{i=16}^{20} \cup \left\{\widetilde{W}_i\right\}_{i=41}^{55} \cup \left\{\widetilde{W}_i\right\}_{i=76}^{80}$

56 letters = 26 at two-loop +30 more at three-loop

Letters: DNA of Feynman integrals

Conjectured in [Chicherin, Henn, Trnka, Zhang '24]

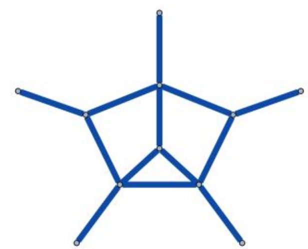
Originate from two topologies



$$\left\{\widehat{W}_i\right\}_{i=1}^{5}$$

$$\widehat{W}_1 = s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{15}$$

New Square Root!



$$\left\{\sqrt{\Delta_4^{(i)}}\right\}_{i=1}^{5}$$

$$\Delta_4^{(1)} = s_{15}^2 s_{12}^2 + s_{23}^2 s_{12}^2 - 2 s_{15} s_{23} s_{12}^2 - 2 s_{23}^2 s_{34} s_{12} + 2 s_{15} s_{23} s_{34} s_{12} + 2 s_{15} s_{34} s_{45} s_{12} + 2 s_{23} s_{34} s_{45} s_{12} + s_{23}^2 s_{34}^2 + s_{34}^2 s_{45}^2 - 2 s_{23} s_{34}^2 s_{45}$$

found by `Baikovletter`

[Jiang, Liu, Xu, Yang '24]

- Analytic boundary values
- Fast numerical evaluation [Hidding '20] DiffExp
- Symbol solutions up to weight-six

$$W_{i_1} \otimes \ldots \otimes W_{i_n} = \int W_{i_n}(t_{i_n}) \int \cdots \int W_{i_2}(t_2) \int W_{i_1}(t_1) \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_{i_n}$$

2220 weight-six symbol $\ll 56^6$ → Symbol Bootstrap

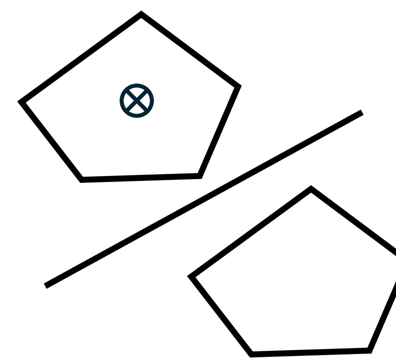
- ....many more discussion....

What can we do with these integrals?

Pentagonal Wilson Loop with Lagrangian Insertion at Three-Loop

$\mathcal{N}=4$ **Super Yang-Mills in planar limit**

$$F_5 = \frac{\langle W[x_1,x_2,x_3,x_4,x_5]\mathscr{L}(x_0)\rangle}{\langle W[x_1,x_2,x_3,x_4,x_5]\rangle} =$$



$$W[x_1,\cdots,x_n] = \mathrm{tr}_{\mathrm{F}}\mathrm{P}\exp\left(\mathrm{i}g_{\mathrm{YM}}t^a\oint A^a_\mu \mathrm{d}x^\mu\right)$$

n-cusp null Wilson loop with a Lagrangian insertion normalized by Wilson loop without Insertion

Theoretical Importance / Simpler Structure

closely related to scattering amplitudes

- Null Wilson Loop ~ amplitudes in N=4 SYM
- Natural in AdS/CFT
- Duality with all-plus pure Yang-Mills amplitudes
- Positivity and geometry
- …

[Alday, Maldacena '07] [Drummond, Korchemsky, Sokatchev '07]
[Arkani-Hamed, Henn, Trnka '21] [Chicherin, Henn '22] [Chicherin, Henn '22]
[Chicherin, Henn, Trnka, Zhang '24] [Brown, Henn, Mazzucchelli, Trnka '24] and so on.

👉See also: talk by Dmitry Chicherin at Amplitudes 2022, Prague

Simpler Structure

- Free of IR divergence
- Uniformly Transcendental of 2L
- Exact dual conformal symmetry
- All Loop Leading Singularity
- All Loop Integrand from Amplituhedron
- …

➡ **A good application of our Three-Loop Five-Point integrals**

Review of perturbative results

Wilson loop with one Lagrangian insertion at weak coupling $F_n = \sum_{L \geq 1} (g^2)^{L+1} F_n^{(L)}$

	Four-Point	Five-Point	Six-Point
Tree-Level	[Alday, Buchbinder, Tseytlin '11]	[Chicherin, Henn '22] n-point Tree-Level and One-Loop	
One-Loop	[Alday, Heslop, Sikorowski '12]		
Two-Loop	[Alday, Henn, Sikorowski '13]	[Chicherin, Henn '22]	[Carrôlo, Chicherin, Henn Yang, Zhang '25]
Three-Loop	[Henn, Korchemsky, Mistlberger ' 19]	[Chicherin, Henn, YX, Zhang, Zhang '25] $F_5^{(3)}$	

See also
[Alday, Buchbinder, Tseytlin '11] Four-Point @ Strong Coupling
[Arkani-Hamed, Henn, Trnka '21] Four-Point Negative geometries expansion

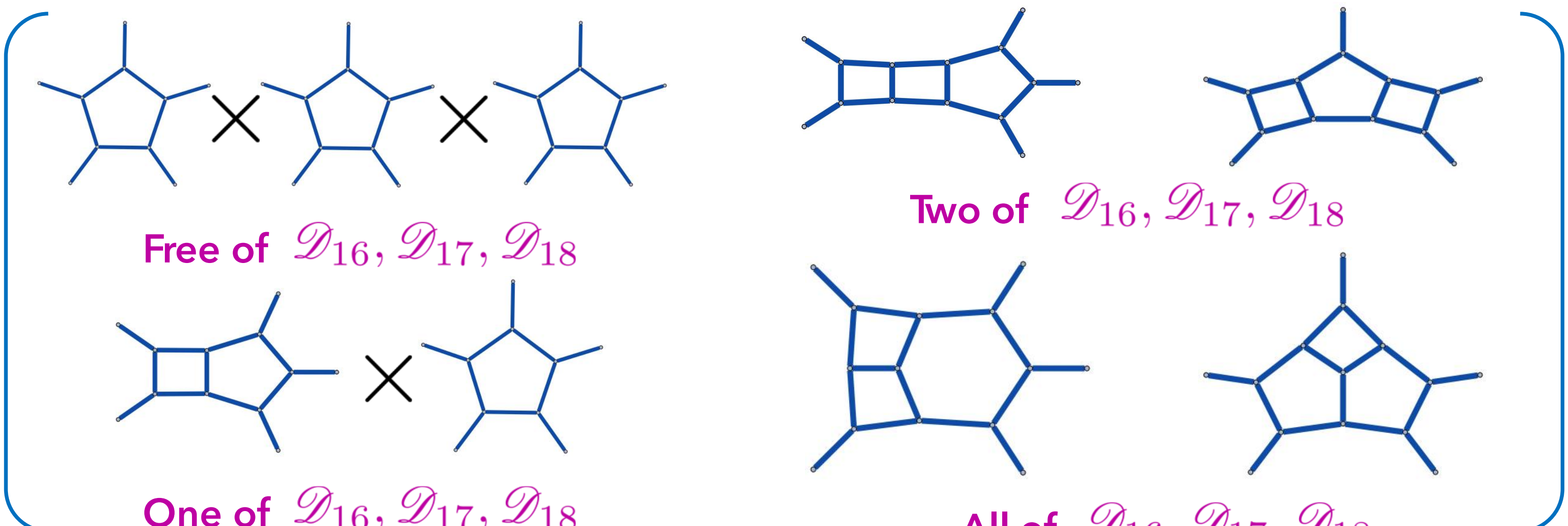
We are now Here

Direct Computation of $F_5^{(3)}$

Step1: Separating Targets from integrand

$\mathscr{D}_1 = (\ell_1 - k_1)^2$	$\mathscr{D}_6 = (\ell_2 - k_1)^2$	$\mathscr{D}_{11} = (\ell_3 - k_1)^2$
$\mathscr{D}_2 = (\ell_1 - k_1 - k_2)^2$	$\mathscr{D}_7 = (\ell_2 - k_1 - k_2)^2$	$\mathscr{D}_{12} = (\ell_3 - k_1 - k_2)^2$
$\mathscr{D}_3 = (\ell_1 - k_1 - k_2 - k_3)^2$	$\mathscr{D}_8 = (\ell_2 - k_1 - k_2 - k_3)^2$	$\mathscr{D}_{13} = (\ell_3 - k_1 - k_2 - k_3)^2$
$\mathscr{D}_4 = (\ell_1 - k_5)^2$	$\mathscr{D}_9 = (\ell_2 - k_5)^2$	$\mathscr{D}_{14} = (\ell_3 - k_5)^2$
$\mathscr{D}_5 = \ell_1^2$	$\mathscr{D}_{10} = \ell_2^2$	$\mathscr{D}_{15} = \ell_3^2$
$\mathscr{D}_{16} = (\ell_1 - \ell_2)^2$	$\mathscr{D}_{17} = (\ell_1 - \ell_3)^2$	$\mathscr{D}_{18} = (\ell_2 - \ell_3)^2$

Loop momenta coupling



with all cyclic permutation S_5

Direct Computation of $F_5^{(3)}$

Step 2: Reduce target integrals into pure basis by $\textsc{NeatIBP}$

Step 3: Assembling with numerator ➡ Results $F_5^{(3)} = r_0 g_0^{(3)} + \sum_{i=1}^{5} r_i g_i^{(3)}$

Leading Singularities:

$$r_0 = \mathrm{tr}_-((p_1+p_2)(p_2+p_3)(p_3+p_4)(p_4+p_5))$$

$$r_i = \frac{s_{i+2,i+3}}{s_{i+1,i+4}} \mathrm{tr}_-(p_{i+1}p_{i+2}p_{i+3}p_{i+4}), \quad i = 1 \cdots 5$$

Weight–Six Symbol:

$$g_i^{(3)} = \sum_{i_1,\ldots,i_6} t_{i_1,\ldots,i_6} W_{i_1} \otimes \ldots \otimes W_{i_6}$$

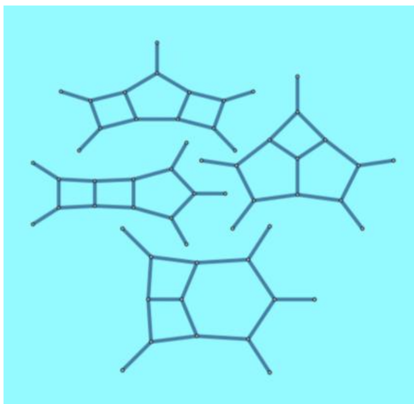
involves 1,373,140 terms…

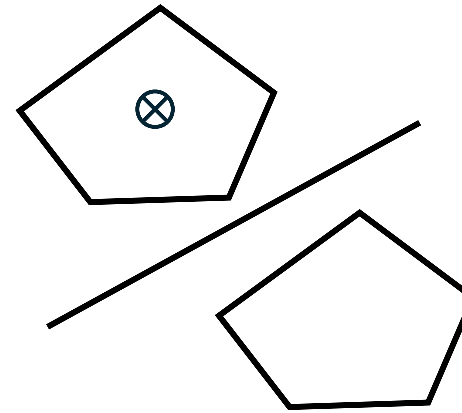
Intermediate expression involves 3,107,425 terms

- ☑ Consistent with Symbol bootstrap
- ☑ Infra Red finite: All epsilon poles vanished
- ☑ Respect soft and collinear limits
- ☑ Free of new square roots $\left\{\sqrt{\Delta_4^{(i)}}\right\}_{i=1}^{5}$
- ☑ Involves totally 30 letters
- ☑ …

[Chicherin, Henn '22] $\times 2$
[Wu, Böhm, Ma, Xu, Zhang '23]
[Chicherin, Henn, Trnka, Zhang '24]
[Brown, Henn, Mazzucchelli, Trnka '24]
[Wu, Böhm, Ma, Usovitsch, Xu, Zhang '25]

Conclusion/Take-home messages





Into the Future

Three-Loop Pentagon functions in Euclidean region?

Functions	Pentagon	PB	PBB	BPB	BHB	PBP
Weight–1	$f_{1,1}^{(i)}$					
Weight–2	$f_{2,1}^{(i)}$					
Weight–3	$f_{3,1}^{(i)}, f_{3,2}^{(i)}, f_{3,3}^{(i)}, f_{3,4}$					
Weight–4	$f_{4,1\ldots11}^{(i)}, f_{4,12}^{(i)}$			$\cup f_{4,13}^{(i)}$	$\cup f_{4,14}^{(i)}$	
Weight–5	$f_{5,0}, f_{5,1}^{(i)}$	$\cup f_{5,2\ldots26}^{(i)}$	$\cup f_{5,27\ldots43}^{(i)} \cup f_{5,44}$			
Weight–6	?	?	?			

Two-Loop Pentagon Functions (Euclidean) [Gehrmann, Henn, Lo Presti '18]

Novel pentagon functions at Three-Loop

Enabling function level Wilson Loop with Lagrangian insertion

$$F_5^{(3)}\Big|_{x_0} = 5982.5 r_0 + \sum_{i=1}^{5} 6039.9 r_i$$

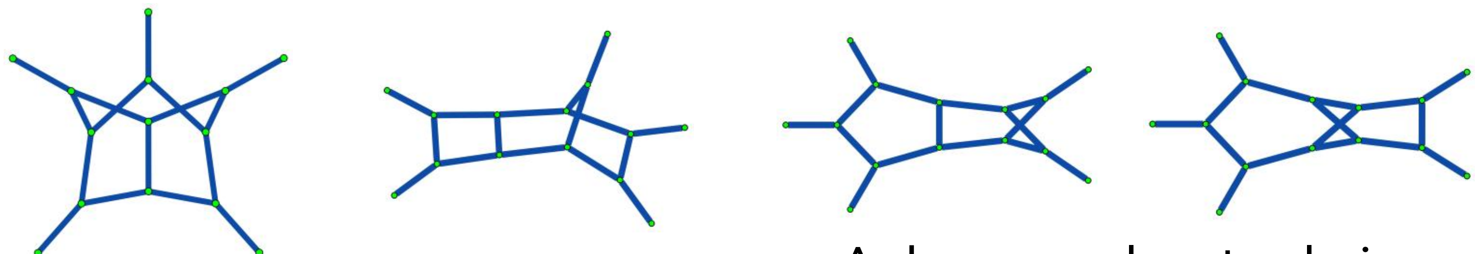
500+ digits

Positivity Properties ? $(-1)^{(L+1)} f_5^{(L)}|_{\text{Eucl}^+} > 0$

[Chicherin, Henn '22] [Chicherin, Henn, Trnka, Zhang '24]

Into the Future

- Toward Non-Planar Three-Loop Five-Point Feynman Integrals



And more non-planar topologies…

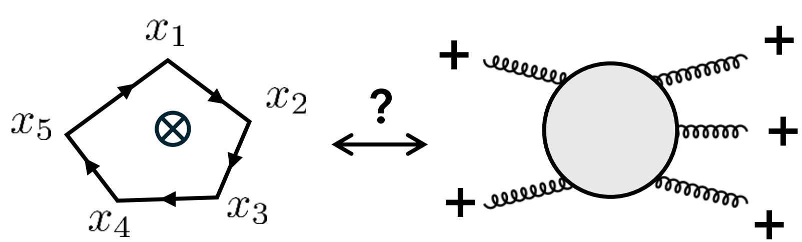
- Three-Loop Five-Point Scattering Amplitudes

e.g. 3L5P all plus amplitudes in Pure Yang-Mills, conjectured duality to $F_5^{(2)}$

[Chicherin, Henn '22] [Chicherin, Henn '22]

$$\langle W[x_1, x_2, x_3, x_4, x_5]\mathscr{L}(x_0)\rangle \quad x_0 \to \infty$$

N=4 Super Yang Mills



$$\mathcal{A}^{(3)}_{\mathrm{YM},5}(1^+, 2^+, 3^+, 4^+, 5^+)$$

All plus pure Yang Mills

- Many more other applications…

List of Publications

Authors are listed in alphabetical order

- **Complete computation of all three-loop five-point massless planar integrals** [2512.17330]

  Dmitry Chicherin, Yu Wu, Zihao Wu, Yongqun Xu, Shun-Qing Zhang, Yang Zhang

- **Three-loop pentagonal Wilson loop with Lagrangian insertion** [2512.17881]

  Dmitry Chicherin, Johannes Henn, Yongqun Xu, Shun-Qing Zhang, Yang Zhang

- **Analytic computation of three-loop five-point Feynman Integrals** [2411.18697]

  Yuanche Liu, Antonela Matijašić, Julian Miczajka, Yingxuan Xu, Yongqun Xu, Yang Zhang — Phys.Rev.D 112 (2025) 1, 016021

- **Two-Loop Spacelike Splitting Amplitude for N=4 Super-Yang-Mills Theory** [2406.14604]

  Johannes Henn, Rourou Ma, Yongqun Xu, Kai Yan, Yang Zhang, Hua Xing Zhu — Phys.Rev.D 112 (2025) 7, 076003

- **An explicit expression of generating function for one-loop tensor reduction** [2308.13336]

  Chang Hu, Tingfei Li, Jiyuan Shen, Yongqun Xu — JHEP 02 (2024) 158

Ongoing Works

- **Multi-Loop Negative Geometries** appear soon [260x.xxxx]

  Lance Dixon, Umut Oktem, Shruti Paranjape, Jaroslav Trnka, Yongqun Xu, Shun-Qing Zhang

- **A first look at the Non-Planar Three-Loop Five-Point Feynman Integrals** [260x.xxxx]

  ......

Thank You!

Backup

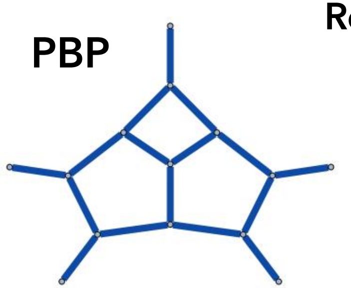
IBP Reduction and Differential Equation

Pentagon-Box-Pentagon

Optimized Differential Operator $$\frac{\partial}{\partial s_{ij}} = \beta^{k\mu}_{ij} \frac{\partial}{\partial p^{\mu}_{k}} \sim \frac{1}{D_i} + \text{Numerator}$$

[Abreu, Page, Zeng '18] $i \in \{1, 7, 9\}$

PBP



top sector MIs #19

Reduce ~ 6800 targets in ~ 2 days by Spanning Cuts with 4 $\times$ 50 CPU

Up to degree 4 / degree 3 + Dot 1

Semi-Analytical Reconstruction
Keep one variable at one time:

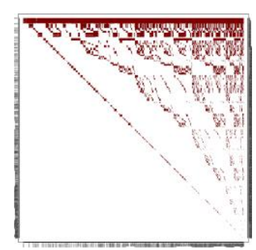
Four dimensionless variables: $$x_i = \left\{ \frac{s_{12}}{s_{45}}, \frac{s_{23}}{s_{45}}, \frac{s_{34}}{s_{45}}, \frac{s_{15}}{s_{45}} \right\}$$

734 master integrals

NeatIBP

Powered by 1.1.0.3

[Wu, Böhm, Ma, Xu, Zhang, '23]
[Wu, Böhm, Ma, Usovitsch, Xu, Zhang '25]



$$\left(n_1, \frac{s_{23}}{s_{45}}, n_3, n_5 \right)$$

734×734

Solving
Analytic Boundary Values
by imposing absence of spurious singularities to Canonical Differential Equation

One Trivial Integral

$$I_1(x_0) = -1 + \frac{\pi^2}{4}\varepsilon^2 + 7\zeta_3\varepsilon^3 + \frac{37\pi^4}{480}\varepsilon^4 + \left(\frac{93\zeta_5}{5} - \frac{7\pi^2\zeta_3}{4}\right)\varepsilon^5 + \left(\frac{943\pi^6}{120960} - \frac{49\zeta_3^2}{2}\right)\varepsilon^6 + O\left(\varepsilon^7\right)$$

$$\mathrm{Res}\left(\frac{\mathrm{d}\widetilde{\mathbf{A}}}{\mathrm{d}x}\right)\Bigg|_{y_i} \cdot \mathbf{I}^{(n)}(y_i) = 0 \;\Longrightarrow\; \mathrm{I}_i^{(n)}(x_0) = c_i \mathrm{I}_1^{(n)}(x_0) + \sum_{j,k,l} d_{ijk} \int_\gamma \mathrm{d}\widetilde{A}_{jl} \cdot \mathrm{I}_l^{(n-1)}(x)$$

$$y_1 = \{-2, -1, -1, -1, -1\}$$

$\Rightarrow s_{35} = 0$ **Collinear Behavior**

$$y_6 = \{1/4, -1, -1, -1, -1\}$$

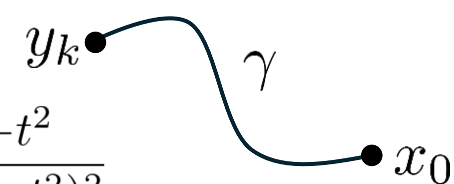
$\Rightarrow G(p_1, p_2, p_3, p_4) = 0$

4-momenta in 3D subspace

Symmetric Point in Euclidean Region

$$x_0 = y_0 = \{-1, -1, -1, -1, -1\}$$

Line Integration



$$\gamma : s_{12} = \frac{-x}{(1-x)^2} \Big/ s_{12} = \frac{-t^2}{(1-t^2)^2}$$

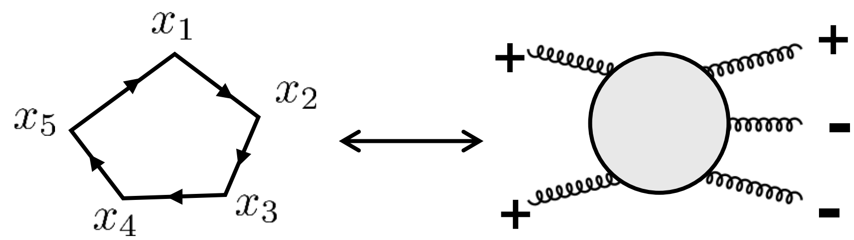
The other four variables were set to -1
And its cyclic permutation. ..

Solve the boundary values up to weight-six

Written in special values of GPL functions

[Gehrmann, Henn, Lo Presti '18] e.t.c

Integrand



planar N=4 sYM

from MHV amplitude Wilson Loop duality

[Alday, Maldacena '07]

[Drummond, Korchemsky, Sokatchev '07]

[Alday, Roiban '08] [Henn '09]

$$\log M_n + \mathcal{O}(1/N) \leftrightarrow \log\langle W_n \rangle$$

Lagrangian insertion: $g^2\partial_{g^2}\langle W_n\rangle = -\mathrm{i}\int \mathrm{d}^d x_0 \langle W_n \mathscr{L}(x_0)\rangle$

Loop expansion

MHV amplitudes integrand

$$-\mathrm{i}\int \mathrm{d}^d x_0 \frac{\langle W[x_1,\cdots,x_n]\mathscr{L}(x_0)\rangle}{\langle W[x_1,\cdots,x_n]\rangle} = g^2\partial_{g^2}\log\langle W_n\rangle \sim g^2\partial_{g^2}\log M_n$$

$$W[x_1,\cdots,x_n] = \mathrm{tr_F P}\exp\left(\mathrm{i}g_{\mathrm{YM}}t^a \oint A^a_\mu \mathrm{d}x^\mu\right)$$

L-loop Integrand

$$F_n^{(L)} \to \int \mathrm{d}^4 y_1 \cdots \int \mathrm{d}^4 y_L \Omega^{(L+1)}$$

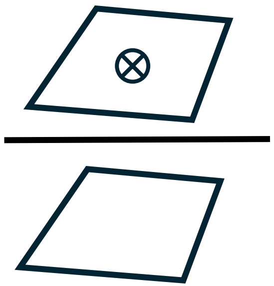
With numerator which suppress all cusp divergence

[Chicherin, Henn '22]

[Chicherin, Henn, Trnka, Zhang '24]

[Brown, Henn, Mazzucchelli, Trnka '24]

(pre) History: A Four-Point Example



$p_i \equiv x_i - x_{i+1}$

[Alday, Buchbinder, Tseytlin '11]
[Engelund, Roiban '11]

$$\frac{\langle W[x_1,x_2,x_3,x_4]\mathscr{L}(x_5)\rangle}{\langle W[x_1,x_2,x_3,x_4]\rangle} = \frac{2}{\pi^2}\frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}F(x) + \mathcal{O}(\varepsilon)$$

Rational Part
Leading singularities

Transcendental Part
Analytic functions

$$F(x) = -\frac{1}{4}g^2 + \left(\frac{1}{8}H_{0,0} + \frac{\pi^2}{16}\right)g^4 + \left(-\frac{\pi^2}{16}H_{0,0} - \frac{3}{16}H_{0,0,0,0} + \cdots\right)g^6 + \left(\frac{323\pi^4}{11520}H_{0,0} + \cdots\right)g^8 + \cdots$$

$$g^2 \equiv \frac{g_{\mathrm{YM}}^2 N_c}{4\pi^2}$$

[Alday, Buchbinder, Tseytlin '11]
[Alday, Heslop, Sikorowski '12]
[Alday, Henn, Sikorowski '13]
[Henn, Korchemsky, Mistlberger '19]

Symbol Bootstrap of $F_5^{(3)}$

Ansatz from function space: $F_5^{(3)} = r_0 g_0^{(3)} + \sum_{i=1}^{5} r_i g_i^{(3)}$

Rational Part: Leading Singularities

$$g_i^{(3)} = \sum_{i_1,\ldots,i_6} t_{i_1,\ldots,i_6} W_{i_1} \otimes \ldots \otimes W_{i_6}$$

2914 unknown coefficients in weight-six symbol

Constraints:

- Soft & collinear limit
- Free of spurious poles
- Duality to all plus amplitudes
- Near-Collinear OPE

Condition on $F^{(3)}$	No. of constraints
collinear limit $p_4 \| \| p_5$	682
soft limit $p_5 \to 0$	237
no spurious poles	1693
dependence on s_{ij}/s_{kl}	778
duality to all-plus	1182
Total	2868

46 more constrains from Near-Collinear OPE

➡ Uniquely fix the $F_5^{(3)}$

[Chicherin, Henn, YX, Zhang, Zhang '25]

[Chicherin, Henn '22] [Chicherin, Henn, Trnka, Zhang '24]

[Brown, Henn, Mazzucchelli, Trnka '24]