Numerical NLO Calculations for Multi-Jet Production in Electron-Positron Annihilation

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HP2: High Precision for Hard Processes
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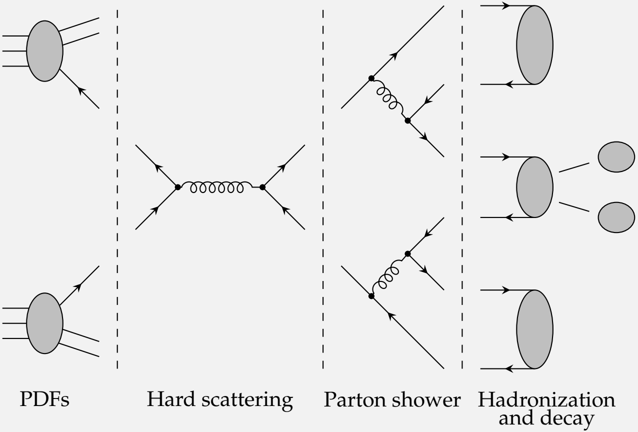
JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Illustration of hadron-hadron collision:



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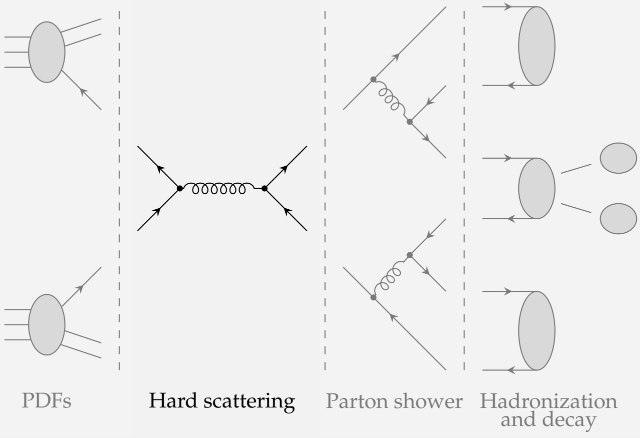

Observable (here: cross section) for hard scattering:

$$\sigma \propto \int \mathrm{d}\phi \, |\mathcal{A}|^2 \equiv \int \mathrm{d}\sigma$$

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Cornerstones:

- NLO accuracy in α_s
- automated computation for many external legs
- fully numerical (including loop integral)

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$$|\mathcal{A}|^2 = \underbrace{\underbrace{\left|\mathcal{A}_n^{(0)}\right|^2}_{\text{tree-level}}}_{\text{LO}} + \underbrace{\underbrace{\left|\mathcal{A}_{n+1}^{(0)}\right|^2}_{\text{real emission}} + \underbrace{2\,\mathfrak{Re}\left(\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}\right)}_{\text{virtual contribution}}}_{\text{NLO}}$$

LO : simple, no divergences, use color decomposition & fast Berends-Giele type recursion relations

NLO : real and virtual contributions separately divergent, sum is finite (KLN theorem)

Rewrite NLO part symbolically:

$$\sigma^{\mathrm{NLO}} = \int\limits_{n+1} \mathrm{d}\sigma^{\mathrm{R}} + \int\limits_{n} \mathrm{d}\sigma^{\mathrm{V}}$$

Problem : Integrations cannot be combined! Different phase space dimensions!

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Problem : Integrations cannot be combined! Different phase space dimensions!

Solution : Subtraction method:

$$\sigma^{\mathrm{NLO}} = \int_{n+1} \left[\mathrm{d}\sigma^{\mathrm{R}} - \mathrm{d}\sigma^{\mathrm{A}}\right] + \int_{n} \left[\mathrm{d}\sigma^{\mathrm{V}} + \int_{1} \mathrm{d}\sigma^{\mathrm{A}}\right]$$

- $\mathrm{d}\sigma^{\mathrm{A}}$ approximates soft & collinear singularities of $\mathrm{d}\sigma^{\mathrm{R}}$.
- $\mathrm{d}\sigma^{\mathrm{V}}$: IR poles canceled by integrated $\mathrm{d}\sigma^{\mathrm{A}}$; UV finite

Renormalized virtual contribution:

$$\int_n \mathrm{d}\sigma^{\mathrm{V}} = \int_n \left[\int_{\mathrm{loop}} \mathrm{d}\sigma^{\mathrm{V}}_{\mathrm{bare}} + \mathrm{d}\sigma^{\mathrm{V}}_{\mathrm{CT}} \right]$$

- different integration dimensions
- both contributions separately divergent

⇒ similar conditions led to subtraction method!

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Idea: Introduce subtraction term $\mathrm{d}\sigma^{\mathrm{L}}$ for UV & IR divergences:

$$\sigma^{\mathrm{NLO}} = \underbrace{\int_{n+1} \left[\mathrm{d}\sigma^{\mathrm{R}} - \mathrm{d}\sigma^{\mathrm{A}} \right]}_{\text{real}} + \underbrace{\int_{\substack{n+\\ \mathrm{loop}}} \left[\mathrm{d}\sigma^{\mathrm{V}}_{\mathrm{bare}} - \mathrm{d}\sigma^{\mathrm{L}} \right]}_{\text{virtual}} + \underbrace{\int_n \left[\mathrm{d}\sigma^{\mathrm{V}}_{\mathrm{CT}} + \int_{\mathrm{loop}} \mathrm{d}\sigma^{\mathrm{L}} + \int_1 \mathrm{d}\sigma^{\mathrm{A}} \right]}_{\text{insertion}}$$

OUTLINE

- VIRTUAL SUBTRACTION
- RESULTS
- RANDOM POLARIZATIONS
- SUMMARY

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Part I

Virtual Subtraction

Amplitude level: $$d\sigma^{\mathrm{V}} \propto 2\,\mathfrak{Re}\left(\mathcal{A}_n^{(0)\,*}\mathcal{A}_n^{(1)}\right)d\phi_n$$

The subtraction term acts on $\mathcal{A}_n^{(1)}$, as follows:

$$\begin{aligned}\mathcal{A}_n^{(1)} &= \mathcal{A}_{\text{bare}}^{(1)} + \mathcal{A}_{\text{CT}}^{(1)} \\ &= \left(\mathcal{A}_{\text{bare}}^{(1)} - \mathcal{A}_{\text{UV}}^{(1)} - \mathcal{A}_{\text{IR}}^{(1)}\right) \qquad + \quad \left(\mathcal{A}_{\text{CT}}^{(1)} + \mathcal{A}_{\text{UV}}^{(1)} + \mathcal{A}_{\text{IR}}^{(1)}\right)\end{aligned}$$

[Becker, Reuschle, Weinzierl — JHEP 1012 (2010) 013]
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&= \underbrace{\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(\mathcal{G}_{\mathrm{bare}}^{(1)} - \mathcal{G}_{\mathrm{UV}}^{(1)} - \mathcal{G}_{\mathrm{IR}}^{(1)}\right)}_{\substack{\textbf{virtual} \\ \text{combine loop \& ps integration}}} \quad + \quad \underbrace{\left(\mathcal{A}_{\mathrm{CT}}^{(1)} + \mathcal{A}_{\mathrm{UV}}^{(1)} + \mathcal{A}_{\mathrm{IR}}^{(1)}\right)}_{\substack{\textbf{part of insertion} \\ \text{sub. terms integrated separately}}}
\end{aligned}$$

where $\mathcal{A}_{\mathrm{x}}^{(1)} = \int \frac{\mathrm{d}^Dk}{(2\pi)^D}\mathcal{G}_{\mathrm{x}}^{(1)}, \quad \mathrm{x} = \text{bare, UV, IR}$

[Becker, Reuschle, Weinzierl — JHEP 1012 (2010) 013]
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Computation of Bare Contribution: $\mathcal{G}^{(1)}_{\text{bare}}$

Computation of amplitudes uses color decomposition:

$$\mathcal{A}^{(1)} = \sum_j C_j \, A_j^{(1)} \qquad \Rightarrow \qquad \mathcal{G}^{(1)}_{\text{bare}} = \sum_j C_j \, G^{(1)}_{\text{bare},j}$$

C_j: color factors, $A_j^{(1)}$: primitive amplitudes, $G_j^{(1)}$: primitive integrand

$\Rightarrow$ formulate method at level of primitive amplitudes

important properties of primitive amplitudes:

- gauge invariant
- fixed cyclic ordering of external legs
- type of each propagator uniquely determined

$\Rightarrow$ use recursive relations to compute $G^{(1)}_{\text{bare},j}$

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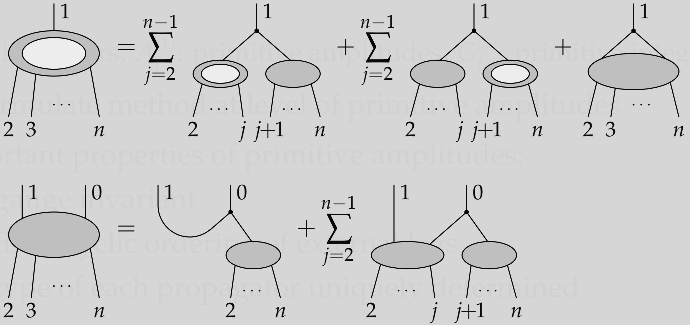
One loop recursion for three-valent toy model:



[Becker, Reuschle, Weinzierl — JHEP 1012 (2010) 013]
[Becker, Reuschle, Weinzierl — JHEP 1207 (2012) 090]

Subtraction of IR Singularities: $G_{\mathrm{IR}}^{(1)}$

Subdivide IR region: $G_{\mathrm{IR}}^{(1)} = G_{\mathrm{soft}}^{(1)} + G_{\mathrm{coll}}^{(1)}$

Soft subtraction term:

$$G_{\mathrm{soft}}^{(1)} = \mathrm{i} \sum_{j \in \mathcal{I}_g} \frac{4 p_j \cdot p_{j+1}}{k_{j-1}^2 \, k_j^2 \, k_{j+1}^2} A_j^{(0)}$$

Collinear subtraction term:

$$G_{\mathrm{coll}}^{(1)} = -2\mathrm{i} \sum_{j \in \mathcal{I}_g} \left(\frac{S_j \, g_{\mathrm{UV}}(k_{j-1}^2, k_j^2)}{k_{j-1}^2 \, k_j^2} + \frac{S_{j+1} \, g_{\mathrm{UV}}(k_j^2, k_{j+1}^2)}{k_j^2 \, k_{j+1}^2} \right) A_j^{(0)}$$

S_j : symmetry factors; 1/2 for gluons, 1 for quarks
g_{UV} : function to ensure there are no UV divergences

$\hookrightarrow$ Integrate algebraically over loop momentum to obtain $A_{\mathrm{IR}}^{(1)}$

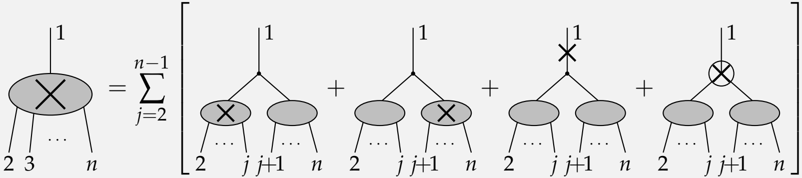
[Becker, Reuschle, Weinzierl — JHEP 1012 (2010) 013]
[Becker, Reuschle, Weinzierl — JHEP 1207 (2012) 090]

SUBTRACTION OF UV SINGULARITIES: $G^{(1)}_{\mathrm{UV}}$

In QCD, only two- to four-point functions are UV divergent!

Idea: compute once and treat them as
local counter terms to propagator and vertex corrections

$\Rightarrow$ use these in tree-level-like recursion to build $G^{(1)}_{\mathrm{UV}}$:



Trick: add finite terms such that integrated term is proportional to pole part,

$$c\left(\frac{1}{\varepsilon} - \ln\frac{\mu^2_{\mathrm{UV}}}{\mu^2}\right) + \mathcal{O}(\varepsilon) \quad \Rightarrow \quad \mathcal{A}^{(1)}_{\mathrm{CT}} + \mathcal{A}^{(1)}_{\mathrm{UV}} \propto \ln\frac{\mu^2_{\mathrm{UV}}}{\mu^2} \times \mathcal{A}^{(0)}$$

[Becker, Reuschle, Weinzierl — JHEP 1012 (2010) 013]
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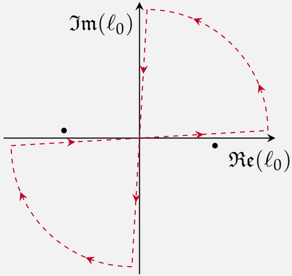
CONTOUR DEFORMATION

Summary: all contributions are now UV & IR finite!
But: loop propagators can still go on shell for finite loop momenta…
Solution: deform integration contour into complex space!

Simplest example: Wick rotation for tadpole

$$\int \frac{d^4\ell}{\ell^2 - m^2 + i\varepsilon}$$



Need sophisticated method for more complicated diagrams!

Contour Deformation

Loop integrand after subtraction:

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left(G^{(1)}_{\text{bare}} - G^{(1)}_{\text{IR}} - G^{(1)}_{\text{UV}}\right) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{R(k)}{\prod_{j=0}^{n-2} \left(k_j^2 - m_j^2\right)}$$

$R(k)$: rational function; no relevant poles!

Direct deformation: choose loop momentum $k = \widetilde{k} + \mathrm{i}\kappa(\widetilde{k}) \Rightarrow$

$$\int \frac{\mathrm{d}^4 \widetilde{k}}{(2\pi)^4} \left|\frac{\partial k^\mu}{\partial \widetilde{k}^\nu}\right| \frac{R\big(k(\widetilde{k})\big)}{\prod_{j=0}^{n-2} \left(\widetilde{k}_j^2 - m_j^2 - \kappa^2 + 2\mathrm{i}\widetilde{k}_j \cdot \kappa\right)}$$

Match Feynman's iε prescription $\Rightarrow$ choose direction of κ such that:

$$\widetilde{k}_j^2 - m_j^2 = 0 \quad \rightarrow \quad \widetilde{k}_j \cdot \kappa \geq 0$$

1 — *[Becker, Reuschle, Weinzierl — JHEP 1207 (2012) 090]*
2 — *[Becker, Weinzierl — arXiv:hep-ph/1208.4088]*

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Simple to find formally correct deformation, but we seek:

- process-independent algorithm
- small Monte Carlo integration error

Error depends strongly on choice of κ!

Available for both massless [1] and massive [2] case!

Match Feynman's iε prescription $\Rightarrow$ choose direction of κ such that:

$$\tilde{k}_j^2 - m_j^2 = 0 \quad \rightarrow \quad \tilde{k}_j \cdot \kappa \geq 0$$

1 — *[Becker, Reuschle, Weinzierl — JHEP 1207 (2012) 090]*
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Part II

Results

COMPARISON WITH ANALYTICAL RESULTS

Durham jet algorithm:

$$y_{ij} = \frac{2 \min\left(E_i^2, E_j^2\right)\left(1 - \cos\theta_{ij}\right)}{Q^2}$$

Jet rate definition:

$$R_n(\mu) = \frac{\sigma_{n\text{-jet}}^{\text{excl}}(\mu)}{\sigma_{\text{tot}}(\mu)}$$

We take σ_{tot} to leading order in α_s:

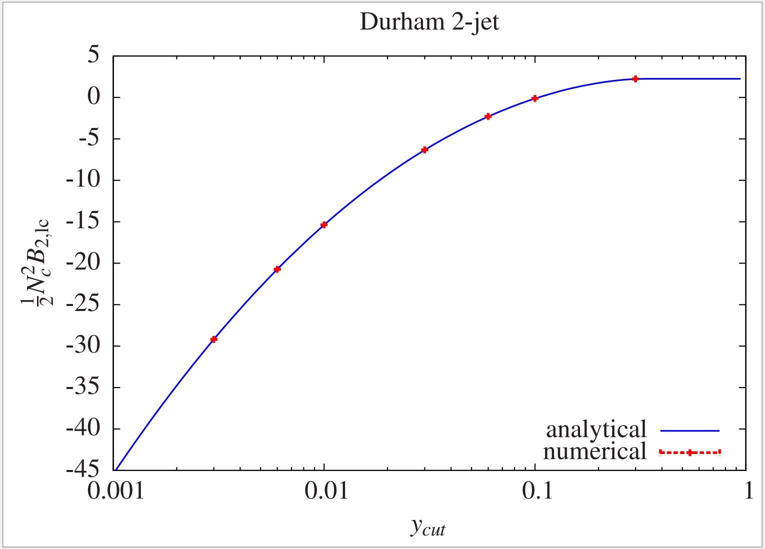
$$\frac{\sigma_{n\text{-jet}}^{\text{excl}}(\mu)}{\sigma_{2\text{-jet}}^{\text{LO}}(\mu)} = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-2} A_n(\mu) + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n-1} B_n(\mu) + \mathcal{O}(\alpha_s^n)$$

At the moment: massless, only leading color (large N_c limit)

[Becker, DG, Reuschle, Schwan, Weinzierl — Phys. Rev. Lett. 108 (2012) 032005]

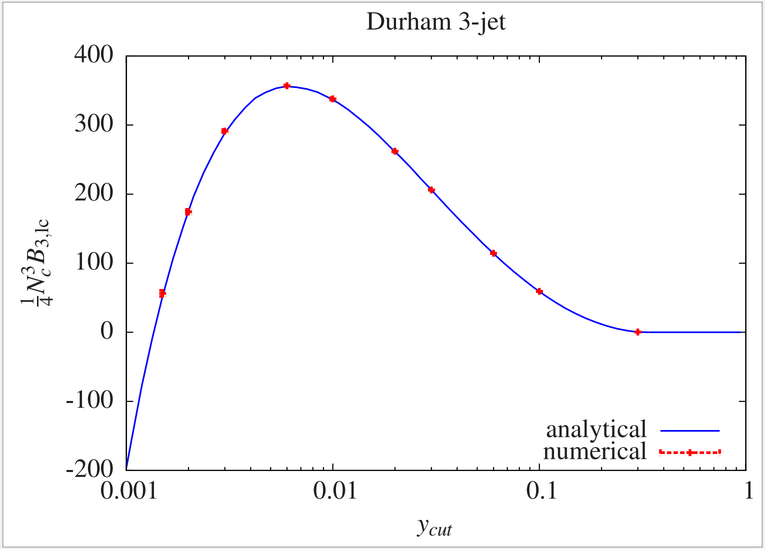
Comparison with Analytical Results

$e^+e^- \to 2$ Jets



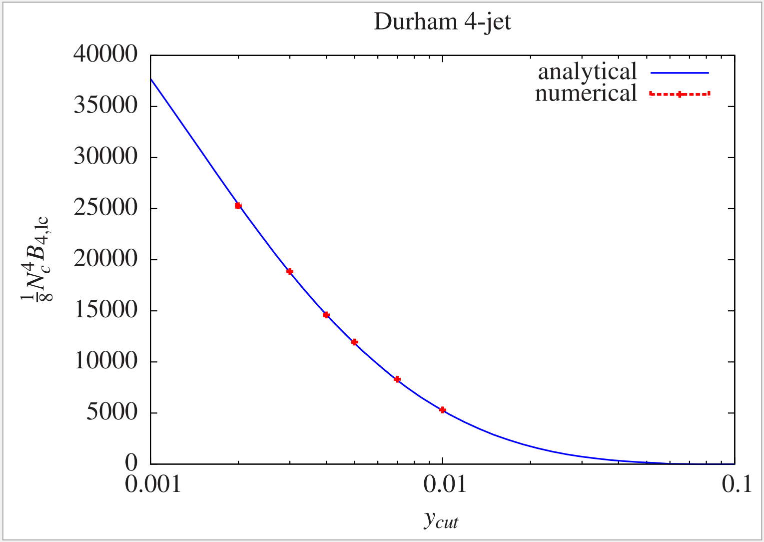
Comparison with Analytical Results

$e^+e^- \to 3$ Jets



Comparison with Analytical Results

$e^+e^- \to 4$ Jets



NEW RESULTS AND COMPUTATION TIME

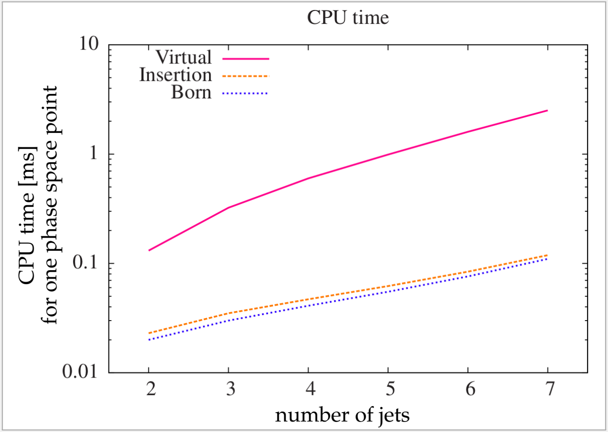
$e^+e^- \to 5,6,7$ JETS

n	y_{cut}	$\frac{N_c^n}{2^{n-1}} B_{n,\text{lc}}$
5	0.002	$(4.275 \pm 0.006) \times 10^5$
	0.001	$(1.050 \pm 0.026) \times 10^6$
	0.0006	$(1.84 \pm 0.15) \times 10^6$
6	0.001	$(1.46 \pm 0.04) \times 10^7$
	0.0006	$(3.88 \pm 0.18) \times 10^7$
7	0.0006	$(5.4 \pm 0.3) \times 10^8$

Computation time for 7 jets:

$\sim$ 5 days on a cluster with $\sim$ 200 cores.

CPU Time and Scaling Behaviour



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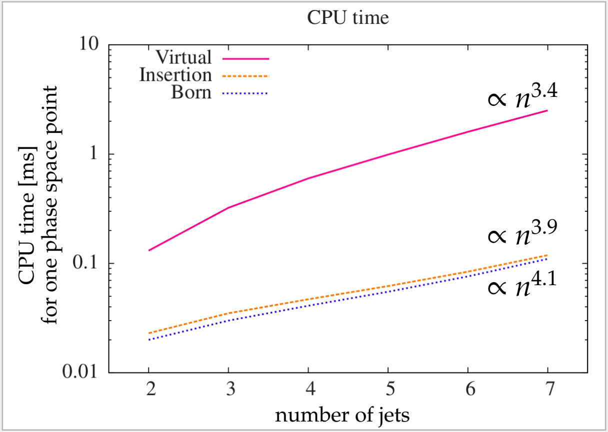

Asymptotic behaviour $\propto n^4 \quad \Rightarrow$
The only practical limitation arises from Monte Carlo statistics!
(already present at Born level)

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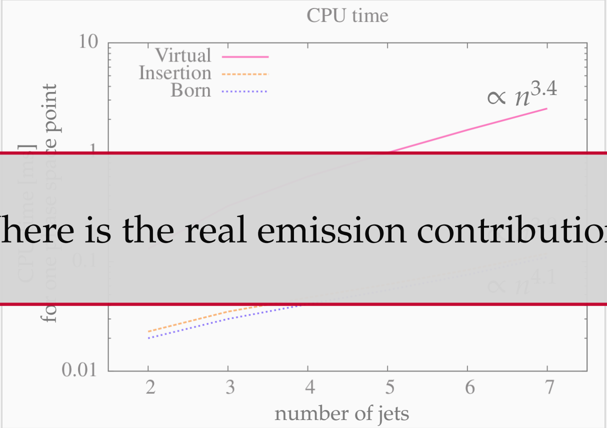

Where is the real emission contribution?

Asymptotic behaviour $\propto n^4$ $\Rightarrow$
The only practical limitation arises from Monte Carlo statistics!
(already present at Born level)

PART III

RANDOM POLARIZATIONS

Our method is based on helicity amplitudes $\Rightarrow$ each PS point involves computation of 2^n squared amplitudes:

$$\sigma \propto \int d\phi\, |\mathcal{A}|^2 = \int d\phi \underbrace{\sum_{\lambda_1,\ldots,\lambda_n} \left|\mathcal{A}(\lambda_1,\ldots,\lambda_n)\right|^2}_{2^n \text{ terms}}$$

Example: $e^+e^- \to 7$ jets has $2^9 = 512$ helicity amplitudes!

Can reduce this number:

- parity conservation,
- vanishing helicity amplitudes (e.g. Parke-Taylor),
- memorize subcurrents with equal helicities

But: we want to reduce this number to one helicity amplitude!

RANDOM POLARIZATIONS

Replace discrete polarizations $\epsilon^\mu_\pm$ with $\epsilon^\mu(\theta) = \mathrm{e}^{\mathrm{i}\theta}\epsilon^\mu_+ + \mathrm{e}^{-\mathrm{i}\theta}\epsilon^\mu_-$

Polarization sum: $$\sum_{\lambda=\pm} \epsilon^\mu_\lambda \left(\epsilon^\nu_\lambda\right)^* = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \epsilon^\mu(\theta)\left(\epsilon^\nu(\theta)\right)^*$$

since: $$\epsilon^\mu(\theta)\left(\epsilon^\nu(\theta)\right)^* = \sum_{\lambda=\pm} \epsilon^\mu_\lambda \left(\epsilon^\nu_\lambda\right)^* + \mathrm{e}^{2\mathrm{i}\theta}\epsilon^\mu_+\left(\epsilon^\nu_-\right)^* + \mathrm{e}^{-2\mathrm{i}\theta}\epsilon^\mu_-\left(\epsilon^\nu_+\right)^*$$

[Draggiotis, Kleiss, Papadopoulos; Phys. Lett. ***B439****, 157 (1998)]*

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since:
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$\Rightarrow$ replace summation by integration; combine with PS by increasing integration dimension!
(note: Monte Carlo error is independent of dimension)

$\Rightarrow$ works similarly for fermion spinors (also massive!)

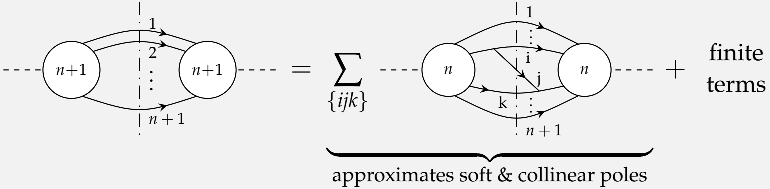
$\Rightarrow$ works without modification for Born, virtual and insertion terms, but not for real subtraction!

[Draggiotis, Kleiss, Papadopoulos; Phys. Lett. ***B439****, 157 (1998)]*

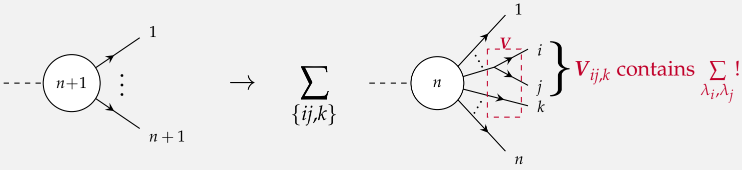
Real Subtraction: Dipole Formalism

formulated at level of squared, helicity summed amplitudes.

External-leg insertion rule: $|\mathcal{A}_{n+1}|^2 \to |\mathcal{A}_n|^2 \otimes \boldsymbol{V}_{ij,k}$



Factorization in dipole formalism:



Formalism & figures: *[Catani, Seymour — Nucl. Phys.* ***B485*** *(1997) 291–419]*

Subtraction for Random Polarizations?

$\boldsymbol{V}_{ij,k}$ obtained from squared, helicity summed soft & collinear limits of splitting vertex
(eikonal currents & DGLAP splitting kernels)

Instead of discrete polarizations, insert random polarizations.

SUBTRACTION FOR RANDOM POLARIZATIONS?

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Instead of discrete polarizations, insert random polarizations.

Recall:

$$\epsilon^{\mu}(\theta)\left(\epsilon^{\nu}(\theta)\right)^{*} = \underbrace{\sum_{\lambda=\pm}\epsilon^{\mu}_{\lambda}\left(\epsilon^{\nu}_{\lambda}\right)^{*}}_{\text{helicity summed result}} + \underbrace{e^{2i\theta}\epsilon^{\mu}_{+}\left(\epsilon^{\nu}_{-}\right)^{*} + e^{-2i\theta}\epsilon^{\mu}_{-}\left(\epsilon^{\nu}_{+}\right)^{*}}_{\text{new terms depending on helicity angles}}$$

New total $\boldsymbol{V}^{\text{tot}}_{ij,k}$ will have similar structure:

$$\boldsymbol{V}^{\text{tot}}_{ij,k} = \underbrace{\boldsymbol{V}_{ij,k}}_{\substack{\text{old term}\\ \text{(e.g. Dipole)}}} + \underbrace{\widetilde{\boldsymbol{V}}_{ij,k}(\theta_i,\theta_j)}_{\substack{\text{new helicity}\\ \text{mixing term}}}$$

SUBTRACTION FOR RANDOM POLARIZATIONS?

Idea: use known real subtraction method, add additional subtraction term $d\sigma^{\widetilde{A}}$ only for helicity mixing terms:

$$\sigma^{\mathrm{NLO}} = \int_{n+1} \left[d\sigma^{R} - \left(d\sigma^{A} + d\sigma^{\widetilde{A}} \right) \right] + \int_{n} \left[d\sigma^{V} + \int_{1} d\sigma^{A} \right]$$

Advantage: no new integrated subtraction term, since helicity mixing terms vanish upon integration:

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta_i \int_0^{2\pi} d\theta_j \; d\sigma^{\widetilde{A}} = 0$$

COMPUTATION OF $\mathrm{d}\sigma^{\widetilde{\mathrm{A}}}$

We need only helicity mixing terms $\Rightarrow$ define operator $\mathcal{R}$

$$\mathcal{R} f(\theta_i, \theta_j) \equiv f(\theta_i, \theta_j) - \sum_{\lambda_i, \lambda_j} f(\lambda_i, \lambda_j)$$

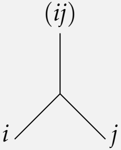
Then, symbolically:

$$\mathrm{d}\sigma^{\widetilde{\mathrm{A}}} = \sum_{i,j} \sum_{k \neq i,j} \widetilde{\mathcal{D}}_{ij,k} + \ldots$$

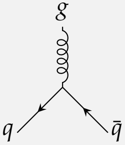
$$\widetilde{\mathcal{D}}_{ij,k} \propto |\mathcal{A}_n^{(0)}|^2 \otimes \frac{\mathbf{T}_{ij} \cdot \mathbf{T}_k}{\mathbf{T}_{ij}^2} \, \widetilde{\boldsymbol{V}}_{ij,k}$$

$$\widetilde{\boldsymbol{V}}_{ij,k} = C_{(ij)\to i,j} \, \mathcal{R}\left[P_{(ij)\to i,j} + S_{(ij)\to i,j}\right]$$

color factor for splitting | splitting vertex (Feynman rules) | additional soft function



example: $g \to q\bar{q}$



REAL SUBTRACTION WITH RANDOM POLARIZATIONS

CONCLUSION

- Previous slides: brief introduction into final-state emitter and spectator; using crossing symmetry one can easily derive other three cases
(final-initial, initial-final, initial-initial)
$\rightarrow$ see publication below!
- all formulas valid for massive case, too; for massive quarks, we need different momentum parametrizations for emitter and spectators
$\rightarrow$ see publication below!
- final-final state method for $e^+e^- \rightarrow$ jets is currently being implemented, we expect results very soon!

[DG, Schwan, Weinzierl — Phys. Rev. D 85, 116011 (2012)]

Summary

- Our approach to numerical NLO computations extends the subtraction method to the virtual corrections, consists of:
  - local subtraction terms approximating UV & IR poles,
  - method for deforming the integration contour to avoid divergences from on-shell loop propagators
- current results include $e^+e^- \to 6$ and 7 jets (massless) in leading color approximation (large N_c limit)
- very moderate growth in computation time ($\propto n^4$)
- newest development: new real subtraction terms for random polarizations

Outlook:

- extension to proton-proton collisions (LHC physics)
- inclusion of full color information

Backup Slides

COMPUTATION OF $\mathrm{d}\sigma^{\widetilde{\mathrm{A}}}$

We need only helicity mixing terms $\Rightarrow$ define operator $\mathcal{R}$

$$\mathcal{R} f(\theta_i, \theta_j) \equiv f(\theta_i, \theta_j) - \sum_{\lambda_i, \lambda_j} f(\lambda_i, \lambda_j)$$

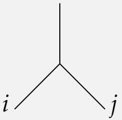
Then, symbolically:

$$\mathrm{d}\sigma^{\widetilde{\mathrm{A}}} = \sum_{i,j} \sum_{k \neq i,j} \widetilde{\mathcal{D}}_{ij,k} + \dots$$

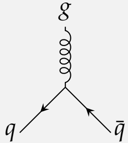
$$\widetilde{\mathcal{D}}_{ij,k} \propto |\mathcal{A}_n^{(0)}|^2 \otimes \frac{\mathbf{T}_{ij} \cdot \mathbf{T}_k}{\mathbf{T}_{ij}^2} \, \widetilde{\boldsymbol{V}}_{ij,k}$$

$$\widetilde{\boldsymbol{V}}_{ij,k} = C_{(ij) \to i,j} \, \mathcal{R} \left[P_{(ij) \to i,j} + S_{(ij) \to i,j} \right]$$

color factor for splitting	splitting vertex (Feynman rules)	additional soft function



example: $g \to q\bar{q}$



DERIVATION OF $\widetilde{V}_{ij,k}$

Soft limit:

$$\lim_{p_j \to 0} \left| \mathcal{A}_{n+1}^{(0)} \right|^2 = 4\pi\alpha_s \mu^{2\epsilon} \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}} \left(\mathcal{A}_n^{(0)} \right)^* \mathbf{T}_j \cdot \mathbf{T}_k \, S_{ij,k}(\epsilon_j) \, \mathcal{A}_n^{(0)}$$

$$S_{ij,k}(\epsilon_j) = \frac{(p_i \cdot \epsilon_j^*)\,(p_i \cdot \epsilon_j)}{(p_i \cdot p_j)^2} - \frac{(p_i \cdot \epsilon_j^*)\,(p_k \cdot \epsilon_j) + (p_k \cdot \epsilon_j^*)\,(p_i \cdot \epsilon_j)}{(p_i \cdot p_j)\,(p_i \cdot p_j + p_j \cdot p_k)}$$

Collinear limit (example for splitting $q \to qg$):

$$\lim_{p_i || p_j} \left| \mathcal{A}_{n+1}^{(0)} \right|^2 = 4\pi\alpha_s \mu^{2\epsilon} \left(\mathcal{A}_n^{(0)} \right)^{\alpha *} \mathbf{T}_{q \to qg} \left[P_{q \to qg}(p, p_i, p_j, h_i, h_j) \right]_{\alpha\beta} \left(\mathcal{A}_n^{(0)} \right)^{\beta}$$

$$\left[P_{q \to qg}(p, p_i, p_j, h_i, h_j) \right]_{\alpha\beta} = \sum_{\lambda, \lambda'} u_\alpha^\lambda(p) \, \mathrm{Split}^{\lambda *} \, \mathrm{Split}^{\lambda'} \, \bar{u}_\beta^{\lambda'}(p),$$

$$\mathrm{Split}^\lambda = \frac{1}{(p_i + p_j)^2 - m_{ij}^2} \, \bar{u}(p_i) \gamma^\mu u^\lambda(p) \, \epsilon_\mu(p_j)$$

Soft limit of collinear limit is identical to collinear limit of soft limit $\Rightarrow$ define

$$\left[S_{q \to qg}(p, p_i, p_j, p_k, h_i, h_j) \right]_{\alpha\beta} = -\frac{(p_i \cdot \epsilon_j^*)\,(p_k \cdot \epsilon_j) + (p_k \cdot \epsilon_j^*)\,(p_i \cdot \epsilon_j)}{(p_i \cdot p_j)\,(p_i \cdot p_j + p_j \cdot p_k)} \, u_\alpha(p_i) \bar{u}_\beta(p_i)$$