Gauge invariant Backreaction in Cosmology

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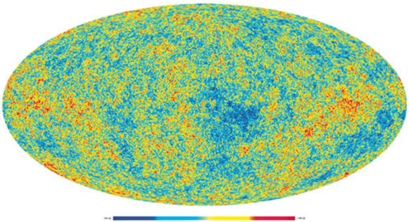
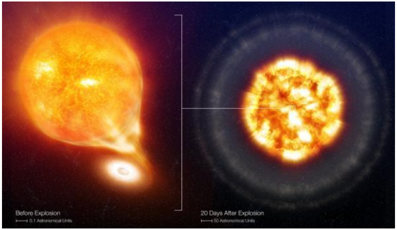
Outline

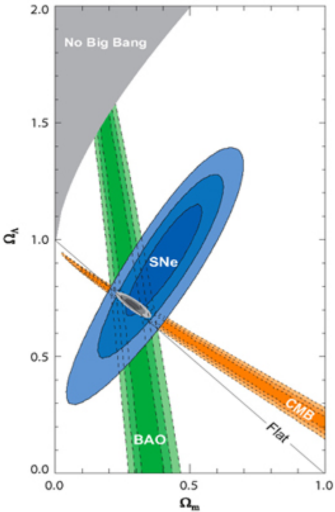
- Isotropy and the standard model of cosmology
- Averaging the scalar parts of Einstein's equations
- The question of gauge invariance
- Experimental perspectives



Isotropy and the standard cosmology

- Supernovae type Ia
- Cosmic microwave background radiation
- Baryon acoustic oscillations





Local and averaged equations

A model with space-time dependent energy density:

$$T_{\mu\nu} = \rho(x)\, u_\mu u_\nu$$

Projected Einstein equations:

$$\tfrac{1}{3}\theta^2 = 8\pi G_N \rho(x) - \tfrac{1}{2}{}^{(3)}R - \sigma^2 \qquad \text{and} \qquad \dot{\theta} + \tfrac{1}{3}\theta^2 + 2\sigma^2 = -4\pi G_N \rho$$

Friedmann equations:

$$3H^2 = 8\pi G_N \rho(t) - \tfrac{k}{3a} \qquad \text{and} \qquad 3\left(\dot{H} + H^2\right) = 3\tfrac{\ddot{a}}{a} = -4\pi G_N \rho$$

Buchert Equations:

$$3\left(\tfrac{\dot{a}}{a}\right)^2 = 8\pi G_N \langle\rho\rangle - \tfrac{1}{2}\langle R\rangle + \tfrac{Q_D}{2} \qquad \text{and} \qquad 3\tfrac{\ddot{a}}{a} = -8\pi G_N \langle\rho\rangle + Q_D$$

$$Q_D = \tfrac{2}{3}\left\langle\theta^2\right\rangle - \tfrac{2}{3}\langle\theta\rangle^2 - 2\left\langle\sigma^2\right\rangle = \tfrac{2}{3}(\Delta\theta)^2 - 2\left\langle\sigma^2\right\rangle$$

Gauge dependence of the average

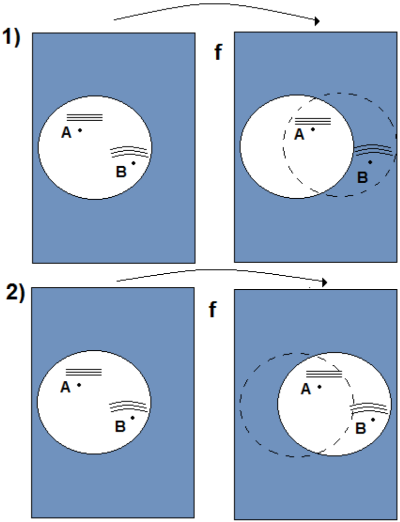
$$F(S,\Omega) = \int_{\mathcal{M}} \sqrt{-g}\, S(x) W_\Omega(x) d^n x$$

Desired transformation property of the Window Function:

$$\tilde{W}(f(x)) = W(x)$$

$$W_\Omega = \delta(T_0 - T(x))\theta(r_0 - B(x))$$

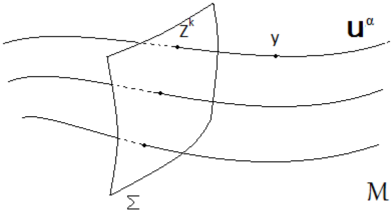
We need matter!

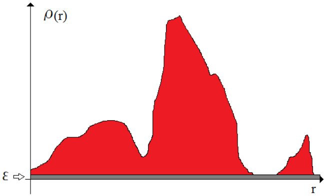


Dust coordinates

$$T_{\mu\nu} = T_{\mu\nu}^{dust} + \tilde{T}_{\mu\nu} \quad \text{and} \quad S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{8\pi G_N} R + \epsilon \left(u^\mu u^\nu g_{\mu\nu} + 1 \right) + \mathcal{L}_{\text{rest}} \right\}$$

$$u^\mu = -\partial^\mu T$$





$\tilde{T} := T(t)$ is the scalar with time-like gradient.

$B(x) := Z^k h_{kl} Z^l$ is the scalar with space-like gradient, where Z^k is the dust-space coordinate.

The new averaging

The slicing is defined by: $n^\mu = \frac{-\partial^\mu T}{\sqrt{-\partial_\nu T \partial^\nu T}}$.

$$\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 = -\frac{1}{6}\left\langle \frac{R_s}{-\partial_\mu T \partial^\mu T} \right\rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\left\langle \frac{\tilde{T}^{\mu\nu} n_\mu n_\nu}{-\partial_\mu T \partial^\mu T} + \epsilon \right\rangle$$

$$\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = \frac{1}{3}Q_D + \frac{1}{6}\left\langle \frac{\partial_\mu T \partial^\mu (\partial_\mu T \partial^\mu T)}{(-\partial_\mu T \partial^\mu T)^{5/2}} \right\rangle + \frac{1}{3}\left\langle \frac{\nabla_\mu (n^\mu \nabla_\mu n_\nu)}{-\partial_\mu T \partial^\mu T} \right\rangle +$$

$$+ \frac{8\pi G_N}{3}\left\{ \left\langle \frac{\tilde{T}_{\mu\nu} n^\mu n^\nu + \frac{1}{2}\tilde{T}}{(-\partial_\mu T \partial^\mu T)^{1/2}} \right\rangle + \frac{1}{2}\left\langle \frac{\epsilon}{\partial_\mu T \partial^\mu T} + 2\epsilon \right\rangle \right\}$$

Choosing the rest frame of the dust (synchronous $N = 1$):

$$ds^2 = -N^2 dt^2 + h_{ij} dx^i dx^j$$

$$\left(\frac{1}{s}\frac{\partial s}{\partial T_0}\right)^2 = -\frac{1}{6}\langle R_s \rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\left\langle \tilde{T}_{\mu\nu} n^\mu n^\nu + \epsilon \right\rangle$$

$$-\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = -\frac{1}{3}Q_D + \frac{8\pi G_N}{3}\left\langle \left(\tilde{T}_{\mu\nu} n^\mu n^\nu + \frac{1}{2}\tilde{T}\right) + \epsilon \right\rangle$$

Comparison to Buchert equations

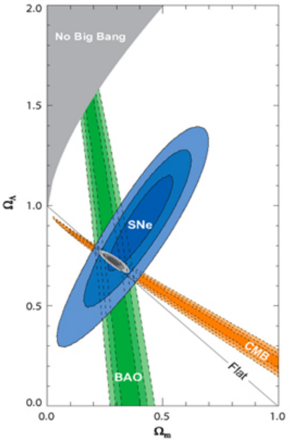
$$\tilde{T}_{\mu\nu} = 0$$

$$H_D^2 = -\frac{1}{6}\langle R_s\rangle - \frac{1}{6}Q_D + \frac{8\pi G_N}{3}\langle\epsilon\rangle$$

$$\Leftrightarrow \Omega_k + \Omega_Q + \Omega_\epsilon = 1$$

$$-\frac{1}{s}\frac{\partial^2 s}{\partial T_0^2} = -\frac{1}{3}Q_D + \frac{8\pi G_N}{3}\langle\epsilon\rangle$$

$$\dot{a} = V(A+B)^{-1}\left(\dot{a}_A V(A) + \dot{a}_B V(B)\right)$$



Experimental perspectives

Considering combined cosmological observations we can conjecture $\langle R \rangle = 0$, hence:

$$Q_D = -6 \left(\frac{1}{s} \frac{\partial s}{\partial T_0} \right)^2 + \frac{8\pi G_N}{3} \langle \epsilon \rangle \quad \text{and} \quad Q_D = 3 \left(\frac{1}{s} \frac{\partial^2 s}{\partial T_0^2} \right) + 16\pi G_N \langle \epsilon \rangle$$

Questions:

1. Are the results for Q_D the same for both equations?
2. Is there a scale on which Q_D is the same in all domains?
3. Is there a scale on which Q_D is zero in all domains?
4. How do those scales depend on the red shift?

Those observations can be done for approximately constant $z < 1$, for large z light-cone has to be considered.

Conclusion

- The gauge dependence of the backreaction has been clarified.
- The relevant inhomogeneity measure Q_D has been identified and is an observable.
- We have to reconsider interpretation of cosmological data.
- Experimental evidence is needed to check whether backreaction can be a cause for acceleration.
- In a toy model it has potential to cause apparent acceleration (S. Räsänen).
- More sophisticated computational models and observations needed in future.
- Light-cone effects have to be considered for larger domains.

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The Friedmann model, its Metric and the ideal fluid:

$$ds^2 = -dt^2 + a(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right\}$$

$$T^{\mu\nu} = \rho(t)u^\mu u^\nu + p(t)(u^\mu u^\nu + g^{\mu\nu}) = \rho u^\mu u^\nu + p\,P_\perp$$

The Einstein equations:

$$G_{\mu\nu}u^\mu u^\nu = 8\pi G_N\,\rho(t) \quad \text{and} \quad Tr(G^\mu_{\ \nu}) = 8\pi G_N\,Tr(T^\mu_{\ \nu})$$

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G_N \rho - 3\frac{k}{a^2} \quad \text{and} \quad -6\frac{\ddot{a}}{a} = 8\pi G_N(\rho + 3p)$$

Scaling:

$$\dot{\rho} = -3\,H\ (\rho + p) = -3\,H\,\rho\,(1+\omega) \Leftrightarrow d\rho = -3\frac{1}{a}da\,(1+\omega)$$

Hence: $\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+\omega)}$

With the definition of the extrinsic curvature, we can connect the curvature to the three curvature of Σ:

$$\theta_{ij} = -K_{ij} = h_i^\mu h_j^\nu \nabla_\nu u_\mu \text{ and } {}^{(3)}R_{abc}{}^d = h_a{}^f h_b{}^g h_c{}^k h_j{}^d R_{fgh}{}^j - K_{ac}K_b{}^d + K_{bc}K_a{}^d$$

$$2G_{ab}n^a n^b =^{(4)} R + 2^{(4)}R_{ac}n^a n^c =^{(4)} R_{abcd}\left(g^{ac} + n^a n^c\right)\left(g^{bd} + n^b n^d\right) =^{(4)} R_{abcd}h^{ac}h^{bd} = \left(K_a{}^a\right)^2 - K_b{}^a K_a{}^b +^{(3)} R$$

Hence: $${}^{(4)}R = 2\left(G_{ab}n^a n^b - R_{ab}n^a n^b\right) =^{(3)} R + K_{ab}K^{ab} - K^2$$

And with the definitions of the expansion rate and shear:

$$\left(K_a{}^a\right)^2 - K_b{}^a K_a{}^b = 2\left(\frac{1}{3}\theta^2 - \sigma^2\right)$$

The Hamiltonian constraint therefore reads:

$$8\pi G_N \rho + \Lambda = \frac{1}{2}((K_a^a)^2 - K_j^i K_i^j +^{(3)} R) = -\sigma^2 + \frac{1}{3}\theta^2 + \frac{1}{2}{}^{(3)}R$$

Backup II

Perturbations of the energy momentum tensor to first order in gauge invariant variables:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\left(g_{\mu\nu} + u_\mu u_\nu\right)$$

$$\delta \bar{T}^0_0 = \bar{\delta} \epsilon, \qquad \delta \bar{T}^0_i = \frac{1}{a}\left(\epsilon_0 + p_0\right)\left(\delta \bar{u}_{\|i} + \delta u_{\perp i}\right), \qquad \delta \bar{T}^i_j = -\bar{\delta} p\, \delta^i_j$$

For dust with $\epsilon_0 << a$ only the diagonal elements are affected.
The metric perturbations in a flat FRW universe can be written in the so called Longitudinal gauge, with ϕ and ψ being the scalar perturbations and η the conformal time, as:

$$ds^2 = a(\eta)^2\left[-\left(1 + 2\phi\right) d\eta^2 + \left(1 - 2\psi\right)\delta_{ij} dx^i dx^j\right]$$

We expect the most relevant contributions to the diagonal elements.
Furthermore the information about perturbations is completely in the trace and the zero zero component.

Backup III

The active diffeomorphisms are generated by the Lie derivative. This shows that any scalar action, invariant under passive diffeomorphisms is also invariant under active diffeomorphism if the generating vector vanishes at the boundary.

$$\mathcal{L}_\xi \Psi = \xi^\mu \nabla_\mu \Psi$$

$$\mathcal{L}_\xi \sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu} = (\nabla_\mu \xi^\mu)\sqrt{-g}$$

$$\delta S = \int_\Omega \mathcal{L}_\xi \left(\Psi\sqrt{-g}\right) d^4x = \int_\Omega \left(\xi^\mu \nabla_\mu \Psi + \Psi \nabla_\mu \xi^\mu\right)\sqrt{-g}\, d^4x =$$

$$= \int_{\partial\Omega} \left(\Psi \xi^\mu\right) d^3\sigma_\mu = 0$$

For the Einstein Hilbert action this yields the Bianchi identity:

$$\delta S = \int_\Omega d^4x \sqrt{-g}\,(G_{\mu\nu})\,\mathcal{L}_\xi g^{\mu\nu} = -\int_\Omega d^4x \sqrt{-g}\,(\xi_\nu)\,(\nabla_\mu G^{\mu\nu}) = 0$$

Backup IV

The projected Einstein equations:

$$G_{\mu\nu}u^\mu u^\nu = \frac{1}{2}((K^a_a)^2 - K^i_j K^j_i +^{(3)} R) = \frac{1}{3}\theta^2 - \sigma^2 + \frac{1}{2}{}^{(3)}R = 8\pi G_N \rho + \Lambda \text{ (1)}$$

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2\sigma^2 + 4\pi G_N \rho - \Lambda = 0 \text{ (2)}$$

With:

$$\sigma_{ij} := \theta_{ij} - \frac{1}{3}\theta h_{ij}\, ; \sigma^2 := \frac{1}{2}\sigma^i_j \sigma^j_i \quad \text{and} \quad \theta_{ij} = -K_{ij} \quad = h^\mu_i h^\nu_j \nabla_\nu u_\mu$$

Average of a scalar Ψ:

$$\left\langle \Psi(t,\, X^i) \right\rangle_D := \frac{1}{V_D}\int_D J\, d^3X\, \Psi(t,\, X^i) \text{ where } \quad J := \sqrt{h} \text{ and } \dot{J} = \theta J$$

Domain scale factor:

$$\text{with } V_D = \langle 1 \rangle_D \quad a_D(t) := \left(\frac{V_D}{V_{D_0}}\right)^{\frac{1}{3}} \Rightarrow \langle \theta \rangle_D = 3\frac{\dot{a}_D}{a_D}$$

Backup V: Active and passive diffeos

- Passive Diffeos. $\leftrightarrow$ pure change of coordinates (corresponds to renaming the locations on a temperature-map).
- Active Diffeos. $\leftrightarrow$ a push-forward followed by a coordinate renaming (corresponds to the action of winds on a temperature-field).
- Scalar, Vector and tensor field, under a diffeomorphism:

$$\tilde{x} = f(x) = x - \xi d\lambda$$

$$\tilde{S}(\tilde{x}) = S(x) \qquad \tilde{S}(x) = S(f^{-1}(x))$$

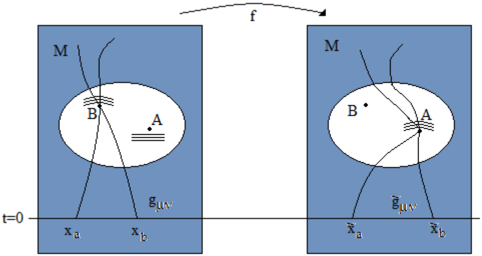
$$\tilde{A}_\mu(\tilde{x}) = A_\alpha(x)\,\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = A_\mu(x) + \{A_\alpha(x)\partial_\mu \xi^\alpha\}\,d\lambda$$

$$\tilde{A}_\mu(x) = A_\alpha(x + \xi d\lambda)\,\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = A_\mu(x) + \{\xi^\alpha \partial_a A_\mu(x) + A_\alpha(x)\partial_\mu \xi^\alpha\}\,d\lambda$$

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \{\xi^\alpha \partial_a g_{\mu\nu}(x) + g_{\alpha\nu}(x)\partial_\mu \xi^\alpha + g_{\mu\alpha}(x)\partial_\nu \xi^\alpha\}\,d\lambda$$

Backup VI: Diffeos and gen. covariance

- Any scalar action under passive diffeo.s is a scalar under active diffeo.s if $\xi = 0$ at the boundary.
- This led to the hole “paradox” and its resolution with material coordinates.



- Active diff. invariance of GR action leads to Bianchi identity, hence active diff.s are gauge trafos!
- Observables are gauge invariant quantities, so called space-time coincidences.

Backup VII: Gauge trafo & average

Apply a gauge transformation (active) $\tilde{x} = f(x)$:

$$F(S,\Omega) = \int_{\mathcal{M}} \sqrt{-g}\, S(x) W_{\Omega}(x) d^n x \to \tilde{F}(\tilde{S},\Omega)$$

$$= \int_{\mathcal{M}} \left| \frac{\partial x}{\partial f} \right|_{f^{-1}(x)} \sqrt{-g(f^{-1}(x))} S(f^{-1}(x))\, W_{\Omega}(x) d^n x$$

Followed by coordinate transformation (passive) $\hat{x} = f^{-1}(x)$:

$$\tilde{F}(\tilde{S},\Omega) = \int_{\mathcal{M}} \sqrt{-g(\hat{x})} S(\hat{x}) W_{\Omega}(f(\hat{x})) d^n \hat{x} = \int_{\mathcal{M}} \sqrt{-g(\hat{x})} S(\hat{x}) \tilde{W}_{\tilde{\Omega}}(\hat{x}) d^n \hat{x}$$

Use a field as reference: $\tilde{W}_{\Omega}(x) = W_{\Omega}(f^{-1}(x)) \Rightarrow W_{\Omega}(f(\hat{x})) = W_{\Omega}(\hat{x})$

$$\tilde{F}(\tilde{S},\Omega) = \int_{\mathcal{M}} \sqrt{-g(\hat{x})} S(\hat{x}) W_{\Omega}(f(\hat{x})) d^n \hat{x} = \int_{\mathcal{M}} \sqrt{-g(\hat{x})} S(\hat{x}) W_{\Omega}(\hat{x}) d^n \hat{x}$$

$$W_{\Omega} = \delta(T_0 - T(x)) \theta(r_0 - B(x))$$

Backup VIII: Observables from matter

Consider a freely falling , its position $\tilde{X}^{\mu}(T(\tau)) := X^{\mu}(\tau)$ is a physical point. A scalar $R(y)$ at the particle's position $\tilde{R} := R(\tilde{X}(0))$ is an observable.

$$\tilde{y} = y + \xi d\lambda$$

$$\delta R(y) = -\xi^{\mu}(y)\partial_{\mu}R(y)d\lambda$$

$$\delta X^{\mu}(\tau) = \xi^{\mu}(X(\tau))d\lambda$$

$$\Rightarrow \delta\tilde{R} = \delta R(\tilde{X}(0)) + \partial_{\mu}R(\tilde{X}(0))\delta\tilde{X}^{\mu} =$$
$$\partial_{\mu}R(\tilde{X}(0))\left[-\xi^{\mu}(\tilde{X}(0)) + \xi^{\mu}(\tilde{X}(0))\right]d\lambda = 0$$

Note: (Luminosity-distance)

$d_L(z) = a_0 r(z)(1+z)$ and for $k=0$: $r(z) = \frac{1}{a_0 H_0}\int_0^z \frac{dx}{\left(\Omega_{\Lambda}+\Omega_m(1+x)^3+\Omega_R(1+x)^4\right)}$

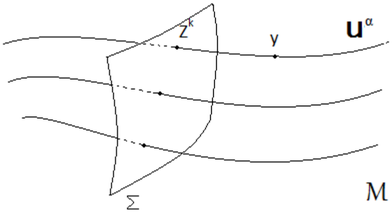
$$F \backsimeq \frac{L\,H_0^2}{4\pi z^2}\left[1 + \left(\left(-\frac{\ddot{a}}{aH}\right)_0 - 1\right)z\right] + \dots$$

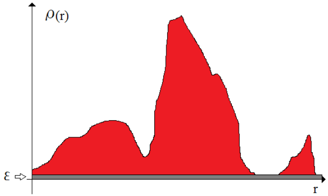
Backup IX: Dust coordinates

- $\tilde{T} := T(t)$ is the scalar with time-like gradient.
- Z^k is a scalar on $\mathcal{M}$, but a vector component on S. We need the tensor on S, $\bar{h}_{ij}(Z\,,\,T(Z,t)) := \tilde{h}_{ij}(Z,t)$ where: $\frac{d}{dt}\tilde{h}_{ab}(Z,t) = \left\{\tilde{h}_{ab}(Z,t)\,,\,H(f)\right\}$.
- Pull back the projector orthogonal to $\partial_\mu T$ on the dust space, under $X\,:\;S \to \mathcal{M}$ to find the induced metric:

$$h_{kl} = \frac{\partial X^\mu}{\partial Z^k}\frac{\partial X^\nu}{\partial Z^l}h_{\mu\nu}$$

- Contract with h_{kl}, now $B(x) := Z^k h_{kl}\, Z^l$ is the scalar with space-like gradient!





Backup X: Variation of dust action

Generalize the idea of a freely falling particle to a field. The result is dust, with the action:

$$S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{8\pi G_N} R + \rho \left(u^\mu u^\nu g_{\mu\nu} + 1 \right) \right\} ; \; u^\mu = -\partial^\mu T + W_i \left(\partial^\mu Z^i \right)$$

$$\delta S =$$

$$\int d^4x \sqrt{-g} \left\{ \left(-\frac{1}{2} g_{\mu\nu} \rho \left(u^\mu u^\nu g_{\mu\nu} + 1 \right) + \frac{-1}{8\pi G_N} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \rho u_\mu u_\nu \right) \delta g^{\mu\nu} \right\} +$$

$$+ \int d^4x \sqrt{-g} \left\{ \left(u^\mu u^\nu g_{\mu\nu} + 1 \right) \delta\rho + 2\rho u^\mu g_{\mu\nu} \delta u^\nu \right\} = 0$$

$$\rho u^\mu g_{\mu\nu} \delta u^\nu = \rho u^\mu u^\lambda g_{\mu\nu} \delta \nabla \left(x^\nu \right)_\lambda = -\nabla_\lambda \left(\rho u^\mu u^\lambda g_{\mu\nu} \right) \delta x^\nu = 0$$

$$\nabla_\lambda \left(\rho u^\mu u^\lambda g_{\mu\nu} \right) = u_\nu \nabla_\lambda \left(\rho u^\lambda \right) + \rho u^\lambda \nabla_\lambda \left(u_\nu \right) = \rho \nabla_u u_\nu = 0 \Rightarrow \nabla_U U = 0$$

$$\delta S = 0 \Leftrightarrow \left\{ G_{\mu\nu} = \rho u_\mu u_\nu \, , \; \mathcal{L}_U Z^i = 0 \, , \; \mathcal{L}_U T = 1 \, , \; \nabla_U U = 0 \right\}$$