RESUMMATION IN PERTURBATIVE QCD

Tiziano Peraro

Max-Planck Insitut für Physik
München

What is resummation?

In perturbative QFT

- a generic physical quantity $y(x, \alpha)$ is computable order by order in its expansion in powers of the coupling constant α

$$y(x, \alpha) = y_0(x) + \alpha\, y_1(x) + \alpha^2\, y_2(x) + \ldots$$

- if there are kinematic regimes where $\alpha^k y_k(x) \gtrsim 1$ a fixed order computation is not a good approximation, even if $\alpha \ll 1$, because large contributions arise at every perturbative order
- in order to obtain reliable predictions in this kinematic regions one must resum all the large contributions to the coefficients $y_k(x)$, to every order in α
- the resummation of these large contributions can, in general, be performed thanks to renormalization group techniques.

A simple example

- The running of the QCD coupling constant $\alpha_s(Q^2)$ is determined by the renormalization group equation

$$\frac{d\alpha_s}{d\ln Q^2} = \beta(\alpha_s) \quad \text{with } \beta(\alpha_s) = -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 + \dots$$

- The leading order solution is

$$\begin{aligned}\alpha_s(Q^2) &= \frac{\alpha_s(\mu^2)}{1+\beta_0\alpha_s(\mu^2)\ln(Q^2/\mu^2)} \\ &= \alpha_s(\mu^2)\left(1-\beta_0\alpha_s(\mu^2)\ln\frac{Q^2}{\mu^2}+\beta_0^2\alpha_s^2(\mu^2)\ln^2\frac{Q^2}{\mu^2}+\dots\right)\end{aligned}$$

  - it resums contributions proportional to $\alpha_s^n \ln^n(Q^2/\mu^2)$ to every perturbative order in α_s (LL approximation)
  - it gives reliable predictions when $\alpha_s \ln(Q^2/\mu^2) \sim 1$.

Resummation in QCD

- A QCD cross section with hadrons in the initial state, e.g. DIS,
  - can be written as a function of
    - an hard scale Q^2
    - a dimensionless variable $x \sim Q^2/s$, such that $0 < x < 1$
  - takes the form of a convolution

$$\sigma(x, Q^2) = \int_x^1 \frac{dz}{z} f(z, \mu_F^2) C\left(\frac{x}{z}, \alpha_s(\mu_R^2), \frac{Q^2}{\mu_F^2}, \frac{Q^2}{\mu_R^2}\right)$$

- Large logarithmic contributions arise at every perturbative order from phase-space collinear integrations of the form

$$\int_{\mu^2}^{Q^2(1-x)/x} \frac{dk_T^2}{k_T^2} \ldots \sim \ln \frac{Q^2}{\mu^2} + \ln \frac{1}{x} + \ln(1-x)$$

  - High-energy or small-x resummation includes the largest $\ln^k(1/x)$ contributions to all perturbative orders in α_s
  - Threshold or soft gluon resummation includes the largest $\ln^k(1-x)$ contributions to all perturbative orders in α_s

Small-x resummation

- Scaling violation: dependence of the parton distributions from the hard scale Q^2, determined by the GLAP equation

$$\frac{df(x,Q^2)}{d\ln Q^2} = \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}, \alpha_s(Q^2)\right) f(y,Q^2)$$

- the splitting function $P(x,\alpha_s) = \alpha_s P_0(x) + \alpha_s^2 P_1(x) + \dots$
  - contains, at small x, large logarithms proportional to $\alpha_s^n \ln^m 1/x$ with $m \leq n$ that must be resummed to all orders when $\alpha_s \ln 1/x \sim 1$
  - at LL level one needs to resum the all order contributions proportional to $\alpha_s^n (\ln^n 1/x)/x$
  - at NLL level one needs to resum the all order contributions proportional to $\alpha_s^{n+1} (\ln^n 1/x)/x$
  - etc...
- The resummation of the largest eigenvalue of P determines the full resummed splitting function matrix

- Mellin transforms of the parton distribution $G(x, Q^2)$

$$G(N, Q^2) = \int_0^1 dx\, x^{N-1} G(x, Q^2)$$
$$G(x, M) = \int_0^\infty \frac{dQ^2}{Q^2} (Q^2)^{-M} G(x, Q^2)$$

- GLAP and BFKL evolution equations

$$Q^2 \frac{d}{dQ^2} G(N, Q^2) = \gamma(N, \alpha_s) G(N, Q^2)$$
$$-x \frac{d}{dx} G(x, M) = \chi(M, \alpha_s) G(x, M),$$

- duality relations

$$\chi(\gamma(N, \alpha_s), \alpha_s) = N, \qquad \gamma(\chi(M, \alpha_s), \alpha_s) = M$$

- The full resummed result contains several contributions
  - leading $\ln 1/x$ contributions correctly resummed thanks to the knowledge of the BFKL kernel χ and duality relations
  - leading $\ln Q^2$ contributions correctly resummed thanks to the knowledge of GLAP anomalous dimension γ
  - symmetrization
  - running coupling corrections
  - ...
- In the limit $n_f = 0$, for example

$$\gamma^{rc}_{\mathrm{NLO}}(N,\alpha_s) = \gamma_\Sigma(N,\alpha_s) - \beta_0\alpha_s\left(\frac{\chi_0''\left(\gamma_s(\alpha_s/N)\right)\chi_0\left(\gamma_s(\alpha_s/N)\right)}{2\left(\chi_0'\left(\gamma_s(\alpha_s/N)\right)\right)^2} - 1\right)$$
$$+ \gamma^B(N,\alpha_s) - \gamma^B_s(N,\alpha_s) - \gamma^B_{ss,0}(N,\alpha_s) - \gamma^B_{ss,1}(N,\alpha_s)$$
$$- \gamma_{\mathrm{match}}(N,\alpha_s) + \gamma_{\mathrm{mom}}(N,\alpha_s)$$

where, e.g., the running coupling contributions

$$\gamma^B(N,\alpha_s) = \frac{1}{2} - \beta_0\bar{\alpha}_s + \frac{1}{A(N,\alpha_s)}\frac{K'_{2B(N,\alpha_s)}\left(1/(\beta_0\bar{\alpha}_s A(N,\alpha_s))\right)}{K_{2B(N,\alpha_s)}\left(1/(\beta_0\bar{\alpha}_s A(N,\alpha_s))\right)}$$

$\gamma_\Sigma(N,\alpha_s)$ is (numerically) found as the the solution of the implicit equation

$$\chi(\gamma_\Sigma, N, \alpha_s) = N$$

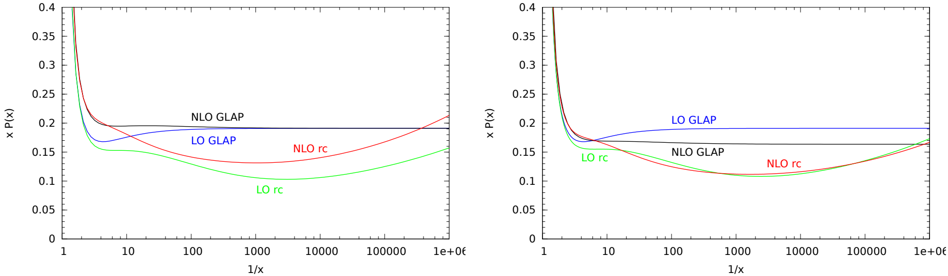
where

$$\chi(M,N,\alpha_s) = \chi_{\Sigma,\mathrm{NLO}}(M,N,\alpha_s) + \alpha_s^2 \chi_1^{\beta_0}(M - \frac{N}{2}, N) + \beta_0 \alpha_s^2 \left(\frac{\bar{\chi}_0\left(M - \frac{N}{2}, N\right)}{M} - \frac{n_c}{\pi M^2} \right)$$

$$\begin{aligned}
&\chi_{\Sigma,\mathrm{NLO}}(M,N,\alpha_s) = \\
&\quad \alpha_s \bar{\chi}_0(M - \tfrac{N}{2}, N) - \frac{\alpha_s^2}{2}\beta_0 \frac{n_c}{\pi}\left(\frac{\pi^2}{n_c^2}\bar{\chi}_0(M - \tfrac{N}{2}, N)^2 - \psi'(M) - \psi'(1-M+N) \right) \\
&\quad + \alpha_s^2 \frac{n_c^2}{4\pi^2}\left[\left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10 n_f}{9 n_c} \right)(\psi(1) - \psi(M)) + 3\zeta(3) + \psi''(M) + 4\left(\frac{\pi^2}{24}\left(\psi\left(\tfrac{1}{2} + \tfrac{M}{2}\right) - \psi\left(\tfrac{M}{2}\right)\right) + \phi_L^+(M) \right) \right. \\
&\quad + \frac{3}{4(1-2M)}\left(\psi'(\tfrac{1}{2} + \tfrac{M}{2}) - \psi'(\tfrac{M}{2}) + \psi'(\tfrac{1}{4}) - \psi'(\tfrac{3}{4}) \right) \left\{ \frac{1}{1-2M}\left(\psi'(\tfrac{1}{2} + \tfrac{M}{2}) - \psi'(\tfrac{M}{2}) + \psi'(\tfrac{1}{4}) - \psi'(\tfrac{3}{4}) \right) \right. \\
&\quad \left. + \frac{1}{2(1+2M)}\left(\psi'(\tfrac{1}{2} + \tfrac{M}{2}) - \psi'(\tfrac{M}{2}) + \psi'(-\tfrac{1}{4}) - \psi'(\tfrac{1}{4}) \right) - \frac{1}{2(3-2M)}\left(\psi'(\tfrac{1}{2} + \tfrac{M}{2}) - \psi'(\tfrac{M}{2}) + \psi'(\tfrac{3}{4}) - \psi'(\tfrac{5}{4}) \right) \right\} \\
&\quad \left. + (M \leftrightarrow 1 - M + N) \right] - \frac{\alpha_s^2}{2}\chi_0(M - \tfrac{N}{2}, N)\frac{n_c}{\pi}\left(2\psi'(1+N) - \psi'(M) - \psi'(1-M+N) \right) \\
&\quad + \mathrm{GLAP} - \mathrm{d.c.} + \mathrm{matching}
\end{aligned}$$

- Resummed splitting functions $x\,P_{gg}(x)$ obtained by Mellin inversion integral, plotted as functions of $1/x$

$\alpha_s = 0.2,\ n_f = 0$ $\qquad\qquad$ $\alpha_s = 0.2,\ n_f = 4$



  - The difference between fixed order NLO (black) and resummed NLO (red) is generally visible but not very large
  - The small-x asymptotic growth of the resummed result becomes very important only at very small x, $x \lesssim 10^{-6}$

Soft gluon resummation

- The partonic cross sections $\hat{\sigma}(x)$ receive logarithmic corrections at every perturbative order due to soft gluon emission

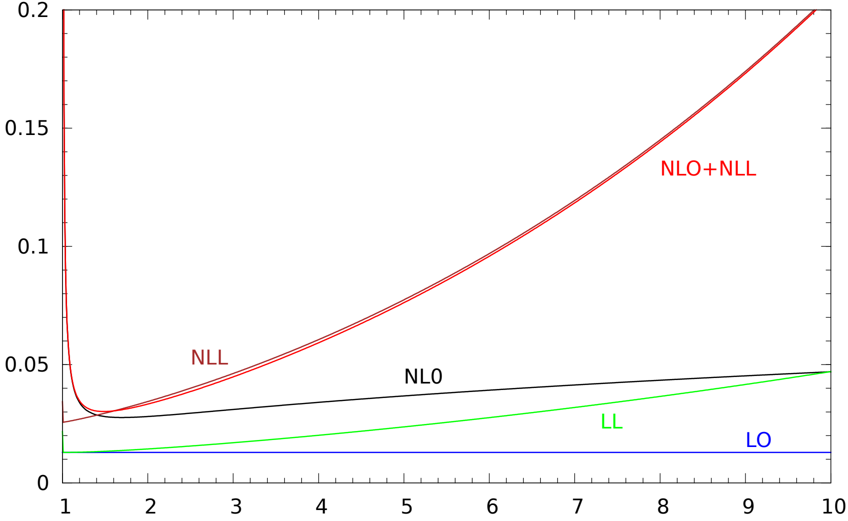
$$\alpha_s^n \left[\frac{\ln^m(1-x)}{1-x}\right]_+ \qquad \text{with } m \leq 2n$$

- these corrections become asymptotically large when $x = Q^2/s \to 1$ (threshold limit) and they need to be resummed when $\alpha_s \ln^2(1-x) \gtrsim 1$.
- The resummed Mellin-space coefficient functions are given by the exponentiation

$$C(N;Q^2)/C_{\mathrm{LO}}(Q^2) = g_0(Q^2)\exp\left[G(N,Q^2)\right] + \mathcal{O}(N^{-1}\ln^n N),$$

- the function $G(N,Q^2)$ is found by comparison with fixed order results

Soft gluon resummation of $C_{gg}(N)$ in Higgs boson production (gluon fusion) for $m_H \sim 120\text{GeV}$



Double resummation: Higgs boson production

- Higgs boson production via gluon fusion: $gg \to H$
  - The dominant contributions come from the gluonic channel gg

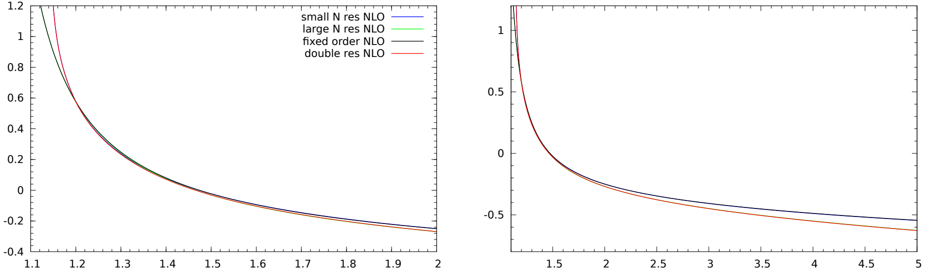
$$\sigma_{gg}(N, m_H^2) = \mathcal{L}_{gg}(N, \mu_F^2) C_{gg}\left(N, \alpha_s(\mu_R^2), \frac{m_H^2}{\mu_F^2}, \frac{m_H^2}{\mu_R^2}\right)$$

  - the definition of $\mathcal{L}$ and C depends on the choice of the factorisation scheme, e.g. in $\overline{\text{MS}}$
    - γ_{gg}^{AP} contains contributions from high energy resummation
    - C_{gg} contains contributions from threshold resummation
  - We can define a scheme-independent physical anomalous dimension

$$\gamma_{gg}^{\text{phys}} \equiv 2\gamma_{gg}^{AP} + \beta(\alpha_s)\frac{d \ln C_{gg}}{d\alpha_s}$$

    - it determines the dependence of $\sigma_{gg}(N, m_H^2)$ from m_H^2
    - it contains contributions from both high energy and threshold resummation
    - it can be used to determine the impact of the two resummations independently from non-perturbative details (like pdfs)

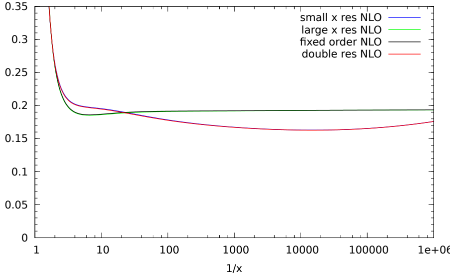
The resummed physical anomalous dimension $\gamma_{gg}^{\text{phys}}$ at NLO



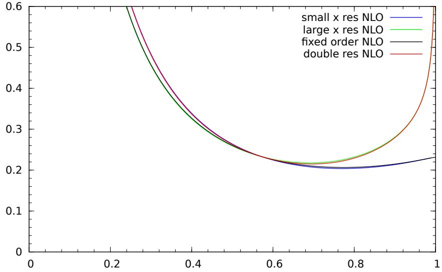
- there's only a very small region, roughly $1.2 \lesssim N \lesssim 1.6$, where the unresummed result offers a very good approximation of the full resummed one

The resummed physical splitting function P_{gg}^{phys} at NLO

$xP(x)$ vs. $1/x$



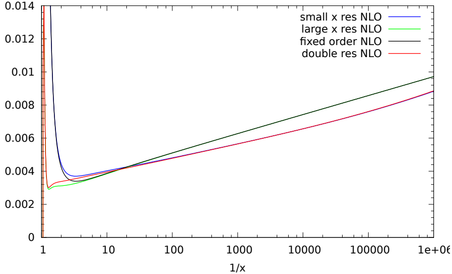
$(1-x)P(x)$ vs. x



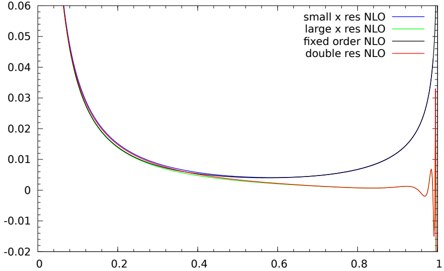
- the full resummed result visibly differs from the fixed order one in the whole space space, aside from a very small region, $0.5 \lesssim x \lesssim 0.6$

After a convolution with $C_{gg}(z)$...

$x(P(x) \otimes C(x))$ vs. $1/x$



$(1 - x)(P(x) \otimes C(x))$ vs. x



...the effects of the double resummation are even more important

- there's no wide intermediate region where the fixed order NLO result offers a very good approximation

- Where a double resummation can be important?
- In the convolution integral

$$\sigma_{gg}(x, m_H^2) = \int_x^1 \frac{dz}{z} \mathcal{L}_{gg}(z, \mu_F^2) C_{gg}\left(\frac{x}{z}, \alpha_s(\mu_R^2), \frac{m_H^2}{\mu_F^2}, \frac{m_H^2}{\mu_R^2}\right)$$

  - the parton luminosity $\mathcal{L}_{gg}(z)$ is peaked at small z and suppressed at large z
  - the convolution integral is dominated by the region $z \sim x$, $x/z \sim 1$, i.e. by the small-x enhanced contributions of the parton distributions and the threshold contributions of the coefficient functions
  - soft gluon resummation is also important away from the threshold region $x \to 1$: by keeping only the soft gluon resummation contributions in C_{gg} one obtains a very good approximation of the complete result
- This is confirmed by
  - phenomenological results (e.g. Catani et al.)
  - saddle point arguments

- A saddle point argument:
  - The Mellin inversion integral

$$\sigma(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} dN\, x^{-N} \sigma(N) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} dN \exp\left[E(x,N)\right]$$

    is dominated by the contributions close to the saddle point N_0 where

$$E'(x, N_0) = \ln \frac{1}{x} + \frac{\sigma'(N_0)}{\sigma(N_0)} = \ln \frac{1}{x} + \frac{\mathcal{L}'(N_0)}{\mathcal{L}(N_0)} + \frac{C'(N_0)}{C(N_0)} = 0$$

  - The saddle point $N_0 = N_0(x)$ is also saddle point of
    - the Mellin inversion integral for $\mathcal{L}(z)$ evaluated in $z = z_{\mathcal{L}}(x)$
    - the Mellin inversion integral for $C(z)$ evaluated in $z = z_C(x)$
  - It's easy to show that: $z_{\mathcal{L}}(x) z_C(x) = x$
  - Since $|\mathcal{L}'(N)/\mathcal{L}(N)| \gg |C'(N)/C(N)|$ the dominant contributions to $\sigma(x)$ come, in the convolution integral, from the region where,
    - the argument of $\mathcal{L}(z)$ is $z \sim z_{\mathcal{L}}(x) \sim x$
    - the argument of $C(z)$ is $z \sim z_C(x) \sim 1$
    - the high energy resummation of $\mathcal{L}$ and the threshold resummation of C can also be simultaneously important

Conclusions

- Perturbative results in QCD receive large higher order corrections in the physical regions $x \to 0$ and $x \to 1$
  - We provided a new implementation of an improved version of the small-x resummation procedure of ABF
  - We combined these results with the soft gluon resummation of the coefficient functions for a specific process
- The cross section for Higgs boson production at LHC
  - is dominated by the small-x enhanced contributions of the parton distributions and the threshold contributions of the coefficient functions
  - the explicit computation of a physical anomalous dimension shows that a double resummation has visible effects in (almost) the whole phase space
- General arguments (convolution, saddle-point, ...) show that
  - the high-energy resummation of the parton distribution functions and the threshold resummation of the coefficient functions can both be expected to by important at the same time