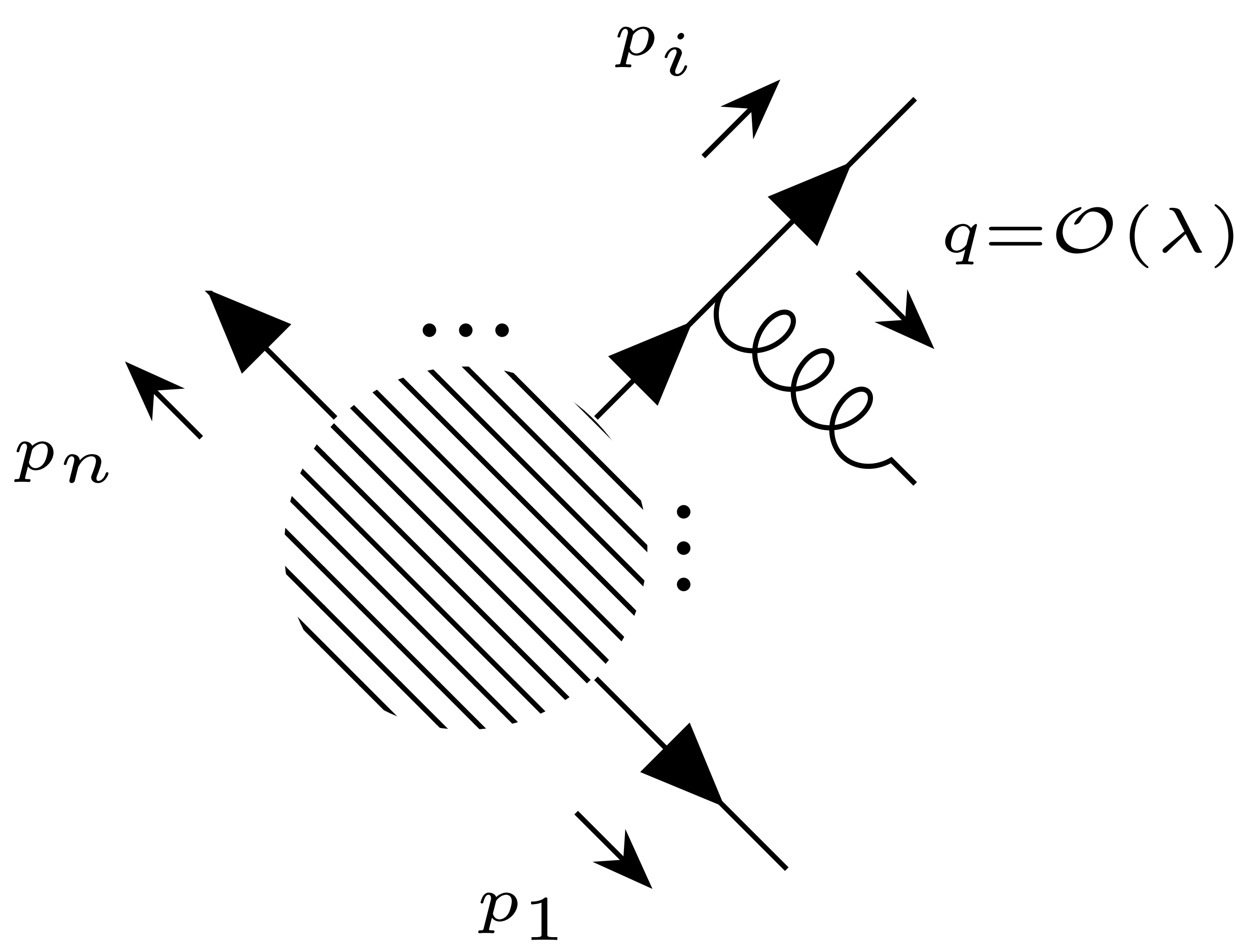
Subleading soft effects in QCD amplitudes

Kilian Erhard Mínguez 13.04.2026

Soft Gluons

Eikonal approximation



$$|M_{n+1}^{(0)}\rangle = -\sum_i \mathbf{T}_i^a \frac{p_i \cdot \varepsilon^*}{p_i \cdot q} |M_n^{(0)}\rangle + \mathcal{O}(\lambda^0)$$

Catani and Seymour, 1997

Subleading Effects in Soft-Gluon Emission at One-Loop in Massive QCD

Michał Czakon, Kilian Erhard Minguez and Felix Eschment

Institut für Theoretische Teilchenphysik und Kosmologie, RWTH Aachen University, D-52056 Aachen, Germany

Motivation

Universal structure of radiative QED amplitudes at one loop

T. Engel[a,b], A. Signer[a,b], Y. Ulrich[c]

Bhabha scattering at NNLO with next-to-soft stabilisation

Pulak Banerjee[a], Tim Engel[a,b], Nicolas Schalch[c], Adrian Signer[a,b], Yannick Ulrich[d]

Muon-electron scattering at NNLO

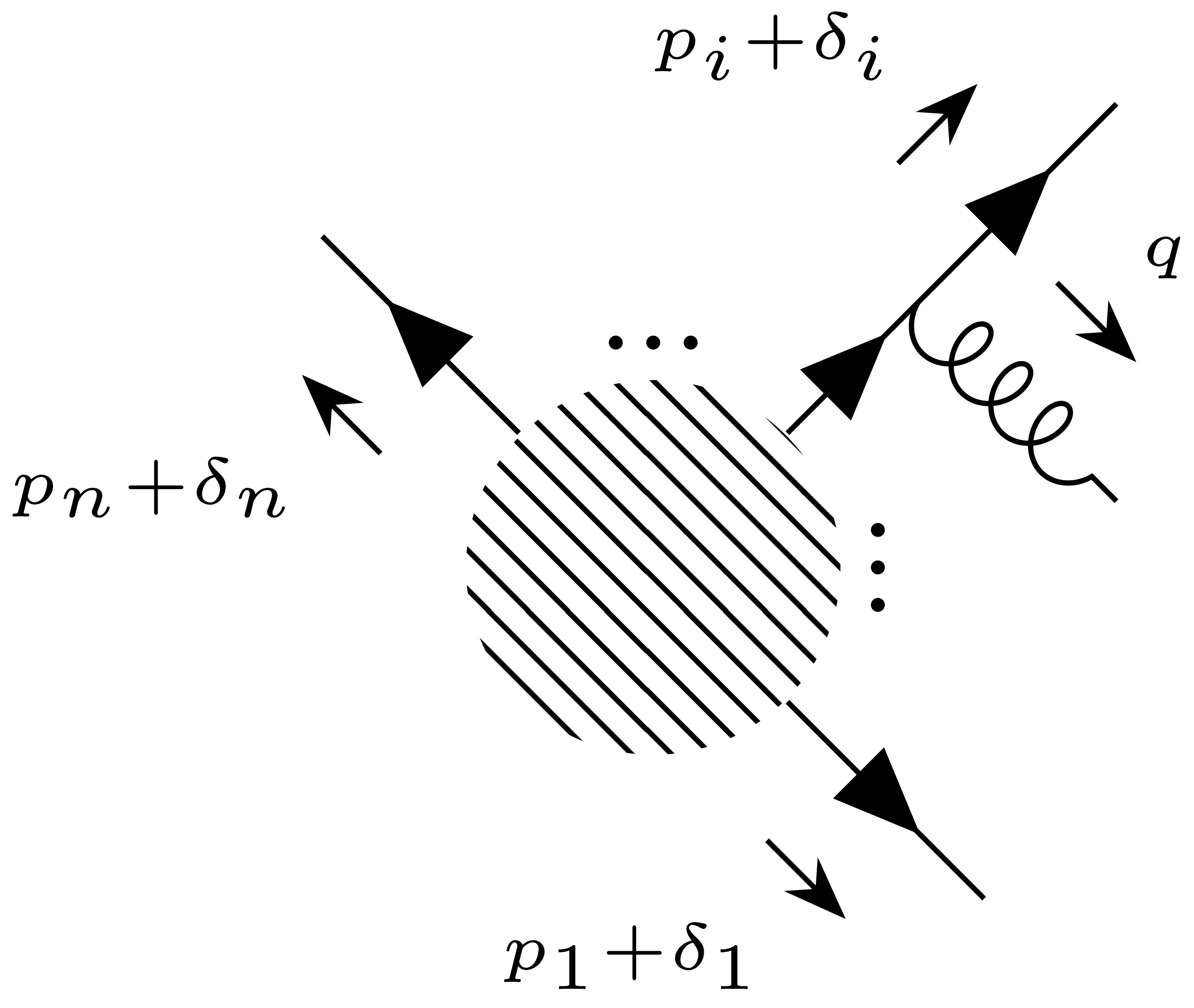
A. Broggio[a], T. Engel[b,c,d], A. Ferroglia[e,f], M. K. Mandal[g,h], P. Mastrolia[i,g], M. Rocco[b], J. Ronca[j], A. Signer[b,c], W. J. Torres Bobadilla[k], Y. Ulrich[l], M. Zoller[b]

Subleading soft currents

LBK Theorem

$$|M_{n+1}^{(0)}\rangle = \mathbf{S}^{(0)}|M_n^{(0)}\rangle + \mathcal{O}(\lambda)$$

$$\mathbf{S}^{(0)} = -\sum_i \mathbf{T}_i^a \otimes \left[\frac{(p_i+\delta_i)\cdot\varepsilon^*}{(p_i+\delta_i)\cdot q}\left(1+\sum_j \delta_j\cdot\partial_j\right) + \frac{1}{2p_i\cdot q}F_{\mu\nu}(\mathrm{J}_i - \mathbf{K}_i)^{\mu\nu}\right]$$



$$F_{\mu\nu}(q,\sigma) = i(q_\mu\varepsilon_\nu^*(q,\sigma) - q_\nu\varepsilon_\mu^*(q,\sigma))$$

Czakon, Eschment, and Schellenberger, 2023

One-loop soft currents

Underlying structure

$$|M_{n+1}^{(1)}\rangle = \mathbf{S}^{(0)}|M_n^{(1)}\rangle + \mathbf{S}^{(1)}|M_n^{(0)}\rangle + \mathcal{O}(\lambda)$$

Hard Region	Soft Region
$l = \mathcal{O}(\lambda^0)$	$l = \mathcal{O}(\lambda)$
$q = \mathcal{O}(\lambda)$	$q = \mathcal{O}(\lambda)$

Smirnov, 2001 & Bonocore et al., 2015

One-loop soft currents

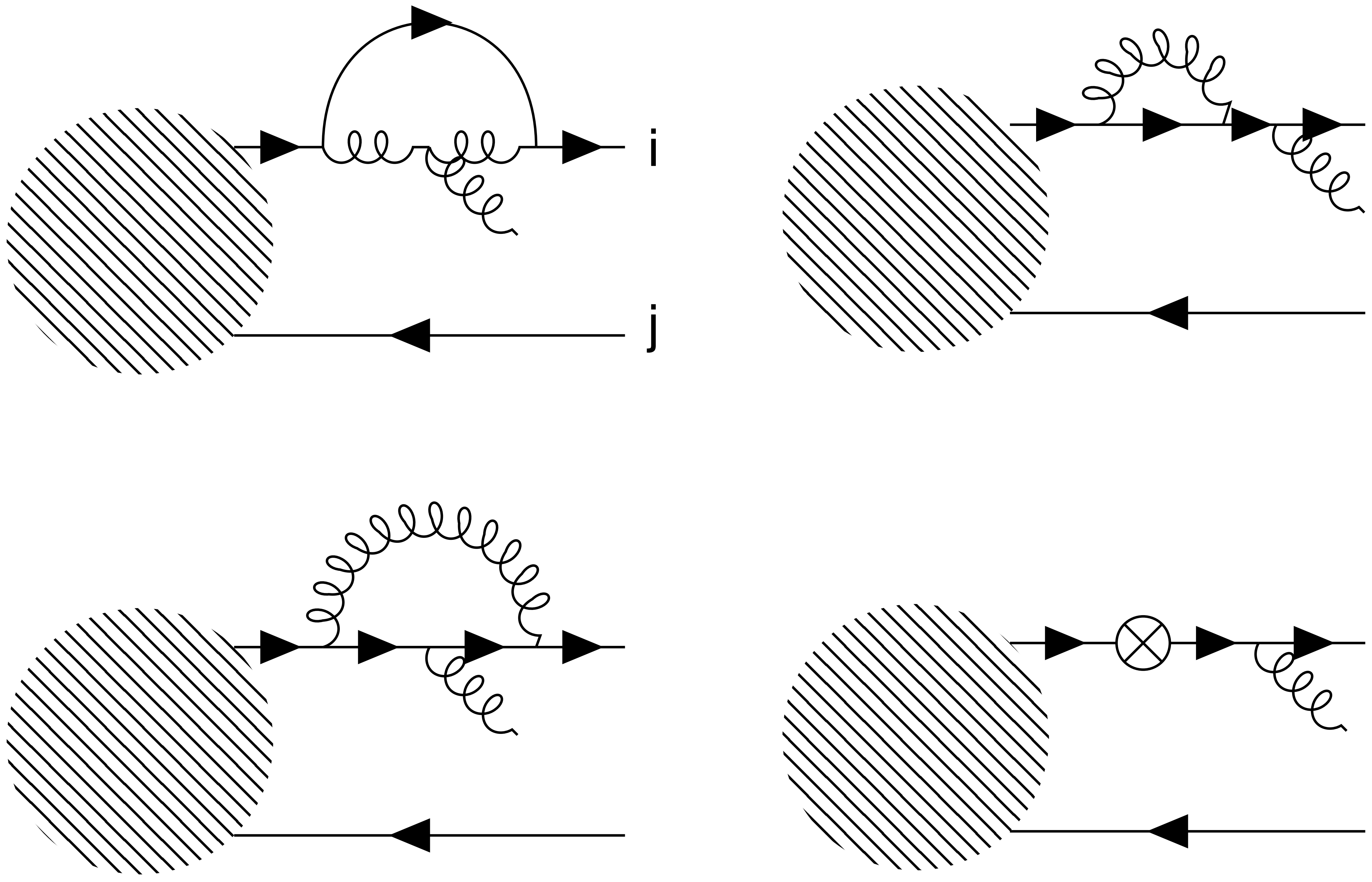
Procedure

• Generate Diagrams	QGRAF
• Apply Feynman Rules and expand in soft factor	FORM
• Passarino-Veltman	MATHEMATICA
• IbP Reduction	KIRA
• Master Integral verification	PYSECDEC
• Numerical verification	RECOLA+CUTTOOLS+ONELOOP

Massive one-loop soft current

Hard Region

$$\mathbf{S}^{(1)}_{Hard} =$$

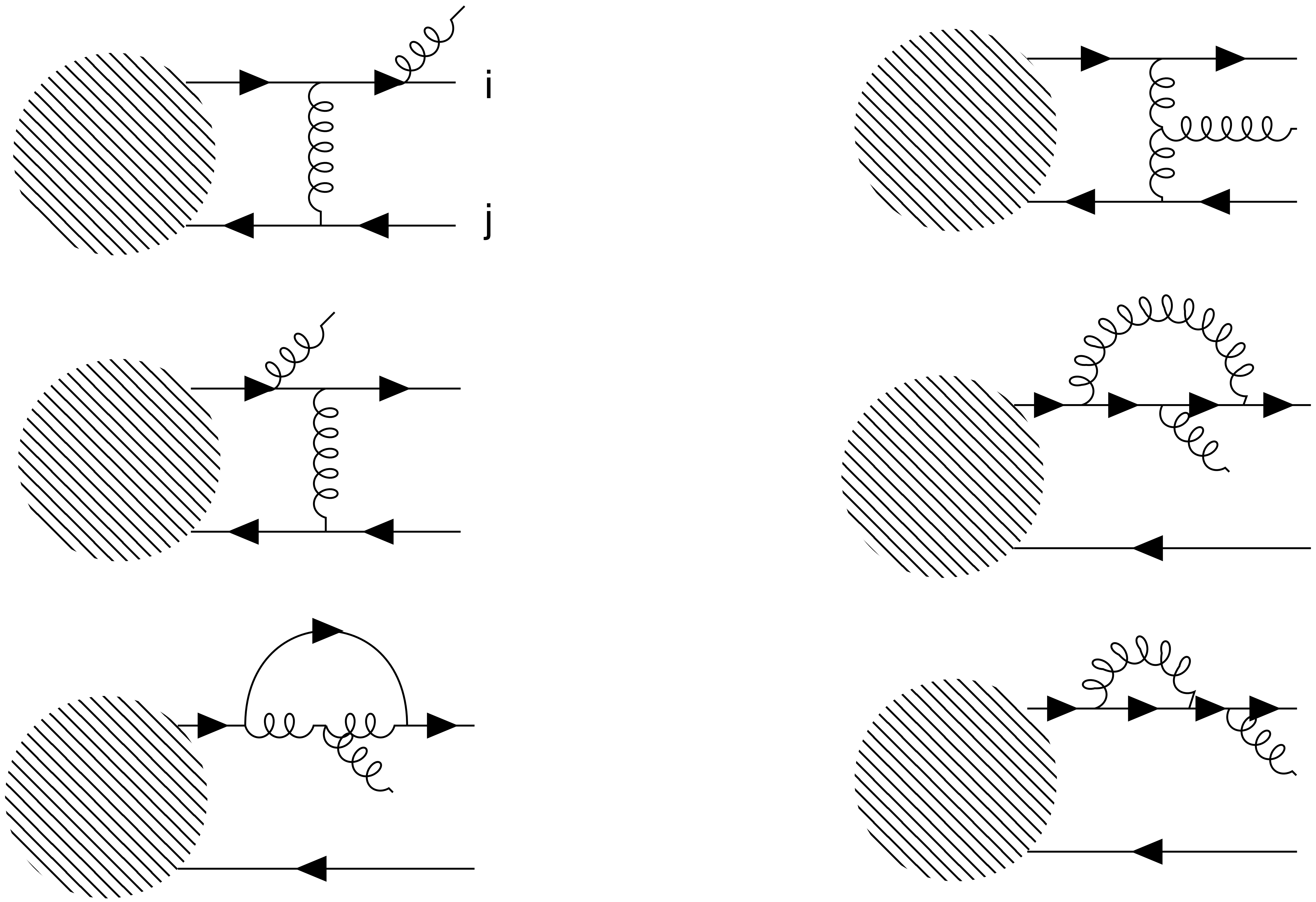


$$l = \mathcal{O}(\lambda^0)$$

One-loop soft currents

Soft region

$$\mathbf{S}^{(1)}_{Soft} =$$



$$l = \mathcal{O}(\lambda)$$

Catani and Grazzini, 2000

One-loop soft currents

Massive Master Integrals

$$M_1 \equiv \frac{i}{p_j \cdot q} \int_l \frac{1}{[-p_j \cdot l][(q+l)^2][l^2]}$$

$$M_2 \equiv \frac{i}{p_i \cdot p_j} \int_l \frac{1}{[p_i \cdot (q+l)][-p_j \cdot l][l^2]}$$

$$M_3 \equiv \frac{i}{p_i \cdot q p_j \cdot q} \int_l \frac{1}{[p_i \cdot (q+l)][-p_j \cdot l][(q+l)^2][l^2]}$$

$$H(m_i^2) = \int_l \frac{1}{(l+p_i)^2 - m_i^2} = -m_i^2 \Gamma(-1+\epsilon) \left(\frac{\mu^2}{m_i^2}\right)^{\epsilon}$$

Bierenbaum, Czakon and Mitov, 2012 & Ellis and Zanderighi, 2008

Massive one-loop soft current

Hard Region

$$\left|M^{(1)}(\{p_i+\delta_i\},q)\right\rangle = \mathbf{S}^{(0)}\left|M^{(1)}(\{p_i\})\right\rangle + \mathbf{S}^{(1)}\left|M^{(0)}(\{p_i\})\right\rangle$$

$$\mathbf{S}^{(1)} = \mathbf{S}^{(1)}_{Soft} + \mathbf{S}^{(1)}_{Hard}$$

$$\mathbf{S}^{(1)}_{Hard} = \sum_{i\neq j} i f^{abc}\mathbf{T}_i^b\mathbf{T}_j^c \frac{4}{s_{iq}} \frac{(-6+d)(30-16d+d^2)}{36(-5+d)} \left(\frac{1}{2}g^{\mu\nu} - \frac{p_i^\mu p_i^\nu}{m_i^2}\right) F_{\mu\rho}(J_i-\mathbf{K}_i)^\rho_\nu \frac{H(m_i^2)}{m_i^2}$$

Massive one-loop soft current

Soft Region

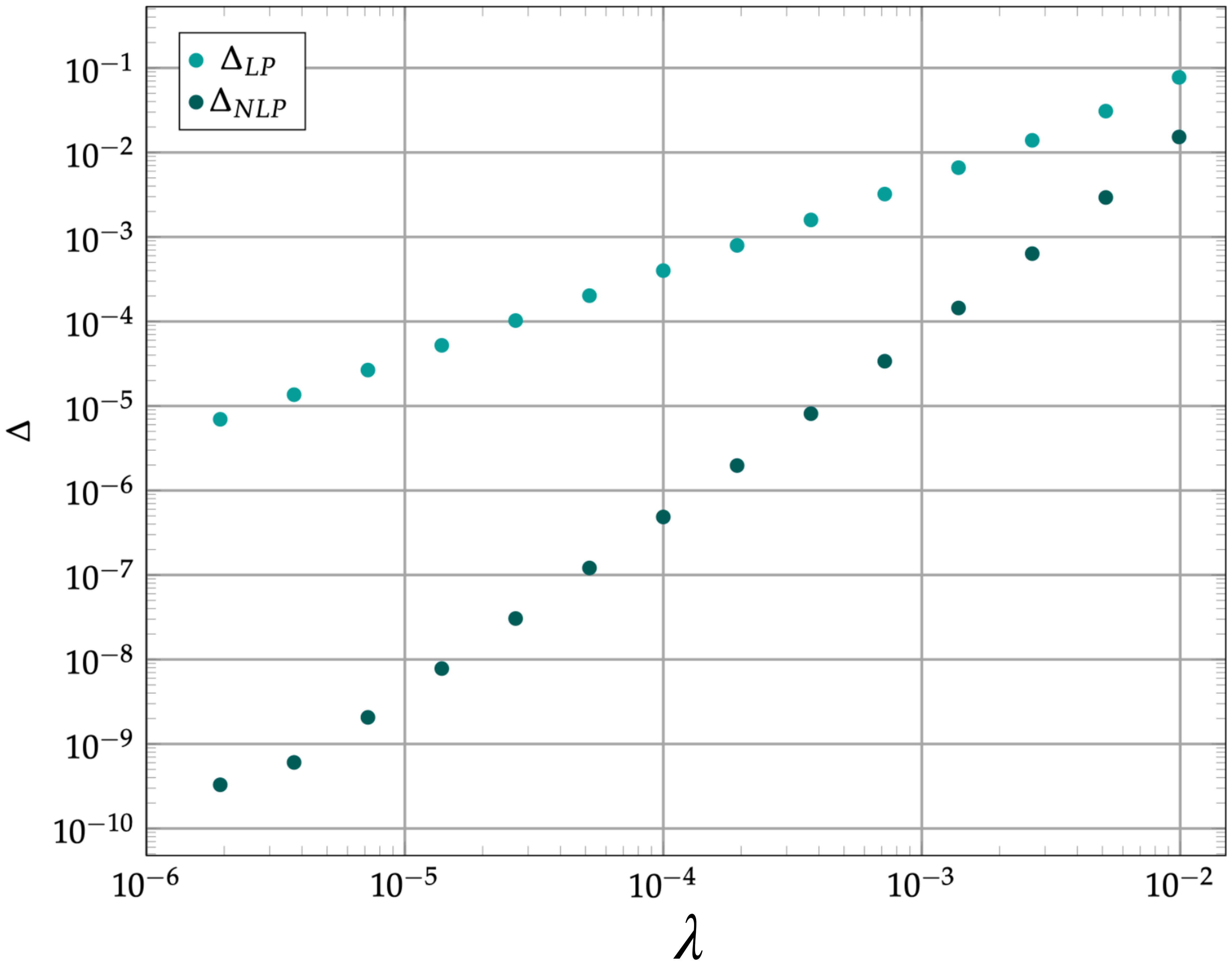
$$\mathbf{S}^{(1)}_{Soft} = \mathbf{S}^{(1)}_{LP}(\{p_i + \delta_i\})(1 + \sum_k \delta_k \cdot \partial_k) + 2if^{abc} \sum_{i \neq j} \mathbf{T}^b_i \mathbf{T}^c_j \Bigg[\frac{F_{\mu\nu} p_i^\mu p_j^\nu}{s_{iq} s_{ij}} A_1 + F_{\mu\rho}(J_i - \mathbf{K}_i)^\mu{}_\nu \times$$
$$\left(\frac{p_i^\rho p_i^\nu}{s_{iq}} A_2 + \frac{p_i^\rho p_j^\nu}{s_{iq} s_{ij}} A_3 + \frac{p_j^\rho p_i^\nu}{s_{iq} s_{ij}} A_4 + \frac{p_j^\rho p_j^\nu}{s_{jq} s_{ij}} A_5 \right) + \frac{1}{s_{iq} s_{ij}^2} (J_i - \mathbf{K}_i)_{\mu\nu} p_i^\mu p_j^\nu F_{\rho\sigma} p_i^\rho p_j^\sigma A_6 +$$
$$(J_i - \mathbf{K}_i)_{\mu\nu} \frac{F^{\mu\nu}}{s_{iq}} A_7 \Bigg] + \sum_{i \neq j} 2\mathbf{T}^a_i \mathbf{T}_i \cdot \mathbf{T}_j \frac{F_{\mu\nu} p_i^\mu p_j^\nu}{s_{ij} s_{iq}} A_8$$

$$A_i = A_i(a_i, a_j, M_1, \hat{M}_1, M_2, \hat{M}_2, M_3)$$

Massive one-loop soft current

Numerical Check: $e^- e^+ \to t\,\bar{t} + soft\,g$

$$\Delta_{LP/NLP} = \frac{1}{N} \sum_{\{c_i\},\{\sigma_i\}} \left| \frac{M_{n+1} - M_{n+1}|_{LP/NLP}}{M_{n+1}} \right|$$

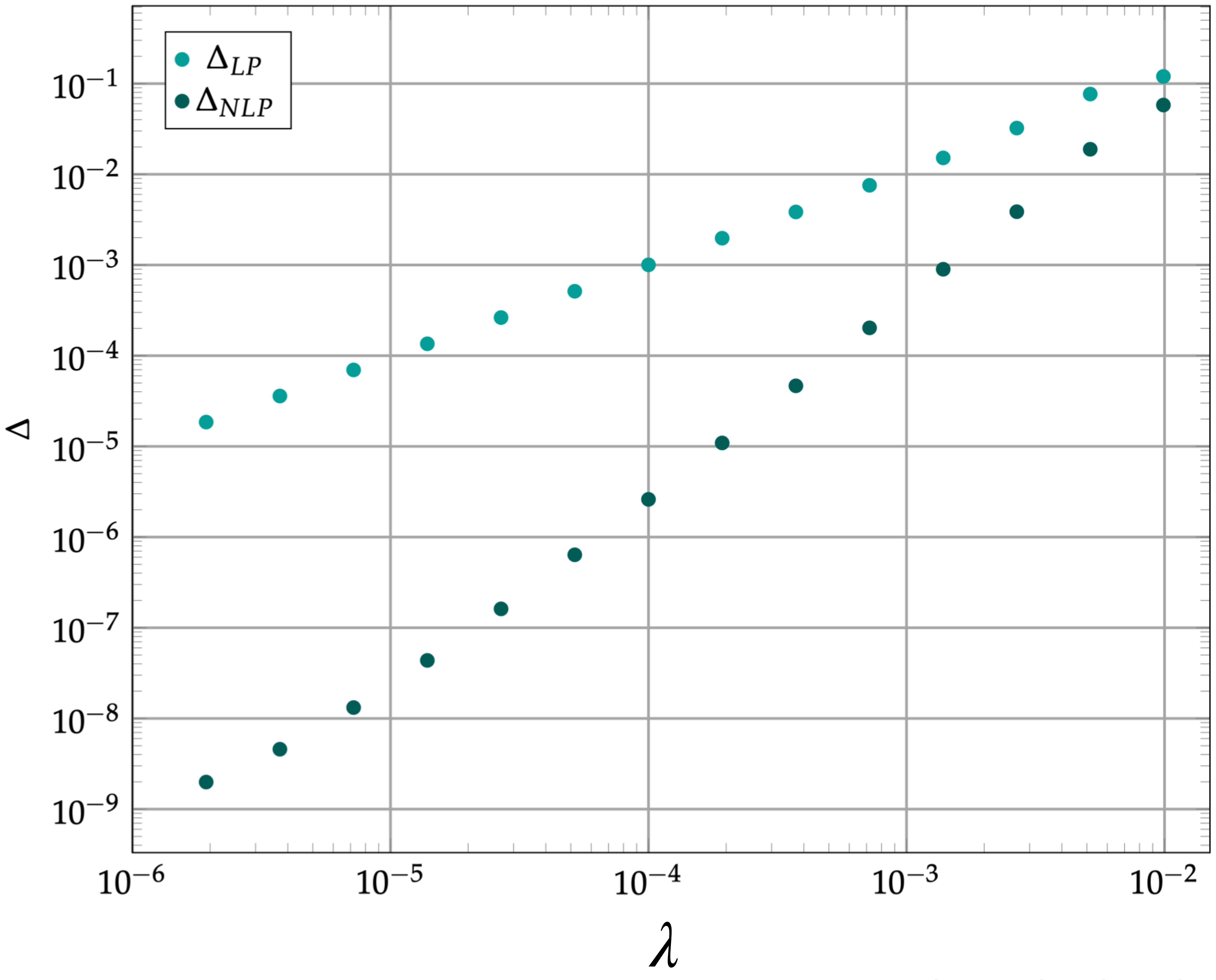


Czakon, Erhard, and Eschment, 2025

Massive one-loop soft current

Numerical Check: $e^- e^+ \to t\,\bar{t}\,b\,\bar{b} + soft\,g$

$$\Delta_{LP/NLP} = \frac{1}{N} \sum_{\{c_i\},\{\sigma_i\}} \left| \frac{M_{n+1} - M_{n+1}|_{LP/NLP}}{M_{n+1}} \right|$$



Czakon, Erhard, and Eschment, 2025

Questions?

Massive one-loop soft current

Pole Check

$$|M^{(1)}\rangle = \left[\frac{n-2}{2}\frac{\beta_0}{\epsilon} + \sum_{i\neq j}\mathbf{T}_i\cdot\mathbf{T}_j\left(\frac{\mu^2}{|s_{jk}|}\right)^{\epsilon}\left(\mathcal{V}(s_{jk},m_j,m_k,\epsilon)+\frac{1}{v_{jk}}\frac{i\pi}{\epsilon}\right) - \sum_i \Gamma_j(m_j,\epsilon)\right]|M^{(0)}\rangle + \mathcal{O}(\epsilon^0),$$

$$\begin{aligned}\boldsymbol{I}_{n+1}\mathbf{S}^{(0)}\left|M^{(0)}\right\rangle =& \mathbf{S}^{(0)}\mathbf{I}_n\left|M^{(0)}\right\rangle - \\ & if^{abc}\sum_{i\neq j}\mathbf{T}_i^b\mathbf{T}_j^c\left[-\frac{2}{\epsilon^2}\left(1+\epsilon\ln\frac{\mu^2 s_{ij}}{s_{iq}s_{jq}}\right)+\frac{1}{\epsilon}\left(-(\ln a_i a_j + 4i\pi)+\frac{\ln x + 2i\pi}{v}\right)\right]\mathbf{S}^{(0)}\left|M^{(0)}\right\rangle - \\ & \frac{1}{\epsilon}\sum_{i\neq j}\mathbf{T}_i^a\mathbf{T}_i\cdot\mathbf{T}_j\frac{iF_{\mu\nu}}{p_i\cdot q}\frac{p_i^\mu p_j^\nu}{p_i\cdot p_j}\left[\frac{2}{v^2}+\frac{1-v^2}{v^3}(\ln x + 2i\pi)\right]\left|M^{(0)}\right\rangle.\end{aligned}$$

Catani, Dittmaier, and Trocsanyi, 2001

Massive one-loop soft current

Full result

$$
\begin{aligned}
A_1 =& 8i \frac{-4+d}{-3+d} \frac{(-4+d)M_1 + 2a_i a_j(-3+d)M_2}{-1+4a_i a_j}, \\
A_2 =& 4 \frac{-4+d}{-3+d} \frac{1}{m_i^2} M_1, \\
A_3 =& -\frac{1}{(-1+a_i+a_j)^2(-3+d)} \Big[2(-1+a_i+a_j)(-4+d)(4M_1 + (-1+2a_j)M_2) + \\
& 2(-4+d+2a_i - 2a_j(-3+d))(\hat{M}_1 + M_1) + \\
& (-4+d+2a_j + a_i(2-4a_j(-2+d))(\hat{M}_2 - M_2) + 2(-4+d+2a_j - 2a_i(-3+d))M_3 \Big], \\
A_4 =& -(A_3+A_5)\frac{-3+d}{-4+d} - 4\hat{M}_2,
\end{aligned}
$$

Massive one-loop soft current

Full result

$$
\begin{aligned}
A_5 =& \hat{A}_3, \\
A_6 =& -\frac{2}{(-1+a_i+a_j)^2(-1+4a_ia_j)(-3+d)}\left[4(-1+4a_ia_j)(-2+d)M_1+\right. \\
&(20-6d+12a_j(-4+d)-8a_j^2(-4+d)+4a_i(-4+4a_j+d)(\hat{M}_1-M_1)+ \\
&(-2+d+4a_j^2(-3+2a_i+d)+a_j(16-8a_i^2(-3+d)-6d+4a_i(-7+2d)))M_2+ \\
&(10-3d+8a_i^2a_j-2a_i(1+2a_j)(4+2a_j(-3+d)-d)+8a_j(-3+d)-4a_j^2(-3+d))\hat{M}_2- \\
&\left.2(-1+4a_ia_j)(-2+d)M_3\right], \\
A_7 =& -2\frac{-4+d}{-3+d}M_1+M_2+\frac{1}{2(-1+a_i+a_j)}\Big[(-2+4a_j)(\hat{M}_1+M_1) \\
&+(-1+4a_ia_j)(\hat{M}_2-M_2)+2(-1+2a_i)M_3\Big], \\
A_8 =& -\frac{1}{-4+d}A_1.
\end{aligned}
$$

Spin operator

$$\omega_{\mu\nu} = q_\mu \epsilon_\nu - q_\nu \epsilon_\mu$$

$$\bar{u}(p', \sigma) = \bar{u}(p, \sigma) + \omega_{\mu\nu} p^\mu \frac{\partial}{\partial p_\nu} \bar{u}(p, \sigma) + \mathcal{O}(q^2)$$

$$\bar{u}'(p, \sigma) = \bar{u}(p, \sigma) + \frac{i}{4} \omega_{\mu\nu} \bar{u}(p, \sigma) \sigma^{\mu\nu} + \mathcal{O}(q^2)$$

Next-to-leading-power collinear asymptotics at tree-level

Collinear Quarks

$$\left| M^{(0)}(\{k_i\}_i^{n+1}) \right\rangle = \mathbf{Split}^{(0)}_{i,n+1\leftarrow i}(k_i, k_{n+1}, p_i) \left| M^{(0)}(\{p_i\}) \right\rangle +$$
$$\sqrt{x(1-x)} \left(\frac{1}{x} |\mathrm{S}^{(0)}\rangle + \frac{1}{1-x} |\overline{\mathrm{S}}^{(0)}\rangle + \sum_I \frac{1}{x_I - x} |\mathrm{R}^{(0)}\rangle + |\mathrm{C}^{(0)}\rangle \right) + \mathcal{O}(l_\perp).$$

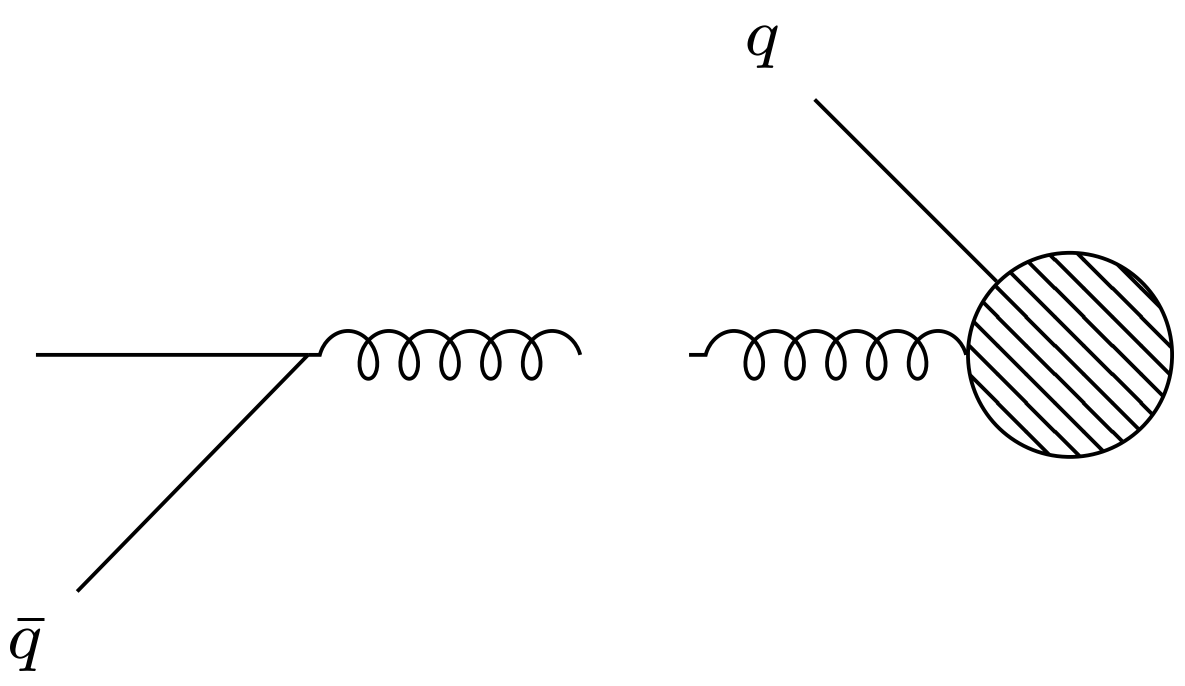
$$a_{n+1} = \bar{q}$$
$$a_i = q$$

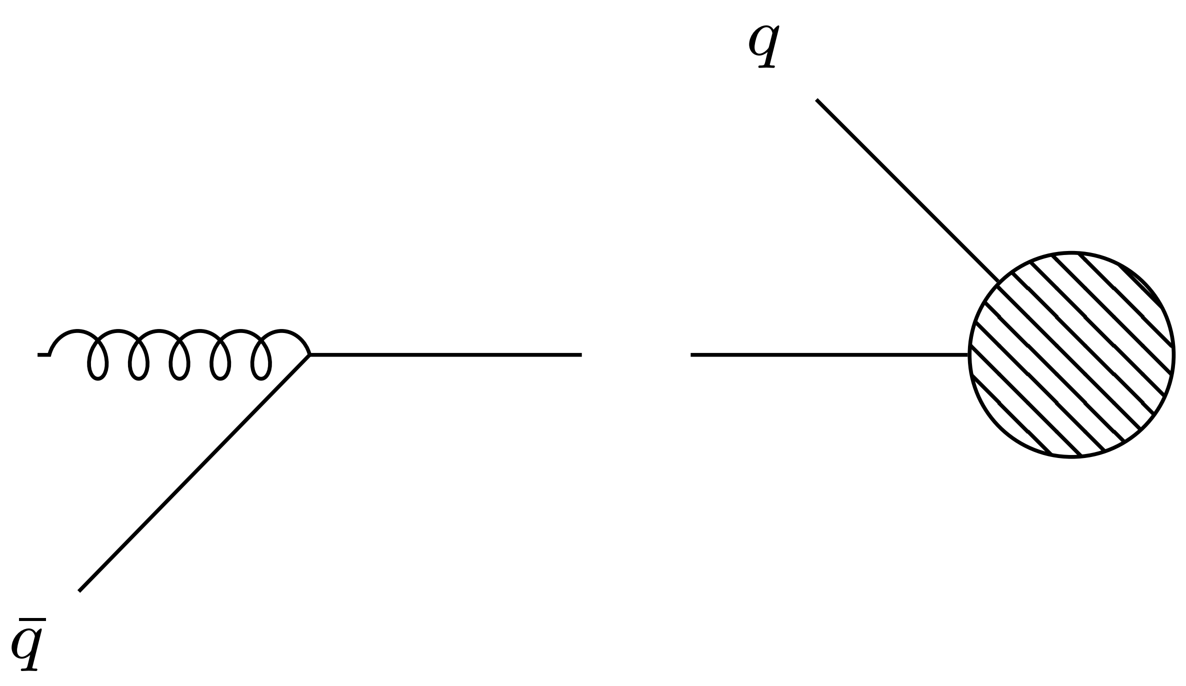
Czakon, Eschment, and Schellenberger, 2023

Next-to-leading-power collinear asymptotics at tree-level

Collinear Quarks

$$\left|M^{(0)}(\{k_i\}_i^{n+1})\right\rangle = \mathbf{Split}^{(0)}_{i,n+1\leftarrow i}(k_i, k_{n+1}, p_i)\left|M^{(0)}(\{p_i\})\right\rangle +$$
$$\sqrt{x(1-x)}\left(\frac{1}{x}|\mathrm{S}^{(0)}\rangle + \frac{1}{1-x}|\overline{\mathrm{S}}^{(0)}\rangle + \sum_I \frac{1}{x_I - x}|\mathrm{R}^{(0)}\rangle + |\mathrm{C}^{(0)}\rangle\right) + \mathcal{O}(l_\perp).$$



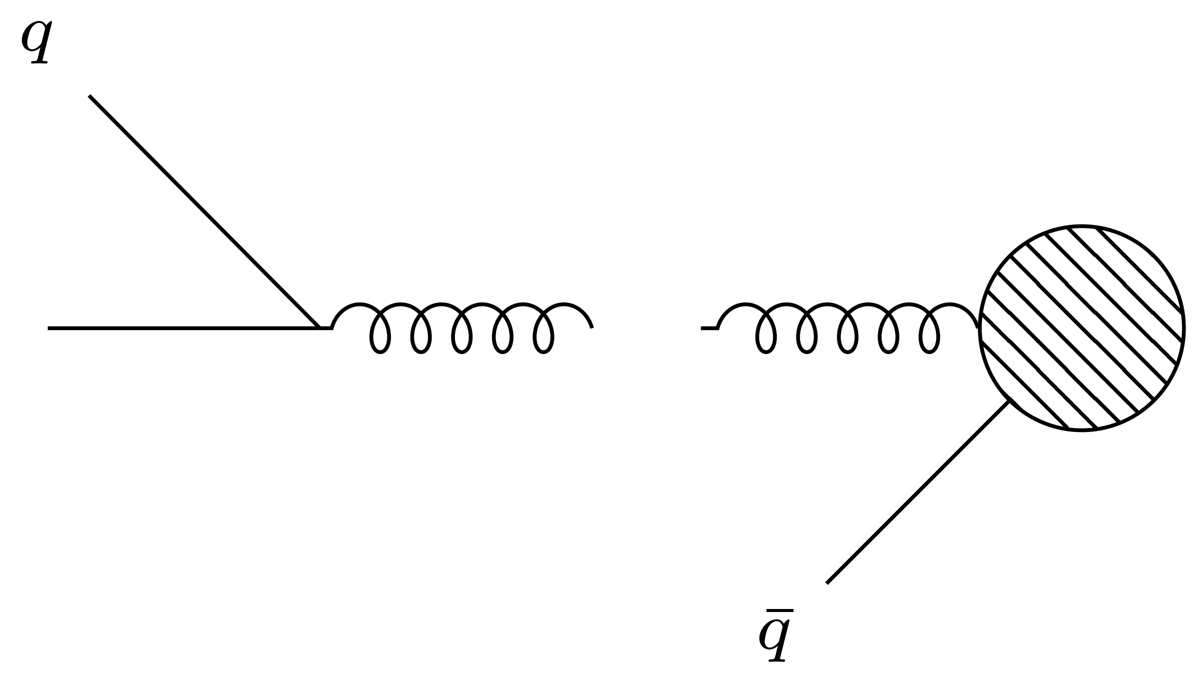


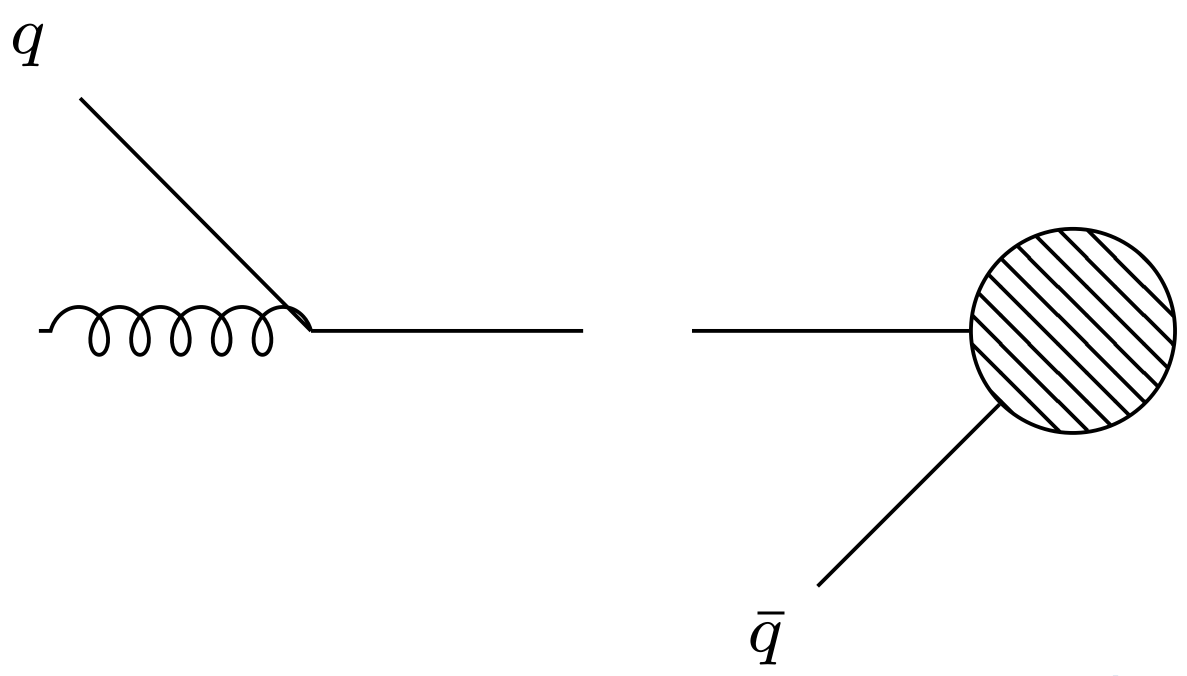
Czakon, Eschment, and Schellenberger, 2023

Next-to-leading-power collinear asymptotics at tree-level

Collinear Quarks

$$\left|M^{(0)}(\{k_i\}_i^{n+1})\right\rangle = \mathbf{Split}^{(0)}_{i,n+1\leftarrow i}(k_i, k_{n+1}, p_i)\left|M^{(0)}(\{p_i\})\right\rangle +$$
$$\sqrt{x(1-x)}\left(\frac{1}{x}|\mathrm{S}^{(0)}\rangle + \frac{1}{1-x}|\overline{\mathrm{S}}^{(0)}\rangle + \sum_I \frac{1}{x_I - x}|\mathrm{R}^{(0)}\rangle + |\mathrm{C}^{(0)}\rangle\right) + \mathcal{O}(l_\perp).$$



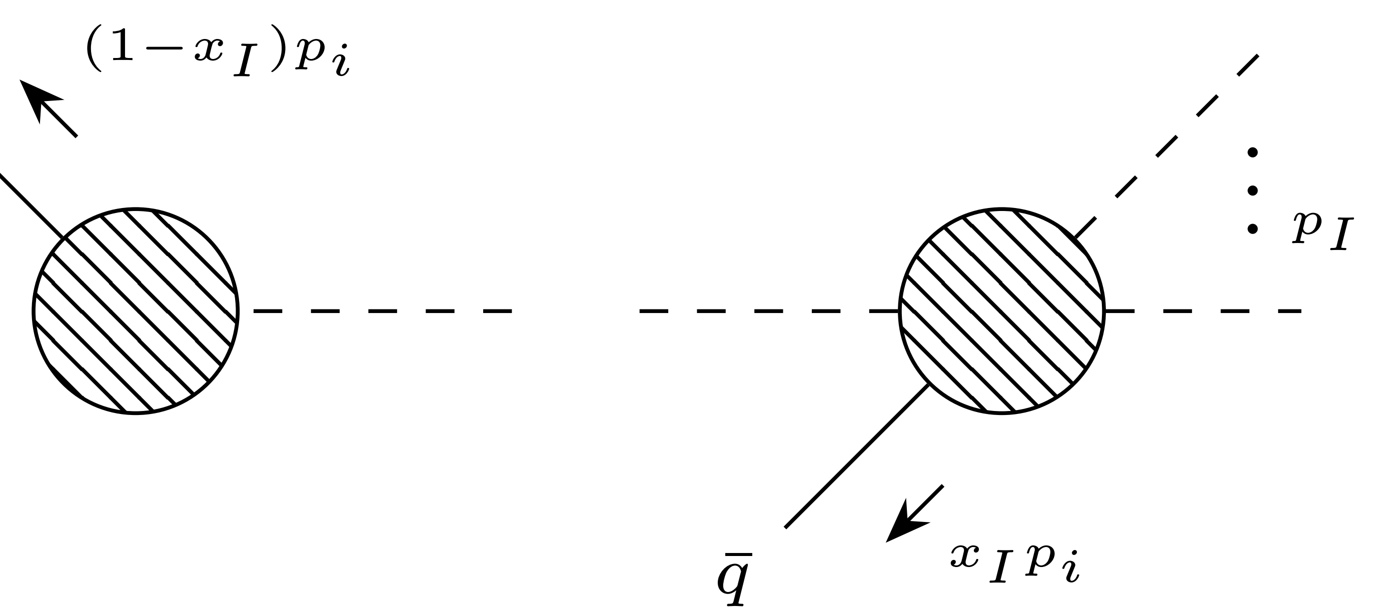


Czakon, Eschment, and Schellenberger, 2023

Next-to-leading-power collinear asymptotics at tree-level

Collinear Quarks

$$\left|M^{(0)}(\{k_i\}_i^{n+1})\right\rangle = \mathbf{Split}^{(0)}_{i,n+1\leftarrow i}(k_i, k_{n+1}, p_i)\left|M^{(0)}(\{p_i\})\right\rangle +$$
$$\sqrt{x(1-x)}\left(\frac{1}{x}|\mathrm{S}^{(0)}\rangle + \frac{1}{1-x}|\bar{\mathrm{S}}^{(0)}\rangle + \sum_I \frac{1}{x_I - x}|\mathrm{R}^{(0)}\rangle + |\mathrm{C}^{(0)}\rangle\right) + \mathcal{O}(l_\perp).$$



$$x_I = \frac{m_I^2 - p_I^2}{2p_i \cdot p_I}$$

Czakon, Eschment, and Schellenberger, 2023

Next-to-leading-power collinear asymptotics at tree-level

Collinear Quarks

$$\left|M^{(0)}(\{k_i\}_i^{n+1})\right\rangle = \mathbf{Split}^{(0)}_{i,n+1\leftarrow i}(k_i, k_{n+1}, p_i)\left|M^{(0)}(\{p_i\})\right\rangle +$$
$$\sqrt{x(1-x)}\left(\frac{1}{x}|\mathrm{S}^{(0)}\rangle + \frac{1}{1-x}|\overline{\mathrm{S}}^{(0)}\rangle + \sum_I \frac{1}{x_I - x}|\mathrm{R}^{(0)}\rangle + |\mathrm{C}^{(0)}\rangle\right) + \mathcal{O}(l_\perp).$$

Czakon, Eschment, and Schellenberger, 2023

Next-to-leading-power collinear asymptotics at tree-level

Collinear Quarks

$$|\mathrm{C}^{(0)}\rangle = -2\delta_{-\sigma,\sigma_i}\sum$$