The anomalous magnetic moment of the muon
A challenge for theory and experiment

Christian Sturm

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)
Föhringer Ring 6
80805 München

Content

Outline of the talk

I. Generalities
- Magnetic moments - definitions
- Anomalous magnetic moments of leptons a_ℓ - definitions
- Why a_μ?

II. Experiment
- Basics for measuring a_μ
- The value

III. Theory
- Calculations to a_ℓ in QED, EW & strong interaction
- Predictions from theory & difference to exp.
- Anomalous magnetic moments of other leptons (electron, tau)

IV. Methods & Calculations
- IBP, MI
- A bit of history – calculations are difficult...
- Own contribution

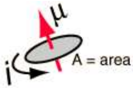
V. Summary & Conclusion

Generalities

Magnetic moments $\vec{\mu}$

- Magnetic moment of any system:

  I.) Motion of el. charges



  II.) Intrinsic mag. moments of elementary particles

- Classically:

$$\vec{\mu} = \frac{q}{2m}\vec{L} \stackrel{q=e}{=} \mu_B \vec{L}, \quad \text{Bohr magneton} : \mu_B = \frac{e}{2m_e}$$

- A magnetic moment in an external magnetic $\vec{B}$ field has a potential energy U:

$$U = -\vec{\mu}\vec{B}$$

- Fundamental particles: intrinsic $\vec{\mu} \leftrightarrow$ Spin $\vec{S}$
  Dirac theory predicts for a lepton $\ell = e, \mu, \tau$:

$$\vec{\mu}_\ell = g_\ell \left(\frac{q_\ell}{2m_\ell}\right)\vec{S}, \quad g_\ell = 2 \qquad \text{(free, non-interacting)}$$

  g for classical orbital rotations would be 1

Generalities

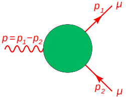
Anomalous magnetic moment of leptons: a_ℓ

- “Switch on” interactions:
  QED, EW interactions, strong interactions
- Quantum fluctuations: $\rightsquigarrow$ deviation from $g_\ell = 2$:
  parametrized by

$$g_\ell = 2(1 + a_\ell) \qquad \rightsquigarrow \text{ precise test of QFT}$$

$\hookrightarrow$ single number!

- a_ℓ can be computed with high accuracy
  a_μ can be measured with high precision
  (in homogenous static mag. field, see later)
- More formally:



$$= \bar{u}(p_1) \left[\gamma^\mu F_E(p^2) + i \frac{\sigma^{\mu\nu} p_\nu}{2m_\ell} F_M(p^2) \right] u(p_2)$$

- In the static limit($p^2 \to 0$): $\quad F_E(0) = 1, \qquad F_M(0) = a_\ell$

- Muon very interesting:
  Quantum fluctuations due to heavier particles M:

$$\delta a_\ell \propto m_\ell^2/M^2 \qquad (M >> m_\ell)$$

  M heavy SM or BSM particle
  ratio: $m_\mu/m_e \sim 200$, $m_\tau/m_\mu \sim 17$

$\rightsquigarrow$ Sensitivity to physics beyond SM through virtual particles in loops

- Loop calculations not only mathematical task to increase precisions of a given observable

$\rightsquigarrow$ also allow to access energy regimes not yet reachable by collider experiments through virtual particles in the loops and open a window for new physics

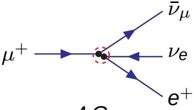
- Constraints on SM particles

Generalities

Example: Electroweak precision measurements

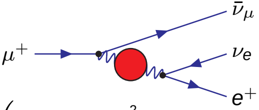
- Exploited in electroweak precision measurements:
  Muon decay: precise measurement of lifetime at PSI

Fermi Model



$\longleftrightarrow$

Standard Model



$$\frac{4\,G_F}{\sqrt{2}} = \frac{2\,\alpha\,\pi}{s_W^2\,M_W^2}\left(1 + \underbrace{\Delta\alpha - \frac{c_W^2}{s_W^2}\,\Delta\rho + \textit{other corrections}}_{\Delta r}\right)$$

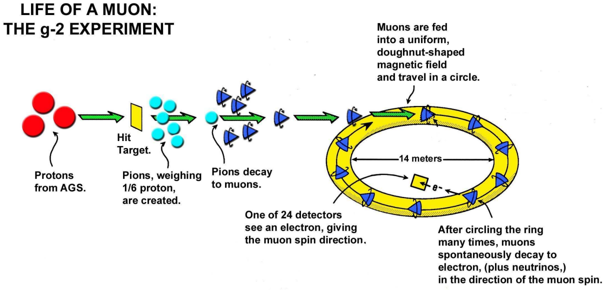
- Determine weak coupling G_F at 0.6 ppm (muLan,FAST)
- Δr depends on SM parameters: $M_t, M_H, \ldots$

  $\Longrightarrow$ prediction of $M_W^{theory} \leftrightarrow M_W^{experiment}$
- Constrain/predict top quark mass (before direct discovery)
- $\Delta\alpha$ running of the fine structure constant
- Consistency constraints on the Higgs mass

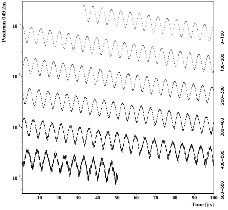
  $\Rightarrow$ Guide line in which mass range to search for it

Experiment

Last experiment carried out at BNL:



http://www.g-2.bnl.gov/

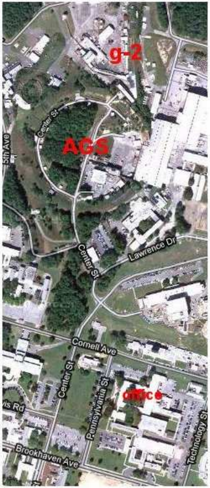


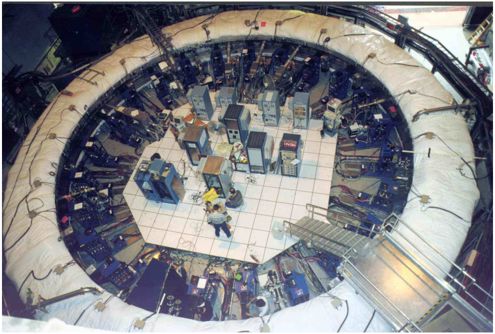
(g-2) Collaboration (H.N. Brown et al.)

Quantum Fluctuations!

- $\vec{\omega_a} = \vec{\omega_s} - \vec{\omega_c} = \frac{a_\mu e\vec{B}}{m_\mu} - \frac{e}{m_\mu}\left[a_\mu - \frac{1}{\gamma^2-1}\right]\vec{v}\times\vec{E}$
  ω_s: spin precession frequency, ω_c: cyclotron frequency
- Experiment done with both polarities (μ^+, μ^-)
- positron time spectrum: $N_0(E)e^{t/\gamma\tau}[1+A(E)\cos(\omega_a t+\phi(E))]$
- Measure magnetic field, NMR

Experiment





http://www.g-2.bnl.gov/

Radius: 7.112 m,

Muon “magic” momentum: 3.094 GeV,

field: 1.45 T

lifetime at rest: 2.1970 μs,

in the ring: 64.435 μs

Experiment

The value

The present experimental value is terrifically accurate!:

$$a_\mu^{\text{exp}} = 116592089(63) \cdot 10^{-11}$$

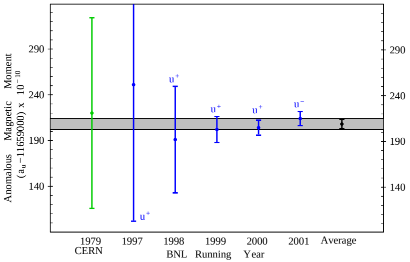
PDG, E821: Final Report: PRD73 (2006)

with statistical error$(54 \cdot 10^{-11})$ and systematic error$(33 \cdot 10^{-11})$

at the level of $\sim$0.5 ppm

- First measurement of a_μ was performed at Columbia 1960
- The result a_μ = 0.00122(8) precision of about 5%
- No difference with a_e

History experiment:

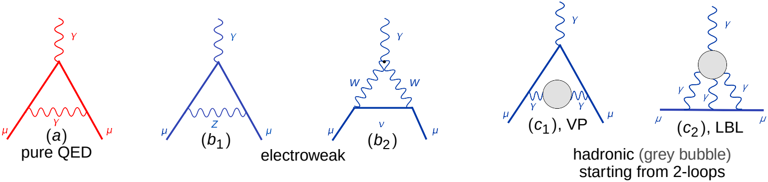


$\rightsquigarrow$ Improvement of a factor of 14 compared to the CERN experiment

Theory

Higher order corrections

- Higher order corrections are classified into 3 classes:



$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{Had}}$$

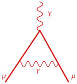
- The QED part is known to 4-loops (and leading terms in 5 loops!)
  (next slide)
- The EW part is known to 2-loops
  R. Jackiw, S. Weinberg; G. Altarelli et al.; I. Bars, M. Yoshimura; A. Czarnecki et al.
- The hadronic part is known but with limited accuracy
  Bouchiat,et al.;M. Gourdin,et al.;Brodsky, de Rafael;Hagiwara et al.;Alemany et al.;Davier et al.;Passera et al.
  Dominant uncertainties to the theory prediction of muon anomaly
  VP traditionally: measurements of cross section and hadronic τ decays(more later)
  Recently: non-pert. lattice calculation from first principles(ETMC)

Theory

Theory: QED contributions to a_μ

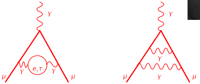
$$a_\mu^{\text{QED}} = \left(\frac{\alpha}{\pi}\right) 0.5$$

Schwinger



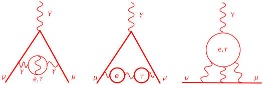
$$+ \left(\frac{\alpha}{\pi}\right)^2 0.765857410(27)$$

Sommerfield; Petermann; Suura & Wichmann; Elend



$$+ \left(\frac{\alpha}{\pi}\right)^3 24.05050964(43)$$

Barbieri, Laporta, Remiddi,..., Czarnecki, Skrzypek; Friot, Greynat, de Rafael,...



$$+ \left(\frac{\alpha}{\pi}\right)^4 130.8055(80)$$

Kinoshita, Lindquist, Nio, Nizic, Okamoto, Aoyama, Hayakawa; Lautrup, de Rafael,...

$$+ \left(\frac{\alpha}{\pi}\right)^5 663(20)$$ In progress

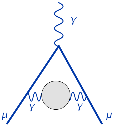
Kinoshita et al.; Kataev; Laporta; Baikov et al.

- Starting from 2-loop: universal(mass independent) vs. mass dependent contr.
- Result $\leq$ 3-loop analytical, $\geq$4-loop numerical (only one complete calculation)
- QED four-loop contributes as much as $380.8 \cdot 10^{-11}$
  (compared to the exp. uncertainty of $\sim 60 - 70 \cdot 10^{-11}$)
  $\Longrightarrow$ five-loop contr. are relevant!
- Log enhancements: $\log(m_\mu/m_e)$ mass dependent contr. important for a_μ!

Theory

The LO hadronic contributions to a_μ

- Can not be computed perturbatively

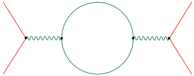


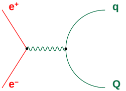
$$a_\mu^{\text{had}}[LO] = 4\frac{\alpha^2}{\pi}\int ds \frac{K(s)}{s}\text{Im}\left[\Pi(s)\right], \quad K(s) \sim 1/s$$

$\Pi(s)$: vacuum polarization function

- Optical theorem:

$$2\,\text{Im}\Big(\quad \Big) = \int d\Pi \Big| \quad \Big|^2$$





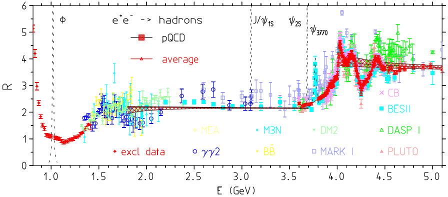
$$12\pi \text{Im}\left[\Pi(q^2 = s)\right] = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = R(s)$$

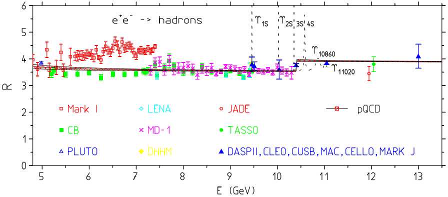
- $K(s)/s \sim 1/s^2$ enhance low energy region of $R(s)$
- R-ratio: $R = N_c \sum_{i=u,d,s,..} Q_i^2$

Theory

The LO hadronic contributions to a_μ, R-ratio

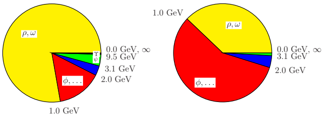
- Measure $R(s)$: energy scan, radiative return (KLOE, BaBar)

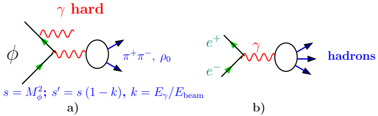




- Contributions to a_μ in %

Jegerlehner et al.





dominant contr. from below 1 GeV, dominant error from 1-2 GeV

- Above data: use continuum pert. theory
- Alternatively: Get information from hadronic τ decays

Theory

The size of the individual contributions to a_μ

Orders of magnitude:[1] (here: muon case)

$$a_\mu^{\text{QED}} = 116\,584\,718.09(0.15) \times 10^{-11}$$
$$a_\mu^{\text{EW}} = 154(1)(2) \times 10^{-11}$$

(suppression m_μ^2/M_W^2)

M_h dependence small

QED dominant contribution

Hadronic contributions:

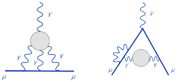
$$a_\mu^{\text{Had}} = a_\mu^{\text{Had}}[\text{LO}] + a_\mu^{\text{Had}}[\text{NLO}]$$
$$= 6955(40)(7) \times 10^{-11}$$
$$+7(26) \times 10^{-11}$$

Compared to recent latt. result:

$$a^{\text{Had}}_{\mu,n_f=2}[\text{LO}] = 5720(160) \times 10^{-11}$$

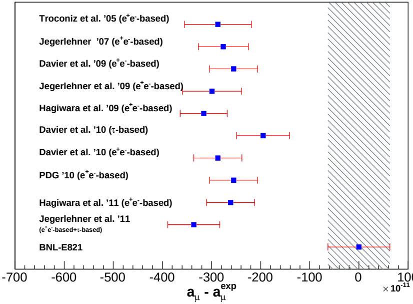
(ETMC) factor$\sim$3

NLO had:



opposite sign

$\rightsquigarrow$ dominant error



theo.<exp.

[1] A. Höcker, W. Marciano, PDG 10

Theory vs. Experiment

The difference

- [2] SM theory prediction:

$$a_\mu^{\text{theo}} = 116591834(2)(41)(26) \cdot 10^{-11}$$

- [2] $\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{theo.}} = 255(63)(49) \cdot 10^{-11}$

  The current theory prediction shows an
  "interesting but not yet conclusive discrepancy" of $\sim 3.2\sigma$

- Deviations with different determinations of hadronic contr.:
  - e^+e^-, based method: 3.6σ Davier et al.'10
  - e^+e^-, based method: 3.3σ Hagiwara et al. '11
  - e^+e^-,τ based method:4.1σ Jegerlehner et al.'11

  using data from SND, CMD-2, KLOE, BaBar, BES, CLEO,...

- Follow up experiment: proposed at Fermilab or JPARC, factor of 4 anticipated ($\delta a_\mu \sim 15 \times 10^{-11}$, 0.14 ppm)

[2] A. Höcker, W. Marciano, PDG 10

$\rightsquigarrow$ This reduction will lead to a more definitive result

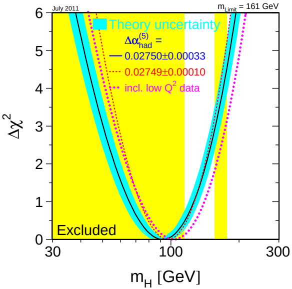
- $> 5\sigma$ “discovery-level” deviation from the SM, if the central value remains unchanged
  $\rightsquigarrow$ enter new physics territory
- January 13th, 2011, P. Oddone: “*Following ... discussions with the Department of Energy on funding projections over the period when we could run the New g-2 Experiment, I grant Stage I approval to g-2*”
- Data taking could start 2015
- Main ingredient would still be muon storage ring from BNL

Theory vs. Experiment

a_μ^{had} and M_H Study by Passera, Marciano, Sirlin:

- The difference could be due to contr. of new BSM physics
- Can difference be explained with errors in a_μ^{had}?
  $\hookrightarrow$ Changes in σ have important consequences on M_H from EW precision measurements
  $\hookrightarrow$ induces changes in $\Delta\alpha_{had}$, key input of EW fits

- Higgs boson mass: EW precision measurements:
  Shifts in σ in for $E \gtrsim 1.2 GeV$
  $\hookrightarrow$ upper bound of M_H inconsistent with LEP lower limit
  Shifts for lower E unlikely, given the small exp. uncertainties



http://lepewwg.web.cern.ch/LEPEWWG/

- Reduce $M_H \lesssim 130$ GeV
  $\hookrightarrow$ leaves only narrow window for M_H in conjunction with 114.4 GeV

- Any precise theoretical prediction requires a precise knowledge of the fundamental parameters
- In QED these are: fine structure constant α
  + lepton masses m_ℓ
- Most important basic parameter for calculating a_μ is: α
- LO result is $\alpha/(2\pi)$ + higher order QED corrections give dominant contribution
- Very precise value for α can be determined using of a_e
- Mass dependent contr. differ for a_e, a_μ and a_τ, such that $a_e \neq a_\mu \neq a_\tau$

Theory

Anomalous magnetic moment of the electron

- Dominant contribution from universal diagrams
- Mass dependent contr. are very small
- Dominant error in prediction: uncertainty in α

contribution	a_e in units 10^{-6}
universal	1159.652 16856(929)(10)(31)
μ–loops	0.000 00271 (0)
τ–loops	0.000 00001 (0)
hadronic	0.000 00168 (2)
weak	0.000 000039 (0)
theory	1159.652 17299(930)
experiment	1159.652 180 73 (28)[1]

- a_e at 0.24 ppb, more precise than a_μ by a factor of $\geq$ 2000
- Thus experimental tests are able to check QED up to 7 digits
- The sensitivity of the latter to "new physics" still about 19 times larger
- [1] $1/\alpha = 137.035999084(51)$ [0.37 ppb]
- Dominant theory error: missing 5-loop QED
  Requires to evaluate the perturbation expansion up to 5 loops

[1] D. Hanneke, S. Fogwell, G. Gabrielse

Theory

Anomalous magnetic moment of the τ-lepton

- No real measurement exists yet for a_τ
- Theory predicts: $a_\tau = 117721(5) \times 10^{-8}$
- The experimental limit from the LEP experiments OPAL and L3 is

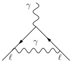
$$-0.052 < a_\tau < 0.013 \quad \text{at} \;\; 95\% \;\; \text{CL}$$

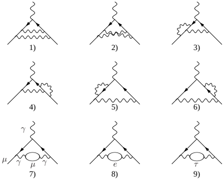
- For a_τ one would have by far the best sensitivity if one could measure it with comparable precision
- ...is beyond present experimental possibilities, because of very short lifetime of the τ

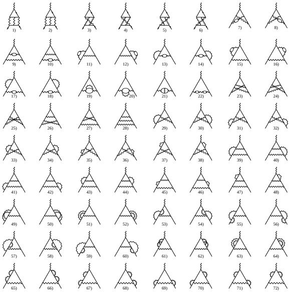
Back to theory of a_μ

calculations, some diagrams 1- to 3-loop...

QED gives dominant contribution







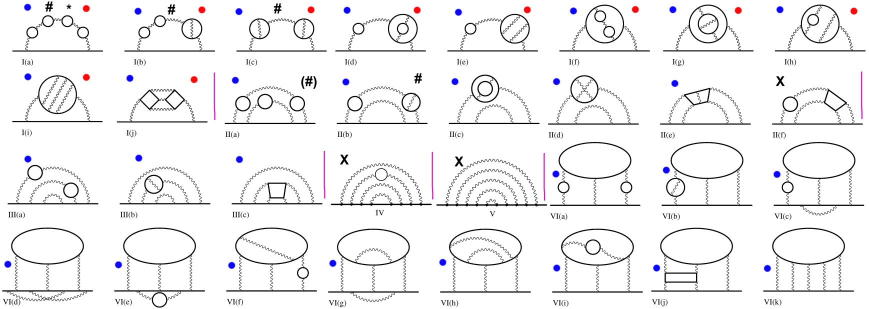
1-loop: 1 diagram (Schwinger),
2-loop: 7(9) diagrams,
3-loop: 72+ diagrams,
4-loop: 891+ diagrams

Each Feynman diagrams leads to many loop integrals

Theory

...calculations, even more diagrams...(currently in progress, frontier)

- 12672 diagrams
- 6 classes (I - VI), 32 gauge invariant subsets
- (For simplicity external photon omitted)



diagrams from M. Nio, T. Aoyama, M. Hayakawa, T. Kinoshita

●: Kinoshita et al. ●: Baikov, Chetyrkin, C.S. #: Laporta *: Aguilar, de Rafael, Greynat

X: still unkown (only 3 out of 32 subsets!)

●: numerical; ●, #, *: (semi) analytical

Methods

...sigh, lots of integrals to solve....

Example: First 10 gauge invariant 5-loop subsets with e: $\sim$ 15000

Integration-by-parts (IBP): K.G. Chetyrkin, F.V. Tkachov

$$0 = \int [d^D \ell_1] \dots [d^D \ell_4] \quad \partial_{(\ell_j)_\mu} \left(\ell_l^\mu \, I_{\alpha\beta} \right) \;, \qquad j, l = 1, \dots, \text{loops}$$

$I_{\alpha\beta}$: Generic integrand with propagator powers $\alpha = \{\alpha_1, \dots\}$
and scalar-product powers $\beta = \{\beta_1, \dots\}$

Laporta-Algorithm: originally developed for g-2 S. Laporta, E. Remiddi

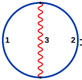
Idea:
– IBP-identities for explicit numerical values of α, β
– Introduction of an order among the integrals
– Solving a linear system of equations

Automation: $\rightarrow$ Computer algebra

Reducible: Diagrams which can be mapped on diagrams with less lines, others masters

Problem: Dramatic growth of number of equations

Here: >100000 IBP-identities generated and solved
$\rightsquigarrow$ Large integral-tables with solutions for several thousand integrals, expressed in terms of few masters(MI)



$$= \int d\ell_1 \int d\ell_2 \frac{1}{(\ell_1^2+m^2)(\ell_2^2+m^2)(\ell_1+\ell_2)^2} = \int d\ell_1 \int d\ell_2 \frac{1}{D_1^1 D_2^1 D_3^1} = f(1,1,1)$$

IBP-identities:

I) $0 = \int d\ell_1 \int d\ell_2 \ \partial_{\ell_1} \ell_2 \ \frac{1}{D_1 D_2 D_3}$

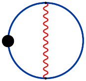
II) $0 = \int d\ell_1 \int d\ell_2 \ \partial_{\ell_1} \ell_1 \ \frac{1}{D_1 D_2 D_3}$

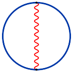
$$0 = f(1,1,1) - f(2,1,0) - 2m^2 f(2,1,1) - f(1,1,1)$$

$$0 = d f(1,1,1) - 2 f(1,1,1) + 2m^2 f(2,1,1) - f(1,1,1)$$

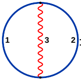
$$\Rightarrow f(2,1,1) = -\frac{1}{2m^2} f(2,1,0)$$

$$\Rightarrow f(1,1,1) = \frac{1}{d-3} f(2,1,0)$$

 $= \frac{-1}{2m^2}$ 

 $= \frac{1}{d-3}$ 

Laporta, Remiddi a_ℓ: 3-loop: 18 MI, 4-loop: $\sim$ 300 MI



$$= \int d\ell_1 \int d\ell_2 \frac{1}{(\ell_1^2+m^2)(\ell_2^2+m^2)(\ell_1+\ell_2)^2} = \int d\ell_1 \int d\ell_2 \frac{1}{D_1^1 D_2^1 D_3^1} = f(1,1,1)$$

IBP-identities:

I) $0 = \int d\ell_1 \int d\ell_2 \; \partial_{\ell_1} \ell_2 \; \frac{1}{D_1 D_2 D_3}$

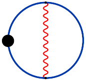
II) $0 = \int d\ell_1 \int d\ell_2 \; \partial_{\ell_1} \ell_1 \; \frac{1}{D_1 D_2 D_3}$

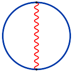
$$0 = f(1,1,1) - f(2,1,0) - 2m^2 f(2,1,1) - f(1,1,1)$$

$$0 = d f(1,1,1) - 2f(1,1,1) + 2m^2 f(2,1,1) - f(1,1,1)$$

$$\Rightarrow f(2,1,1) = -\frac{1}{2m^2} f(2,1,0)$$

$$\Rightarrow f(1,1,1) = \frac{1}{d-3} f(2,1,0)$$

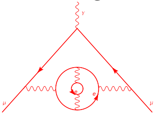
 $= \frac{-1}{2m^2}$  $= \frac{d-2}{4m^4}$ 

 $= \frac{1}{d-3}$  $= \frac{(2-d)}{(d-3)2m^2}$ 

Laporta, Remiddi a_ℓ: 3-loop: 18 MI, 4-loop: $\sim$ 300 MI

Some historical examples at 4-loop

- Analytical and numerical results help each other in mutually testing current and future calculations
- Using results from[3] R. Faustov et al. '91 found the contribution to:



$$a_\mu = \left(\frac{\alpha}{\pi}\right)^4 \left[0.92374 + \mathcal{O}\left(\frac{m_e}{M_\mu}\right)\right]$$

is in disagreement to the numerical result by T. Kinoshita, B. Nizic, Y. Okamoto '91

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^4 1.4416(18)$$

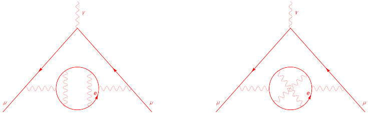
- Problem came from theoretical error in[3]; after correction T. Kinoshita, H. Kawai, Y. Okamoto the new result was in good agreement:

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^4 \left[1.452570 + \mathcal{O}\left(\frac{m_e}{M_\mu}\right)\right]$$

[3] J. Calmet, E. de Rafael '75

An other historical example at 4-loop

- Quenched approximation of the photon propagator



+...

- Asymptotic result Broadhurst, Kataev, Tarasov '94 was in disagreement to the numerical result by T. Kinoshita, B. Nizic, Y. Okamoto '90

$$a_\mu^{asymp.} = \left(\frac{\alpha}{\pi}\right)^4 (-0.29087 + \mathcal{O}(\frac{m_e}{M_\mu}))$$

$$a_\mu^{num.} = \left(\frac{\alpha}{\pi}\right)^4 (-0.7945(202))$$

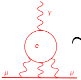
- After recalculation with improved integration routine (VEGAS instead of RIWIAD + much better statistics) result changed to T. Kinoshita '93

$$a_\mu^{num.} = \left(\frac{\alpha}{\pi}\right)^4 (-0.2415(19))$$

Calculation

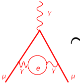
...do the same comparison at 5-loops, logarithmically enhanced contributions

There are 2 sources of numerically leading enhanced logs of large ratio $\frac{M_\mu}{m_e} = 206.7682838$ pure QED contributions:



$$\sim \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{2\pi^2}{3} \ln\left(\frac{M_\mu}{m_e}\right) + \ldots \right),$$

LBL: light by light scattering



$$\sim \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{1}{3} \ln\left(\frac{M_\mu}{m_e}\right) + \ldots \right)$$

VP: vacuum polarization

$\rightarrow$ We will consider mixed VP contributions: photon propagator composed from electron loops and photon exchanges only!

$$a_\mu = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \left[d_R \left(\frac{-x^2}{1-x} \frac{M_\mu}{m_e}, \alpha \right) - 1 \right]$$

B. Lautrup, de Rafael '74

with $d_R(q^2/m^2, \alpha) = 1/(1 + \alpha \Pi^{OS}(q^2/m_e^2, \alpha^{OS}))$

Π^{OS} proper photon VP in OS-scheme

Calculation

Vacuum polarization function

Required ingredients for the calculation:

- Vacuum polarization function
  - High-energy limit $\rightarrow$ massless propagators
    The value of $\overline{\Pi}(Q^2, m = 0, \overline{\alpha})$ is known at 4-loops
    with Baicer (P. Baikov, 2000-2008) in $\overline{\text{MS}}$-scheme

    P.A. Baikov, K.G. Chetyrkin, J.H. Kühn

Traditionally in calculations of a_ℓ everybody uses the classical OS-scheme: α and all lepton masses are on-shell and:

$$\Pi^{\text{OS}}(Q = 0, m, \alpha^{\text{OS}}) = 0$$

$\rightsquigarrow$ Transform massless propagator from $\overline{\text{MS}} \rightarrow$ on-shell scheme

- Need: $\overline{\text{MS}} \leftrightarrow$ On-shell relation at 4-loop
  for fine structure constant conversion: $\overline{\alpha} \rightarrow \alpha^{\text{OS}}$

Result

$\overline{\text{MS}}$-OS-relation for conversion of fine structure constant

$$\overline{\alpha} = \alpha^{\text{OS}} \left(1 + \sum_{i \geq 1} C^{(i)}_{\overline{\alpha}\alpha} \left(\frac{\alpha^{\text{OS}}}{\pi} \right)^i \right)$$

$$\begin{aligned} C^{(4)}_{\overline{\alpha}\alpha} = {} & \frac{14327767}{9331200} + \frac{8791}{3240}\pi^2 + \frac{204631}{259200}\pi^4 - \frac{175949}{4800}\zeta_3 + \frac{1}{24}\pi^2\zeta_3 + \frac{9887}{480}\zeta_5 - \frac{595}{108}\pi^2 \ln 2 \\ & - \frac{106}{675}\pi^4 \ln 2 + \frac{6121}{2160}\pi^2 \ln^2 2 - \frac{32}{135}\pi^2 \ln^3 2 - \frac{6121}{2160}\ln^4 2 + \frac{32}{225}\ln^5 2 - \frac{6121}{90} a_4 - \frac{256}{15} a_5 \\ & + \ell_{\mu m} \left[-\frac{383}{31104} + \frac{23}{108}\pi^2 - \frac{41}{144}\zeta_3 - \frac{2}{9}\pi^2 \ln 2 \right] + \frac{43}{144}\ell^2_{\mu m} + \frac{13}{108}\ell^3_{\mu m} + \frac{1}{81}\ell^4_{\mu m}, \end{aligned}$$

$$\ell_{\mu m} = \ln \frac{\mu}{m}, \quad a_n = \text{Li}_n\left(\frac{1}{2}\right)$$

Baikov, Chetyrkin, C.S.

Important: $\Pi(q^2/m^2, \alpha) = \Pi^{\infty}(q^2/m^2, \alpha) + \mathcal{O}(m^2/q^2)$

then the resulting error in a_μ will be:

$$a_\mu = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \left[d_R^{\infty} \left(\frac{-x^2}{1-x} \frac{M_\mu}{m_e}, \alpha \right) - 1 \right] + \mathcal{O}\left(\frac{m_e}{M_\mu} \right)$$

of order m_e/M_μ with $d_R^{\infty} = 1/(1 + \alpha \Pi^{\infty})$

Result

Analytical 5-loop contributions to a_μ

The resulting contributions to a_μ coming from 4-loop terms in the photon propagator read :

$$a_\mu^{\text{asymp.}} = \sum_{i\geq 2} a_\mu^{\text{asymp.},(i)} \left(\frac{\alpha}{\pi}\right)^i$$

$$\begin{aligned}
a_\mu^{\text{asymp.},(5)} = & -\frac{296496193}{41990400} + \frac{45709}{58320}\pi^2 + \frac{212701}{518400}\pi^4 - \frac{4488523}{259200}\zeta_3 + \frac{35}{144}\pi^2\zeta_3 + \frac{4}{3}\zeta_3^2 + \frac{10909}{720}\zeta_5 \\
& + \frac{35}{8}\zeta_7 - \frac{55}{24}\pi^2\ln 2 - \frac{53}{675}\pi^4\ln 2 + \frac{6121}{4320}\pi^2\ln^2 2 - \frac{16}{135}\pi^2\ln^3 2 - \frac{6121}{4320}\ln^4 2 \\
& + \frac{16}{225}\ln^5 2 - \frac{6121}{180}a_4 - \frac{128}{15}a_5 + \ell_{\mu e}\left[\frac{1416095}{279936} + \frac{41}{972}\pi^2 - \frac{1855}{432}\zeta_3 - \frac{10}{3}\zeta_5 - \frac{2}{9}\pi^2\ln 2\right] \\
& + \ell_{\mu e}^2\left[-\frac{1507}{1944} + \frac{8}{81}\pi^2 + \frac{4}{3}\zeta_3\right] - \frac{83}{243}\ell_{\mu e}^3 + \frac{8}{81}\ell_{\mu e}^4 + \mathcal{O}\left(\frac{m_e}{M_\mu}\right), \qquad \ell_{\mu e} = \ln\frac{M_\mu}{m_e}, \quad a_n = \text{Li}_n\left(\frac{1}{2}\right)
\end{aligned}$$

Baikov, Chetyrkin, C.S.

$\Longrightarrow$ Logarithmically enhanced terms: $\ell_{\mu e}$

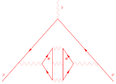
Numerically: $\left(\frac{\alpha}{\pi}\right)^5 a_\mu^{asymp.,(5)} = \left(\frac{\alpha}{\pi}\right)^5 62.2667 = 0.42105 \cdot 10^{-11}$

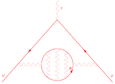
(compared to $\left(\frac{\alpha}{\pi}\right)^5 663(20)$, about $\sim$10%)

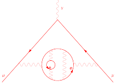
log vs. non-log terms: 87% $\longleftrightarrow$ 13%

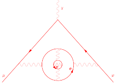
Comparison

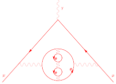
Numerical results $\longleftrightarrow$ Analytical results Baikov, Chetyrkin, C.S.

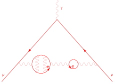

$I(j)$

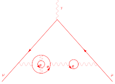

$I(i)$

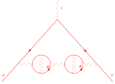

$I(h)$

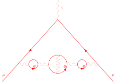

$I(g)$

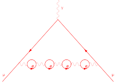

$I(f)$


$I(e)$


$I(d)$


$I(c)$


$I(b)$


$I(a)$

Subset	analytical	numerical	$\delta(a-n)$
$I(j)$	$-1.21429+\mathcal{O}(\frac{m_e}{m_\mu})$	-1.24726(12)	+0.033
$I(i)$	$+0.25237+\mathcal{O}(\frac{m_e}{m_\mu})$	+0.0871(59)	+0.165
$I(g)+I(h)$	$+1.50112+\mathcal{O}(\frac{m_e}{m_\mu})$	+1.56070(64)	-0.060
$I(f)$	$+2.89019+\mathcal{O}(\frac{m_e}{m_\mu})$	+2.88598(9)	+0.004
$I(c)$	$+4.81759+\mathcal{O}(\frac{m_e}{m_\mu})$	+4.74212(14)	+0.075
$I(d)$	$+7.44918+\mathcal{O}(\frac{m_e}{m_\mu})$	+7.45270(88)	-0.004
$I(e)$	$-1.33141+\mathcal{O}(\frac{m_e}{m_\mu})$	-1.20841(70)	-0.123
$I(b)$	$+27.7188+\mathcal{O}(\frac{m_e}{m_\mu})$	+27.69038(30)	+0.028
$I(a)$	$+20.1832+\mathcal{O}(\frac{m_e}{m_\mu})$	+20.14293(23)	+0.040

-Numerics from: T. Kinoshita, M. Nio (06); T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, N. Watanabe (08); T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio (07,08,10)

-Agreement with Kataev, where available

-Remaining differences should come from power suppressed corrections to the asymptotic result of $\mathcal{O}(m_e/M_\mu)$

Summary & Conclusion

- a_μ has already been measured with impressive precision 0.5 ppm
- Dominant contribution to theory prediction comes from pure QED
- Dominant theory error arises from hadronic contributions
- Currently a discrepancy of $> 3\sigma$ between theory and experiment
- Discrepancy not yet understood, sign of new physics?
- Future experiment will give a more definite answer
- To conclude: After more than $\sim$50 years, the anomalous magnetic moment of the muon is still a challenge for experiment and theory