Resummed small-x and first moment evolution of fragmentation functions in pQCD

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Collaborators

Talk based on

- arXiv:1207.5631 (to appear in JHEP)
- See also 1108.2993 Vogt 11

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Outline of Topics

Semi-inclusive e^+e^- annihilation (SIA)

Fragmentation functions $F_a^h(x, Q^2)$ in $e^+e^- \to \gamma/Z/\phi(q) \to h(p) + X$ $(x = 2pq/Q^2)$:

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dx\,d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)\,F_T^h + \frac{3}{4}\sin^2\theta\,F_L^h + \frac{3}{4}\cos\theta F_A^h$$

Factorisation formula ($\mathcal{O}(1/Q)$ terms neglected)

$$F_a^h(x, Q^2) = \sum_{j=q,\bar{q},g}\int_x^1 \frac{dz}{z} c_{a,j}\left(z, \alpha_S(Q^2)\right) D_j^h\left(\frac{x}{z}, Q^2\right)$$

Coefficient functions $c_{a,j}$ known at α_S^2 Rijken,van Neerven 96, Mitov,Moch 06
Fragmentation distributions D_j^h evolve via time-like splitting functions P^T

$$\frac{d}{d\ln Q^2} D_i^h(x, Q^2) = \sum_{j=q,\bar{q},g}\int_x^1 \frac{dz}{z} P_{ji}^T(z, \alpha_S) D_j^h\left(\frac{x}{z}, Q^2\right)$$

P^T known at α_S^2 (NLO) Curci,Furmanski,Petronzio 80, Floratos,Kounnas,Lacaze 81, ...
NNLO computed via analytic continuation Mitov,Moch,Vogt 06, Moch,Vogt 07, Almasy,Moch,Vogt 11

SIA in the small-x limit

In the small-x limit need to resum logarithms of the form:

$$
\begin{aligned}
xP^{(n)T}_{gi} &\sim \alpha_S^{n+1}\ln^{2n}x + \ldots \quad , \quad xP^{(n)T}_{qi} \sim \alpha_S^{n+1}\ln^{2n-1}x + \ldots ,\\
xc^{(n)}_{T,g} &\sim \alpha_S^{n}\ln^{2n-1}x + \ldots \quad , \quad xc^{(n)}_{T,q} \sim \alpha_S^{n}\ln^{2n-2}x + \ldots ,\\
xc^{(n)}_{L,g} &\sim \alpha_S^{n}\ln^{2n-2}x + \ldots \quad , \quad xc^{(n)}_{L,q} \sim \alpha_S^{n}\ln^{2n-3}x + \ldots .
\end{aligned}
$$

P^T: LL Mueller 81, Bassetto,Ciafaloni,Marchesini,Mueller 82 , NLL (MG scheme) Mueller 83

$C_{T(\phi),g}$: LL Mueller 83, Albino,Bolzoni,Kniehl,Kotikov 11

Mellin transform: $M\left[\frac{1}{x}\ln^k x\right](N) \equiv \int_0^1 dx x^{N-1}\frac{1}{x}\ln^k x \;=\; \frac{(-1)^k k!}{\bar{N}^{k+1}}$, $\bar{N} = N-1$

Goal: Finite Mellin $N=1$ limit (required for multiplicity calculations) at NLO (in $\overline{\text{MS}}$). Need NLL for $c_{T,i}$ and NNLL for P^T_{ij}, $c_{L,i}$.

Resum by analysing D-dimensional structure of unfactorised fragmentation functions. Highest three logs now available, including Higgs exchange (ϕ) in the heavy top limit.

pFF before factorisation

Unfactorised pFF in $D = 4 - 2\epsilon$ dimensions ($a_s = \alpha_S/4\pi$):

$$\widehat{F}_{a,k} = \widetilde{C}_{a,i} Z^T_{ik}\,, \quad P^T = \frac{dZ^T}{d\ln Q^2}(Z^T)^{-1},\ \frac{da_s}{d\ln Q^2} = -\epsilon a_s + \beta_{D=4}$$

- Z^T expressed in terms of P^T:

$$Z^T\Big|_{a_s^n \epsilon^{-n}} : P_0^T, \beta_0\,; \quad Z^T\Big|_{a_s^n \epsilon^{-n+1}} : P_0^T, \beta_0, P_1^T, \beta_1\,; \ldots \quad Z^T\Big|_{a_s^n \epsilon^{-1}} : P_{n-1}^T$$

- $\widetilde{C}$ contains only positive powers of ϵ:

$$\widetilde{C}_{a,i} = (1) + \sum_{n=1}^{\infty}\sum_{l=0}^{\infty} a_s^n \epsilon^l c_{a,i}^{(n,l)}$$

At a_s^n: $\epsilon^{-n}, \ldots, \epsilon^{-2}$: lower order terms. ϵ^{-1}: n-loop splitting fnc. + ...

ϵ^0: n-loop coeff. fnc. + ... $\epsilon^{0<k<l}$: required for order a_s^{n+l}

NNLO results fix the highest three powers of $1/\epsilon$ to all orders in a_s.

D-dim. structure of unfactorised observables

KLN means all $a_s^n \epsilon^{-m}$ terms are zero for $m > n$ (zero also for $m = n$ for $a = L$ and 'off-diagonal' terms).

Decompose the D-dim. pFF with according to # external legs Vogt 11 .
E.g. for $a = T, \phi$:

$$\begin{aligned} \widehat{F}^{(n)}_{a,g} &= \frac{1}{\epsilon^{2n-1}} \sum_{k=1}^{n} x^{-1-2k\epsilon} \left\{ A^{(k,n)}_{a,g} + \epsilon B^{(k,n)}_{a,g} + \epsilon^2 C^{(k,n)}_{a,g} + \cdots \right\} \\ x^{-k\epsilon} &= 1 - k\epsilon \ln x + \frac{(k\epsilon)^2}{2} \ln^2 x + \cdots \\ &\to A^{(k,n)} : \mathrm{LL},\ B^{(k,n)} : \mathrm{NLL},\ C^{(k,n)} : \mathrm{NNLL} \quad \mathrm{etc.} \end{aligned}$$

(Motivated by phase space structure for SIA in small-x:

$$2 \to \gamma/Z \to 2\,(+\,k\,\mathrm{extra\,partons})\ :\ x^{-k\epsilon} \cdots \int_0^1 \mathrm{other\ variables.}$$

N^2LO: Matsuura,van Neerven 88, Rijken,vN 95)

KLN cancellation and fixed order results provide constraints to fix A, B ...

D-dim. structure of unfactorised observables

LL: $\widehat{F}^{(n)}_{a,g}$ includes terms of the form $x^{-1}\ln^{n+\delta-1}x$ at all orders $\epsilon^{-n+\delta}$ with $\delta = 0, 1, 2, \cdots$, and is decomposed into n contributions of the form

$$\epsilon^{-2n+1}x^{-1-2k\epsilon}A^{(k,n)}_{a,g} = \epsilon^{-2n+1}x^{-1}\left[1 - 2k\epsilon\ln x + \tfrac{1}{2}(2k\epsilon)^2\ln^2 x + \ldots\right]A^{(k,n)}_{a,g}\,,\qquad k = 1, 2, \ldots, n$$

$n-1$ (KLN) $+$ 3 (NNLO) constraints $\rightarrow$ The n $A^{(k,n)}_{a,g}$ coefficients are overconstrained by $n+2$ linear eqns.

$n+1(n)$ linear eqns for NLL(NNLL) $B^{(l,n)}_{a,g}(C^{(l,n)}_{a,g})$. $N^2LO \rightarrow N^2LL$ resummation.

Quark case similar: ϵ^{-2n+2} prefactor, one fewer term in sum:

$$\widehat{F}^{(n)}_{a,q} = \frac{1}{\epsilon^{2n-2}}\sum_{l=0}^{n-2} x^{-1-2(n-l)\epsilon}\left\{B^{(l,n)}_{a,q} + \epsilon C^{(l,n)}_{a,q} + \epsilon^2 D^{(l,n)}_{a,q} + \cdots\right\}$$

Also here highest three logs at all orders fixed by NNLO results.

Obtain NNLL timelike splitting functions and coefficient functions.

pFF before factorisation

Longitudinal case:

$$\widehat{F}_{L,\mathrm{g}}^{(n)} = \frac{1}{\epsilon^{2n-2}} \sum_{\ell=0}^{n-1} x^{-1-2(n-\ell)\epsilon} \left(A_{L,\mathrm{g}}^{(\ell,n)} + \epsilon\, B_{L,\mathrm{g}}^{(\ell,n)} + \ldots \right) , \qquad (1)$$

$$\widehat{F}_{L,\mathrm{q}}^{(n)} = \frac{1}{\epsilon^{2n-3}} \sum_{\ell=0}^{n-2} x^{-1-2(n-\ell)\epsilon} \left(B_{L,\mathrm{q}}^{(\ell,n)} + \epsilon\, C_{L,\mathrm{q}}^{(\ell,n)} + \ldots \right) \qquad (2)$$

c_L known at α_S^2 Rijken,van Neerven 96, Mitov,Moch 06 $\rightarrow$ highest two logs.

We obtain the highest three logs at NNLO (α_S^3) CHK,Vogt,Yeat 12 via analytic continuation of the physical evolution kernel

$$K_{ab}^{T}(x,\alpha_S) = \mathcal{AC}\left[K_{ab}^{S}(x,\alpha_S)\right]$$

between the SIA system $(F_T,\ \widetilde{F}_L)$ with $\widetilde{F}_L(N) = F_L(N)/(\alpha_S\, c_{L,\mathrm{q}}^{(1)}(N))$ and the DIS counterpart $(F_1,\ \widetilde{F}_L^{S})$ for which NNLO is known $\rightarrow F_{L,g}|_{\mathrm{NNLL}}$ $(F_{L,q}|_{\mathrm{NNLL}} \leftrightarrow$ 2nd highest log)

Results

NNLL completely known. $\widehat{F}$ calculated to α_S^{18} CHK,Vogt,Yeat 12 (see also Vogt 11) using FORM Vermaseren 00, Tentyukov,Vermaseren 07

Analytical form inferred from the 'all-order' results.

Building blocks: $S = (1-4\xi)^{1/2}$, $\mathcal{L} = \ln\left(\frac{1}{2}(1+S)\right)$, $\xi = -8C_A a_s/\bar{N}^2$
(Also $F \equiv S^{-1/2}$ for the coeff. fcn.), e.g.

$$P_{\mathrm{gg}}^T(N) = \frac{1}{4}\bar{N}(S-1) - \frac{1}{6C_A}a_s(11C_A^2+2C_An_f-4C_Fn_f)(S^{-1}-1) - P_{\mathrm{qq}}^T(N) + \text{other NNLL terms}$$

$$\begin{aligned}P_{\mathrm{qq}}^T(N) = {} & \frac{4}{3}\frac{C_Fn_f}{C_A}a_s\left\{\frac{1}{2\xi}(S-1)(\mathcal{L}+1)+1\right\} + \frac{1}{18}\frac{C_Fn_f}{C_A^3}a_s\bar{N}\Big\{(-11C_A^2+6C_An_f \\ & -20C_Fn_f)\frac{1}{2\xi}(S-1+2\xi)+10C_A^2\frac{1}{\xi}(S-1)\mathcal{L}-(51C_A^2-6C_An_f \\ & +12C_Fn_f)\frac{1}{2}(S-1)+(11C_A^2+2C_An_f-4C_Fn_f)S^{-1}\mathcal{L}+(5C_A^2 \\ & -2C_An_f+6C_Fn_f)\frac{1}{\xi}(S-1)\mathcal{L}^2+(51C_A^2-14C_An_f+36C_Fn_f)\mathcal{L}\Big\}\end{aligned}$$

$$C_{T,\mathrm{q}}(N) = 1 + \frac{1}{3}\frac{C_Fn_f}{C_A^2}\bar{N}(F^{-1}-1-F\mathcal{L}) + \text{NNLL terms}$$

Results

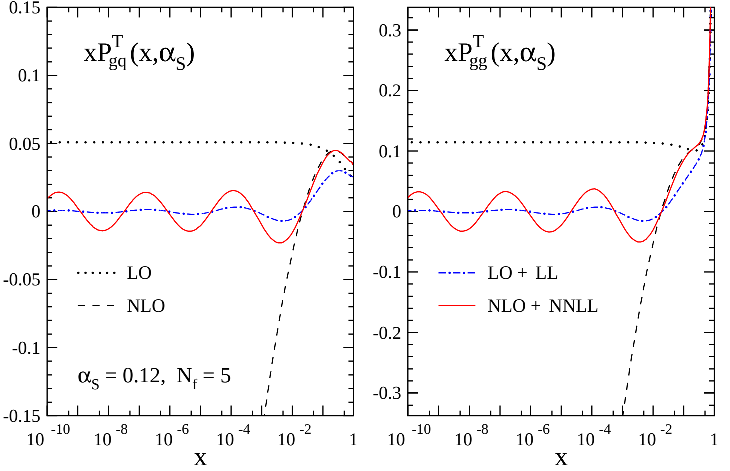
Finite first moments ($N=1$): 'NLO + resummed' results.

For P^T: NLO + NNLL. For $C_{T,\phi}$: NLO + NLL. For C_L: NLO + NNLL. E.g.

$$
\begin{aligned}
P^T_{\rm gg}(N=1) &= (2C_A a_s)^{1/2} - \frac{1}{6C_A}(11C_A^2 + 2C_A n_f + 12 C_F n_f) a_s \\
&\quad + \frac{1}{144 C_A^3}\Big([1193 - 576\zeta_2] C_A^4 - 140 C_A^3 n_f + 4 C_A^2 n_f^2 + 760 C_A^2 C_F n_f \\
&\quad - 80 C_A C_F n_f^2 + 144 C_F^2 n_f^2\Big)(2C_A a_s^3)^{1/2} + \mathcal{O}(a_s^2)\,, \\
P^T_{\rm gq}(N=1) &= \frac{C_F}{C_A}\left(P^T_{\rm gg}(N=1) + \frac{4}{3}\frac{C_F n_f}{C_A} a_s\right) + \mathcal{O}(a_s^2). \\
P^T_{\rm qg}(N=1) &= \frac{8}{3} n_f a_s - \frac{1}{3\,C_A^2}\left(17 C_A^2 n_f - 2 C_A n_f^2 + 4 C_F n_f^2\right)(2C_A a_s^3)^{1/2} + \mathcal{O}(a_s^2)\,, \\
P^T_{\rm qq}(N=1) &= \frac{C_F}{C_A}\left(P^T_{\rm qg}(N=1) - \frac{4}{3} n_f a_s\right) + \mathcal{O}(a_s^2),
\end{aligned}
$$

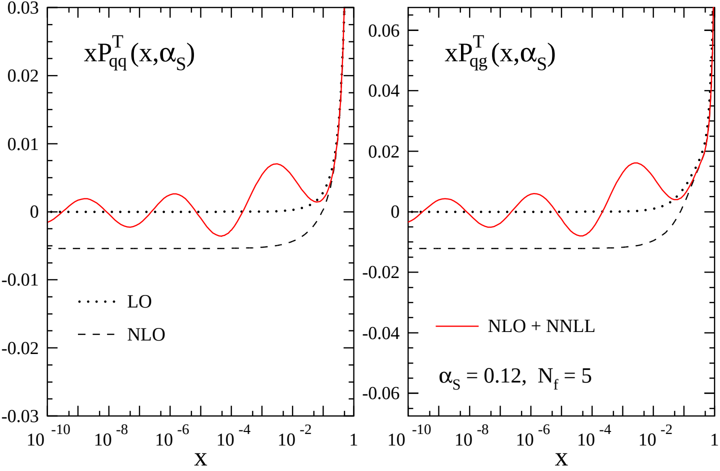
Corresponding 'NLO+resummed' first moments for coefficient functions also calculated CHK,Vogt,Yeats 12

Small-x gluon-parton splitting functions



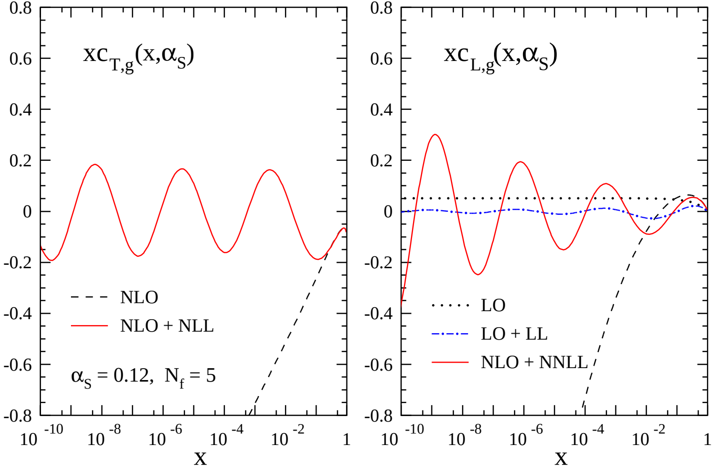
NB: similar resummed curves: approximate 'Casimir scaling' of P^T_{gi}.

Small-x quark-parton splitting functions

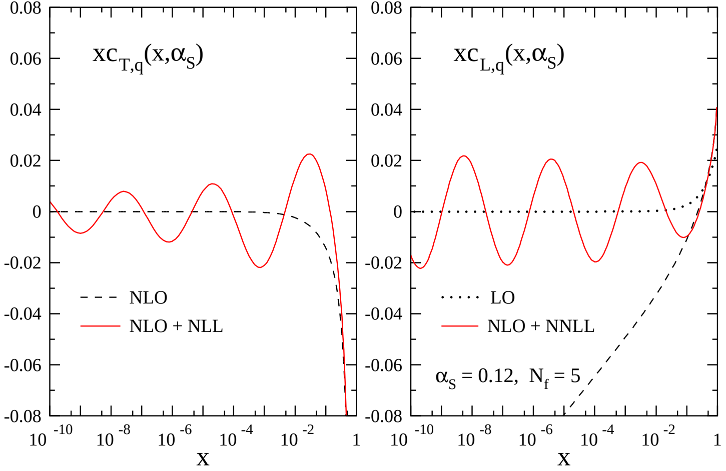


NB: similar resummed curves: approximate ‘Casimir scaling’ of P^T_{qi}.

Small-x coefficient functions for gluons



Small-x coefficient functions for quarks



Analytical results in x-space

Non-log parts (ie. the $\mathcal{S}(N)$'s) for splitting functions can be expressed in terms of Bessel functions of the first kind, $J_1(z)$, $J_2(z)$, $z = (32 C_A a_s)^{1/2} \ln\frac{1}{x}$, e.g.

$$
\begin{aligned}
xP^T_{\rm gg} + xP^T_{\rm qq}\Big|_{\rm NNLL} &= \left\{4C_A a_s + \tfrac{8}{3}(11C_A^2 + 2C_A n_f - 4C_F n_f)a_s^2 \ln\tfrac{1}{x}\right\} \tfrac{2}{z} J_1(z) \\
&+ \left\{\tfrac{4}{9}(26C_F n_f - 23C_A n_f)a_s^2 + \tfrac{8}{9C_A}(11C_A^2 + 2C_A n_f - 4C_F n_f)^2 a_s^3 \ln^2\tfrac{1}{x}\right\} \tfrac{2}{z} J_1(z) \\
&+ \tfrac{32}{9C_A}\Big([134-72\zeta_2]C_A^4 + 23C_A^3 n_f - 48C_A^2 C_F n_f + 4C_A C_F n_f^2 - 8C_F^2 n_f^2\Big) a_s^3 \ln^2\tfrac{1}{x}\, \tfrac{4}{z^2} J_2(z)
\end{aligned}
$$

NB: hint of ‘second resummation’ ?

NB: $J_n(z) \to \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{2}(n + \frac{1}{2})\right)$ for $z \to \infty$.

Log parts ($\mathcal{L}(\mathcal{S})$) as convolutions of J_0:
$\int_0^1 dx\, x^{N-2} \frac{1}{\ln x}\left(J_0(2\sqrt{a}\ln x) - 1\right) = \ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4a}{(N-1)^2}}\right)$

NB: $\mathcal{L}(\mathcal{S})$ do not appear in physical kernels $\to$ scheme dependent feature.

Coefficient functions as convolutions of generalised hypergeometric functions (see also Albino,Bolzoni,Kniehl,Kotikov 11)

Beyond NNLL

Recap of what we already know:

- P^T_{gi}: NNLL; P^T_{qi}: N^{3}LL
- $C_{a,g}$: NNLL; $C_{a,q}$: N^{3}LL (closed form at NNLL known) for $a = T, \phi$
- $C_{L,i}$: NNLL $i = q, g$ ($C_{L,q}$ at N^{3}LL also possible from physical kernel)

$P^T_{gg}\big|_{C_F=0}$ at N^{3}LL and N^{4}LL using DMS relation Dokshitzer,Marchesini,Salam 05 :

$$\begin{aligned}
\partial_{\ln Q^2} f_\sigma(N, Q^2) &= P_\sigma(N)\, f_\sigma(N, Q^2) = P_u(N + \sigma\, \partial_{\ln Q^2})\, f_\sigma(N, Q^2) \\
P_\sigma(N) &= P_u(N) + \sum_{n=1}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^{n-1}}{\partial N^{n-1}} \left(\frac{\partial P_u}{\partial N} [P_u(N)]^n \right)
\end{aligned}$$

$\sigma = +1$: time-like. $\sigma = -1$: space-like. P_u: universal splitting fnc

$\rightarrow$ Time-Space difference given by lower order quantities.

Beyond NNLL

P^S single log enhanced BFKL . At LL:

$$P^S_{\rm gg}(N)\Big|_{\rm LL} \;=\; \frac{C_A\alpha_S}{\pi}\,\bar N^{\,-1} + 2\zeta_3\left(\frac{C_A\alpha_S}{\pi}\right)^4 \bar N^{\,-4} + \mathcal{O}(\alpha_S^6\,\bar N^{\,-6})\,.$$

Recall:

$$P^T_{\rm gg}(N)\Big|_{\rm LL} \;\sim\; \mathcal{O}(\alpha_S\bar N^{-1}) + \mathcal{O}(\alpha_S^2\bar N^{-3}) + \mathcal{O}(\alpha_S^3\bar N^{-5}) + \mathcal{O}(\alpha_S^4\bar N^{-7}) + \cdots$$

- $P^T_{\rm gg}|_{C_F=0}$ at LL,NLL,NNLL completely fixed without any space-like input.
- $P^T_{\rm gg}|_{C_F=0}$ at N^{3}LL requires $P^{S\,(3)}_{\rm gg}$ at LL (known).
- $P^T_{\rm gg}|_{C_F=0}$ at N^{4}LL requires $P^{S\,(3)}_{\rm gg}$ at NLL (not known in $\overline{\rm MS}$).

Also leads to additional cross checks with N^{2}LL results obtained from D-dim structure.

See CHK,Vogt,Yeats 12 for the analytic form of $P^T_{\rm gg}|_{C_F=0}$ and corresponding first moments at N^{3}LL and at N^{4}LL (up to one parameter)

Summary and outlook

- D-dimensional structure of unfactorised SIA fragmentation functions

  $\rightarrow$ NNLO $\leftrightarrow$ highest three logs.

  $\rightarrow$ NLO + resummed (finite N=1 limit) time-like splitting and coeff. fnc.
- Results towards NNLO + resummed (require N^4LL) partially available.
- Application to small-x DIS CHK,Vogt and large-x SIA Almasy,LoPresti,Vogt : soon.
- See also large-x DIS Almasy,Soar,Vogt 10