Nonassociative Geometry in String Theory

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Contents

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"Noncommutative" and "nonassociative" geometry

- The spacetime M can be characterized by functions $f : M \to \mathbb{R}$ on it and how to compose them (algebra)
- Points in M are characterized by coordinates $X^i : M \to \mathbb{R}$
- Product of algebra of functions usually $(f \cdot g)(x) = f(x)\, g(x)$

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Noncommutative geometry

- Product on functions noncommutative, i.e. $f \star g \neq g \star f$
- In particular, coordinates noncommutative; $[X^i, X^j] \neq 0$
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Nonassociativity

- A nonassociative product, i.e. $f \star (g \star h) \neq (f \star g) \star h$
- $[X^i, [X^j, X^k]] + \mathrm{cycl.} \neq 0$, i.e. Jacobi identity fails
- Right objects to replace points?

The Bosonic String in Background Fields

Bosonic string action with Kalb-Ramond field

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2z\, \sqrt{h}\, \left(G_{\mu\nu}(X) + B_{\mu\nu}(X)\right)\, \partial X^\mu\, \overline{\partial} X^\nu$$

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Properties

- The spacetime is characterized geometrically by the metric G and a torsion $H = dB$ – the backgrounds
- Quantization spoils Weyl invariance $\Rightarrow$ recovered by consistency equations relating G and H

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Some spacetime directions $\mathbb{T}^n$ with isometries: **T-duality**

- T-duality changes background fields $\Rightarrow$ very different spacetime (non-)geometries
- Equivalent quantum theories

Emergence of noncommutative geometry

Open string with $B = \mathrm{const.}$ governed by S with $\partial\Sigma \neq \emptyset$.

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Dirac quantisation along the D-brane [Chu,Ho (1999)]

$$\left[X^a(\tau, \sigma = 0), X^b(\tau, \sigma' = 0)\right] = 2\pi i \alpha' \left(B^{-1}\right)^{ab} =: i\theta^{ab}$$

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CFT correlation functions: NC parameter as phase [Seiberg,Witten (1999)]

- Related to correlators of the free theory by a phase factor

$$C^n_{g,\theta} = \exp\left[-\tfrac{i}{2}\textstyle\sum_{i<j}^n \theta^{ab}\, p^i_a\, p^j_b\right] C^n_{g,\theta=0}$$

- Phase captured by **Moyal star-product** on functions

$$(f_1\star\cdots\star f_n)(X) := e^{\frac{i}{2}\sum_{i<j}^n \theta^{ab}\frac{\partial}{\partial X^a_i}\frac{\partial}{\partial X^b_j}} f_1(X_1)\ldots f_n(X_n)\Big|_{X_i=X}$$

The Moyal star-product

Properties of the Moyal star-product [Cornalba,Schiappa (2001)]

- Captures non-vanishing commutator
- $\star_{\mathrm{n}}$ can be obtained by successive application of $\star_2$.
- If $d\phi = 0$ then $\star$ is associative. Otherwise $\star$ is nonassociative.
- The phase is independent of worldsheet coordinates $\Rightarrow$ genuine spacetime property

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The guiding principle

Product of the algebra of functions on the spacetime was deduced from the structure of CFT correlation functions.

Closed string with H-flux

[R.Blumenhagen, A.Deser, D.Lüst, E.Plauschinn, F.R. (2011)]; arXiv:1106.0316 [hep-th]

The stage: Closed string ($\Sigma = S^2$) with

- Three spacetime dimensions compactified on flat $\mathbb{T}^3$
- $B_{ab} = H_{abc}X^c$ with H_{abc} constant H-flux $\Rightarrow$ more difficult than open string case since constant B can be "gauged away"
- Admissible background: only linear in H

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What was done?

- We constructed a CFT for the system and computed the basic three-point function

$$\left\langle \mathcal{X}^a(z_1,\bar{z}_1)\mathcal{X}^b(z_2,\bar{z}_2)\mathcal{X}^c(z_3,\bar{z}_3)\right\rangle_R^H = \theta^{abc}\left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) \mp \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right)\right]$$

  with R denoting the **R-flux** background
- This can be used to compute CFT correlation functions

(Non-)geometry of R-flux background

The R-flux

- Three (formal) T-dualities on $\mathbb{T}^3$ with H-flux $\Rightarrow$ non-geometric R-flux background [Shelton,Taylor,Wecht (2005)]
- Basic three-point function implies a non-vanishing Jacobi identity for the spacetime coordinates only for R-flux $\Rightarrow$ nonassociative geometry [Blumenhagen,Plauschinn (2010)]

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A phase in CFT correlation functions

Since well understood, we scattered tachyons $\mathcal{V}^{\mp}$ in H- resp. R-flux background. A non-trivial phase emerges upon permuting operator insertions *only* for the R-flux:

$$C(\mathcal{V}^{\mp})_{\theta}^{\sigma(n)} = \exp\left(i\, \frac{1-\operatorname{sgn}(\sigma)}{2} \pi^2\, \theta^{abc} \sum_{i<j<k} p_{i,a} p_{j,b} p_{k,c} \right) C(\mathcal{V}^{\mp})_{\theta}^{n}$$

A nonassociative product

Properties of the phase

- Vanishes after imposing momentum conservation
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Phase captured by the product

$$(f_1 \triangle_n \ldots \triangle_n f_n)(x) :=$$

$$\exp\left(\frac{\pi^2}{2}\theta^{abc}\sum_{i<j<k}^{n}\frac{\partial}{\partial x_i^a}\frac{\partial}{\partial x_j^b}\frac{\partial}{\partial x_k^c}\right) f_1(x_1)\ldots f_n(x_n)\Big|_{x_i=x}$$

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An "intrinsically" nonassociative product

- $\triangle_n$ cannot be obtained from successive application of $\triangle_3$

Conclusion

We have

- found explicit hints for nonassociative geometries in string theory
- gained explicit insights on the structure of poorly understood R-flux backgrounds
- hints that generic string backgrounds correspond to more complicated (non-)geometries

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Thank you!

Backup

Background details

String beta-functionals [Callan,Friedan,Martinec,Perry (1985)]

$$\beta^G_{\mu\nu} = \alpha'\, R_{\mu\nu} - \frac{\alpha'}{4}\, H_{\mu\sigma\rho}\, H^{\sigma\rho}{}_{\nu} + \mathcal{O}(\alpha'^2) = 0$$

$$\beta^B_{\mu\nu} = \frac{\alpha'}{2}\, \nabla^\sigma H_{\sigma\mu\nu} + \mathcal{O}(\alpha'^2) = 0$$

$$\beta^\phi_{\mu\nu} = \frac{D-26}{6} - \frac{\alpha'}{24}\, H_{\mu\nu\sigma}\, H^{\mu\nu\sigma} + \mathcal{O}(\alpha'^2) = 0$$

with $H = dB$.

T-duality

To obtain $D = 4$ theory surplus dimensions will be compactified:

$$M = \mathbb{R}^{1,3} \times \mathbb{T}^{22}$$

If the compact manifold has isometries: T-duality [Buscher (1987); Roček,Verlinde (1992)]

- String has momentum $p = \frac{n}{R}$ *and* winding $w = \frac{mR}{\alpha'}$ along a cycle; $p \overset{T}{\longleftrightarrow} w$
- CFT point of view: $X = X_L + X_R \xrightarrow{T} X = X_L - X_R$
- T-dual theories are equivalent quantum theories
- The dual theories have very different spacetime (non-)geometries

CFT_H

Consistency equations predict CFT at $\mathcal{O}(H) \Rightarrow \mathrm{CFT}_H$

- Anti-/holomorphic currents

$$\mathcal{J}^a(z) := i\partial X^a(z,\bar z) - \frac{i}{2} H_{abc}\, \partial X^b(z)\, X^c_R(\bar z)$$

$$\overline{\mathcal{J}}^a(\bar z) := i\bar\partial X^a(z,\bar z) - \frac{i}{2} H_{abc}\, X^b_L(z)\, \bar\partial X^c(\bar z)$$

- Chiral three-point functions from *conformal perturbation theory*

$$\Big\langle \mathcal{J}^a(z_1)\, \mathcal{J}^b(z_2)\, \mathcal{J}^c(z_3) \Big\rangle = -\frac{i(\alpha')^2}{8} \frac{H^{abc}}{z_{12}\, z_{13}\, z_{23}}$$

$$\Big\langle \overline{\mathcal{J}}^a(z_1)\, \overline{\mathcal{J}}^b(z_2)\, \overline{\mathcal{J}}^c(z_3) \Big\rangle = +\frac{i(\alpha')^2}{8} \frac{H^{abc}}{\bar z_{12}\, \bar z_{13}\, \bar z_{23}}$$

- Current algebra

$$\mathcal{J}^a(z)\, \mathcal{J}^b(w) = \frac{\alpha'}{2} \frac{\delta^{ab}}{(z-w)^2} - i\frac{\alpha'}{4} \frac{H^{ab}{}_c}{z-w} \mathcal{J}^c(w) + \text{reg.}$$

Tachyon scattering

Tachyon vertex operator

$$\mathcal{V}(z,\overline{z}) = :\exp\left(ik_L \cdot \mathcal{X}_L + ik_R \cdot \mathcal{X}_R\right):$$

is a vertex operator of the perturbed theory and corresponds to the degenerate groundstate $|k_L, k_R\rangle$ provided $H^a{}_{bc} p^b w^c = 0$.

Correlation functions

- Structure of n-tachyon correlator

$$C(\mathcal{V}^{\mp})^n_\theta = \exp\Big\{ -i\theta^{abc} \sum_{i<j<k} p_{i,a} p_{j,b} p_{k,c} \Big[\mathcal{L}\Big(\frac{z_{ij}}{z_{ik}}\Big) \mp \mathcal{L}\Big(\frac{\overline{z}_{ij}}{\overline{z}_{ik}}\Big) \Big] \Big\} \times C(\mathcal{V}^{\mp})^n_{\theta=0}$$

- Permuting the operator insertions

$$C(\mathcal{V}^{\mp})^{\sigma(n)}_\theta = \exp\Big(i\epsilon \tfrac{1-\mathrm{sgn}(\sigma)}{2} \pi^2 \, \theta^{abc} \sum_{i<j<k} p_{i,a} p_{j,b} p_{k,c} \Big) C(\mathcal{V}^{\mp})^n_\theta$$