Holographic Renormalization

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Motivations

Strongly-coupled systems described by Quantum Field Theories.
Possible applications: Condensed Matter Physics.

Example:

Two-point function	$\longleftrightarrow$	Conductivity
$G_{\mu\nu}(x-y) = i\theta(x^0 - y^0)\langle[J_\mu(x), J_\nu(y)]\rangle$		$\sigma(\omega) \sim \frac{\hat{G}(\omega, k)}{i\omega}$

How do we compute the two-points function?

Gravity/Gauge correspondence

Weak/strong coupling correspondence

Bulk: Asymptotically Anti-de Sitter spacetime (AAdS)

Bulk Fields		Boundary Operators
$\mathcal{F} \sim \rho^m f_{(0)} + \cdots$	$\longleftrightarrow$	$\mathcal{O}$

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$\mathcal{F} \sim \rho^m f_{(0)} + \cdots$		$\mathcal{O}$

$f_{(0)} \longleftrightarrow$ source for the dual operator $\mathcal{O}$.

$$\int_M D\mathcal{F}\ exp(-S[\mathcal{F}])|_{\mathcal{F}\sim f_{(0)}} = \langle exp(-\int_{\partial M} f_{(0)}\mathcal{O})\rangle_{QFT}$$

$$\langle \mathcal{O} \rangle = \frac{1}{\sqrt{-g_{(0)}}} \frac{\delta S}{\delta f_{(0)}}$$

Matching Fields/Operators

$$g_{\mu\nu}^{(0)} \longleftrightarrow \langle T_{\mu\nu} \rangle \Longrightarrow \langle T_{\mu\nu} \rangle = \frac{-2}{\sqrt{-g_{(0)}}} \frac{\delta S}{\delta g_{(0)}^{\mu\nu}}$$

$$A_\mu^{(0)} \longleftrightarrow \langle J^\mu \rangle \Longrightarrow \langle J^\mu \rangle = \frac{1}{\sqrt{-g_{(0)}}} \frac{\delta S}{\delta A_\mu^{(0)}}$$

$$\phi_{(0)} \longleftrightarrow \langle \mathcal{O}_\phi \rangle \Longrightarrow \langle \mathcal{O}_\phi \rangle = \frac{1}{\sqrt{-g_{(0)}}} \frac{\delta S}{\delta \phi_{(0)}}$$

Why do we need renormalization in AdS/CFT?

Heuristic argument:

$$S \sim \int_M d^{d+1}x\sqrt{-G}\,(R+2\Lambda)+\;\cdots\;\sim\quad Vol(M)\quad+\;\cdots$$

$$\downarrow$$

The action itself is divergent ∞

Infrared divergences in the bulk $\longrightarrow$ Ultraviolet divergences for the QFT

2+1 dimensions - Vector Field coupled to BTZ black hole

T. Andrade, M. Bañados, R. Benguria, A. Gomberoff *The 2+1 charged black hole in topologically massive Electrodynamics* (2005) arXiv: hep-th/0503095v2

Maxwell and Chern Simons term for the U(1) Vector field

$$S = \frac{1}{16\pi G_3}\int_M d^2x d\rho \sqrt{-G}(R + 2 - \frac{\kappa}{4}F_{MN}F^{MN}) - \frac{\alpha}{2}\frac{1}{16\pi G_3}\int_M d^2x d\rho\ \epsilon^{MNP}A_M F_{NP} + \frac{1}{16\pi G_3}\int_{\partial M} d^2x \sqrt{-\gamma}2K.$$

Rejecting the solution that diverges at the boundary ($A_\mu \sim r^\alpha$) we can find only a BTZ black hole with Coulomb charge C=0.

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Which is the dual theory of this system?

Can we restore a charged BTZ using Holographic Renormalization?

The procedure: Action and Equations of Motion

AAdS in $D = d + 1$ dim. (Graham-Fefferman coordinates):

$$ds^2 = G_{MN} dx^M dx^N = \frac{d\rho^2}{4\rho^2} + \gamma_{\mu\nu} dx^\mu dx^\nu ,$$

$$\gamma_{\mu\nu}(\rho, x) = \frac{g_{\mu\nu}(\rho, x)}{\rho}, \qquad \text{Boundary: } \rho = 0.$$

$$S = \frac{1}{16\pi G_D} \int_M d^d x d\rho \sqrt{-G}(R + d(d-1) + \ldots) + \frac{1}{16\pi G_D} \int_{\partial M} d^d x \sqrt{-\gamma} 2K,$$

K = second fundamental form.

$$\text{Einstein eqs: } \frac{\delta S}{\delta G^{MN}} = 0, \left(\text{Maxwell eqs: } \frac{\delta S}{\delta A^M} = 0, \cdots \right)$$

The procedure: the Radial Behaviour

Generic Field

$$\mathcal{F}(x,\rho) = \rho^m \left(f_{(0)}(x) + \rho f_{(2)}(x) + \ldots + \rho^n f_{(2n)}(x) + \rho^n \log\rho \; \tilde{f}_{(2n)}(x) + \ldots \right)$$

$f_{(2k)}(k<n), \tilde{f}_{(2n)}$ algebraically determined by $f_{(0)}$ (using eom)

$f_{(2n)}$ undetermined.

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The metric

$$g_{\mu\nu} = \underbrace{g^{(0)}_{\mu\nu}}_{\text{source}} + \rho g^{(2)}_{\mu\nu} + \cdots + \rho^{d/2} \underbrace{g^{(d)}_{\mu\nu}}_{\text{VEV}} + \rho^{d/2}\log\rho \; h^{(d)}_{\mu\nu} + \cdots$$

$$Tr(g_{(d)}) = \cdots \,, \;\; \nabla^\nu g^{(d)}_{\mu\nu} = \cdots \,.$$

The procedure: the Regulated Action

Radial cut-off: $\rho \geq \varepsilon$

$$
\begin{aligned}
S_{reg} \quad &= \quad \int_{\rho\geq\varepsilon} d^d x d\rho \, \sqrt{-G}(R + d(d-1) + \ldots) + \\
&\quad + \int_{\rho=\varepsilon} d^d x \, \sqrt{-\gamma} 2K = \\
\overset{rad.beh.}{\underset{sol.}{=}} \quad & \int_{\rho=\varepsilon} d^d x \, \sqrt{-g_{(0)}} \big[\varepsilon^{-d/2}(a_{(0)} + \varepsilon a_{(2)} + \cdots) + \\
&\quad + a_{(2d)} \log \varepsilon + \mathcal{O}(\varepsilon^0)\big]
\end{aligned}
$$

The procedure: the Counterterm Action

We translate back, using the boundary fields:

$$\begin{array}{ccc} f_{(0)} & \rightarrow & \mathcal{F} \\ g^{(0)}_{\mu\nu} & \rightarrow & \gamma_{\mu\nu} \\ & \dots & \end{array}$$

$$S_{ct}[\mathcal{F},\varepsilon] \equiv -\text{divergent terms of } (S_{reg}[f_{(0)}[\mathcal{F}],\varepsilon])$$

The procedure: the Counterterm Action

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$$S_{ct}[\mathcal{F},\varepsilon] \equiv -\text{divergent terms of } (S_{reg}[f_{(0)}[\mathcal{F}],\varepsilon])$$

$$S_{sub}[\mathcal{F},\varepsilon] \equiv S_{reg}[\mathcal{F},\varepsilon] + S_{ct}[\mathcal{F},\varepsilon]$$

$$S_{Ren} = \lim_{\varepsilon\to 0} S_{sub}$$

The procedure: Renormalized Operators

$$\langle T_{\mu\nu}\rangle = \frac{-2}{\sqrt{-g_{(0)}}}\frac{\delta S_{Ren}}{\delta g_{(0)}^{\mu\nu}} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{d/2-1}}\frac{-2}{\sqrt{-\gamma}}\frac{\delta S_{sub}}{\delta\gamma^{\mu\nu}}$$

$$\langle J^{\mu}\rangle = \frac{1}{\sqrt{-g_{(0)}}}\frac{\delta S_{Ren}}{\delta A_{\mu}^{(0)}} = \lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\varepsilon^{d/2}}\frac{1}{\sqrt{-\gamma}}\frac{\delta S_{sub}}{\delta A_{\mu}}$$

$$\langle J^{\mu}(x)J^{\nu}(y)\rangle = \frac{-1}{\sqrt{-g_{(0)}(y)}}\frac{\delta\langle J^{\mu}(x)\rangle}{\delta A_{\nu}^{(0)}(y)}$$

Check: Ward Identities

$$\delta S_{Ren} = \int d^d x \left(-\frac{1}{2} T_{\mu\nu} \delta g_{(0)}^{\mu\nu} + J^{\mu} \delta A_{\mu}^{(0)} + \cdots \right)$$

Weyl transformations

$$\delta \varepsilon = 2\sigma\varepsilon,$$

$$\delta g_{(0)}^{\mu\nu} = 2\sigma g_{(0)}^{\mu\nu},$$

$$\ldots$$

$$\langle Tr(T) \rangle = \cdots + \mathcal{A}$$

Boundary Diffeomorphisms

$$\delta \varepsilon = 0,$$

$$\delta g_{(0)}^{\mu\nu} = -(\nabla^{\mu}\xi^{\nu} + \nabla^{\nu}\xi^{\mu}),$$

$$\ldots$$

$$\langle \nabla^{\nu} T_{\mu\nu} \rangle = \cdots$$

2+1 dimensions - Vector Field coupled to gravity

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$$+\frac{1}{16\pi G_3}\int_{\partial M} d^2x \sqrt{-\gamma}2K.$$

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Maxwell and Chern Simons term for the U(1) Vector field

$$S = \frac{1}{16\pi G_3}\int_M d^2x d\rho\, \sqrt{-G}(R + 2 - \frac{\kappa}{4}F_{MN}F^{MN}) - \frac{\alpha}{2}\frac{1}{16\pi G_3}\int_M d^2x d\rho\, \epsilon^{MNP}A_M F_{NP} + \frac{1}{16\pi G_3}\int_{\partial M} d^2x\, \sqrt{-\gamma}2K.$$

$$\begin{aligned}
-2\rho\kappa\partial_\mu(\sqrt{-g}g^{\mu\nu}A'_\nu) - \alpha\epsilon^{\mu\nu}F_{\mu\nu} &= 0,\\
2\kappa\partial_\rho(\sqrt{-g}\rho g^{\mu\nu}A'_\nu) + \frac{\kappa}{2}\partial_\sigma(\sqrt{-g}F_{\lambda\nu}g^{\sigma\lambda}g^{\mu\nu}) + 2\alpha\epsilon^{\mu\nu}A'_\nu &= 0,\\
-\frac{1}{2}g^{\alpha\beta}g''_{\alpha\beta} + \frac{1}{4}g^{\alpha\beta}g'_{\beta\gamma}g^{\gamma\delta}g'_{\delta\alpha} + \frac{\kappa}{16}F_{\alpha\beta}F_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta} &= 0,\\
\nabla^\alpha g'_{\mu\alpha} - \nabla_\mu(g^{\alpha\beta}g'_{\alpha\beta}) - \kappa\rho g^{\alpha\beta}A'_\alpha F_{\mu\beta} &= 0,\\
R_{\mu\nu}[g] - 2\rho g''_{\mu\nu} + g_{\mu\nu}(g^{\alpha\beta}g'_{\alpha\beta}) - \rho g'_{\mu\nu}(g^{\alpha\beta}g'_{\alpha\beta}) + 2\rho g'_{\mu\alpha}g^{\alpha\beta}g'_{\beta\nu} &\\
+\kappa g_{\mu\nu}(2\rho^2 A'_\alpha A'_\beta g^{\alpha\beta} + \frac{\rho}{4}F_{\alpha\beta}F_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta}) - 2\kappa\rho^2 A'_\mu A'_\nu + \frac{\kappa}{2}\rho F_{\mu\alpha}F_{\beta\nu}g^{\alpha\beta} &= 0.
\end{aligned}$$

The BTZ black hole and the exploding vector field

Bañados et al.: radially symmetric solutions.
We want functions also of the *boundary* coordinates.

New Ansatz for the radial expansion ($0 < \alpha < 1/2$)

$$
\begin{aligned}
g_{\mu\nu} &= g^{(0)}_{\mu\nu} + \rho g^{(2)}_{\mu\nu} + \rho \log \rho \; h^{(2)}_{\mu\nu} + \rho^{-2\alpha+1} g^{(-)}_{\mu\nu} + \rho^{2\alpha+1} g^{(+)}_{\mu\nu} + \ldots, \\
A_\mu &= A^{(0)}_\mu + \rho A^{(2)}_\mu + \rho^\alpha A^{(+)}_\mu + \rho^{-\alpha} A^{(-)}_\mu + \rho \log \rho \; B_\mu + \ldots
\end{aligned}
$$

Solutions

$$
\begin{aligned}
F_{(0)}^{\mu\nu} &= 0, \\
\left(\kappa\sqrt{-g_{(0)}}g_{(0)}^{\mu\nu} \pm \epsilon^{\mu\nu}\right) A_{\nu}^{(\pm)} &= 0, \\
A_{\sigma}^{(\pm)} A_{\lambda}^{(\pm)} g_{(0)}^{\sigma\lambda} &= 0, \\
Tr g_{(2)} &= -\frac{R[g_{(0)}]}{2} + 2\kappa\alpha^2 A_{\sigma}^{(+)} A_{\lambda}^{(-)} g_{(0)}^{\sigma\lambda}, \\
Tr h_{(2)} &= 0, \\
Tr g_{(\pm)} &= 0.
\end{aligned}
$$

Present status and future work

$$S_{reg} = \frac{1}{16\pi G_3}\int_{\rho=\varepsilon} d^2x\,\sqrt{-g_{(0)}}\Big[2\,\varepsilon^{-1} + \varepsilon^{-2\alpha}\Big(-\kappa\alpha A^{(-)}_{\sigma}A^{(-)}_{\lambda}g^{\sigma\lambda}_{(0)} + +(4\alpha - \frac{1}{2\alpha})Tr g_{(-)}\Big) + \log\varepsilon\,\Big(Tr g_{(2)} - 4\kappa\alpha^2 A^{(+)}_{\sigma}A^{(-)}_{\lambda}g^{\sigma\lambda}_{(0)}\Big)\Big] + \mathcal{O}(\varepsilon^0).$$

Present status and future work

$$
\begin{aligned}
S_{reg} \quad = \quad & \frac{1}{16\pi G_3} \int_{\rho=\varepsilon} d^2x \, \sqrt{-g_{(0)}} \Big[2\, \varepsilon^{-1} + \varepsilon^{-2\alpha} \Big(-\kappa\alpha A_\sigma^{(-)} A_\lambda^{(-)} g_{(0)}^{\sigma\lambda} + \\
& + (4\alpha - \frac{1}{2\alpha}) Tr g_{(-)} \Big) + \log\varepsilon \, \Big(Tr g_{(2)} - 4\kappa\alpha^2 A_\sigma^{(+)} A_\lambda^{(-)} g_{(0)}^{\sigma\lambda} \Big) \Big] + \mathcal{O}(\varepsilon^0).
\end{aligned}
$$

Future questions:

- which are the correct *covariant* counterterms?
- is the system renormalizable even with $A_\mu^{(-)} \neq 0$ ($\iff$ $C \neq 0$)?