Feynman Diagrams For Pedestrians

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Abstract

This set of interleaved lectures and exercises will (re)introduce working experimental particle physicists to the techniques used for computing simple cross sections in the standard model and its extensions. The approach is deliberately pedestrian with an emphasis on real world applications.

After an introduction, gradually more and more time will be spent actually computing stuff in order to build confidence and gain intuition for (new) physics signals in cross sections.

`CAVEAT EMPTOR:`
`The typesetting has not been optimized yet,`
`in particular for the solutions.`

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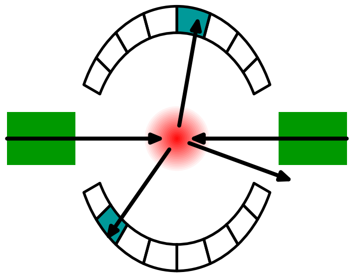
1 Introduction

1.1 Scattering Amplitudes

Basic principle of quantum mechanics

- an accelerator prepares an initial state $|\mathrm{in}\rangle$
- that is transformed by an interaction S
- and a detector measures the overlap of the resulting state with a final state $|\mathrm{out}\rangle$.



- the transition probability P is given by the absolute square of the transition amplitude A

$$A_{\mathrm{in}\to\mathrm{out}} = \langle \mathrm{out}|S|\mathrm{in}\rangle \tag{1a}$$
$$P_{\mathrm{in}\to\mathrm{out}} = |A_{\mathrm{in}\to\mathrm{out}}|^2 \tag{1b}$$

- if the initial and final states $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ are not pure states, the corresponding transition probabilities must be added (e. g. spins and flavors) or integrated (e. g. angles, energies and momenta)

Task(s):

1. describe $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$:
   - pure states: completetely polarized electrons, muons, photons

   ⇒ Dirac equation, Klein-Gordon equation, &c.
   - mixtures: protons, partially polarized or unpolarized electrons, muons, photons, . . .
2. compute S (i. e. the part of S that contributes to $\langle \mathrm{out}|S|\mathrm{in}\rangle$)
   - quantum electro dynamics (QED)
   - quantum chromo dynamics (QCD)
   - standard model (SM)
   - "new physics", "beyond the SM" (BSM)

   ⇒ Feynman rules
3. square $A_{\mathrm{in}\to\mathrm{out}}$ and integrate $P_{\mathrm{in}\to\mathrm{out}}$
   - Monte Carlo

1.2 Lorentz Transformations

Basic principle of special relativity:

- the velocity of light c is the same in each inertial system.

$\therefore$ the wavefronts of a spherical light wave is for every observer located at

$$|\vec{x}| = ct \tag{2}$$

- introducing the notation $x_0 = ct$, this means that the solutions of

$$x_0^2 - \vec{x}^2 = 0 \tag{2'}$$

  are the same in every inertial reference frame

- adding homogeneity and isotropy of space, this means that

$$x^2 = x_0^2 - \vec{x}^2 \tag{3}$$

  must be the same in every inertial reference frame.

- useful notations:
  - 3 vectors: (covariant w. r. t. rotations)

$$\vec{x} = (x^1, x^2, x^3) \tag{4}$$

  - 4 vectors: (covariant w. r. t. rotations and boosts into a moving inertial frame)

$$x = (x^0; \vec{x}) = (x^0; x^1, x^2, x^3) = (x_0; -x_1, -x_2, -x_3) \tag{5}$$

- introduce Minkowski metric

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

  to shift indices

$$x_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu, \qquad x^\mu = \sum_{\nu=0}^{3} g^{\mu\nu} x_\nu \tag{7}$$

- convenient summation convention

$$\begin{aligned} xp = \sum_{\mu=0}^{3} x_\mu p^\mu = x_\mu p^\mu = x^\mu p_\mu = g^{\mu\nu} x_\mu p_\nu = g_{\mu\nu} x^\mu p^\nu \\ = x_0 p_0 - \sum_{i=1}^{3} x_i p_i = x_0 p_0 - x_i p_i = x_0 p_0 - \vec{x}\vec{p} \end{aligned} \tag{8}$$

- a Lorentz transformation Λ must leave xp invariant because $2xp = (x+p)^2 - x^2 - p^2$:

$$x_\mu \to x'_\mu = \Lambda_\mu{}^\nu x_\nu \ (\text{mit } x'^2 = x^2) \iff g_{\mu\mu'} = \Lambda_\mu{}^\nu \Lambda_{\mu'}{}^{\nu'} g_{\nu\nu'} \tag{9}$$

- derivatives:

$$\frac{\partial}{\partial x_\mu} f(x) = \partial_x^\mu f(x) = \partial^\mu f(x)\,,\quad \frac{\partial}{\partial x^\mu} f(x) = \partial_\mu f(x) \tag{10}$$

for example

$$\partial_x^\mu (xp) = \partial(x_\nu p^\nu)/\partial x_\mu = p^\mu \tag{11}$$

Problem 1. *Compute the partial derivative w. r. t.* x

$$\partial_\mu e^{-ipx}\,,\quad (a\partial)(b\partial)e^{-ipx}\,,\quad \partial^2 e^{-ipx} \tag{12}$$

for constant four vectors a, b *and* p.

Problem 2. *Show that*

$$\partial_\mu x^\mu = 4 \tag{13a}$$

(NB: $\partial_\mu x^\mu = g_\mu{}^\nu \partial x^\mu/\partial x^\nu$ *and* $g_\mu{}^\nu = \delta_\mu{}^\nu$*) and compute*

$$\partial^2 e^{-x^2/2} \tag{13b}$$

1.3 Schrödinger Equation

- Wave functions satisfy the Schrödinger equation

$$i\hbar \frac{d}{dt}\Psi(t) = H\Psi(t) \tag{14a}$$

with solution (for infinitesimal time intervals)

$$\Psi(t+\delta t) = e^{-iH\cdot\delta t/\hbar}\Psi(t) \tag{14b}$$

$\therefore$ scattering amplitude (infinite time intervals)

$$A_{\text{in}\to\text{out}} = \langle \text{out}|S|\text{in}\rangle = \lim_{\substack{t_1\to-\infty\\ t_2\to+\infty}} \left\langle \text{out}(t_2)\middle| e^{-iH\cdot(t_2-t_1)/\hbar} \middle| \text{in}(t_1)\right\rangle \tag{15}$$

- Problems with this approach
  - particle production and decay has been observed, but can not be described by wave functions (without "2nd quantization"), because probability is conserved ("unitarity")
  - Schrödinger equation (14) not manifestly Lorentz covariant
  - free single particle equation

$$i\hbar \frac{d}{dt}\Psi(t) = \frac{1}{2m}\left(\frac{1}{i\hbar c}\vec{\nabla}\right)^2 \Psi(t) \tag{16}$$

    is manifestly not Lorentz covariant!

1.4 Units

- from now on, we will use units which will give us numbers with natural order of magnitude for quantum mechanics and relativistic kinematics
$$\hbar = c = 1\,. \tag{17}$$

- velocities and actions are dimensionless and therefore
$$[\text{energy}] = [\text{momentum}] = [\text{mass}] = \left[\frac{1}{\text{length}}\right]\,. \tag{18}$$

- in particular, our Feynman rules, will later yield cross sections in units of $[\text{energy}^{-2}]$, e. g.
$$\sigma = \frac{4\pi\alpha^2}{3E^2} \tag{19}$$

- the relevant conversion factors are
$$\hbar c = 197.327\,053(59)\,\text{MeV fm} \tag{20}$$
$$(\hbar c)^2 = 0.389\,379\,66(23)\,\text{TeV}^2\,\text{nb} \tag{21}$$
($\text{TeV}^2\,\text{nb} = \text{GeV}^2\,\text{mb}$) and therefore
$$\sigma = \frac{4\pi\alpha^2}{3(E/\text{TeV})^2} 0.39\,\text{nb} \tag{19$'$}$$

2 Asymptotic States

- described by wave equations, that are
  1. linear: superposition principle of quantum mechanics
  2. relativistic: matrix elements of observables must transform under rotations and Lorentz boosts like scalars, four vectors, tensors, &c.
  3. and have the correct dispersion relation: $E^2 = \vec{p}^2 + m^2$

- objects of interest
  - spin-0 particles: not yet(?) observed as an elementary particle, but possible (e. g. Higgs)
    * one invariant component
  - spin-1/2 particles: leptons, quarks
    * at least two components: spinor under rotations
  - spin-1 particles: gauge bosons
    * massive three components (polarizations)
    * massless two components (polarizations)

2.1 Klein-Gordon Equation

$$(i\partial_0)^2\phi(x) = \left[(-i\vec{\partial})^2 + m^2\right]\phi(x) \tag{22}$$

- is obviously a covariant wave equation, because

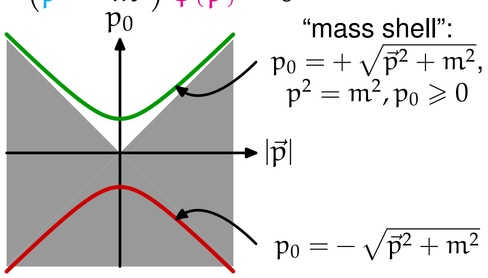
$$\left(\Box + m^2\right)\phi(x) = 0 \tag{23}$$

- fourier transform

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\phi}(p)\,,\quad i\partial_\mu\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} p_\mu\tilde{\phi}(p)\,,\quad \text{etc} \tag{24}$$

$\therefore$ algebraic equation

$$\left(p^2 - m^2\right)\tilde{\phi}(p) = 0 \tag{23'}$$



- correct relativistic dispersion relation $E = +\sqrt{\vec{p}^2 + m^2}$
- but what about the other solution $E = -\sqrt{\vec{p}^2 + m^2}$?

2.2 Free Spin-0 Particles

- general solution of the Klein-Gordon equation

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} 2\pi\Theta(p_0)\delta(p^2 - m^2)\left(\phi^{(+)}(\vec{p})e^{-ipx} + \phi^{(-)}(\vec{p})e^{ipx}\right) \tag{25}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}\bigg|_{p_0=\sqrt{\vec{p}^2+m^2}} \left(\phi^{(+)}(\vec{p})e^{-ipx} + \phi^{(-)}(\vec{p})e^{ipx}\right) \tag{26}$$

$$= \int \widetilde{dp}\,\left(\phi^{(+)}(\vec{p})e^{-ipx} + \phi^{(-)}(\vec{p})e^{ipx}\right) \tag{27}$$

- conserved current $\partial_0 j_0(x) - \vec{\nabla}\vec{j}(x) = \partial_\mu j^\mu(x) = 0$ out of two solutions ϕ_1 and ϕ_2 of the Klein-Gordon equation with the same mass:

$$j_\mu(x) = \phi_1^*(x) i\overleftrightarrow{\partial_\mu}\phi_2(x) = \phi_1^*(x)[i\partial_\mu\phi_2(x)] - [i\partial_\mu\phi_1^*(x)]\phi_2(x) \tag{28}$$

- indeed

$$\begin{aligned}\partial_\mu j^\mu(x) &= \partial^\mu\Big(\phi_1^*(x)[i\partial_\mu\phi_2(x)]\Big) - \partial^\mu\Big([i\partial_\mu\phi_1^*(x)]\phi_2(x)\Big)\\ &= i[\partial^\mu\phi_1^*(x)][\partial_\mu\phi_2(x)] + i\phi_1^*(x)[\partial^2\phi_2(x)] - i[\partial^2\phi_1^*(x)]\phi_2(x)\\ &\quad -i[\partial_\mu\phi_1^*(x)][\partial^\mu\phi_2(x)] = -i\phi_1^*(x)m^2\phi_2(x) + im^2\phi_1^*(x)\phi_2(x) = 0\end{aligned} \tag{29}$$

- invariant and time independent overlap out of the conserved current

$$
Q = \int_{x_0=t} d^3\vec{x}\, j_0(x) = \int_{x_0=t} d^3\vec{x}\, \phi_1^*(x) i \overleftrightarrow{\partial_0} \phi_2(x) =
$$

$$
\int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \frac{1}{2p_0} \Big(\phi_1^{(+)*}(\vec{p})\phi_2^{(+)}(\vec{p}) \cdot e^{ip_0t} i\overleftrightarrow{\partial_0} e^{-ip_0t} + \phi_1^{(-)*}(-\vec{p})\phi_2^{(+)}(\vec{p}) \cdot e^{-ip_0t} i\overleftrightarrow{\partial_0} e^{-ip_0t}
$$

$$
+ \phi_1^{(+)*}(\vec{p})\phi_2^{(-)}(-\vec{p}) \cdot e^{ip_0t} i\overleftrightarrow{\partial_0} e^{ip_0t} + \phi_1^{(-)*}(-\vec{p})\phi_2^{(-)}(-\vec{p}) \cdot e^{-ip_0t} i\overleftrightarrow{\partial_0} e^{ip_0t} \Big)
$$

$$
= \int \widetilde{dp} \left(\phi_1^{(+)*}(\vec{p})\phi_2^{(+)}(\vec{p}) - \phi_1^{(-)*}(\vec{p})\phi_2^{(-)}(\vec{p}) \right) \quad (30)
$$

- normalization only positive on the positive mass shell!

∴ $j_\mu(x)$ must not be interpreted as a probability current!

- ... anyway: the existence of the negative mass shell makes the energy unbounded from below and no ground state exists!

∴ $\phi(x)$ must not be interpreted as Schrödinger wave function!

2.3 Anti Particles

Observations:

- the positive and negative mass shells of free (i. e. not interacting) particles are independent

∴ one can simply project out the negative mass shell in this case ...

- ... unfortunately, all local and Lorentz invariant interactions couple positive and negative mass shell (see below)

- ... however, asymptotic states are assumed to be noninteracting

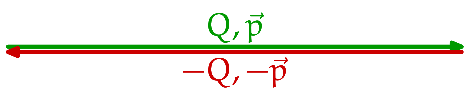
∴ we can at least reinterpret the negative mass shell

however:

- the amplitude $\phi^{(+)}(\vec{p})e^{-ipx}$ on the positive mass shell corresponds to the momentum $+\vec{p}$, but the amplitude $\phi^{(-)}(\vec{p})e^{+ipx}$ on the negative mass shell corresponds to the reversed momemtum $-\vec{p}$

∴ the formalism is consistent, if all quantum numbers are reversed on the negative mass shell

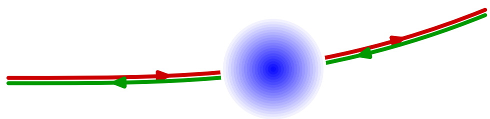
∵ in a stationary, i. e. time independent, state, the cases



can not be distinguished.

∴ the states on the negative mass shell describe not particles with "negative energy", but anti particles with opposite quantum numbers instead!

- in stationary states, this can be taken a ste further: instead of anti particles moving forward in time . . .



- . . . one can use particles moving backward in time, without noticing a difference in the overall balance!
- caveat: it is not (yet) obvious that this makes sense of interactions are switched on . . .
- . . . will be shown later.
- NB: this is a calculationally convenient interpretation — there is no time travel going on, since we're in a steady state
- The equivalent picture with anti particles moving forward in time requires the full machinery of quantum field theory

2.4 Dirac Equation

- task: find "objects" γ^μ, such that

$$(\gamma^\mu \partial_\mu)^2 = \partial^2 \tag{31}$$

- because the solutions of the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0 \tag{32}$$

automagically satisfy the die Klein-Gordon equation as well

$$(i\gamma^\mu \partial_\mu + m)\,(i\gamma^\mu \partial_\mu - m)\,\psi(x) = \left(-\partial^2 - m^2\right)\psi(x) = 0 \tag{33}$$

- the Dirac equation is obviously linear and its solutions satisfy the proper relativistic dispersion relation.
- can we construct "objects" γ^μ, that satisfy (31)?
- a sufficient condition is

$$[\gamma_\mu, \gamma_\nu]_+ := \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \cdot \mathbf{1} \tag{34}$$

because partial derivatives commute $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$.

- using a useful and ubiquitous notation, the Feynman slash

$$\not{a} = \gamma_\mu a^\mu = \gamma^\mu a_\mu \tag{35}$$

this reads equivalently $[\not{a}, \not{b}]_+ := \not{a}\not{b} + \not{b}\not{a} = 2 \cdot ab = 2 \cdot a_\mu b^\mu$

2.5 Gamma Matrices

- recall the Pauli matrices with the defining property

$$\left[\sigma^k,\sigma^l\right] = \sigma^k\sigma^l - \sigma^l\sigma^k = 2i\sum_{m=1}^{3}\epsilon^{klm}\sigma^m \tag{36a}$$

$$\left(\sigma^k\right)^\dagger = \sigma^k \tag{36b}$$

using the totally antisymmetric tensor ϵ

$$\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1,\ \epsilon^{213} = \epsilon^{321} = \epsilon^{132} = -1 \tag{37}$$

- concrete realisation

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{38}$$

- with

$$\sigma^k\sigma^l = \delta^{kl}\mathbf{1} + i\sum_{m=1}^{3}\epsilon^{klm}\sigma^m \tag{39}$$

- in particular

$$\left[\sigma^k,\sigma^l\right]_+ = 2\delta^{kl}\mathbf{1} \tag{40}$$

- Dirac realisation of the gamma (a. k. a. Dirac) matrices:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{41}$$

- there are (infinitely) many more realisations, but no smaller one
- we can verify the anti commutation relations by explicit calculation.

NB: block matrices are multiplied just like ordinary matrices with non commuting matrix elements

$$\left(\gamma^0\right)^2 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \mathbf{1} \tag{42a}$$

$$\begin{aligned}[\gamma^0,\gamma^i]_+ = \gamma^0\gamma^i + \gamma^i\gamma^0 &= \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{1}\cdot\sigma^i \\ (-\mathbf{1})\cdot(-\sigma^i) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (-\sigma^i)\cdot\mathbf{1} \\ \sigma^i\cdot(-\mathbf{1}) & 0 \end{pmatrix} \end{aligned} \tag{42b}$$

$$= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 0 \tag{42c}$$

Problem 3. *Verify the remaining* $(k,l = 1,2,3)$ *anti commutation relations (34):*

$$[\gamma^k,\gamma^l]_+ = -2\delta^{kl}\cdot\mathbf{1}\,. \tag{43}$$

- in the Dirac realization (41) we have obviously

$$\gamma_0^\dagger = \gamma_0\,, \quad \gamma_i^\dagger = -\gamma_i \tag{44}$$

- however, since $(\gamma_0)^2 = \mathbf{1}$ and $(\gamma_i)^2 = -\mathbf{1}$

∴ γ_0 must have only real eigenvalues, and

∴ all γ_i must have only imaginary eigenvalues

∴ this must be true in all realizations.

- Another useful and ubiquitous notation is therefore the Dirac adjoint for matrices

$$\overline{A} = \gamma_0 A^\dagger \gamma_0\,, \quad \overline{\gamma_\mu} = \gamma_0 \gamma_\mu^\dagger \gamma_0 = \gamma_\mu \tag{45}$$

- NB: on the next page we will meet the related, but different Dirac adjoint for column vectors

$$\bar{v} = v^\dagger \gamma_0 \tag{46}$$

- Don't mix them up!

2.6 Free Spin-1/2 Particles

- Ansatz:

$$\psi(x) = \int \widetilde{dp}\, \left(\psi^{(+)}(p)e^{-ipx} + \psi^{(-)}(p)e^{ipx}\right) \tag{47}$$

$$(i\partial\!\!\!/ - m)\,\psi(x) = 0 \Leftrightarrow \begin{cases} (p\!\!\!/ - m)\,\psi^{(+)}(p) = 0 \\ (p\!\!\!/ + m)\,\psi^{(-)}(p) = 0 \end{cases} \tag{48}$$

- the adjoint solution

$$\bar\psi(x) = \psi(x)^\dagger \gamma_0 = \int \widetilde{dp}\, \left(\bar\psi^{(+)}(p)e^{ipx} + \bar\psi^{(-)}(p)e^{-ipx}\right) \tag{49}$$

satisfies

$$\begin{aligned}\bar\psi(x) i \overleftarrow{\partial\!\!\!/} &= i\partial_\mu \bar\psi(x)\gamma^\mu = i\partial_\mu \psi(x)^\dagger \gamma_0 \gamma^\mu \gamma_0 \gamma_0 \\ &= i\partial_\mu \psi(x)^\dagger \gamma^{\mu\dagger}\gamma_0 = \left(-i\partial_\mu \gamma^\mu \psi(x)\right)^\dagger \gamma_0 = \overline{\left(-i\partial\!\!\!/ \psi(x)\right)} = -m\bar\psi(x)\end{aligned} \tag{50}$$

∴

$$\bar\psi(x)\left(i\overleftarrow{\partial\!\!\!/} + m\right) = 0\,, \tag{51}$$

or

$$\bar\psi^{(+)}(p)\,(p\!\!\!/ - m) = 0 \tag{52a}$$

$$\bar\psi^{(-)}(p)\,(p\!\!\!/ + m) = 0 \tag{52b}$$

- general solution of the Dirac equation

$$\psi^{(+)}(p) = \sum_{k=1}^{2} u_k(p) b_k(p) \tag{53a}$$

$$\psi^{(-)}(p) = \sum_{k=1}^{2} v_k(p) d_k(p) \tag{53b}$$

with four independent basis solutions $u_1(p)$, $u_2(p)$, $v_1(p)$, $v_2(p)$ satisfying

$$(\not{p} - m) u_k(p) = 0 \tag{54a}$$

$$(\not{p} + m) v_k(p) = 0 \tag{54b}$$

and the corresponding expansion coefficients $b_1(p)$, $b_2(p)$, $d_1(p)$, $d_2(p)$.

- in the rest frame of the particle, i. e. for $p = (m, \vec{0})$, we have $\not{p} = m\gamma_0$ and the Dirac equation simplifies to

$$m(\gamma_0 - \mathbf{1}) u_k(\vec{0}) = \begin{pmatrix} 0 & 0 \\ 0 & -2m \cdot \mathbf{1} \end{pmatrix} u_k(\vec{0}) = 0 \tag{55a}$$

$$m(\gamma_0 + \mathbf{1}) v_k(\vec{0}) = \begin{pmatrix} 2m \cdot \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} v_k(\vec{0}) = 0 \tag{55b}$$

- the independent solutions in the rest frame are therefore

$$u_1(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2(\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{56a}$$

$$v_1(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{56b}$$

- from which we can construct solutions for arbitrary on shell momenta with $p^2 = m^2$

$$u_k(p) = \frac{\not{p} + m}{\sqrt{p_0 + m}} u_k(\vec{0}) \tag{57a}$$

$$v_k(p) = \frac{\not{p} - m}{\sqrt{p_0 + m}} v_k(\vec{0}) \tag{57b}$$

- since $(\not{p} + m)(\not{p} - m) = p^2 - m^2$, these are obviously solutions of the Dirac equation for on-shell momenta

- the motivation for the not obviously covariant normalization will be apparent after problem 4 mathematical reminder:

- compare the inner product of a row vector with a column vector

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i b_i \tag{58}$$

to the outer product of a column vector with a row vector

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 & \cdots & b_m \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_m \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_m \\ \vdots & \vdots & & \vdots \\ a_n b_1 & a_n b_2 & \dots & a_n b_m \end{pmatrix} \tag{59}$$

- these are two very different operations
  - the inner product produces a number
  - the outer product produces a matrix

Problem 4. *Determine* $\bar{u}_k(p)$ *und* $\bar{v}_k(p)$ *from the definitions and show that for* $p^2 = m^2$

$$\sum_{k=1}^{2} u_k(p)\bar{u}_k(p) = \not{p} + m\,, \quad \sum_{k=1}^{2} v_k(p)\bar{v}_k(p) = \not{p} - m \tag{60}$$

Problem 5. *Compute (always assuming* $p^2 = m^2$*)*

$$\bar{u}_k(p)u_l(p)\,,\ \bar{v}_k(p)v_l(p)\,,\ \bar{u}_k(p)v_l(p)\,,\ \bar{v}_k(p)u_l(p)\,. \tag{61}$$

Problem 6. *Compute (always assuming* $p^2 = m^2$*)*

$$\bar{u}_k(p)\gamma_\mu u_l(p)\,,\ \bar{v}_k(p)\gamma_\mu v_l(p)\,,\ \bar{u}_k(\vec{p})\gamma_0 v_l(-\vec{p})\,,\ \bar{v}_k(-\vec{p})\gamma_0 u_l(\vec{p})\,. \tag{62}$$

- this is a special case of the Gordon decomposition for arbitrary solutions of the Dirac equation:

$$\begin{aligned} \bar{u}_k(p)\gamma_\mu u_l(q) &= \bar{u}_k(p)\frac{\not{p}\gamma_\mu + \gamma_\mu\not{q}}{2m}u_l(q) \\ &= \bar{u}_k(p)\frac{p_\nu(g_{\nu\mu} + \frac{1}{2}[\gamma_\nu,\gamma_\mu]_-) + q_\nu(g_{\mu\nu} + \frac{1}{2}[\gamma_\mu,\gamma_\nu]_-)}{2m}u_l(q) \\ &= \frac{p_\mu + q_\mu}{2m}\bar{u}_k(p)u_l(q) + \frac{q_\nu - p_\nu}{2im}\bar{u}_k(p)\frac{i}{2}[\gamma_\mu,\gamma_\nu]_- u_l(q) \end{aligned} \tag{63}$$

- there are alltogether 16 independent (anti-)hermitian 4×4-matrices:

$$\mathbf{1} \qquad 1 \quad \text{“scalar”} \tag{64a}$$

$$\gamma_\mu \qquad 4 \quad \text{“vector”} \tag{64b}$$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]_- \qquad 6 \quad \text{“tensor”} \tag{64c}$$

$$\gamma_5\gamma_\mu \qquad 4 \quad \text{“axial vector”} \tag{64d}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad 1 \quad \text{“pseudo scalar”} \tag{64e}$$

- NB: the bare gamma matrices do not transform like vectors, tensors, axial vectors, or pseudo scalars!

- there are additional nontrivial transformations $L(\Lambda)$ that have to be applied on the left and right, e. g.

$$\gamma_\mu \to \Lambda_\mu{}^\nu L(\Lambda)\gamma_\nu L^{-1}(\Lambda) \tag{65}$$

- however since

$$\psi(x) \to L(\Lambda)\psi(\Lambda^{-1}x), \quad \bar\psi(x) \to \bar\psi(\Lambda^{-1}x)L^{-1}(\Lambda) \tag{66}$$

the $L(\Lambda)$ compensate each other in matrix elements (a. k. a. “sandwiches”)

$$\begin{aligned}\bar\psi(x)\gamma_\mu\psi(y) \to \bar\psi(\Lambda^{-1}x)L^{-1}(\Lambda)\Lambda_\mu{}^\nu L(\Lambda)\gamma_\nu L^{-1}(\Lambda)L(\Lambda)\psi(\Lambda^{-1}x)\\ = \Lambda_\mu{}^\nu\bar\psi(\Lambda^{-1}x)\gamma_\nu\psi(\Lambda^{-1}x)\end{aligned} \tag{67}$$

∴ and the $L(\Lambda)$ can be ignored in the computation of matrix elements

∴ the characterization as vector, tensor, axial vector, or pseudo scalars is meaningful

Problem 7. *Compute* $[\gamma_5, \gamma_\mu]_+$

Problem 8. *Show the conservation of the vector current for two solutions* $\psi_1(x)$ *und* $\psi_2(x)$ *of the Dirac equation* (32) *and* (51)

$$\partial^\mu\left[\bar\psi_1(x)\gamma_\mu\psi_2(x)\right] = 0\,. \tag{68}$$

- invariant overlap from conserved current:

$$\begin{aligned}Q &= \int\limits_{x_0=t} d^3\vec{x}\, j_0(x) = \int\limits_{x_0=t} d^3\vec{x}\,\bar\psi_1(x)\gamma_0\psi_2(x) = \int\limits_{x_0=t} d^3\vec{x}\,\psi_1^\dagger(x)\psi_2(x)\\ &= \int\frac{d^3\vec{p}}{(2\pi)^3 2p_0}\frac{1}{2p_0}\Big(\psi_1^{(+)\dagger}(\vec{p})\psi_2^{(+)}(\vec{p}) + \psi_1^{(-)\dagger}(-\vec{p})\psi_2^{(+)}(\vec{p})\cdot e^{-2ip_0t}\\ &\qquad + \psi_1^{(+)\dagger}(\vec{p})\psi_2^{(-)}(-\vec{p})\cdot e^{2ip_0t} + \psi_1^{(-)\dagger}(-\vec{p})\psi_2^{(-)}(-\vec{p})\Big)\\ &= \int\frac{d^3\vec{p}}{(2\pi)^3 2p_0}\sum_{k=1}^{2}\Big(b_{1,k}^\dagger(p)b_{2,k}(\vec{p}) + d_{1,k}^\dagger(p)d_{2,k}(\vec{p})\Big)\end{aligned} \tag{69}$$

$\therefore$ normalization positive everywhere!

Problem 9. *Show the Partial Conservation of the Axial Current (PCAC) for solutions $\psi_1(x)$ und $\psi_2(x)$ of the Dirac equation*

$$\partial^\mu\left[\bar\psi_1(x)\gamma_\mu\gamma_5\psi_2(x)\right] = 2im\bar\psi_1(x)\gamma_5\psi_2(x)\,. \tag{70}$$

Problem 10. *Compute $\sigma_{kl} = \frac{i}{2}[\gamma_k, \gamma_l]_-$ for $k, l = 1, 2, 3$ in the Dirac realization of the Gamma matrices.*

$\therefore$ the matrices $\{\sigma_{23}, \sigma_{31}, \sigma_{12}\}$ can take over the rôle of the Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ and distinguish spin up from spin down in the rest frame

$$\frac{1}{2}\sigma_{12}u_1(\vec 0) = +\frac{1}{2}u_1(\vec 0)\,, \qquad \frac{1}{2}\sigma_{12}u_2(\vec 0) = -\frac{1}{2}u_2(\vec 0) \tag{71a}$$

$$\frac{1}{2}\sigma_{12}v_1(\vec 0) = +\frac{1}{2}v_1(\vec 0)\,, \qquad \frac{1}{2}\sigma_{12}v_2(\vec 0) = -\frac{1}{2}v_2(\vec 0) \tag{71b}$$

- just as before in the case of scalar particles, we can interpret solutions on the negative energy mass shell for spin-1/2 particles as anti particles that move in the opposite direction in space time:
  - $u_k(p)$ amplitude for a particles in the initial state
  - $\bar u_k(p)$ amplitude for a particle in the final state
  - $v_k(p)$ amplitude for an anti particle in the final state
  - $\bar v_k(p)$ amplitude for an anti particle in the initial state
- in addition to the other quantum numbers, we must flip the spins so that the overall balance is maintained

$$\xrightarrow{\quad Q, \vec p, s \quad}$$
$$\xleftarrow[\quad -Q, -\vec p, -s \quad]{}$$

2.7 Free Spin-1 Particles

- combine $\vec E$ and $\vec B$ into field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{72}$$

- manifestly covariant Maxwell equations

$$\partial^\mu F_{\mu\nu} = j_\nu\,, \ \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \tag{73}$$

- current j_μ necessarily conserved: $\partial_\mu j^\mu = 0$

- equation of motion for the vector potential A_μ

$$(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu = j^\mu \tag{73'}$$

- gauge invariance: $F_{\mu\nu}$ does not change, if

$$A_\mu(x) \to A_\mu(x) - \partial_\mu\omega(x) \tag{74}$$

- with special gauge condition $\partial_\mu A^\mu = 0$: $\partial^2 A_\mu = j_\mu$
- more, but not most, general case

$$(g^{\mu\nu}\partial^2 - (1-\xi)\partial^\mu\partial^\nu)A_\nu = j^\mu \tag{75}$$

- explicit mass term

$$\partial^\mu F_{\mu\nu} + M^2 A_\nu = j_\nu \tag{76}$$

or

$$(g^{\mu\nu}(\partial^2 + M^2) - \partial^\mu\partial^\nu)A_\nu = j^\mu \tag{76'}$$

- contraction with ∂_μ

$$M^2\partial^\nu A_\nu = 0 \tag{77}$$

- massless vector bososns have two degrees of freedom, massive ones three
- polarization vectors for massless vector bosons with momentum $k = (k_0; 0, 0, k_0)$

$$\epsilon_\pm = \epsilon^*_\mp = \frac{1}{\sqrt{2}}(0; 1, \pm i, 0) \tag{78}$$

- properties

$$\epsilon^\mu_\lambda \epsilon^*_{\lambda',\mu} = -\delta_{\lambda\lambda'} \tag{79a}$$

$$\epsilon^\mu_\lambda k_\mu = 0 \tag{79b}$$

and with $c = (1; 0, 0, -1)$

$$\sum_{\lambda=-1,+1} \epsilon^\mu_\lambda \epsilon^{\nu,*}_\lambda = -g^{\mu\nu} + \frac{c_\mu k_\nu + c_\nu k_\mu}{ck} \tag{80}$$

- polarization vectors for massive vector bosons with momentum

$k = (k_0; |\vec{k}| \sin\theta\cos\phi, |\vec{k}|\sin\theta\sin\phi, |\vec{k}|\cos\theta)$:

$$\epsilon_\pm = \epsilon^*_\mp = \frac{e^{\mp i\phi}}{\sqrt{2}}(0; \cos\theta\cos\phi \mp i\sin\phi, \cos\theta\sin\phi \pm i\cos\phi, -\sin\theta) \tag{81a}$$

$$\epsilon_0 = \frac{k_0}{M}(|\vec{k}|/k_0; \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \epsilon^*_0 \tag{81b}$$

- properties (79) and

$$\sum_{\lambda=-1,0,+1} \epsilon^\mu_\lambda \epsilon^{\nu,*}_\lambda = -g^{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \tag{82}$$

3 Interactions

3.1 Propagators

- Photons far from all electrical charges satisfy

$$\partial^2 A^{(0)}_\mu(x) = 0 \tag{83}$$

  and we have already seen the solutions.

- In the presence of electrical charges, the photons couple to the electromagnetic current

$$\partial^2 A_\mu(x) = j_\mu(x) = -e\bar{\psi}(x)\gamma_\mu\psi(x) + \ldots \tag{84}$$

  and the solutions turn out to be more complicated.

- Assumption: there is a "function" D, that solves

$$(\partial^2 + m^2)D(x, m) = -\delta^4(x) \tag{85}$$

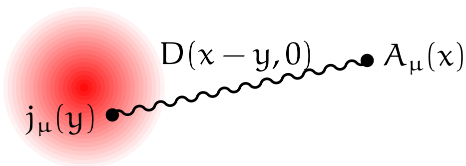
- Then

$$A_\mu(x) = A^{(0)}_\mu(x) - \int d^4y D(x-y, 0) j_\mu(y) \tag{86}$$

  is a solution of the inhomogeneous equation (84) for each solution of the homogeneous equation (83), since

$$\begin{aligned}\partial^2 A_\mu(x) &= \partial^2 A^{(0)}_\mu(x) - \int d^4y \left[\partial^2 D(x-y,0)\right] j_\mu(y) \\ &= 0 - \int d^4y \left[-\delta^4(x-y)\right] j_\mu(y) = j_\mu(x)\end{aligned} \tag{87}$$

- interpretation: the current $j_\mu(y)$ acts at the space time point y as a source of photons, that are "propagated" by the propagator $D(x-y,0)$ to the space time point x



(86′)

- NB: retardation is built in in (86), because we integrate over the four dimensional space time and not over the three dimensional space at a given instant.

Problem 11. *Show that*

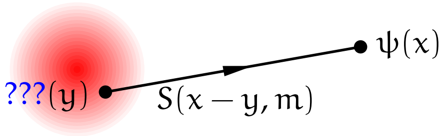
$$S(x, m) = (i\not{\partial} + m)\, D(x, m) \tag{88}$$

is the Feynman propagator S for Dirac particles of mass m*, i. e. that*

$$(i\not{\partial} - m)\, S(x, m) = \delta^4(x)$$

if $(\partial^2 + m^2)D(x, m) = -\delta^4(x)$*, as in* (85) *above.*

- but what is the source ??? of the Dirac field?



(89)

- consider the Dirac equation with (electromagnetic) interaction

$$(i\not{\partial} - e\not{A}(x) - m)\, \psi(x) = 0 \tag{90}$$

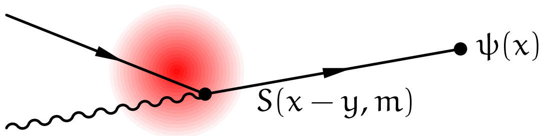
or

$$(i\not{\partial} - m)\, \psi(x) = e\not{A}(x)\psi(x) \tag{90'}$$

- with the formal solution

$$\psi(x) = \psi^{(0)}(x) + \int d^4y S(x-y, m) e\not{A}(y)\psi(y) \tag{91}$$

that can be represented graphically as

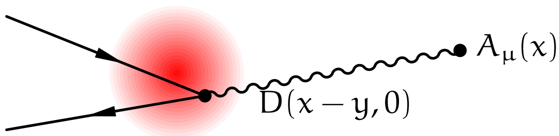


(91′)

- (91) is analogous to (86), when we use the current $j_\mu(y) = -e\bar{\psi}(y)\gamma_\mu\psi(y)$

$$A_\mu(x) = A_\mu^{(0)}(x) - \int d^4y D(x-y, 0) e\bar{\psi}(y)\gamma_\mu\psi(y) \tag{92}$$

which can be represented graphically as



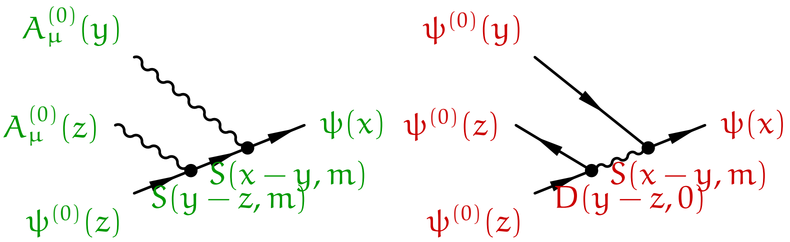
(92′)

- caveat: the equations (91) and (92) are not explicit solutions, but coupled integral equations

- that can be solved recursively by mutual series expansion

$$
\begin{aligned}
\psi(x) = \psi^{(0)}(x) + \int d^4y S(x-y,m) e\not{A}(y)\psi(y) \\
= \psi^{(0)}(x) + e\int d^4y S(x-y,m)\not{A}^{(0)}(y)\psi^{(0)}(y) \\
+ e^2 \int d^4y d^4z \Big(S(x-y,m)\not{A}^{(0)}(y)S(y-z,m)\not{A}^{(0)}(z)\psi^{(0)}(z) \\
- S(x-y,m)\gamma^\mu\psi^{(0)}(y)D(y-z,m)\bar{\psi}(z)\gamma_\mu\psi(z)\Big) + O(e^3)
\end{aligned} \tag{93}
$$

- these recursive expansions become very big very fast and a graphical representation is useful:



(93′)

- remaining open questions:
  - does $D(x-y,m)$ exist?
  - can we compute it?
- fortunately, we only need to solve

$$
\left(\partial^2 + m^2\right) D(x,m) = -\delta^4(x) \tag{85′}
$$

because (most) other propagators can be obtained by taking derivatives

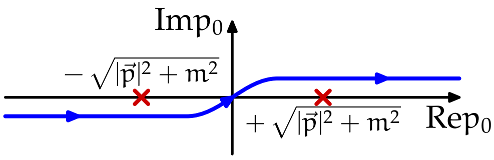
- since the equation is translation invariant, we should use Fourier transformation
- formally (with $\epsilon \to 0+$)

$$
D(x,m) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{1}{p^2 - m^2 + i\epsilon} \tag{94}
$$

$$
\left(\partial^2 + m^2\right) D(x,m) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{-p^2 + m^2}{p^2 - m^2 + i\epsilon} = -\int \frac{d^4p}{(2\pi)^4} e^{-ipx} = -\delta^4(x) \tag{95}
$$

- singularities in the integral over p_0 at $\pm\sqrt{|\vec{p}|^2 + m^2}$ ($+i\epsilon$ is a convenient shorthand for the choice of integration contour):

- compare

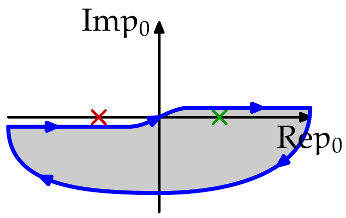


$$\frac{1}{p^2-m^2+i\epsilon} = \frac{1}{(p_0)^2-(|\vec{p}|^2+m^2)+i\epsilon} \overset{E=+\sqrt{|\vec{p}|^2+m^2}}{=} \frac{1}{(p_0)^2-E^2+i\epsilon}$$
$$\overset{E>0}{=} \frac{1}{(p_0)^2-(E-i\epsilon)^2} = \frac{1}{p_0-E+i\epsilon}\,\frac{1}{p_0+E-i\epsilon} = \frac{1}{2E}\left(\frac{1}{p_0-E+i\epsilon}-\frac{1}{p_0+E-i\epsilon}\right) \tag{96}$$

- forward in time:

$$x_0>0: \lim_{p_0\to -i\infty} e^{-ip_0x_0}\to 0 \tag{97a}$$

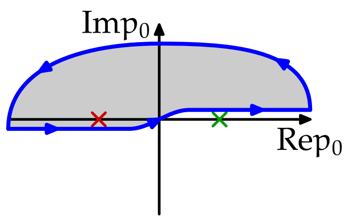
the integration contour in (94) can be closed below



- backward in time:

$$x_0<0: \lim_{p_0\to +i\infty} e^{-ip_0x_0}\to 0 \tag{97b}$$

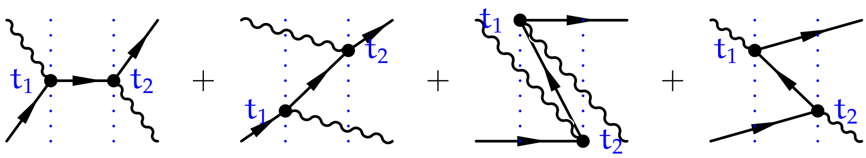
the integration contour in (94) can be closed above



$\therefore$ in

$$\Phi'(x) = \int d^4y\, D(x-y,m)\Phi(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\frac{1}{p^2-m^2+i\epsilon}\tilde{\Phi}(p) \tag{98}$$

- the part of $\tilde{\Phi}(p)$ with $p_0=+\sqrt{|\vec{p}|^2+m^2}$ is propagated into the future
- the part of $\tilde{\Phi}(p)$ with $p_0=-\sqrt{|\vec{p}|^2+m^2}$ is propagated into the past
- Compton scattering in nonrelativistic perturbation theory contains scattering as well as pair creation and pair annihilation contributions:



(99)

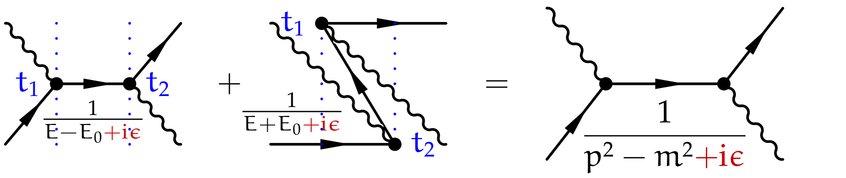
∵ intermediate states violate energy conservation and vertices can have space like distances

∴ temporal order of t_1 and t_2 depends in general on the reference frame, i. e. is undefined

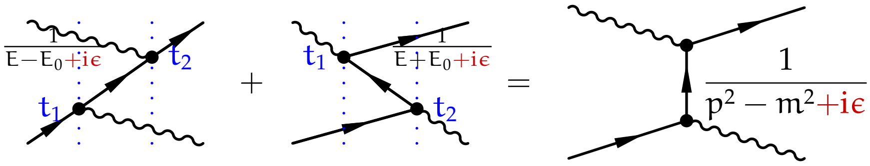
- Feynman's brilliant (re-)interpretation:
  - particles with $p_0 = +\sqrt{|\vec{p}|^2 + m^2}$ are propagated into the future
  - anti particles with $p_0 = -\sqrt{|\vec{p}|^2 + m^2}$ and opposite charges are propagated into past

  ∴ charges are conserved along the arrows in (99)!

  ∴ the four nonrelativistic diagrams in (99) can be combined to two covariant expressions using Feynman propagators



(100a)



(100b)

∴ the Feynman propagator allows to extend our interpretation of external, non-interacting anti particles as particles traveling backward in time to interacting particles.

Problem 12. *Compute the propagator* $S(x, m)$ *for Dirac particles in momentum space!*

- propagator for massless spin-1 particles

$$\frac{-ig_{\mu\nu} + i(1-\xi)\frac{k_\mu k_\nu}{k^2+i\epsilon}}{k^2 + i\epsilon} \tag{101}$$

- the gauge parameter ξ ist arbitrary and must cancel in the final result
- partial results can depend on ξ
- the propagator for massive spin-1 particles

$$\frac{-ig_{\mu\nu} + i\frac{k_\mu k_\nu}{M^2}}{k^2 - M^2 + i\epsilon} \tag{102}$$

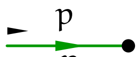
  is not gauge dependent, because (76′) can be inverted, contrary to (73′)

- NB: this is not what happens in the standard model, where the mass comes solely from the coupling to the Higgs sector and a gauge freedom remains. However (102) is in lowest order equivalent to the Higgs mechanism in unitarity gauge

- NB: propagators and external states are universal and models differ only in the particle content and interaction vertices!

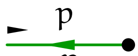
3.2 Feynman Rules

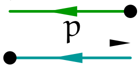
external spin-1/2 particles:

incoming:  $\Longleftrightarrow \cdots u(p)$ (103a)

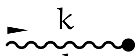
outgoing:  $\Longleftrightarrow \bar{u}(p) \cdots$ (103b)

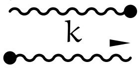
external spin-1/2 anti particles:

incoming:  $\Longleftrightarrow \bar{v}(p) \cdots$ (103c)

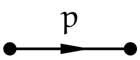
outgoing:  $\Longleftrightarrow \cdots v(p)$ (103d)

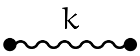
external spin-1 particles:

incoming:  $\Longleftrightarrow \epsilon_\mu(k)$ (103e)

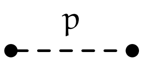
outgoing:  $\Longleftrightarrow \epsilon^*_\mu(k)$ (103f)

- internal particles and anti particles with sign of momentum relative to the arrow (i. e. charge) direction:

spin-1/2:  $\Longleftrightarrow \dfrac{i}{\not{p} - m + i\epsilon}$ (104a)

spin-1 ($m = 0$):  $\Longleftrightarrow \dfrac{-ig_{\mu\nu} + i(1-\xi)\frac{k_\mu k_\nu}{k^2+i\epsilon}}{k^2 + i\epsilon}$ (104b)

- finally

spin-0:  $\Longleftrightarrow \dfrac{i}{p^2 - m^2 + i\epsilon}$ (105)

$\because$ the S-matrix always contains an uninteresting diagonal piece (no interaction) and the global momentum conservation δ-distribution

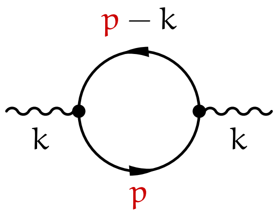
$$S = \mathbf{1} + (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2 \ldots - \ldots q_n) iT \tag{106}$$

$\therefore$ we can focus on T

- the following Feynman rules produce an expression for iT:

1. draw all diagrams using propagators and interaction vertices that connect the initial state with the final state
2. assign momenta and external wave functions accordingly
3. use momentum conservation at each vertex to fix the momenta of the internal lines
4. follow each connected fermion line against the direction of the arrows and collect wave functions, propagators and vertices along the way
5. complete iT with the remaining wave functions, propagators and vertices
6. add all diagrams with the relative signs such that that sum is anti symmetric under the exchange of identical external (anti-)fermions (and symmetric for bosons)

- in diagrams with closed loops not all momenta are fixed by momentum conservation, e. g.:



- in this case, one must integrate over these loop momenta with

$$\int \frac{d^4 p}{(2\pi)^4} \cdots \tag{107}$$

- unfortunately, there are infinitely many loop diagrams for each process
- fortunately, each loop comes with additional powers of the coupling constants
- in weakly interacting theories, we can expand the scattering amplitudes in the couplings constants or the number of loops

3.3 Cross Section

- definition in terms of observables

$$\sigma(\Delta\Phi) = \frac{R(\Delta\Phi)}{j} \qquad \text{where} \tag{108}$$

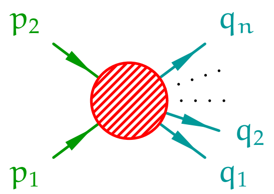
$$\begin{aligned} \Delta\Phi &= \text{region of phase space} && \text{(109a)} \\ \sigma(\Delta\Phi) &= \text{cross section for scattering into } \Delta\Phi && \text{(109b)} \\ R(\Delta\Phi) &= \text{event rate in } \Delta\Phi && \text{(109c)} \\ j &= \text{incoming flux} && \text{(109d)} \end{aligned}$$

- for fixed target experiments, the incoming flux j is just the number of incoming particles per time and area
- differential cross section

$$\sigma(\Delta\Phi) = \int_{\Delta\Phi} \frac{d\sigma}{d\Phi}(\Phi)d\Phi \tag{110}$$

with phase space element $d\Phi$, e. g. $d\Omega = \sin\theta d\theta d\phi$ for $2 \to 2$ processes

- the differential cross section can be computed from the scattering amplitude T and the (a. k. a. "Fermi's golden rule")
- general formula for $2 \to n$ processes



$$d\sigma = \frac{|T|^2}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}} \frac{\widetilde{dq_1}\dots\widetilde{dq_n}}{\prod_i n_i!}(2\pi)^4\delta^4(p_1 + p_2 - q_1 - \dots - q_n) \tag{111}$$

- where we have used again

$$\widetilde{dp} = \left.\frac{d^3\vec{p}}{(2\pi)^3 2p_0}\right|_{p_0=\sqrt{\vec{p}^2+m^2}} \tag{112}$$

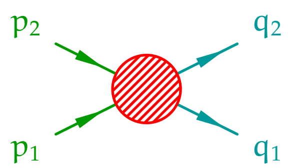
NB: in the old days, people use a different normalization for fermions (an additional factor $2m$), that is only useful for heavy (slow) particles ...

- symmetry factor

$$n_i = \begin{cases}\text{number of identical particles of the species } i \text{ in the final state}\end{cases} \tag{113}$$

3.4 Kinematics

- simplest example: $2 \to 2$



- invariants: Mandelstam variables

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \qquad \text{(total energy)} \tag{114a}$$
$$t = (q_1 - p_1)^2 = (q_2 - p_2)^2 \qquad \text{(momentum transfer)} \tag{114b}$$
$$u = (q_1 - p_2)^2 = (q_2 - p_1)^2 \tag{114c}$$

- momentum conservation:

$$s + t + u = p_1^2 + p_2^2 + q_1^2 + q_2^2 = \sum_{i=1}^{4} m_i^2 \tag{115}$$

- center of mass system (CMS) at high energies

$$p_{1/2} = (E; 0, 0, \pm E) \tag{116a}$$
$$q_{1/2} = (E; \pm E \sin\theta \cos\phi, \pm E \sin\theta \sin\phi, \pm E \cos\theta) \tag{116b}$$

$$s = 4E^2\,,\; t = -2E^2(1 - \cos\theta)\,,\; u = -2E^2(1 + \cos\theta) \tag{117}$$

3.5 Phase Space

- two particles in the final state:

$$\begin{aligned}\int \frac{d^3\vec{q}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{q}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(q_1 + q_2 - P) \\ = \frac{1}{16\pi^2} \int \frac{|\vec{q}_1|^2 d|q_1| d\Omega_1}{E_1 E_2} \delta(E_1(|\vec{q}_1|) + E_2(|\vec{q}_1|) - E) \\ = \frac{1}{16\pi^2} \int \frac{|\vec{q}_1| E_1 dE_1 d\Omega_1}{E_1 E_2} \delta(E_1 + E_2(E_1) - E) = \frac{1}{16\pi^2} \int d\cos\theta_1 d\phi_1 \frac{|\vec{q}_1|}{E}\end{aligned} \tag{118}$$

- the second equality in (118) follows from $E^2 = |\vec{q}|^2 + m^2$, which yields $|\vec{q}| d|\vec{q}| = E dE$ independent of the masses

- in the third equality we have used that E_2 depends on E_1 through momentum conservation and dispersion relations

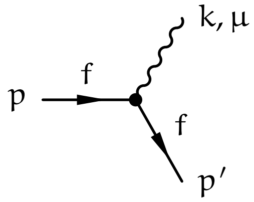
$$\frac{d(E_1 + E_2(E_1) - E)}{dE_1} = 1 + E_1/E_2 = E/E_2 \tag{119}$$

- special case: high energy limit in the center of mass frame: $|\vec{q}_1| = |\vec{q}_2| = E/2 + O(m/|\vec{q}_2|^2)$.

$$d\cos\theta_1\, d\phi_1 \,/\, (32\pi^2) \tag{118'}$$

4 QED

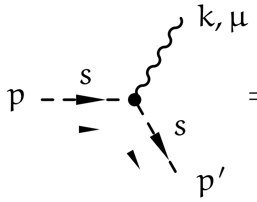
- Quantum Electro Dynamics: interacting electrons, positrons and photons (and other charged particles)
- single interaction vertex for fermions:



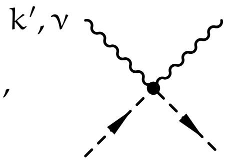
$$= iQ_f e\gamma_\mu \tag{120}$$

with the electrical charge Q_f of the particle, e. g. $Q_e = -1$ for electrons

- charged bosons require an additional ("sea gull") vertex



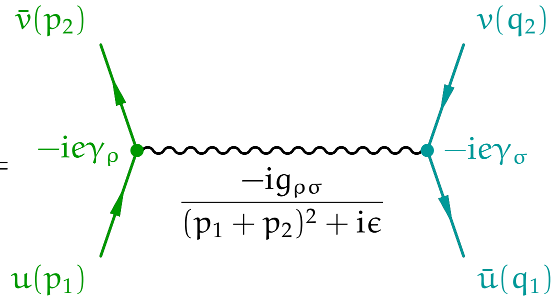
$$= iQ_s e(p_\mu + p'_\mu),$$



$$= 2iQ_s^2 e^2 g_{\mu\nu} \tag{121}$$

4.1 $e^+e^- \to \mu^+\mu^-$

$$iT =$$



(122)

$$\begin{aligned} iT &= \bar{v}(p_2)(-ie\gamma^\rho)u(p_1)\frac{-ig_{\rho\sigma}}{(p_1+p_2)^2+i\epsilon}\bar{u}(q_1)(-ie\gamma^\sigma)v(q_2) \\ &= i\frac{e^2}{s}\left[\bar{v}(p_2)\gamma_\rho u(p_1)\right]\left[\bar{u}(q_1)\gamma^\rho v(q_2)\right] \end{aligned} \tag{123}$$

$$TT^\dagger = \frac{e^2}{s}\frac{e^2}{s}\left[\bar{v}(p_2)\gamma_{\rho_1}u(p_1)\bar{u}(p_1)\gamma_{\rho_2}v(p_2)\right]\left[\bar{u}(q_1)\gamma^{\rho_1}v(q_2)\bar{v}(q_2)\gamma^{\rho_2}u(q_1)\right] \tag{124}$$

$$\begin{aligned} \sum_{\text{spins}} TT^\dagger &= \frac{e^4}{s^2}\mathrm{tr}\left[(\not{p}_2 - m_e)\gamma_{\rho_1}(\not{p}_1 + m_e)\gamma_{\rho_2}\right]\mathrm{tr}\left[(\not{q}_1 + m_\mu)\gamma^{\rho_1}(\not{q}_2 - m_\mu)\gamma^{\rho_2}\right] \\ &= \frac{e^4}{s^2}L_{\rho_2\rho_1}(p_1, p_2, m_e)L^{\rho_1\rho_2}(q_1, q_2, m_\mu) \end{aligned} \tag{125}$$

4.2 Trace Techniques

- consider a matrix element

$$\bar{u}(p)\Gamma u(q) = \sum_{k,l=1}^{4} \bar{u}_k(p)\Gamma_{kl}u_l(q) = \sum_{k,l=1}^{4} \Gamma_{kl}u_l(q)\bar{u}_k(p) \tag{126}$$

- using the tensor product

$$u(q) \otimes \bar{u}(p) = \begin{pmatrix} u_1(q)\bar{u}_1(p) & \cdots & u_1(q)\bar{u}_4(p) \\ \vdots & \ddots & \vdots \\ u_4(q)\bar{u}_1(p) & \cdots & u_4(q)\bar{u}_4(p) \end{pmatrix} \tag{127}$$

we can write

$$\bar{u}(p)\Gamma u(q) = \sum_{k,l=1}^{4} \Gamma_{kl}[u(q) \otimes \bar{u}(p)]_{lk} = \sum_{k=1}^{4} (\Gamma[u(q) \otimes \bar{u}(p)])_{kk} \tag{128}$$

- using the trace of a matrix

$$\mathrm{tr}(A) = \sum_{k=1}^{4} A_{kk} \tag{129}$$

we can express a matrix element equivalently as a trace

$$\bar{u}(p)\Gamma u(p) = \mathrm{tr}(\Gamma[u(p) \otimes \bar{u}(p)]) \tag{130}$$

- independent of the concrete realization of the Dirac matrices, we can compute their traces using their anti commutation relations (34) alone

$$\mathrm{tr}(\mathbf{1}) = 4 \tag{131a}$$

$$\mathrm{tr}\left(\not{a}\not{b}\right) = \frac{1}{2}\Big(\mathrm{tr}\left(\not{a}\not{b}\right) + \mathrm{tr}\left(\not{b}\not{a}\right)\Big) \stackrel{(34)}{=} \mathrm{tr}(\mathbf{1}) \cdot ab = 4 \cdot ab \tag{131b}$$

$$\mathrm{tr}\left(\not{a}_1\right) = \mathrm{tr}\left(\not{a}_1\not{a}_2\not{a}_3\right) = \mathrm{tr}\left(\not{a}_1\not{a}_2\cdots\not{a}_{2n+1}\right) \stackrel{(\gamma_5\gamma_5=1)}{=} 0 \tag{131c}$$

$$\mathrm{tr}\left(\not{a}_1\not{a}_2\cdots\not{a}_n\right) = \mathrm{tr}\left(\not{a}_n\cdots\not{a}_2\not{a}_1\right) \tag{131d}$$

$$\mathrm{tr}(\gamma_5) = \mathrm{tr}(\gamma_5\not{a}) = \mathrm{tr}(\gamma_5\not{a}\not{b}) = \mathrm{tr}(\gamma_5\not{a}\not{b}\not{c}) = 0 \tag{131e}$$

$$\mathrm{tr}(\gamma_5\not{a}\not{b}\not{c}\not{d}) = 4i \cdot \epsilon(a, b, c, d) \tag{131f}$$

only (131d) depends on the existence of a charge conjugation matrix

- also from the anti commutation relations alone, we can prove contraction formulae:

$$\gamma^\mu\not{a}\gamma_\mu = -2 \cdot \not{a} \tag{132a}$$

$$\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2 \cdot \not{c}\not{b}\not{a} \tag{132b}$$

$$\gamma^\mu\gamma_\mu = 4 \tag{132c}$$

$$\gamma^\mu\not{a}\not{b}\gamma_\mu = 4 \cdot ab \tag{132d}$$

- it's instructive to prove at least one of the trace theorems yourselves, because it helps to memorize the result.

Problem 13. *Compute*

$$\mathrm{tr}\left(\not{a}\not{b}\not{c}\not{d}\right) \tag{133}$$

using the cyclic invariance of the trace

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) \tag{134}$$

and subsequently the anti commutation relations (34).

Problem 14. *Compute the trace* $L_{\mu\nu}(p,q,m) = \mathrm{tr}\left[(\not{p}+m)\gamma_\mu(\not{q}-m)\gamma_\nu\right]$.

Problem 15. *Compute* $L_{\rho_2\rho_1}(p_1,p_2,0)L^{\rho_1\rho_2}(q_1,q_2,0)$ *as a function of the Mandelstam variables in the high energy limit* $s,-t,-u \gg m_i^2$.

4.3 Cross Section

Problem 16. *Compute the differential cross section*

$$\frac{d\sigma}{d\Omega}(\cos\theta, E_{CM}) \tag{135}$$

for $e^+e^- \to \mu^+\mu^-$ *in the region* $s,-t,-u \gg m_e^2$.

Problem 17. *Compute the integrated cross section*

$$\sigma(E_{CM}) \tag{136}$$

for $e^+e^- \to \mu^+\mu^-$ *in the region* $s \gg m_e^2$.

4.4 FORM

- very efficient computation using programs for symbolic manipulation, e. g. FORM.
- declare variables

```
vector p1, p2, q1, q2;
symbol s, t, u, me, mq;
indices rho1, rho2;
```

- expressions

```
local [TT*] =
    (g_(1, p2) - me*g_(1)) * g_(1, rho1)
  * (g_(1, p1) + me*g_(1)) * g_(1, rho2)
  * (g_(2, q1) + mq*g_(2)) * g_(2, rho1)
  * (g_(2, q2) - mq*g_(2)) * g_(2, rho2);
```

- traces

```
trace4, 1;
trace4, 2;
print;
.sort;
```

- reduction to Mandelstam variables (114)

$$s = (p_1 + p_2)^2 = 2m_e^2 + 2p_1p_2 \tag{137a}$$

$$t = (q_2 - p_1)^2 = m_q^2 + m_e^2 - 2q_2p_1 \tag{137b}$$

  etc.

```
id p1.p2 = 1/2 * (s - 2*me^2);
id q1.q2 = 1/2 * (s - 2*mq^2);
id p1.q1 = - 1/2 * (t - me^2 - mq^2);
id p2.q2 = - 1/2 * (t - me^2 - mq^2);
id p1.q2 = - 1/2 * (u - me^2 - mq^2);
id p2.q1 = - 1/2 * (u - me^2 - mq^2);
```

- human intelligence (i. e. experience, cf. (211)): the result can be written most compactly as a function of t und u:

```
id s = - u - t + 2*me^2 + 2*mq^2;
bracket me, mq;
print;
.sort;
```

running the program:

```
ohl@thopad2:~fdfp$ form mumu.frm

FORM by J.Vermaseren,version 3.3(Mar 28 2009) Run at: Thu Jul  7 18:37:42 2011
   [TT*] =
      64*me^2*mq^2 + 32*p1.p2*mq^2 + 32*p1.q1*p2.q2 + 32*p1.q2*p2.q1 + 32*
      q1.q2*me^2;
   [TT*] =
       + mq^2 * (  - 32*u - 32*t )
       + mq^4 * ( 48 )
       + me^2 * (  - 32*u - 32*t )
       + me^2*mq^2 * ( 96 )
       + me^4 * ( 48 )
       + 8*u^2 + 8*t^2;
  0.00 sec out of 0.00 sec
```

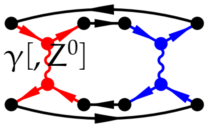
4.5 Bhabha Scattering

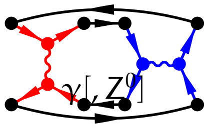
Problem 18. *Draw all Feynman diagrams for* $e^+e^- \to e^+e^-$ *("Bhabha scattering")*

- do we need to add or subtract the diagrams?
- more general: what's the relative phase of the diagrams?

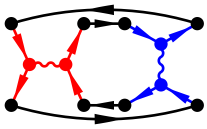
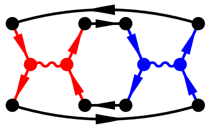
$$TT^\dagger = (T_t - T_s)(T_t - T_s)^\dagger = T_t {T_t}^\dagger - T_t {T_s}^\dagger - T_s {T_t}^\dagger + T_s {T_s}^\dagger \tag{138}$$

- relative signs of the diagrams from permuting the endpoints of the fermion lines
- equivalently: relative signs from the number of closed fermion lines in squared diagrams

$T_t {T_t}^\dagger = (-1)^2 \times$ 

, $T_t {T_s}^\dagger = (-1)^1 \times$ 

(139a)

$T_s {T_t}^\dagger = (-1)^1 \times$ 
, $T_s {T_s}^\dagger = (-1)^2 \times$ 
(139b)

4.6 FORM

- declaration of variables as above
- expressions

$$T_s = e^2 \frac{1}{s} \left[\bar{v}(p_2)\gamma_\rho u(p_1)\right] \left[\bar{u}(q_1)\gamma^\rho v(q_2)\right] \tag{140}$$

$$T_t = e^2 \frac{1}{t} \left[\bar{v}(p_2)\gamma_\rho v(q_2)\right] \left[\bar{u}(q_1)\gamma^\rho u(p_1)\right] \tag{141}$$

- $T_s T_s^\dagger$ just like in $e^+e^- \to \mu^+\mu^-$

```
local [SS*] =
     (g_(1, p2) - me*g_(1)) * g_(1, rho1)
   * (g_(1, p1) + me*g_(1)) * g_(1, rho2)
   * (g_(2, q1) + me*g_(2)) * g_(2, rho1)
   * (g_(2, q2) - me*g_(2)) * g_(2, rho2);
```

- $T_t T_t^\dagger$ similar

```
local [TT*] =
     (g_(1, q1) + me*g_(1)) * g_(1, rho1)
   * (g_(1, p1) + me*g_(1)) * g_(1, rho2)
   * (g_(2, p2) - me*g_(2)) * g_(2, rho1)
   * (g_(2, q2) - me*g_(2)) * g_(2, rho2);
```

- the interference terms are more complicated and contain traces of eight Dirac matrices

$$T_s T_t{}^* = e^4 \frac{1}{st} \left[\bar{v}(p_2)\gamma_{\rho_1} u(p_1)\right] \left[\bar{u}(p_1)\gamma^{\rho_2} u(q_1)\right] \left[\bar{u}(q_1)\gamma^{\rho_1} v(q_2)\right] \left[\bar{v}(q_2)\gamma_{\rho_2} v(p_2)\right] \tag{142}$$

```
local [ST*] =
    (g_(1, p2) - me*g_(1)) * g_(1, rho1)
  * (g_(1, p1) + me*g_(1)) * g_(1, rho2)
  * (g_(1, q1) + me*g_(1)) * g_(1, rho1)
  * (g_(1, q2) - me*g_(1)) * g_(1, rho2);
```

$$T_t T_s{}^* = e^4 \frac{1}{st} \left[\bar{v}(p_2)\gamma_{\rho_1} v(q_2)\right] \left[\bar{v}(q_2)\gamma^{\rho_2} u(q_1)\right] \left[\bar{u}(q_1)\gamma^{\rho_1} u(p_1)\right] \left[\bar{u}(p_1)\gamma_{\rho_2} v(p_2)\right] \tag{143}$$

```
local [TS*] =
    (g_(1, p2) - me*g_(1)) * g_(1, rho1)
  * (g_(1, q2) - me*g_(1)) * g_(1, rho2)
  * (g_(1, q1) + me*g_(1)) * g_(1, rho1)
  * (g_(1, p1) + me*g_(1)) * g_(1, rho2);
```

- `FORM` does the traces just the same . . .

```
trace4, 1;
trace4, 2;
.sort;
```

- reduction to Mandelstam variables again as above, but all masses are equal

```
id p1.p2 = 1/2 * (s - 2*me^2);
id q1.q2 = 1/2 * (s - 2*me^2);
id p1.q1 = - 1/2 * (t - 2*me^2);
id p2.q2 = - 1/2 * (t - 2*me^2);
id p1.q2 = - 1/2 * (u - 2*me^2);
id p2.q1 = - 1/2 * (u - 2*me^2);
```

- human intelligence and experience: the expression for $|T_s|^2$ is most compact as function of t and u

```
id s = - u - t + 4*me^2;
bracket me;
print;
.sort;
```

- the expression for $|T_t|^2$ is most compact as function of s and u

- ... two different reductions in the same FORM program require advanced tricks ...

- running the program:

```
ohl@thopad2:~fdfp$ form bhabha.frm

FORM by J.Vermaseren,version 3.3(Mar 28 2009) Run at: Thu Jul  7 18:37:42 2011
    [SS*] =
        + me^2 * (  - 64*u - 64*t )
        + me^4 * ( 192 )
        + 8*u^2 + 8*t^2;
    [TT*] =
        + me^2 * (  - 64*u )
        + me^4 * ( 64 )
        + 16*u^2 + 16*t*u + 8*t^2;
    [ST*] =
        + me^2 * ( 64*u )
        + me^4 * (  - 96 )
        - 8*u^2;
    [TS*] =
        + me^2 * ( 64*u )
        + me^4 * (  - 96 )
        - 8*u^2;
  0.00 sec out of 0.00 sec
```

- NB: $|T_s|^2(t,u) = |T_t|^2(s,u)$

- final result very symmetrical

$$\sum_{\text{spins}} TT^* = 8e^4 \left(\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st} \right) + O(m_e^2) \tag{144}$$

Problem 19. *Compute the differential cross section*

$$\frac{d\sigma}{d\Omega}(\cos\theta) \tag{145}$$

for Bhabha scattering in the region $s, -t, -u \gg m_e^2$.

5 QCD

5.1 Feynman Rules

∵ full account requires another full set of lectures

∴ just a few SU(N) formulae

- generators T_a and totally anti symmetric structure constants f_{abc}:

$$[T_a, T_b] = i f_{abc} T_c \tag{146}$$

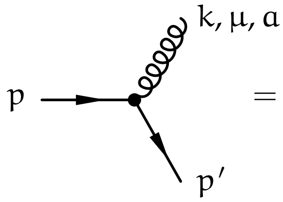
  (summations for $c = 1, 2, \ldots, (N_c^2 - 1)$ implied).
- normalization and contractions

$$\mathrm{tr}\,(T_a T_b) = \frac{1}{2}\delta_{ab} \tag{147a}$$

$$T_a T_a = C_F \cdot \mathbf{1} = \frac{N_C^2 - 1}{2N_C} \cdot \mathbf{1} \tag{147b}$$

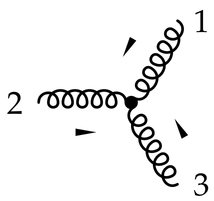
$$f_{acd} f_{bcd} = C_F \cdot \delta_{ab} \tag{147c}$$

- physical degrees of freedom: quarks and gluons:



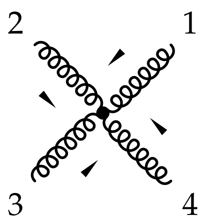
$$= \qquad i g \gamma_\mu T_a \tag{148}$$

- triple gluon couplings



$$\begin{aligned} = \qquad & g f_{a_1 a_2 a_3} g_{\mu_1 \mu_2} (k^1_{\mu_3} - k^2_{\mu_3}) \\ & + g f_{a_1 a_2 a_3} g_{\mu_2 \mu_3} (k^2_{\mu_1} - k^3_{\mu_1}) \\ & + g f_{a_1 a_2 a_3} g_{\mu_3 \mu_1} (k^3_{\mu_2} - k^1_{\mu_2}) \end{aligned}$$

- quartic gluon couplings

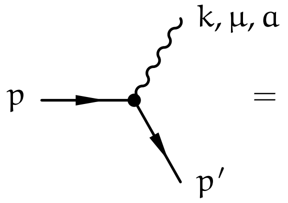


$$\begin{aligned} = \qquad & -i g^2 f_{a_1 a_2 b} f_{a_3 a_4 b} \times \\ & \quad (g_{\mu_1 \mu_3} g_{\mu_4 \mu_2} - g_{\mu_1 \mu_4} g_{\mu_2 \mu_3}) \\ & -i g^2 f_{a_1 a_3 b} f_{a_4 a_2 b} \times \\ & \quad (g_{\mu_1 \mu_4} g_{\mu_2 \mu_3} - g_{\mu_1 \mu_2} g_{\mu_3 \mu_4}) \\ & -i g^2 f_{a_1 a_4 b} f_{a_2 a_3 b} \times \\ & \quad (g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} - g_{\mu_1 \mu_3} g_{\mu_4 \mu_2}) \end{aligned}$$

- Faddeev-Popov ghosts appear only in loop diagrams

5.2 3-Jet Production

- quarks are electrically charged



$$= \quad -ieQ_q\gamma_\mu$$

and couple in the standard model (cf. (176) and (178)) to the Z^0 and $W^\pm$ as well.

Problem 20. *Draw all Feynman diagrams without closed loops, that follow from the Feynman rules of QED und QCD for the electroproction of 3 jets*

$$e^+ + e^- \to q + \bar{q} + g$$

(at higher energies, there are additional contributions from Z^0 bosons).

Write down the corresponding algebraic expression for the T-matrix.

Problem 21. *Verify the Ward identity for the gluon*

$$T_1\Big|_{\epsilon^*_\mu(k)=k_\mu} + T_2\Big|_{\epsilon^*_\mu(k)=k_\mu} = 0\,, \tag{152}$$

i. e. the fact that no gluons with unphysical polarization $\epsilon^*_\mu(k) = k_\mu$ are produced. Hint: use the Dirac equation for the external quarks, e. g.

$$\frac{1}{-\not{p}_1 - \not{k} - m + i\epsilon}\,(\not{k} + \not{p}_1 + m)\,v(p_1) = -v(p_1) \qquad \textit{(für } k \neq 0\textit{)} \tag{153}$$

- down the road, some additional shorthands will be useful

$$T_n = e^2Qg\,[\bar{v}(p_2)\gamma_\mu u(p_1)]\,\frac{1}{s}J^\mu_n(q_1,q_2,k,\epsilon) \tag{154}$$

with the current matrix element

$$J^\mu(q_1,q_2,k,\epsilon) = J^\mu_1(q_1,q_2,k,\epsilon) + J^\mu_2(q_1,q_2,k,\epsilon) \tag{155a}$$

$$J^\mu_1(q_1,q_2,k,\epsilon) = \frac{\bar{u}(q_1)T_a\not{\epsilon}^*(k)(\not{q}_1+\not{k}+m)\gamma^\mu v(q_2)}{(q_1+k)^2 - m^2 + i\epsilon} \tag{155b}$$

$$J^\mu_2(q_1,q_2,k,\epsilon) = \frac{\bar{u}(q_1)\gamma^\mu(-\not{q}_2-\not{k}+m)T_a\not{\epsilon}^*(k)v(q_2)}{(q_2+k)^2 - m^2 + i\epsilon} \tag{155c}$$

- this allows to write

$$\sum_{\text{spins, pol.}} |T_1 + T_2|^2 = \frac{e^4g^2Q^2}{s^2}L_{\mu\nu}(p_1,p_2,0)H^{\mu\nu}(q_1,q_2,k) \tag{156}$$

with the "hadronic tensor"

$$H^{\mu\nu}(q_1, q_2, k) = \sum_{\text{spins},\epsilon} J^{\mu}(q_1, q_2, k, \epsilon) J^{\nu,*}(q_1, q_2, k, \epsilon) \tag{157}$$

$\because$ with the same trick as in problem 21, we can show that

$$[q_1^{\mu} + q_2^{\mu} + k^{\mu}]\, J_{\mu}(q_1, q_2, k, \epsilon) = 0 \tag{158}$$

$\therefore$ using the center of mass momentum $p = p_1 + p_2 = q_1 + q_2 + k$

$$p^{\mu} H^{\mu\nu}(q_1, q_2, k) = p^{\nu} H^{\mu\nu}(q_1, q_2, k) = 0 \tag{159}$$

- angular dependence of $H^{\mu\nu}(q_1, q_2, k)$ contains a lot of information ...

$\because$ ... but the energy dependence is much simpler

$$x_1 = 2q_1 p/p^2, \quad x_2 = 2q_2 p/p^2, \quad x_3 = 2kp/p^2 \tag{160}$$

$\therefore$ integrate over the angles

$$\int \widetilde{dq_1}\widetilde{dq_2}\widetilde{dk}\,(2\pi)^4 \delta^4(q_1 + q_2 + k - p)\, f(x_1, x_2, x_3)$$
$$= \frac{s}{128\pi^3} \int dx_1 dx_2\, f(x_1, x_2, 2 - x_1 - x_2) \tag{161}$$

afterwards, thre result will depend only on p and the x_i.

$\because$ from (159) we find

$$\int d\tilde{\Omega}\, H^{\mu\nu}(q_1, q_2, k) = \left(\frac{p^{\mu} p^{\nu}}{p^2} - g^{\mu\nu}\right) \tilde{H}(p, x_1, x_2) \tag{162}$$

$\therefore$

$$\tilde{H}(p, x_1, x_2) = -\frac{1}{3} \int d\tilde{\Omega}\, H^{\mu}{}_{\mu}(q_1, q_2, k) \tag{163}$$

- energy conservation

$$x_1 + x_2 + x_3 = \frac{2(q_1 + q_2 + k)p}{p^2} = 2 \tag{164}$$

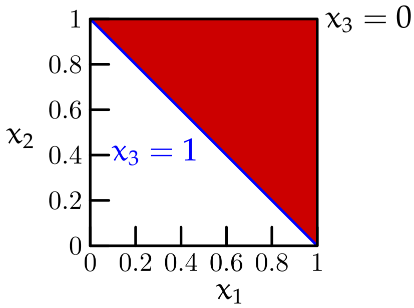
- momentum conservation for massless particles

$$x_1 + x_2 \geqslant x_3 = 2 - x_1 - x_2 \tag{165}$$

with equality for parallel q_1 und q_2

$\therefore$

$$x_1 + x_2 \geqslant 1 \tag{165'}$$



- for massless particles

$$\sum_{\text{spins, pol.}} \int d\tilde{\Omega}\, |T_1 + T_2|^2 = \frac{4e^4 g^2 Q^2}{s^2} \left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - p_1 p_2 g^{\mu\nu}\right)$$
$$\times \left(\frac{p_\mu p_\nu}{s} - g_{\mu\nu}\right) \tilde{H}(p, x_1, x_2) = \frac{4e^4 g^2 Q^2}{s} \tilde{H}(p, x_1, x_2) \quad (166)$$

- add phase space factor

$$\frac{d^2\sigma}{dx_1 dx_2} = \frac{1}{2s} \frac{s}{128\pi^3} \frac{1}{4} \sum_{\text{spins, pol.}} \int d\tilde{\Omega}\, |T_1 + T_2|^2 = \frac{\alpha_s \alpha^2 Q^2}{4s} \tilde{H}(p, x_1, x_2) \quad (167)$$

Problem 22. *Express the invariants $q_1 q_2$, $q_1 k$ and $q_2 k$ by s and the x_i, assuming all particles to be massless.*

Problem 23. *Compute*

$$\frac{d^2\sigma}{dx_1 dx_2}(x_1, x_2) \quad (168)$$

for massless particles.

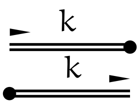
- *start with computing $H^\mu{}_\mu(q_1, q_2, k)$ as a function of $q_1 q_2$, $q_1 k$ and $q_2 k$*
- *since (152), you may use $\sum_\epsilon \epsilon_\mu \epsilon^*_\nu = -g_{\mu\nu}$*
- *use the contraction identities (132) before calculating the traces!*

6 Standard Model

			$(\mathbf{C}, \mathbf{T})_Y$	$Q = T_3 + \frac{Y}{2}$
	leptons			
$\nu_{e,R}$	$\nu_{\mu,R}$	$\nu_{\tau,R}$	$(\mathbf{1}, \mathbf{1})_0$	0
e_R	μ_R	τ_R	$(\mathbf{1}, \mathbf{1})_{-2}$	-1
$\begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu,L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau,L} \\ \tau_L \end{pmatrix}$	$(\mathbf{1}, \mathbf{2})_{-1}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
	quarks			
u_R	c_R	t_R	$(\mathbf{3}, \mathbf{1})_{4/3}$	$\frac{2}{3}$
d_R	s_R	b_R	$(\mathbf{3}, \mathbf{1})_{-2/3}$	$-\frac{1}{3}$
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	$(\mathbf{3}, \mathbf{2})_{1/3}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
	gauge bosons			
A	Z^0	$W^\pm$	g	
	Higgs			
	$\Phi\ (?)$		$(\mathbf{1}, ?)_?$	$?$

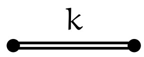
6.1 Propagators

- external spin-1 particles



$$\text{incoming:} \quad \Longleftrightarrow \epsilon_\mu(k) \tag{169a}$$
$$\text{outgoing:} \quad \Longleftrightarrow \epsilon^*_\mu(k) \tag{169b}$$

- internal particles and anti particles



$$\text{Spin-1 } (m \neq 0): \quad \Longleftrightarrow \frac{-ig_{\mu\nu} + i\frac{k_\mu k_\nu}{M^2}}{k^2 - M^2 + i\Gamma M} \tag{170}$$

- polarization sum:

$$\sum_{\text{pol.}} \epsilon_\mu(k)\epsilon^*_\nu(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \tag{171}$$

- unitarity gauge: $\partial^\mu D_{\mu\nu} = 0$, other options

$$\frac{-ig_{\mu\nu} + i(1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M^2}}{k^2 - M^2 + i\Gamma M} \tag{172}$$

better for radiative corrections, but larger intermediate terms

- finite width Γ: (tricky for charged particles, but required)

$$|D(p, M)|^2 \propto \frac{1}{(p^2 - M^2)^2 + \Gamma^2 M^2} \tag{173}$$

6.2 Feynman Rules

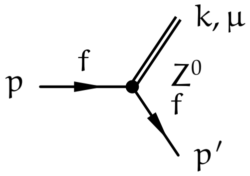
- vector and axial vector couplings

$$g_V = T_3 - 2Q\sin^2\theta_w \tag{174a}$$
$$g_A = T_3 \tag{174b}$$

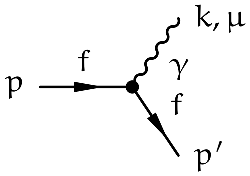
- e. g. for electrons

$$g_V = -\frac{1-4\sin^2\theta_w}{2}, \quad g_A = -\frac{1}{2} \tag{175}$$

- neutral currents

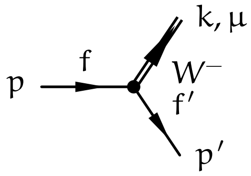


$$= \quad -i\frac{g}{2\cos\theta_w}\left(g_V^f\gamma_\mu - g_A^f\gamma_\mu\gamma_5\right) \tag{176}$$



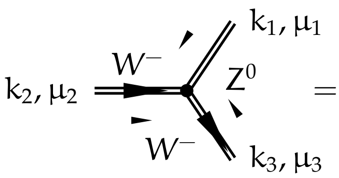
$$= \quad -ieQ_f\gamma_\mu \tag{177}$$

- charged currents ($V_{ff'}$ is the CKM matrix)

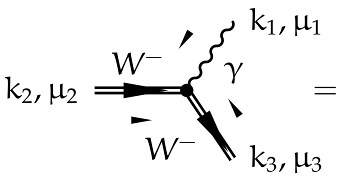


$$= \quad -i\frac{g}{\sqrt{2}}V_{ff'}\tau^+\gamma_\mu\frac{1-\gamma_5}{2} \tag{178}$$

- triple gauge couplings

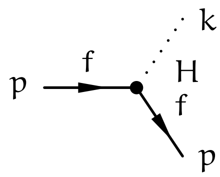


$$= \begin{aligned} &ie\cot\theta_w g_{\mu_1\mu_2}(k^1_{\mu_3} - k^2_{\mu_3}) \\ +&ie\cot\theta_w g_{\mu_2\mu_3}(k^2_{\mu_1} - k^3_{\mu_1}) \\ +&ie\cot\theta_w g_{\mu_3\mu_1}(k^3_{\mu_2} - k^1_{\mu_2}) \end{aligned} \tag{179}$$

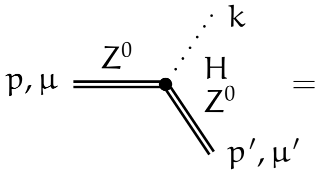


$$= \begin{aligned} &ieg_{\mu_1\mu_2}(k^1_{\mu_3} - k^2_{\mu_3}) \\ +&ieg_{\mu_2\mu_3}(k^2_{\mu_1} - k^3_{\mu_1}) \\ +&ieg_{\mu_3\mu_1}(k^3_{\mu_2} - k^1_{\mu_2}) \end{aligned} \tag{180}$$

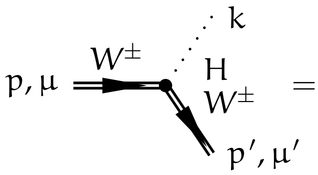
- Yukawa couplings



$$= \quad -i\frac{g m_f}{2M_W} \tag{181}$$



$$= \quad i\frac{g M_Z}{\cos\theta_w} g_{\mu\mu'} \tag{182}$$

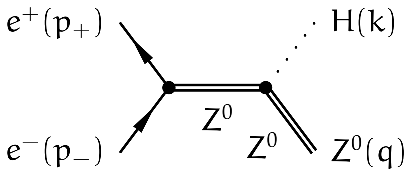


$$= \quad i g M_W g_{\mu\mu'} \tag{183}$$

- a lot more vertices: quartic couplings, Higgs selfcouplings, etc.

6.3 Higgs Strahlung

Problem 24. *Compute the scattering amplitude for Higgs strahlung* $e^+e^- \to ZH$



(184)

ignoring all terms of $O(m_e/M_Z)$, $O(m_e/M_H)$ *and* $O(m_e/\sqrt{s})$. *You may assume that* $\sqrt{s} \ll M_Z^2$ *and ignore the width of the* Z.

Problem 25. *Show that the four momenta*

$$
\begin{aligned}
p_- &= (E;0,0,E) && \text{(185a)}\\
p_+ &= (E;0,0,-E) && \text{(185b)}\\
k &= (E_H; p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta) && \text{(185c)}\\
q &= (E_Z; -p\sin\theta\cos\phi, -p\sin\theta\sin\phi, -p\cos\theta) && \text{(185d)}
\end{aligned}
$$

with

$$p = \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{2\sqrt{s}} \tag{186a}$$

$$E_H = \frac{s + M_H^2 - M_Z^2}{2\sqrt{s}} \tag{186b}$$

$$E_Z = \frac{s + M_Z^2 - M_H^2}{2\sqrt{s}} \tag{186c}$$

and

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \tag{187}$$

satisfy energy and momentum conservation and the mass shell conditions for $e^+e^- \to ZH$ *with* $m_e = 0$. *Compute*

$$16(p_+q)(p_-q)\,. \tag{188}$$

Problem 26. *Compute the differential cross section*

$$\frac{d\sigma}{d\Omega}(\theta_{e^-H}) \tag{189}$$

for unpolarized Higgs strahlung $e^+e^- \to ZH$.

NB: use the fact that $\sum \epsilon_\mu \epsilon^*_\nu$ *is symmetric in* $\mu \leftrightarrow \nu$.

Problem 27. *Compute the integrated cross section for Higgs strahlung* $e^+e^- \to ZH$.

7 Solutions

Solution 1.

$$\partial_\mu e^{-ipx} = -ip_\mu e^{-ipx} \tag{190a}$$
$$(a\partial)(b\partial)e^{-ipx} = -(ap)(bp)e^{-ipx} \tag{190b}$$
$$\partial^2 e^{-ipx} = -i(p\partial)e^{-ipx} = -p^2 e^{-ipx} \tag{190c}$$

Solution 2.

$$\partial_\mu x^\mu = g_\mu{}^\nu \partial x^\mu/\partial x^\nu = g_\mu{}^\nu \delta^\mu{}_\nu = g_\mu{}^\mu = 4 \tag{191a}$$
$$\begin{aligned}\partial^2 e^{-x^2/2} &= -\partial^\mu\left(x_\mu e^{-x^2/2}\right) = -x_\mu \partial^\mu e^{-x^2/2} - (\partial^\mu x_\mu)\, e^{-x^2/2} \\ &= x_\mu x^\mu e^{-x^2/2} - g_\mu{}^\mu e^{-x^2/2} = (x^2-4)e^{-x^2/2}\end{aligned} \tag{191b}$$

Solution 3.

$$\begin{aligned}[\gamma^k,\gamma^l]_+ &= \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0\end{pmatrix} + (k \longleftrightarrow l) = \begin{pmatrix} -\sigma^k\sigma^l & 0 \\ 0 & -\sigma^k\sigma^l\end{pmatrix} + (k \longleftrightarrow l) \\ &= \begin{pmatrix} -[\sigma^k,\sigma^l]_+ & 0 \\ 0 & -[\sigma^k,\sigma^l]_+\end{pmatrix} = -2\delta^{kl}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1}\end{pmatrix}\end{aligned} \tag{192a}$$

Solution 4.

$$\bar{u}_k(p) = u_k^\dagger(p)\gamma_0 = u_k^\dagger(\vec{0})\gamma^0\gamma^0\frac{\not{p}^\dagger + m}{\sqrt{p_0+m}}\gamma_0 = \bar{u}_k(\vec{0})\frac{\not{p}+m}{\sqrt{p_0+m}} \tag{193a}$$
$$\bar{v}_k(p) = \bar{v}_k(\vec{0})\frac{\not{p}-m}{\sqrt{p_0+m}} \tag{193b}$$

Using the definition and multiplying with γ_0 from the right

$$\sum_{k=1}^{2} u_k(\vec{0})\bar{u}_k(\vec{0}) = \frac{\gamma_0+1}{2}, \quad \sum_{k=1}^{2} v_k(\vec{0})\bar{v}_k(\vec{0}) = \frac{\gamma_0-1}{2} \tag{194}$$

Therefore

$$\sum_{k=1}^{2} u_k(p)\bar{u}_k(p) = \frac{(\not{p}+m)(\gamma_0+1)(\not{p}+m)}{2(p_0+m)} \tag{195a}$$
$$\sum_{k=1}^{2} v_k(p)\bar{v}_k(p) = \frac{(\not{p}-m)(\gamma_0-1)(\not{p}-m)}{2(p_0+m)} \tag{195b}$$

and (60) follows from

$$\begin{aligned}(\not{p}\pm m)(\gamma_0\pm 1)(\not{p}\pm m) &= \not{p}\gamma_0\not{p} \pm (m\gamma_0\not{p} + m\not{p}\gamma_0) + m^2\gamma_0 \pm (\not{p}\pm m)^2 \\ &= -p^2\gamma_0 + 2p_0\not{p} \pm 2p_0 m + m^2\gamma_0 + 2m(\not{p}\pm m) = 2(p_0+m)(\not{p}\pm m)\,.\end{aligned} \tag{196}$$

Solution 5.

$$\begin{aligned}\bar{u}_k(p)u_l(p) = \bar{u}_k(\vec{0})\frac{\not{p}+m}{\sqrt{p_0+m}}\frac{\not{p}+m}{\sqrt{p_0+m}}u_l(\vec{0})\\ = \frac{2m}{p_0+m}\bar{u}_k(\vec{0})(\not{p}+m)u_l(\vec{0})\\ = \frac{2m}{p_0+m}\bar{u}_k(\vec{0})(p_0+m)u_l(\vec{0}) = 2m\bar{u}_k(\vec{0})u_l(\vec{0}) = 2m\delta_{kl}\end{aligned} \tag{197}$$

$$\begin{aligned}\bar{v}_k(p)v_l(p) = \bar{v}_k(\vec{0})\frac{\not{p}-m}{\sqrt{p_0+m}}\frac{\not{p}-m}{\sqrt{p_0+m}}v_l(\vec{0})\\ = \frac{-2m}{p_0+m}\bar{v}_k(\vec{0})(\not{p}-m)v_l(\vec{0})\\ = \frac{2m}{p_0+m}\bar{v}_k(\vec{0})(p_0+m)v_l(\vec{0}) = 2m\bar{v}_k(\vec{0})v_l(\vec{0}) = -2m\cdot\delta_{kl}\end{aligned} \tag{198}$$

$$\bar{u}_k(p)v_l(p) = \bar{u}_k(\vec{0})\frac{\not{p}+m}{\sqrt{p_0+m}}\frac{\not{p}-m}{\sqrt{p_0+m}}v_l(\vec{0}) = 0 \tag{199a}$$

$$\bar{v}_k(p)u_l(p) = \bar{v}_k(\vec{0})\frac{\not{p}-m}{\sqrt{p_0+m}}\frac{\not{p}+m}{\sqrt{p_0+m}}u_l(\vec{0}) = 0 \tag{199b}$$

Solution 6.

$$\begin{aligned}\bar{u}_k(p)\gamma_\mu u_l(p) = \bar{u}_k(\vec{0})\frac{\not{p}+m}{\sqrt{p_0+m}}\gamma_\mu\frac{\not{p}+m}{\sqrt{p_0+m}}u_l(\vec{0})\\ = \frac{1}{p_0+m}\bar{u}_k(\vec{0})2p_\mu(\not{p}+m)u_l(\vec{0})\\ = \frac{2p_\mu}{p_0+m}\bar{u}_k(\vec{0})(p_0+m)u_l(\vec{0}) = 2p_\mu\bar{u}_k(\vec{0})u_l(\vec{0}) = 2p_\mu\delta_{kl}\end{aligned} \tag{200a}$$

$$\begin{aligned}\bar{v}_k(p)\gamma_\mu v_l(p) = \bar{v}_k(\vec{0})\frac{\not{p}-m}{\sqrt{p_0+m}}\gamma_\mu\frac{\not{p}-m}{\sqrt{p_0+m}}v_l(\vec{0})\\ = \frac{1}{p_0+m}\bar{v}_k(\vec{0})2p_\mu(\not{p}-m)v_l(\vec{0})\\ = \frac{-2p_\mu}{p_0+m}\bar{v}_k(\vec{0})(p_0+m)v_l(\vec{0}) = -2p_\mu\bar{v}_k(\vec{0})v_l(\vec{0}) = 2p_\mu\delta_{kl}\end{aligned} \tag{200b}$$

$$\begin{aligned}\bar{u}_k(\vec{p})\gamma_0 v_l(-\vec{p}) = \bar{u}_k(\vec{0})\frac{\not{p}+m}{\sqrt{p_0+m}}\gamma_0\frac{\not{p}^\dagger-m}{\sqrt{p_0+m}}v_l(\vec{0})\\ = \bar{u}_k(\vec{0})\frac{\not{p}+m}{\sqrt{p_0+m}}\frac{\not{p}-m}{\sqrt{p_0+m}}\gamma_0 v_l(\vec{0}) = 0\end{aligned} \tag{201a}$$

$$\bar{v}_k(-\vec{p})\gamma_0 u_l(\vec{p}) = 0 \tag{201b}$$

Solution 7.

$$\gamma_5\gamma^2 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = i\gamma^0\gamma^2\gamma^1\gamma^2\gamma^3 = -i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma_5 \quad (202)$$

since every γ_μ appears exaclty once and anti-commutes with the other three, we have for every μ

$$[\gamma_5, \gamma_\mu]_+ = 0 \quad (203)$$

Solution 8. *product rule*

$$i\partial^\mu \left[\bar\psi_1(x)\gamma_\mu\psi_2(x)\right] = \bar\psi_1(x)\Big(i\overleftarrow{\not\partial} + i\overrightarrow{\not\partial}\Big)\psi_2(x) = \bar\psi_1(x)\Big(-m+m\Big)\psi_2(x) = 0\,. \quad (204)$$

Solution 9. *product rule and* $[\gamma_5, \gamma_\mu]_+ = 0$

$$\begin{aligned} i\partial^\mu \left[\bar\psi_1(x)\gamma_\mu\gamma_5\psi_2(x)\right] &= \bar\psi_1(x)\Big(i\overleftarrow{\not\partial}\gamma_5 + i\overrightarrow{\not\partial}\gamma_5\Big)\psi_2(x) = \bar\psi_1(x)\Big(i\overleftarrow{\not\partial}\gamma_5 - \gamma_5 i\overrightarrow{\not\partial}\Big)\psi_2(x) \\ &= \bar\psi_1(x)\Big(-m\gamma_5 - \gamma_5 m\Big)\psi_2(x) = -2m\bar\psi_1(x)\gamma_5\psi_2(x)\,. \end{aligned} \quad (205)$$

Solution 10.

$$\begin{aligned} -2i\sigma_{kl} &= \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{pmatrix} - (k \longleftrightarrow l) \\ &= \begin{pmatrix} -[\sigma^k, \sigma^l]_- & 0 \\ 0 & -[\sigma^k, \sigma^l]_- \end{pmatrix} = -2i\epsilon^{klm}\begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix} \end{aligned} \quad (206)$$

therefore

$$\sigma_{kl} = \epsilon^{klm}\begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix} \quad (207)$$

Solution 11.

$$(i\not\partial - m)\,S(x,m) = \left(-\partial^2 - m^2\right)D(x,m) = \delta^4(x) \quad (208)$$

Solution 12.

$$\begin{aligned} S(x,m) &= (i\not\partial + m)\int \frac{d^4p}{(2\pi)^4}\,e^{-ipx}\frac{1}{p^2 - m^2 + i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4}\,e^{-ipx}\frac{\not p + m}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4}\,e^{-ipx}\frac{1}{\not p - m + i\epsilon} \end{aligned} \quad (209)$$

Solution 13.

$$\begin{aligned} \mathrm{tr}\,(\not a\not b\not c\not d) = +\,\mathrm{tr}\,(\not b\not c\not d\not a) &= -\,\mathrm{tr}\,(\not b\not c\not a\not d) + 2\cdot ad\cdot \mathrm{tr}\,(\not b\not c) \\ &= +\,\mathrm{tr}\,(\not b\not a\not c\not d) - 2\cdot ac\cdot \mathrm{tr}\,(\not b\not d) + 2\cdot ad\cdot 4\cdot bc \\ &= -\,\mathrm{tr}\,(\not a\not b\not c\not d) + 2\cdot ab\cdot \mathrm{tr}\,(\not c\not d) - 2\cdot ac\cdot 4\cdot bd + 2\cdot ad\cdot 4\cdot bc \end{aligned}$$

therefore

$$\mathrm{tr}\,(\not a\not b\not c\not d) = 4\cdot(ab\cdot cd - ac\cdot bd + ad\cdot bc) \quad (133')$$

Solution 14.

$$\begin{aligned} L_{\mu\nu} &= \mathrm{tr}\,[(\not p + m)\gamma_\mu(\not q - m)\gamma_\nu] \;= \mathrm{tr}\,[\not p\gamma_\mu\not q\gamma_\nu] - m^2\,\mathrm{tr}\,[\gamma_\mu\gamma_\nu] \\ &= 4\cdot(p_\mu q_\nu - g_{\mu\nu}pq + p_\nu q_\mu) - 4m^2\cdot g_{\mu\nu} = 4\cdot\left(p_\mu q_\nu + p_\nu q_\mu - (pq+m^2)g_{\mu\nu}\right) \end{aligned} \tag{210}$$

Solution 15.

$$\begin{aligned} L_{\rho_2\rho_1}L^{\rho_1\rho_2} &= 16\cdot(p_{1,\rho_2}p_{2,\rho_1} + p_{1,\rho_1}p_{2,\rho_2} - p_1p_2g_{\rho_2\rho_1})(q_1^{\rho_1}q_2^{\rho_2} + q_1^{\rho_2}q_2^{\rho_1} - q_1q_2g^{\rho_1\rho_2}) \\ &= 8\cdot\Big(2(p_1q_2)2(p_2q_1) + 2(p_1q_1)2(p_2q_2)\Big) = 8\cdot\Big(u^2+t^2\Big) \end{aligned} \tag{211}$$

NB: cross terms cancel:

$$(p_{1,\rho_2}p_{2,\rho_1} + p_{1,\rho_1}p_{2,\rho_2})q_1q_2g^{\rho_1\rho_2} + p_1p_2g_{\rho_2\rho_1}(q_1^{\rho_1}q_2^{\rho_2} + q_1^{\rho_2}q_2^{\rho_1}) = p_1p_2g_{\rho_2\rho_1}q_1q_2g^{\rho_1\rho_2}$$

Solution 16.

$$t = -\frac{s}{2}(1-\cos\theta)\,,\quad u = -\frac{s}{2}(1+\cos\theta) \tag{212}$$

therefore

$$\frac{t^2+u^2}{s^2} = \frac{1+\cos^2\theta}{2} \tag{213}$$

$$\begin{aligned} d\sigma &= \frac{1}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}}\frac{1}{\prod_i n_i!}\frac{1}{4}|T|^2\widetilde{dq_1}\widetilde{dq_2}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2) \\ &= \frac{1}{4\sqrt{(s/2)^2}}\frac{1}{4}|T|^2\frac{1}{32\pi^2}d\cos\theta d\phi = \frac{1}{64\pi^2 s}\frac{1}{4}|T|^2 d\Omega \end{aligned} \tag{214}$$

therefore

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}e^4\,(1+\cos^2\theta) = \alpha^2\frac{1}{4s}\,(1+\cos^2\theta) \tag{215}$$

Solution 17.

$$\sigma = \int d\Omega\,\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}2\pi\int_{-1}^{1}d\cos\theta\,(1+\cos^2\theta) = \frac{\alpha^2}{4s}2\pi\left(2+\frac{2}{3}\right) = \frac{4\pi\alpha^2}{3s} \tag{216}$$

compare

$$\sigma = \frac{4\pi\alpha^2}{3(\sqrt{s}/\mathrm{TeV})^2}0.39\,\mathrm{nb} \tag{19$''$}$$

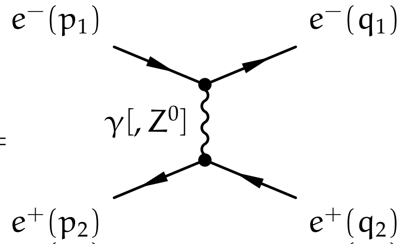
with

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.0359895(61)} \tag{217}$$

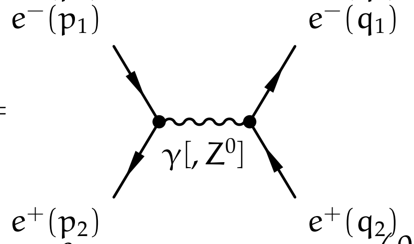
$$\sigma = \frac{87\,\mathrm{fb}}{(\sqrt{s}/\mathrm{TeV})^2} = \frac{8.7\,\mathrm{pb}}{(\sqrt{s}/100\,\mathrm{GeV})^2} \tag{19$''$}$$

Solution 18.

$$iT_t = \quad (218a)$$



$$iT_s = \quad (218b)$$



Solution 19. $\because$

$$t = -\frac{s}{2}(1-\cos\theta) = -s\sin^2\left(\frac{\theta}{2}\right) \tag{219a}$$

$$u = -\frac{s}{2}(1+\cos\theta) = -s\cos^2\left(\frac{\theta}{2}\right) \tag{219b}$$

$\therefore$

$$\frac{s^2+u^2}{t^2} = \frac{1+\cos^4\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)} \tag{220a}$$

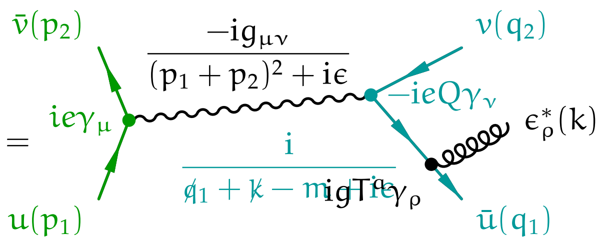
$$\frac{u^2}{st} = -\frac{\cos^4\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} \tag{220b}$$

- *finally*

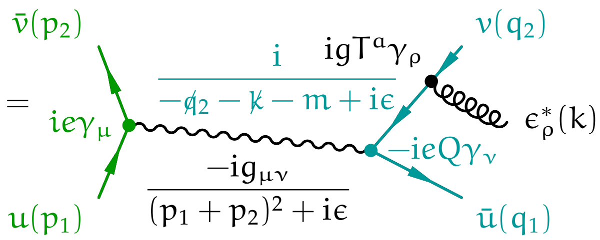
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left(\frac{1+\cos^2\theta}{2} + \frac{1+\cos^4\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)} - 2\frac{\cos^4\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\right) = \frac{\alpha^2}{4s}\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2 \tag{221}$$

Solution 20.

$$iT_1 = \quad (222a)$$



$$iT_2 = \quad (222b)$$



$$iT_1 = \left[\bar{u}(q_1)(igT_a\not{\epsilon}^*(k))\frac{i}{\not{q}_1+\not{k}-m+i\epsilon}(-ieQ\gamma^\mu)v(q_2)\right]\frac{-i}{(p_1+p_2)^2+i\epsilon}\left[\bar{v}(p_2)(ie\gamma_\mu)u(p_1)\right] \tag{223a}$$

$$iT_2 = \left[\bar{u}(q_1)(-ieQ\gamma^\mu)\frac{i}{-\not{q}_2-\not{k}-m+i\epsilon}(igT_a\not{\epsilon}^*(k))v(q_2)\right]\frac{-i}{(p_1+p_2)^2+i\epsilon}\left[\bar{v}(p_2)(ie\gamma_\mu)u(p_1)\right] \tag{223b}$$

Solution 21.

$$T_1\big|_{\epsilon^*_\mu(k)=k_\mu} = e^2 Qg\,[\bar v(p_2)\gamma_\mu u(p_1)]\,\frac{1}{s}\bar u(q_1)T_a(\not k + \not q_1 - m)\frac{1}{\not q_1 + \not k - m + i\epsilon}\gamma^\mu v(q_2)$$
$$= e^2 Qg\,[\bar v(p_2)\gamma_\mu u(p_1)]\,\frac{1}{s}\bar u(q_1)T_a\gamma^\mu v(q_2) \quad (224)$$

on the other hand

$$T_2\big|_{\epsilon^*_\mu(k)=k_\mu} = e^2 Qg\,[\bar v(p_2)\gamma_\mu u(p_1)]\,\frac{1}{s}\bar u(q_1)\gamma^\mu\frac{1}{-\not q_2 - \not k - m + i\epsilon}(\not k + \not q_2 + m)T_a v(q_2)$$
$$= -e^2 Qg\,[\bar v(p_2)\gamma_\mu u(p_1)]\,\frac{1}{s}\bar u(q_1)T_a\gamma^\mu v(q_2) = -T_1\big|_{\epsilon^*_\mu(k)=k_\mu} \quad (225)$$

Solution 22.

$$2q_1q_2 = (q_1+q_2)^2 = (p-k)^2 = s(1-x_3) \quad (226a)$$
$$2q_1k = (q_1+k)^2 = (p-q_2)^2 = s(1-x_2) \quad (226b)$$
$$2q_2k = (q_2+k)^2 = (p-q_1)^2 = s(1-x_1) \quad (226c)$$

Solution 23. *four traces:*

$$H^{\mu\nu}(q_1,q_2,k) =$$
$$\sum_\epsilon \frac{\mathrm{tr}[\not q_1 T_a \not\epsilon^*(\not q_1+\not k)\gamma^\mu\not q_2\gamma^\nu(\not q_1+\not k)\not\epsilon T_a]}{(2q_1k)^2} + \sum_\epsilon \frac{\mathrm{tr}[\not q_1 T_a\not\epsilon^*(\not q_1+\not k)\gamma^\mu\not q_2\not\epsilon T_a(-\not q_2-\not k)\gamma^\nu]}{(2q_1k)(2q_2k)}$$
$$+\sum_\epsilon \frac{\mathrm{tr}[\not q_1\gamma^\mu(-\not q_2-\not k)T_a\not\epsilon^*\not q_2\gamma^\nu(\not q_1+\not k)\not\epsilon T_a]}{(2q_1k)(2q_2k)} + \sum_\epsilon \frac{\mathrm{tr}[\not q_1\gamma^\mu(-\not q_2-\not k)T_a\not\epsilon^*\not q_2\not\epsilon T_a(-\not q_2-\not k)\gamma^\nu]}{(2q_2k)^2} \quad (227)$$

color part of the quark traces $\mathrm{tr}(T_aT_a) = C_F\,\mathrm{tr}(\mathbf{1}) = C_FN_C$ *and polarization sum*

$$H^{\mu\nu}(q_1,q_2,k) = 2C_FN_c\frac{\mathrm{tr}[\not q_1(\not q_1+\not k)\gamma^\mu\not q_2\gamma^\nu(\not q_1+\not k)]}{(2q_1k)^2} + 2C_FN_c\frac{\mathrm{tr}[\not q_1\not q_2\gamma^\mu(\not q_1+\not k)(-\not q_2-\not k)\gamma^\nu]}{(2q_1k)(2q_2k)}$$
$$+2C_FN_c\frac{\mathrm{tr}[\not q_1\gamma^\mu(-\not q_2-\not k)(\not q_1+\not k)\gamma^\nu\not q_2]}{(2q_1k)(2q_2k)} + 2C_FN_c\frac{\mathrm{tr}[\not q_1\gamma^\mu(-\not q_2-\not k)\not q_2(-\not q_2-\not k)\gamma^\nu]}{(2q_2k)^2} \quad (228)$$

contraction

$$H^\mu{}_\mu(q_1,q_2,k) = -4C_FN_c\frac{\mathrm{tr}[\not q_1(\not q_1+\not k)\not q_2(\not q_1+\not k)]}{(2q_1k)^2} + 8C_FN_c(q_1+k)(-q_2-k)\frac{\mathrm{tr}[\not q_1\not q_2]}{(2q_1k)(2q_2k)}$$
$$+8C_FN_c(-q_2-k)(q_1+k)\frac{\mathrm{tr}[\not q_1\not q_2]}{(2q_1k)(2q_2k)} - 4C_FN_c\frac{\mathrm{tr}[\not q_1(-\not q_2-\not k)\not q_2(-\not q_2-\not k)]}{(2q_2k)^2} \quad (229)$$

final traces

$$H^\mu{}_\mu(q_1,q_2,k) = -16C_FN_c\frac{2(q_1k)(q_2k)}{(2q_1k)^2} - 64C_FN_c\frac{(q_1q_2+q_1k+q_2k)(q_1q_2)}{(2q_1k)(2q_2k)} - 16C_FN_c\frac{2(q_1k)(q_2k)}{(2q_2k)^2}$$
$$-8C_FN_c\frac{1-x_1}{1-x_2} - 16C_FN_c\frac{(1-x_3)}{(1-x_1)(1-x_2)} - 8C_FN_c\frac{1-x_2}{1-x_1} = -8C_FN_c\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} \quad (230)$$

result

$$\frac{d^2\sigma}{dx_1 dx_2} = N_c \frac{4\pi\alpha^2 Q^2}{3s} \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \tag{231}$$

Solution 24. *with* $p = p_+ + p_-$*:*

$$i T = -i \frac{g}{2\cos\theta_w} \left[\bar{v}(p_+)\gamma_\mu (g_V^e - g_A^e \gamma_5) u(p_-)\right] \frac{-ig^{\mu\nu} + ip^\mu p^\nu / M_Z^2}{p^2 - M_Z^2} (i \frac{g M_Z}{\cos\theta_w} g_{\nu\rho}) \epsilon^{*,\rho}(q) \tag{232}$$

and from current conservation $\left[\bar{v}(p_+)\gamma_\mu (g_V^e - g_A^e \gamma_5) u(p_-)\right] p^\mu = -2i g_A^e m_e \bar{v}(p_+)\gamma_5 u(p_-) = O(m_e)$

$$T = -\frac{g^2 M_Z}{2\cos^2\theta_w} \frac{1}{s - M_Z^2} \left[\bar{v}(p_+) \not{\epsilon}^*(q) (g_V^e - g_A^e \gamma_5) u(p_-)\right] \tag{233}$$

Solution 25.

- *momentum conservation is obvious and energy conservation follows from*

$$E_H + E_Z = \sqrt{s} = 2E \tag{234}$$

- *the Higgs mass shell follows from*

$$E_H^2 - p^2 = \frac{s^2 + M_H^4 + M_Z^4 + 2sM_H^2 - 2sM_Z^2 - 2M_H^2 M_Z^2}{4s} - \frac{s^2 + M_H^4 + M_Z^4 - 2sM_H^2 - 2sM_Z^2 - 2M_H^2 M_Z^2}{4s} = M_H^2 \,, \tag{235}$$

  and the Z *mass shell analogously:* $E_Z^2 - p^2 = M_Z^2$.

- *finally*

$$4p_\mp q = 4E(E_Z \pm p\cos\theta) = s + M_Z^2 - M_H^2 \pm \sqrt{\lambda(s, M_H^2, M_Z^2)} \cos\theta \,, \tag{236}$$

  i. e.

$$\begin{aligned} 16(p_+ q)(p_- q) &= (s + M_Z^2 - M_H^2)^2 - \lambda(s, M_H^2, M_Z^2)\cos^2\theta \\ &= 4sM_Z^2 + \lambda(s, M_H^2, M_Z^2)\sin^2\theta \end{aligned} \tag{237}$$

Solution 26.

$$\begin{aligned} \sum_{spins} |T|^2 &= \frac{g^4 M_Z^2}{4\cos^4\theta_w} \frac{1}{(s - M_Z^2)^2} \operatorname{tr}\left(\not{p}_+ \not{\epsilon}^*(q) (g_V^e - g_A^e \gamma_5) \not{p}_- (g_V^e + g_A^e \gamma_5) \not{\epsilon}(q)\right) \\ &= \frac{g^4 M_Z^2}{4\cos^4\theta_w} \frac{1}{(s - M_Z^2)^2} ({g_V^e}^2 + {g_A^e}^2) \operatorname{tr}\left(\not{p}_+ \not{\epsilon}^*(q) \not{p}_- \not{\epsilon}(q)\right) \\ &= \frac{g^4 M_Z^2 ({g_V^e}^2 + {g_A^e}^2)}{4\cos^4\theta_w} \frac{1}{(s - M_Z^2)^2} L^{\mu\nu}(p_+, p_-, 0) \epsilon_\mu^*(q) \epsilon_\nu(q) \end{aligned} \tag{238}$$

polarization sum

$$\sum_{pol.} \epsilon_\mu(q)\epsilon^*_\nu(q) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2} \tag{239}$$

$$\begin{aligned}\sum_{spins,\,pol.} |T|^2 &= \frac{g^4 M_Z^2({g_V^e}^2 + {g_A^e}^2)}{\cos^4\theta_w} \frac{1}{(s-M_Z^2)^2} \left(p_+^\mu p_-^\nu + p_-^\mu p_+^\nu - p_+ p_- g^{\mu\nu}\right) \times \\ &\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right) = \frac{g^4({g_V^e}^2 + {g_A^e}^2)}{\cos^4\theta_w} \frac{p_+p_- M_Z^2 + 2(p_+q)(p_-q)}{(s-M_Z^2)^2} \\ &= \frac{g^4({g_V^e}^2 + {g_A^e}^2)}{8\cos^4\theta_w} \frac{8sM_Z^2 + \lambda(s, M_H^2, M_Z^2)\sin^2\theta}{(s-M_Z^2)^2}\end{aligned} \tag{240}$$

phase space (118)

$$\frac{d\sigma}{d\Omega}(\theta_{e-H}) = \frac{1}{2s}\frac{1}{16\pi^2}\frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{2s}\frac{1}{4}\sum_{spins,\,pol.} |T|^2 \tag{241}$$

i.e.

$$\frac{d\sigma}{d\Omega}(\theta_{e-H}) = \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{s} \frac{\alpha^2({g_V^e}^2 + {g_A^e}^2)}{128 s \sin^4\theta_w \cos^4\theta_w} \frac{8sM_Z^2 + \lambda(s, M_H^2, M_Z^2)\sin^2\theta}{(s-M_Z^2)^2} \tag{242}$$

Solution 27. *with*

$$\int d\Omega\,(a + b\sin^2\theta) = 4\pi a + \frac{8\pi}{3} b \tag{243}$$

follows

$$\sigma = \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{s} \frac{\pi\alpha^2({g_V^e}^2 + {g_A^e}^2)}{48 s \sin^4\theta_w \cos^4\theta_w} \frac{12sM_Z^2 + \lambda(s, M_H^2, M_Z^2)}{(s-M_Z^2)^2} \tag{244}$$

or

$$\sigma = \frac{\sqrt{\lambda(s, M_H^2, M_Z^2)}}{s} \frac{\pi\alpha^2(1 + (1 - 4\sin^2\theta_w)^2)}{192 s \sin^4\theta_w \cos^4\theta_w} \frac{12sM_Z^2 + \lambda(s, M_H^2, M_Z^2)}{(s-M_Z^2)^2} \tag{245}$$