Numerical Evaluation of Multi-loop Integrals

Sophia Borowka

MPI for Physics, Munich

Max-Planck-Institut für Physik (Werner-Heisenberg-Institut) IMPRS EPP

In collaboration with G. Heinrich

Based on arXiv:1204.4152 [hep-ph]

HP^8:Workshop on High Precision for Hard Processes, Munich
September 5th, 2012

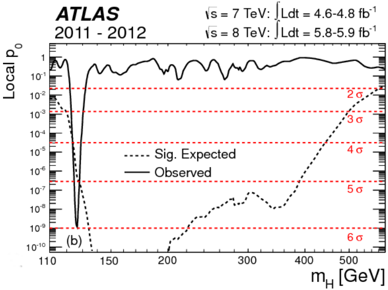
http://secdec.hepforge.org

The LHC Era has begun

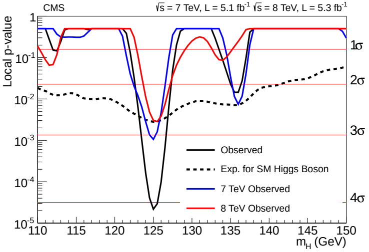
- We are probing energies which have never been reached at colliders before
- High experimental precision is possible due to high luminosities
- Highly precise theoretical predictions are necessary

New particle consistent with the Higgs found

- Announcement of a new particle finding on July 4th 2012



ATLAS '12



CMS '12

Making Predictions in the LHC Era

- At NLO: lot of progress (multi-leg, automation,...)

Making Predictions in the LHC Era

- At NLO: lot of progress (multi-leg, automation,...)
- Beyond NLO: calculations progressing well, but automation is difficult

Making Predictions in the LHC Era

- At NLO: lot of progress (multi-leg, automation,...)
- Beyond NLO: calculations progressing well, but automation is difficult
  - Analytic methods to calculate e.g. two-loop integrals involving massive particles reach their limit

Making Predictions in the LHC Era

- At NLO: lot of progress (multi-leg, automation,...)
- Beyond NLO: calculations progressing well, but automation is difficult
  - Analytic methods to calculate e.g. two-loop integrals involving massive particles reach their limit
  - Numerical methods are in general easier to automate, problems mainly are
    - Extraction of IR and UV singularities
    - Numerical convergence in the presence of integrable singularities (e.g. thresholds)
    - Speed/accuracy

Making Predictions in the LHC Era

- At NLO: lot of progress (multi-leg, automation,...)
- Beyond NLO: calculations progressing well, but automation is difficult
  - Analytic methods to calculate e.g. two-loop integrals involving massive particles reach their limit
  - Numerical methods are in general easier to automate, problems mainly are
    - Extraction of IR and UV singularities (solved with SecDec 1.0)
    - Numerical convergence in the presence of integrable singularities (e.g. thresholds)
    - Speed/accuracy

Making Predictions in the LHC Era

- At NLO: lot of progress (multi-leg, automation,...)
- Beyond NLO: calculations progressing well, but automation is difficult
  - Analytic methods to calculate e.g. two-loop integrals involving massive particles reach their limit
  - Numerical methods are in general easier to automate, problems mainly are
    - Extraction of IR and UV singularities (solved with SecDec 1.0)
    - Numerical convergence in the presence of integrable singularities (e.g. thresholds) (solved with SecDec 2.0)
    - Speed/accuracy

Making Predictions in the LHC Era

- At NLO: lot of progress (multi-leg, automation,...)
- Beyond NLO: calculations progressing well, but automation is difficult
  - Analytic methods to calculate e.g. two-loop integrals involving massive particles reach their limit
  - Numerical methods are in general easier to automate, problems mainly are
    - Extraction of IR and UV singularities (solved with SecDec 1.0)
    - Numerical convergence in the presence of integrable singularities (e.g. thresholds) (solved with SecDec 2.0)
    - Speed/accuracy

Many people are/have been working on PURELY numerical methods, e.g. Soper/Nagy et al., Binoth/Heinrich et al., Kurihara et al., Passarino et al., Lazopoulos et al., Anastasiou et al., Freitas et al., Weinzierl et al., ...

Public Implementations of the Sector Decomposition Method on the Market

- sector_decomposition (uses GiNaC) C. Bogner & S. Weinzierl '07
- FIESTA (uses Mathematica, C) A. Smirnov, V. Smirnov & M. Tentyukov '08 '09
- SecDec (uses Mathematica, Perl, Fortran/C++) J. Carter & G. Heinrich '10

Limitation until recently:
Multi-scale integrals were limited to the Euclidean region (i.e., no thresholds)

Public Implementations of the Sector Decomposition Method on the Market

- sector_decomposition (uses GiNaC) C. Bogner & S. Weinzierl '07
- FIESTA (uses Mathematica, C) A. Smirnov, V. Smirnov & M. Tentyukov '08 '09
- SecDec (uses Mathematica, Perl, Fortran/C++) J. Carter & G. Heinrich '10

Limitation until recently:
Multi-scale integrals were limited to the Euclidean region (i.e., no thresholds)

NOW:
Extension of SecDec to general kinematics! SB, J. Carter & G. Heinrich '12

SecDec 2.0 Computes ...

- Feynman graphs for arbitrary kinematics, and more general parametric functions with no poles within the integration region

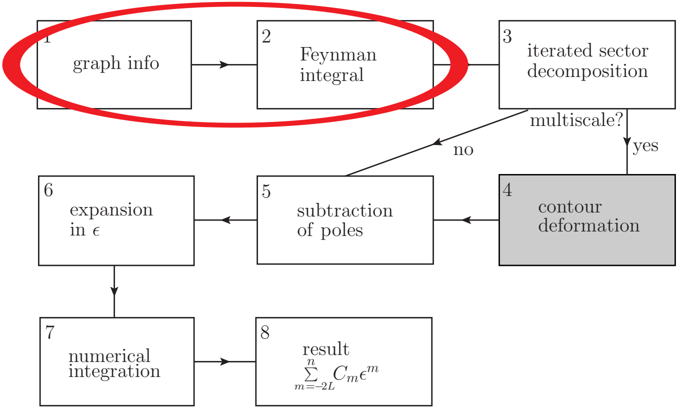
Feynman graph or parametric function

Parametric Functions

A general parametric function can be

- a phase space integral where IR divergences are regulated dimensionally
- polynomial functions, e.g. hypergeometric functions ${}_pF_{p-1}(a_1, ..., a_p; b_1, ..., b_{p-1}; \beta)$

Operational Sequence of the SecDec Program



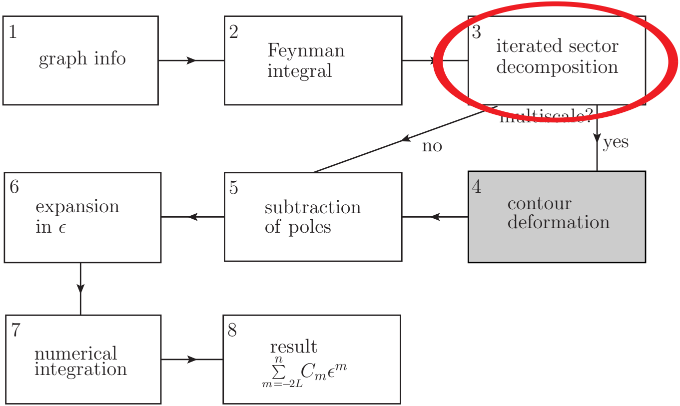
General Feynman Integral

- Graph infos are converted into (scalar or contracted tensor) **Feynman integral** in D dimensions at L loops with N propagators to power ν_j of rank R
- After loop momentum integration, generic scalar **Feynman integral** reads

$$G = \frac{(-1)^{N_\nu}}{\prod_{j=1}^{N}\Gamma(\nu_j)}\Gamma(N_\nu - LD/2)\int_0^\infty \prod_{j=1}^{N} \mathrm{d}x_j \; x_j^{\nu_j - 1}\,\delta(1-\sum_{l=1}^{N} x_l)\frac{\mathcal{U}^{N_\nu-(L+1)D/2}(\vec{x})}{\mathcal{F}^{N_\nu - LD/2}(\vec{x})}$$

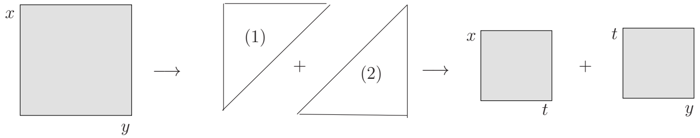
where $N_\nu = \sum_{j=1}^{N}\nu_j$ and where $\mathcal{U}$ and $\mathcal{F}$ can be constructed via **topological cuts**

Operational Sequence of the SecDec Program



Sector Decomposition

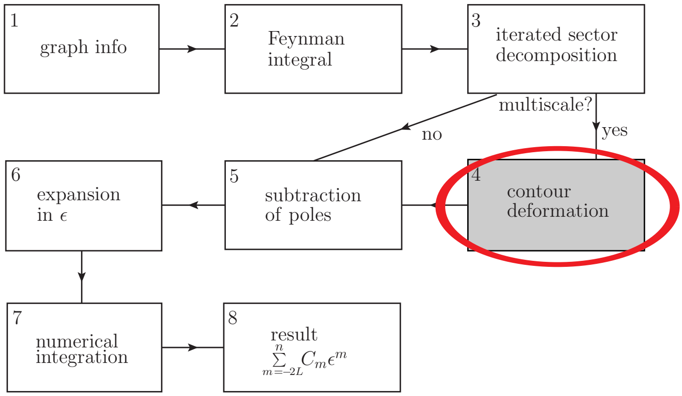
- Overlapping divergences are factorized



$$\int_0^1 dx \int_0^1 dy \, \frac{1}{(x+y)^{2+\epsilon}} = \int_0^1 dx \int_0^1 dt \, \frac{1}{x^{1+\epsilon}(1+t)^{2+\epsilon}} + \int_0^1 dt \int_0^1 dy \, \frac{1}{y^{1+\epsilon}(1+t)^{2+\epsilon}}$$

- Iterated **sector decomposition** is done, where dimensionally regulated soft, collinear and UV singularities are factored out
  Hepp '66, Denner & Roth '96, Binoth & Heinrich '00

Operational Sequence of the SecDec Program

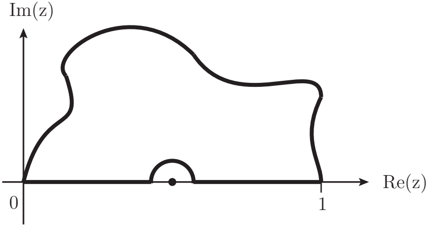


Contour Deformation I

- For kinematics in the physical region, $\mathcal{F}$ can still vanish

$$\mathcal{F}_{Bubble} = -s\, t_1(1 - t_1) + m^2 - i\delta$$

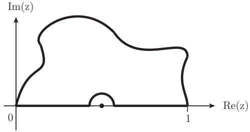
but a deformation of the integration contour



and Cauchy's theorem can help

$$\oint_c f(t)\mathrm{d}t = \int_0^1 f(t)\mathrm{d}t + \int_1^0 \frac{\partial z(t)}{\partial t} f(z(t))\mathrm{d}t = 0$$

Contour Deformation II



- The integration contour is deformed by

$$\vec{t} \to \vec{z} = \vec{t} + \mathrm{i}\vec{y} \ ,$$

$$y_j(\vec{t}) = -\lambda t_j(1-t_j)\frac{\partial \mathcal{F}(\vec{t})}{\partial t_j}$$

Soper '99

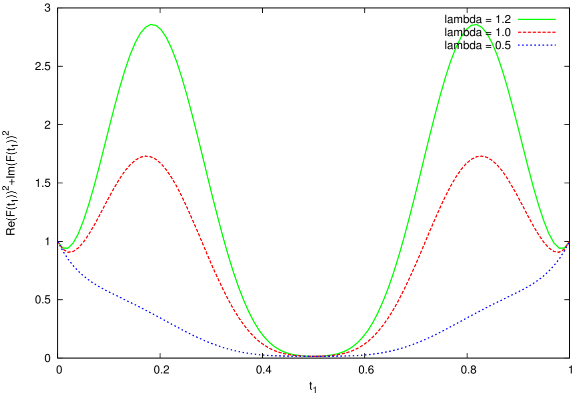
- Integrand is analytically continued into the complex plane

$$\mathcal{F}(\vec{t}) \to \mathcal{F}(\vec{t} + \mathrm{i}\vec{y}(\vec{t})) = \mathcal{F}(\vec{t}) + \mathrm{i}\sum_j y_j(\vec{t})\frac{\partial \mathcal{F}(\vec{t})}{\partial t_j} + \mathcal{O}(y(\vec{t})^2)$$

Soper, Nagy, Binoth; Kurihara et al., Anastasiou et al., Freitas et al., Becker et al.

Find the Optimal Deformation Parameter λ I

- Robust method: check the maximally allowed λ for $\mathcal{F}$ and maximize the modulus at critical points

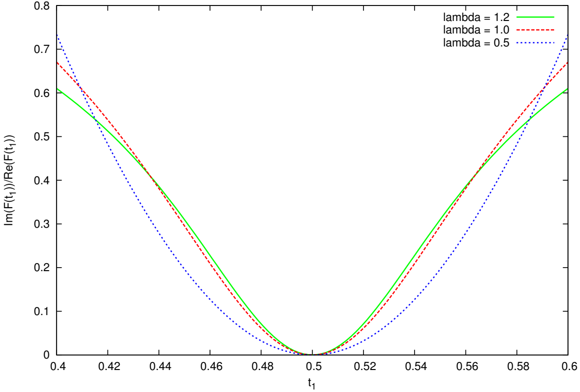


example is for 1-loop bubble, $m^2 = 1.0$, $s = 4.5$ with Feynman parameter t_1

- robust method default: *smalldefs=0, largedefs=0*

Find the Optimal Deformation Parameter λ II

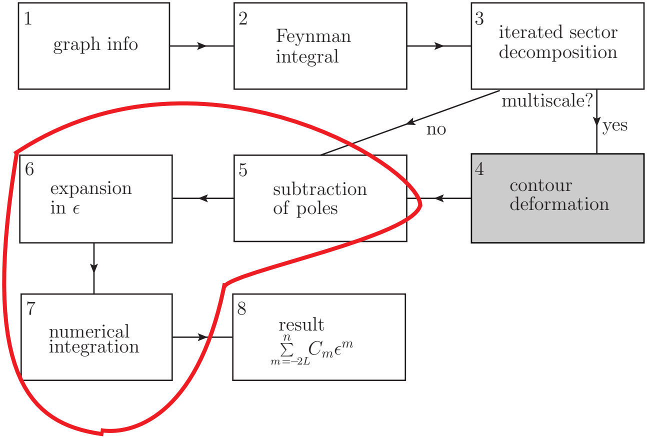
- Faster convergence: minimize the complex argument of $\mathcal{F}$



example is for 1-loop bubble, $m^2 = 1.0$, $s = 4.5$ with Feynman parameter t_1

- Singular points lie far from endpoints (0 and 1) of integration region, use *smalldefs=1*

Operational Sequence of the SecDec Program



Subtraction, Expansion, Numerical Integration

Subtraction

- The factorized poles in a subsector integrand $\mathcal{I} \propto \mathcal{U}, \mathcal{F}$ are extracted by subtraction (e.g. logarithmic divergence)

$$\int_0^1 \mathrm{d}t_j \, t_j^{-1-b_j\epsilon} \mathcal{I}(t_j, \epsilon) = -\frac{\mathcal{I}(0,\epsilon)}{b_j\epsilon} + \int_0^1 \mathrm{d}t_j \, t_j^{-1-b_j\epsilon} (\mathcal{I}(t_j,\epsilon) - \mathcal{I}(0,\epsilon))$$

Expansion

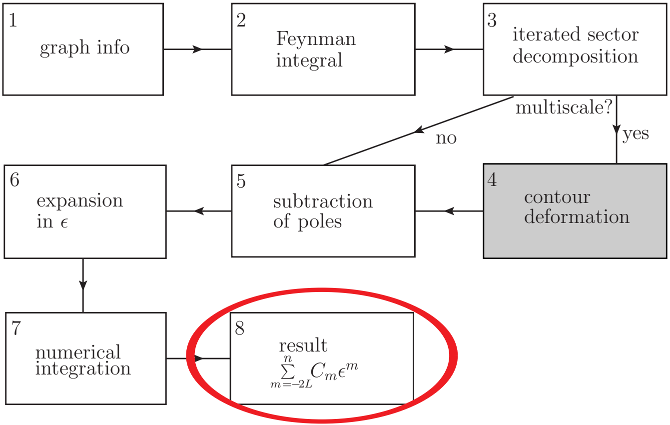
- After the extraction of poles, an expansion in the regulator ϵ is done

Numerical Integration

- Monte Carlo integrator programs containted in CUBA library or BASES can be used for numerical integration

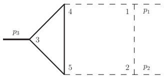
  Hahn et al. '04 '11, Kawabata '95

Operational Sequence of the SecDec Program



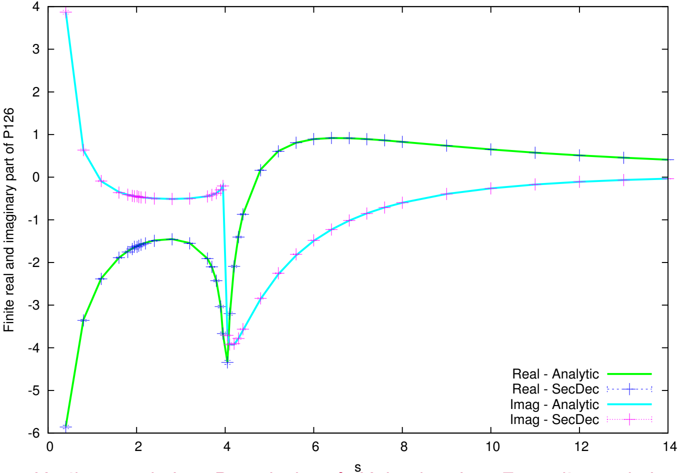
Results

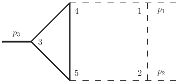
- Successful application of **SecDec 1.0** to massless multi-loop diagrams up to 5-loop 2-point functions and 4-loop 3-point functions for Euclidean kinematics
- Successful application of the public **SecDec 2.0** to various multi-scale examples, e.g., the massive 2-loop vertex graph, planar and non-planar 6- and 7-propagator massive 2-loop box diagrams
- Timings for the 2-loop vertex diagram and a relative accuracy of 1% using the CUBA 3.0 library on an Intel(R) Core i7 CPU at 2.67GHz



s/m^2	timing (finite part)
3.9	9.5 secs
14.0	3.6 secs

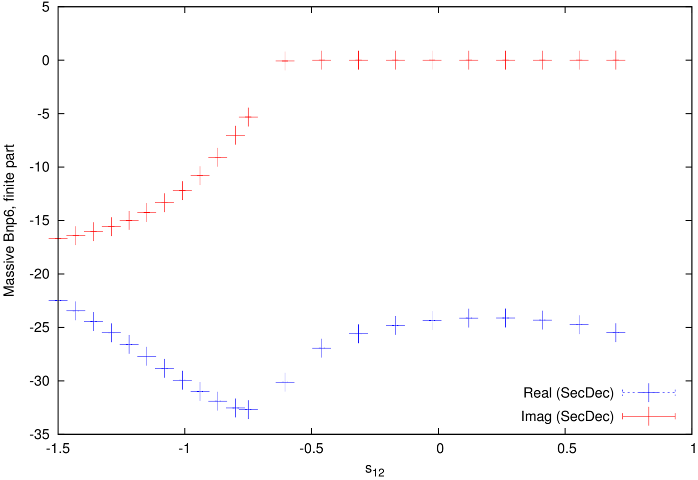
Results II: Massive Two-loop Vertex Graph G

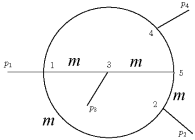




Kotikov et al. '97, Davydychev & Kalmykov '04, Ferroglia et al. '04, Bonciani et al. '04

Results III: Massive Non-planar 6-propagator Graph

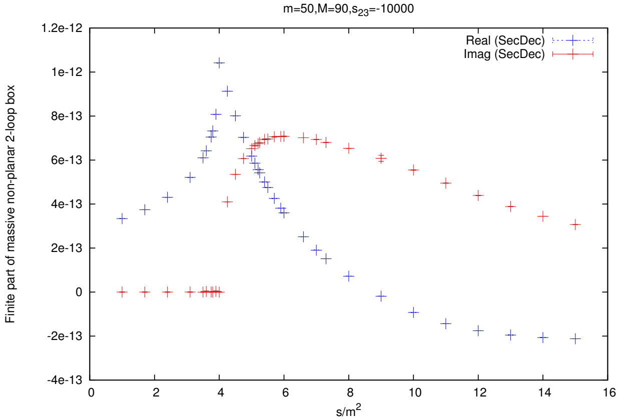


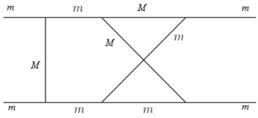


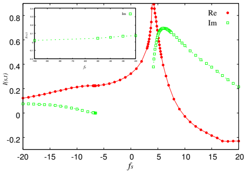
$m = 0.5$

massless case: Tausk '99

Results IV: Non-planar Massive Two-loop Box

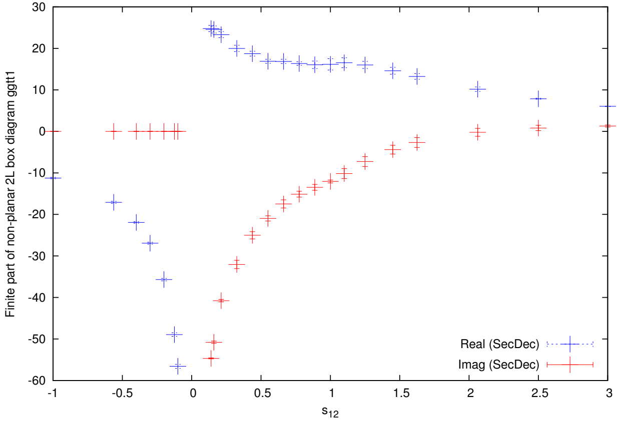


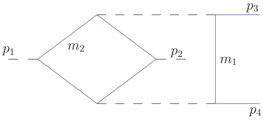




Fujimoto et al. '11

Results V: Non-planar ggtt Contribution





$$m_1, m_2 = 0.5$$
$$s_{23} = -0.4$$

Install SecDec 2.0

- **Download**:
  http://secdec.hepforge.org

- **Install**:
  tar xzvf SecDec.tar.gz
  cd SecDec-2.0
  ./install

- **Prerequisites**:
  Mathematica (version 6 or above), Perl, Fortran and/or C++ compiler

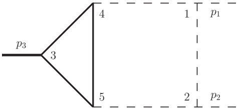
User Input I

- param.input: parameters for integrand specification and numerical integration

```
################ input parameters for sector decomposition ################
#-----------------------------------------------------------------------------------------------
# subdirectory for the mathematica output files (will be created if non-existent) :
# if not specified, a directory with the name of the graph given below will be created by default
subdir=2loop
#-----------------------------------------------------------------------------------------------
# if outputdir is not specified: default directory for
# the output will have integral name (given below) appended to directory above,
# otherwise specify full path for Mathematica output files here
outputdir=
#-----------------------------------------------------------------------------------------------
# graphname (can contain underscores, numbers, but should not contain commas)
graph=P126
#-----------------------------------------------------------------------------------------------
# number of propagators:
propagators=6
#-----------------------------------------------------------------------------------------------
# number of external legs:
legs=3
#-----------------------------------------------------------------------------------------------
# number of loops:
loops=2
#-----------------------------------------------------------------------------------------------
# construct integrand (F and U) via topological cuts (only possible for scalar integrals)
# default is 0 (no cut construction used)
cutconstruct=1
########################################################
# parameters for subtractions and epsilon expansion
########################################################
```

User Input II

- template.m: definition of the integrand (Mathematica syntax)




```
(******* USER INPUT for construction of integrand ****************************)

(******* Use with cutconstruct=1 *********************************************)

proplist={{ms[1],{3,4}},{ms[1],{4,5}},{ms[1],{5,3}},
          {0,{1,2}},{0,{1,4}},{0,{2,5}}};

(******* Use with cutconstruct=0 *********************************************)
(*
momlist={k1,k2};
proplist={k1^2-ms[1],(k1+p3)^2-ms[1],(k1-k2)^2-ms[1],
         (k2+p3)^2,(k2+p1+p3)^2,k2^2};
numerator={1};
*)

(******* Propagator powers (optional) ****************************************)

powerlist=Table[1,{i,Length[proplist]}];

(******* On-shell conditions (optional) **************************************)
onshell={ssp[1]->0,ssp[2]->0,ssp[3]->sp[1,2],sp[1,3]->0,sp[2,3]->0};

(******* Set Dimension *******************************************************)

Dim=4-2*eps;
(****************************************************************************)
```

Program Test Run

- ./launch -p param.input -t template.m

```
********** This is SecDec version 2.0 **********
Authors: Sophia Borowka, Jonathon Carter, Gudrun Heinrich
**************************************************
graph = P126
primary sectors 1,2,3,4,5,6, will be calculated
calculating F and U . . .
done
written to /home/pcl335a/sborowka/Work/SecDecBeta/loop/2loop/P126/FUN.m

results of the decomposition will be written to
/home/pcl335a/sborowka/Work/SecDecBeta/loop/2loop/P126
doing sector decomposition . . .
done

working on pole structure: 2 logarithmic poles, 0 linear poles, 0 higher poles
C++ functions created for pole structure 2l0h0
compiling 2l0h0/epstothe0 ...
doing numerical integrations in P126/2l0h0/epstothe0
compiling 2l0h0/epstothe-1 ...
doing numerical integrations in P126/2l0h0/epstothe-1
compiling 2l0h0/epstothe-2 ...
doing numerical integrations in P126/2l0h0/epstothe-2
working on pole structure: 1 logarithmic pole, 0 linear poles, 0 higher poles
C++ functions created for pole structure 1l0h0
compiling 1l0h0/epstothe0 ...
doing numerical integrations in P126/1l0h0/epstothe0
compiling 1l0h0/epstothe-1 ...
doing numerical integrations in P126/1l0h0/epstothe-1
working on pole structure: 0 logarithmic poles, 0 linear poles, 0 higher poles
C++ functions created for pole structure 0l0h0
compiling 1l0h0/epstothe0 ...
doing numerical integrations in P126/0l0h0/epstothe0
Output written to /home/pcl335a/sborowka/Work/SecDecBeta/loop/2loop/P126/P126_pfull.res

To remove intermediate files, execute the command /home/pcl335a/sborowka/Work/SecDecBeta/loop/launchcleanP126
```

Get the Result

- resultfile P126_full.res

```
************************************************
***OUTPUT: P126 p5 ***************
 point: 7.0
 ext. legs: 0.0 0.0 7.0
 prop. mass: 1.0 0. 0. 0. 0. 0.
Prefactor=-Exp[-2EulerGamma*eps]
************************************************
****** eps^-2 coeff ******

result       =0.07563683
             +0.1003924148 I
error        =0.000493522517701388
             + 0.00139691015080074 I
CPUtime (all eps^-2 subfunctions)  =0.04
CPUtime (longest eps^-2 subfunction)  =0.01
.
.
.

****** eps^0 coeff ******

result       =0.906978296750816
             -0.908781551612644 I
error        =0.00754504726896407
             + 0.0442867373250588 I
CPUtime (all eps^0 subfunctions)  =2.44
CPUtime (longest eps^0 subfunction)  =0.51

************************************************

Time taken for decomposition = 2.005725

Total time for subtraction and eps expansion = 41.5057 secs
Time taken for longest subtraction and eps expansion = 17.8613 secs
```

Conclusion

Summary

- With SecDec the numerical evaluation of multi-loop integrals is possible for arbitrary kinematics
- SecDec can also be used for more general parametric functions (e.g. phase space integrals)
- Useful to check analytic results for multi-loop master integrals, e.g. 2-loop boxes, 3-loop form factors, ...

Outlook

- Implement contour deformation for more general parametric functions
- Implement further variable transformation to tackle singularities very close to pinch singularities
- Application to 2-loop processes involving several mass scales, e.g. QCD/EW/MSSM corrections