# Cosmic microwave background and large-scale structure

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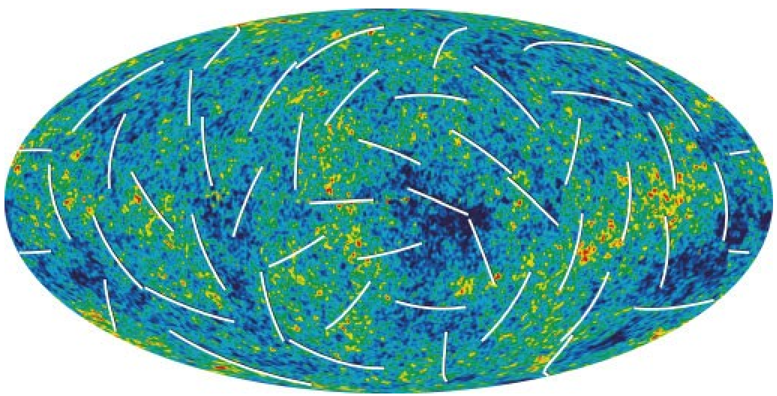
IMPRS Block course, MPP, March 5 – 8, 2012

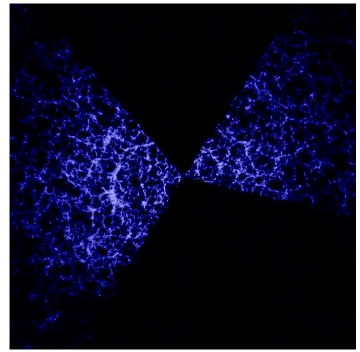
# CMB and LSS...



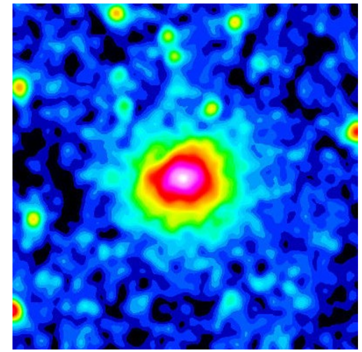
> 0.5 deg: COBE, WMAP, Planck

< 0.5 deg: DASI, CBI, ACBAR, Boomerang, VSA, QuaD, QUIET, BICEP, ACT, SPT, etc.

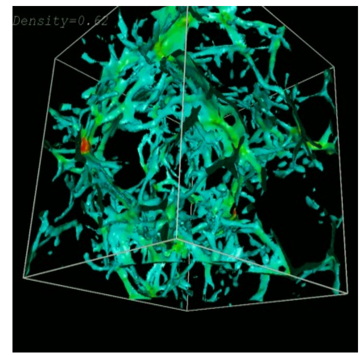
Galaxy clustering



Cluster abundance

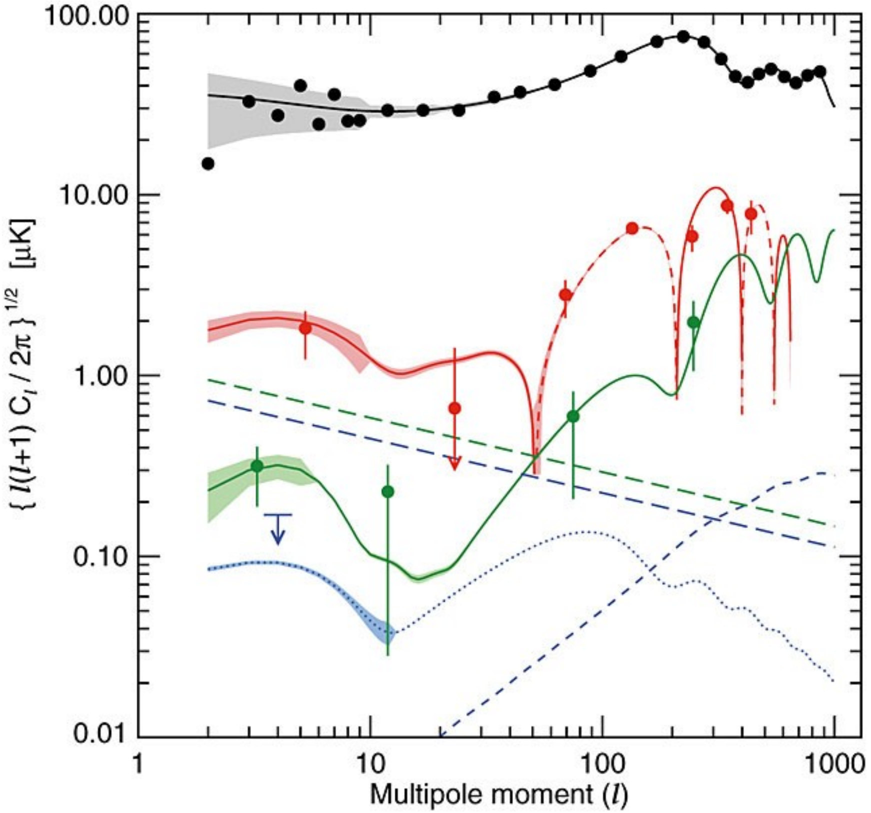


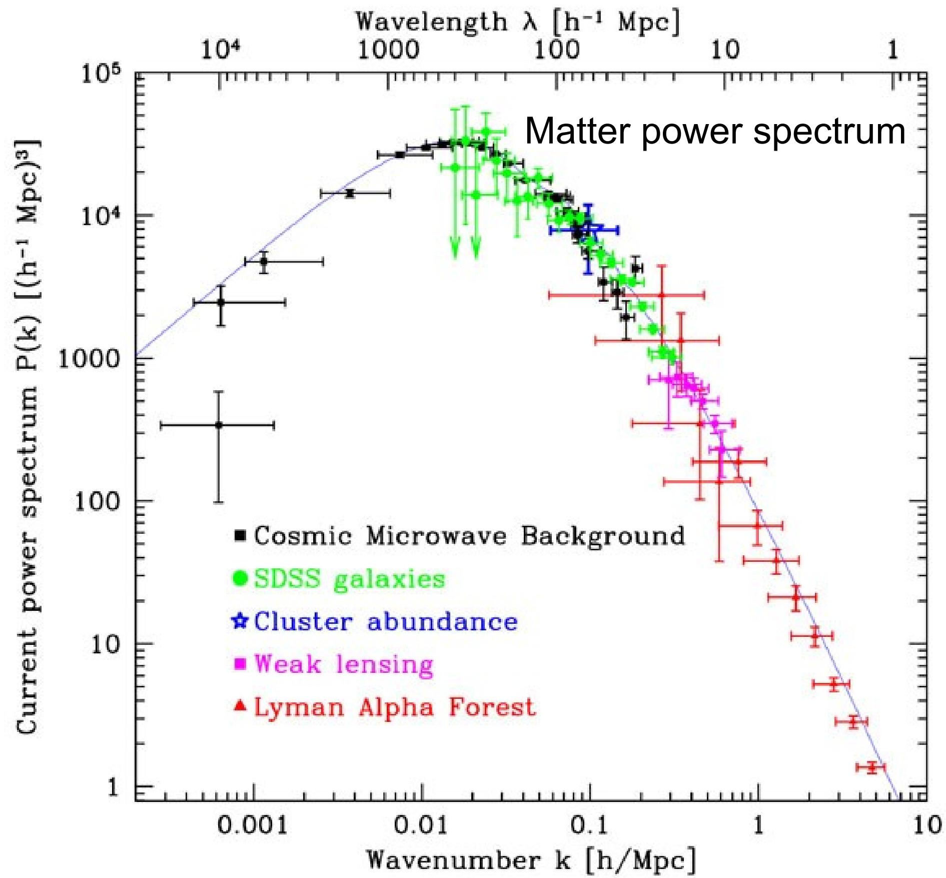
Gravitational lensing



Intergalactic hydrogen clumps; Lyman-α

# CMB and LSS 2-point spectra (power spectra)...

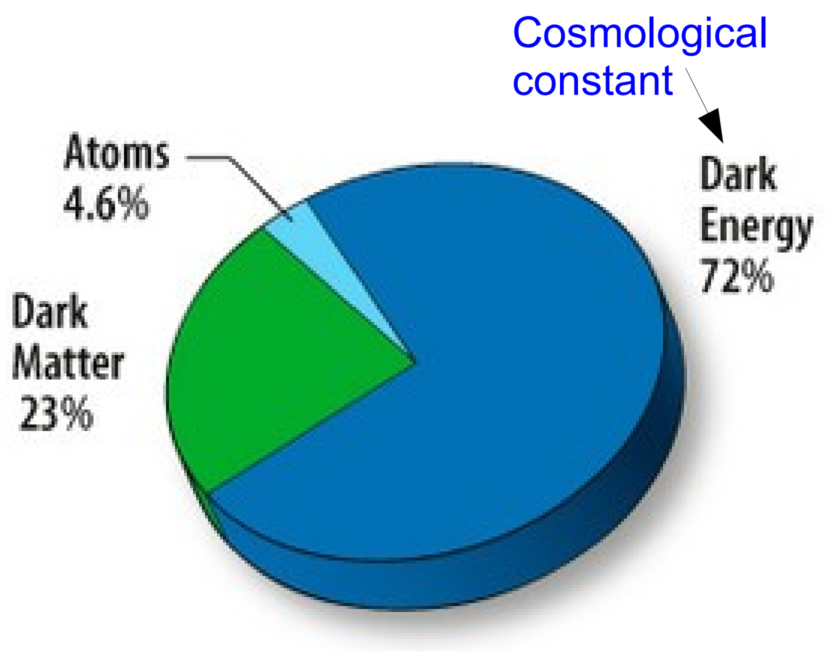


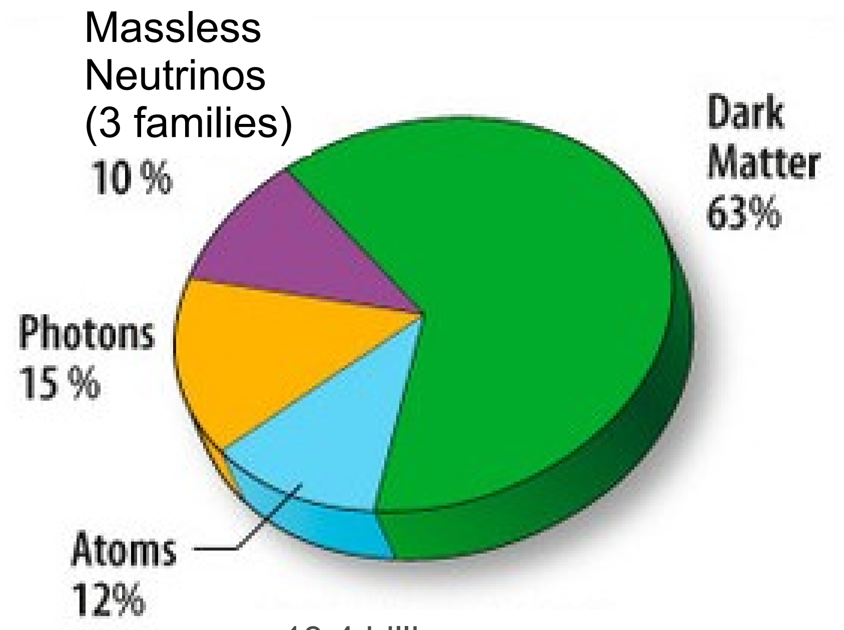

3-point spectrum (bispectrum), 4-point spectrum (trispectrum)...

# The concordance flat ΛCDM model...

- The simplest model consistent with **present observations**.




Composition today




13.4 billion years ago
(at photon decoupling)

Plus flat spatial geometry+initial conditions
from single-field inflation

- Six-parameter vanilla model:

Optical depth to reionisation, CMB only

$$P_{\text{vanilla}} = (\Omega_c h^2, \Omega_b h^2, \Omega_\Lambda, n_s, A_s, \tau)$$

Energy content

Initial conditions

Flat geometry

$$\sum_i \Omega_i = 1$$

SUMMARY OF THE COSMOLOGICAL PARAMETERS OF ΛCDM MODEL[a]

| Class | Parameter | *WMAP* 7-year ML[b] | *WMAP*+BAO+$H_0$ ML | *WMAP* 7-year Mean[c] | *WMAP*+BAO+$H_0$ Mean |
|---|---|---|---|---|---|
| Primary | $100\Omega_b h^2$ | 2.227 | 2.253 | $2.249^{+0.056}_{-0.057}$ | $2.255 \pm 0.054$ |
| | $\Omega_c h^2$ | 0.1116 | 0.1122 | $0.1120 \pm 0.0056$ | $0.1126 \pm 0.0036$ |
| | $\Omega_\Lambda$ | 0.729 | 0.728 | $0.727^{+0.030}_{-0.029}$ | $0.725 \pm 0.016$ |
| | $n_s$ | 0.966 | 0.967 | $0.967 \pm 0.014$ | $0.968 \pm 0.012$ |
| | $\tau$ | 0.085 | 0.085 | $0.088 \pm 0.015$ | $0.088 \pm 0.014$ |
| | $\Delta^2_{\mathcal{R}}(k_0)$[d] | $2.42 \times 10^{-9}$ | $2.42 \times 10^{-9}$ | $(2.43 \pm 0.11) \times 10^{-9}$ | $(2.430 \pm 0.091) \times 10^{-9}$ |
| Derived | $\sigma_8$ | 0.809 | 0.810 | $0.811^{+0.030}_{-0.031}$ | $0.816 \pm 0.024$ |
| | $H_0$ | 70.3 km/s/Mpc | 70.4 km/s/Mpc | $70.4 \pm 2.5$ km/s/Mpc | $70.2 \pm 1.4$ km/s/Mpc |
| | $\Omega_b$ | 0.0451 | 0.0455 | $0.0455 \pm 0.0028$ | $0.0458 \pm 0.0016$ |
| | $\Omega_c$ | 0.226 | 0.226 | $0.228 \pm 0.027$ | $0.229 \pm 0.015$ |
| | $\Omega_m h^2$ | 0.1338 | 0.1347 | $0.1345^{+0.0056}_{-0.0055}$ | $0.1352 \pm 0.0036$ |
| | $z_{\text{reion}}$[e] | 10.4 | 10.3 | $10.6 \pm 1.2$ | $10.6 \pm 1.2$ |
| | $t_0$[f] | 13.79 Gyr | 13.76 Gyr | $13.77 \pm 0.13$ Gyr | $13.76 \pm 0.11$ Gyr |

Komatsu et al. (WMAP7), 2010

# From theory to phenomenological observables...

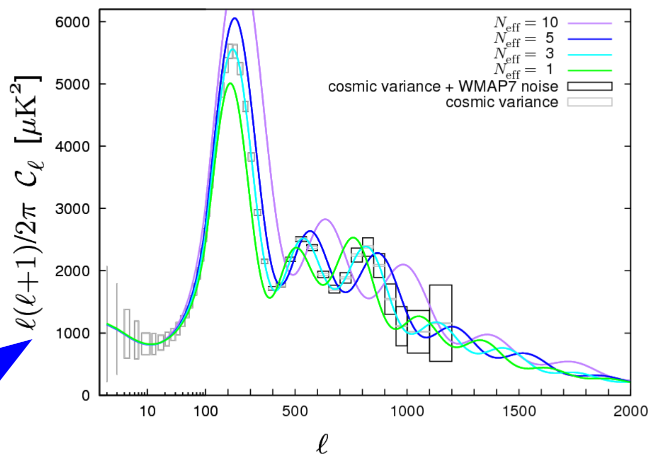
Initial conditions set by an inflation theory

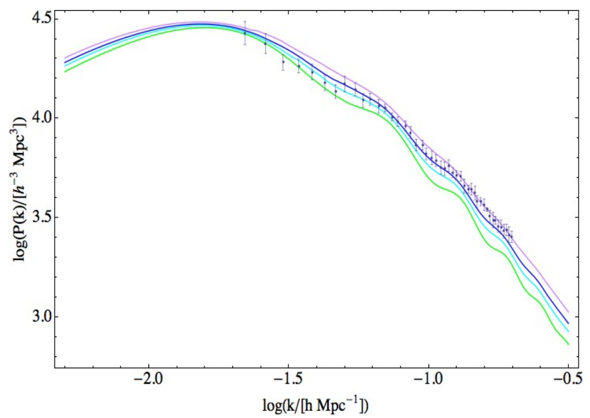
Dark matter
Baryons
Neutrinos
Photons
Dark energy
Spatial geometry
etc..

Boltzmann-Einstein black box

**This course**

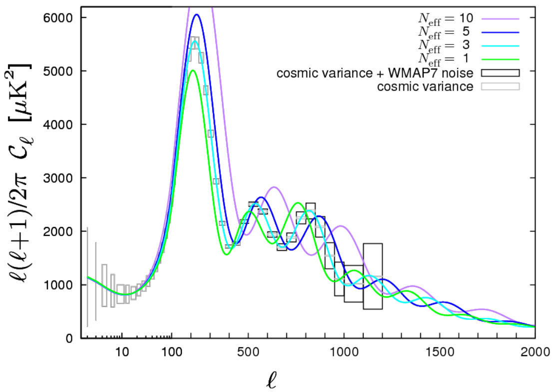
In gory detail



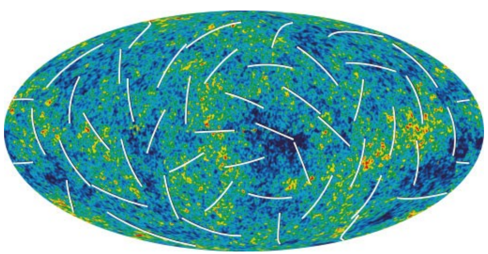



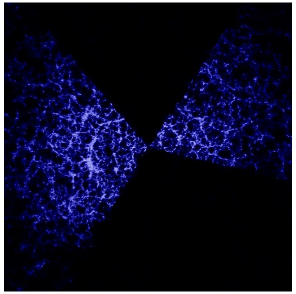
# From predictions+data to cosmological parameters...

**Theoretical predictions**

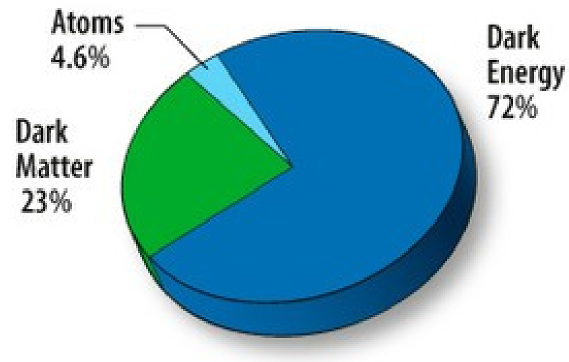



**Observational data**





Statistics black box




**Some other time... but**

… once the theoretical predictions have been understood, it should be clear how observations can be used to constrain cosmological parameters.

# Plan...

1. Review: Homogeneous and isotropic universe
2. Inhomogeneities I: cosmological perturbation theory
3. Inhomogeneities II: Boltzmann equation
4. Initial conditions
5. Approximate solutions I: matter density perturbations
6. Approximate solutions II: CMB temperature fluctuations
7. Cosmological parameters from CMB temperature anisotropies

# Useful references...

- **Textbooks**
  - S. Dodelson, *Modern cosmology*
  - R. Durrer, *The cosmic microwave background*
- **Lecture notes**
  - U. Seljak, *Lectures on dark matter* (Google it!)
- **Research papers**
  - C.-P. Ma and E. Bertschinger, *Cosmological perturbation theory in the synchronous and conformal Newtonian gauges*, Astrophys.J. **455** (1995) 7-25 [astro-ph/9506072]

# Warning...

- There are many different conventions around!
- My S,V,T decomposition convention here is a tad different (for simplicity and consistency).
- When in doubt, rederive the equations yourself!

# 1. Review: Homogeneous and isotropic universe...

# 1.1 Friedmann-Lemaître-Robertson-Walker universe...

- Modern cosmology is based on the hypothesis that our universe is homogeneous and isotropic on sufficiently large length scales.
  - Homogeneous → same everywhere
  - Isotropic → same in all directions
  - Sufficiently large scales → > O(100 Mpc)

- I pc = 1 parsec = 3.0856 x $10^{18}$ cm
  - Distance from Sun to Galactic centre ~ 10 kpc
  - Distance to the Virgo cluster ~ 20 Mpc
  - Size of the visible universe ~ O(10 Gpc)

# 1.1 Friedmann-Lemaître-Robertson-Walker universe...

- Homogeneity and isotropy imply maximally symmetric 3-spaces (3 translational and 3 rotational symmetries).
  - A spacetime metric that satisfies these requirements:

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)[-d\eta^2 + \gamma_{ij} dx^i dx^j]$$

FLRW metric

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

  - $a(\eta)$ = scale factor; $\eta$ = conformal time
  - $K$ = 0,+1, -1 → flat, positively and negatively curved spatial geometry

# 1.2 Matter/energy content...

- Matter/energy content is encoded in the stress-energy tensor $\bar{T}_{\mu\nu}$.
- Homogeneity and isotropy imply only **one viable form**:

$$\bar{T}^{\mu\nu}_{(\alpha)} = \begin{pmatrix} -\bar{\rho}_\alpha(\eta)\,\bar{g}^{00} & 0 \\ 0 & \bar{P}_\alpha(\eta)\,\bar{g}^{ij} \end{pmatrix}$$

  - Fluids at rest with respect to the FLRW coordinates (or comoving frame)
  - $\bar{\rho}_\alpha$ = energy density of fluid i in the comoving frame
  - $\bar{P}_\alpha$ = pressure of fluid i in the comoving frame

# 1.2 Matter/energy content: conservation...

- Local conservation of energy-momentum: $\nabla_\mu T^{\mu\nu}_{(\alpha)} = 0$
- In a FLRW universe:

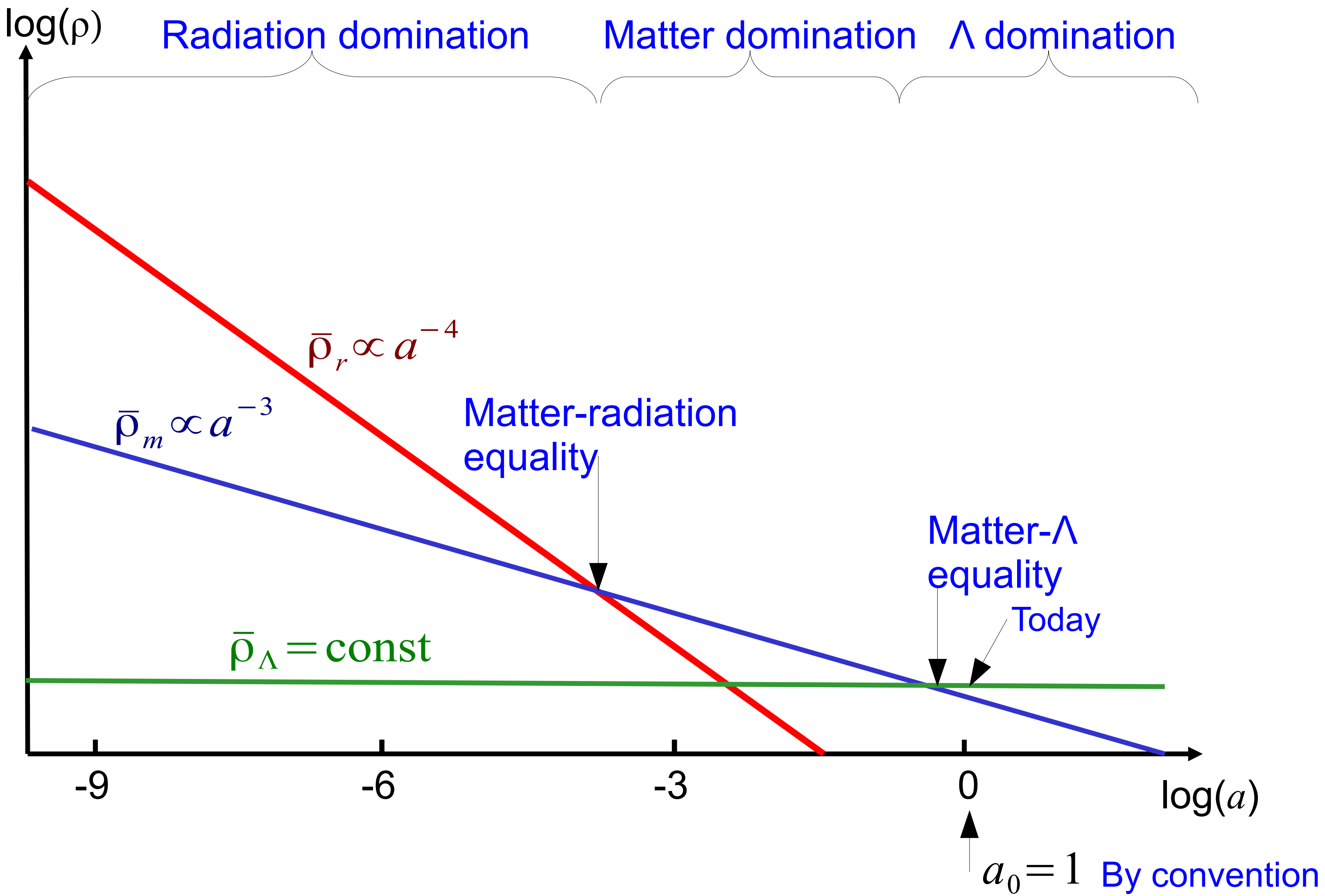
$$\frac{d\bar{\rho}_\alpha}{d\eta} + 3\frac{\dot{a}}{a}\left(\bar{\rho}_\alpha + \bar{P}_\alpha\right) = 0$$

Continuity equation

- A general fluid can be specified by an equation of state parameter:

$$w_\alpha(\eta) \equiv \bar{P}_\alpha(\eta)/\bar{\rho}_\alpha(\eta)$$

  - **Nonrelativistic matter** ($w_m$ ~ 0): $\bar{\rho}_m \propto a^{-3}$
  - **Radiation** ($w_r$ = 1/3): $\bar{\rho}_r \propto a^{-4}$
  - **Vacuum energy** ($w_\Lambda$ = -1): $\bar{\rho}_\Lambda \propto \text{constant}$



log(ρ)
Radiation domination
Matter domination
Λ domination
$\bar{\rho}_r \propto a^{-4}$
$\bar{\rho}_m \propto a^{-3}$
Matter-radiation equality
Matter-Λ equality
Today
$\bar{\rho}_\Lambda = \text{const}$
-9
-6
-3
0
log($a$)
$a_0 = 1$
By convention

# 1.3 Friedmann equation...

- Derived from the Einstein equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

- An evolution equation for the scale factor $a(\eta)$:

$$\mathscr{H}^2(\eta) = (aH)^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G a^2}{3} \sum_\alpha \bar{\rho}_\alpha - K$$

Friedmann equation

  - $H$ = Hubble parameter
  - ($\mathscr{H}$ = conformal/comoving Hubble parameter)

- **Friedmann+continuity** equations → specify the whole system.

# 1.3 Friedmann equation...

- You may also have seen the Friedmann equation in this form:

$$\mathcal{H}^2(\eta) = a^2 H^2(\eta_0)\left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_K a^{-2}\right]$$

$$\Omega_\alpha = \frac{\bar{\rho}_\alpha(\eta_0)}{\rho_{\mathrm{crit}}(\eta_0)}, \qquad \rho_{\mathrm{crit}}(\eta) \equiv \frac{3\mathcal{H}^2(\eta)}{8\pi G a^2}, \qquad \Omega_K \equiv -\frac{K}{\mathcal{H}^2(\eta_0)}$$

Critical density

- **Current observations**:

$$\Omega_m \sim 0.3, \quad \Omega_\Lambda \sim 0.7, \quad \Omega_r \sim 10^{-5}$$

$$|\Omega_K| < 0.01$$

$$H_0 \equiv H(\eta_0) \sim 70 \ \ \mathrm{km\,s^{-1}\,Mpc^{-1}}$$

e.g., Komatsu et al. [WMAP7] 1001.4538

# 1.3 Friedmann equation: solutions...

- Solutions in some limits:

  - Radiation domination: $a \propto \eta, \quad \mathscr{H} = 1/\eta$
  - Matter domination: $a \propto \eta^2, \quad \mathscr{H} = 2/\eta$
  - ( Vacuum energy domination: $a \propto 1/\eta$ )

# 1.4 Redshift...

- From the geodesic equation: $$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{d x^\nu}{d\tau} \frac{d x^\rho}{d\tau} = 0$$
- Cosmological redshift:

  Proper momentum of a point particle measured by a comoving observer → $|\vec{p}| \propto a^{-1}$

- For photons:

  Measured by comoving observer → $\lambda_0$; Measured at comoving emitter → $\lambda_e$; today → $\eta_0$; Redshift parameter → $z$

$$\frac{\lambda_0}{\lambda_e} = \frac{E_e}{E_0} = \frac{a(\eta_0)}{a(\eta_e)} \equiv 1 + z$$

- In a FRLW universe, there is a one-to-one correspondence between $\eta$, $a$, and $z$ → We use them interchangeably as a measure of time.

# 1.4 Distances: comoving distance...

- The comoving distance is the **coordinate distance** travelled by a light ray between emission and observation:

$$\chi(a_e) = \eta_0 - \eta_e = \int_{\eta_e}^{\eta_0} d\eta = \int_{a_e}^{a_0} \frac{da}{a\,\mathscr{H}(a)}$$

# 1.4 Distances: comoving distance...

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$$\chi(a_e) = \eta_0 - \eta_e = \int_{\eta_e}^{\eta_0} d\eta = \int_{a_e}^{a_0} \frac{da}{a\,\mathscr{H}(a)}$$

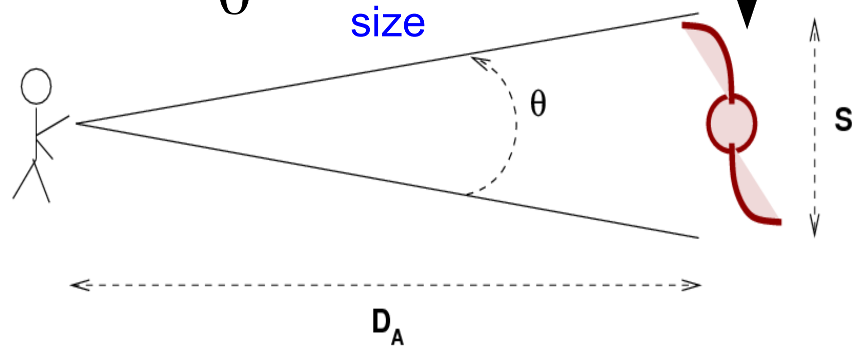
- **Observable**: angular diameter distance:

Physical size

$$d_A = \frac{s}{\theta}$$

Angular size

Standard ruler




Theory

$$d_A(a(z)) \equiv a \begin{pmatrix} \sinh\chi(a) \\ \chi(a) \\ \sin\chi(a) \end{pmatrix} \begin{matrix} K=-1 \\ K=0 \\ K=+1 \end{matrix}$$

# 1.4 Distances: comoving distance...

- The comoving distance is the **coordinate distance** travelled by a light ray between emission and observation:

$$\chi(a_e) = \eta_0 - \eta_e = \int_{\eta_e}^{\eta_0} d\eta = \int_{a_e}^{a_0} \frac{da}{a\,\mathscr{H}(a)}$$

- **Observable**: luminosity distance:

Theory

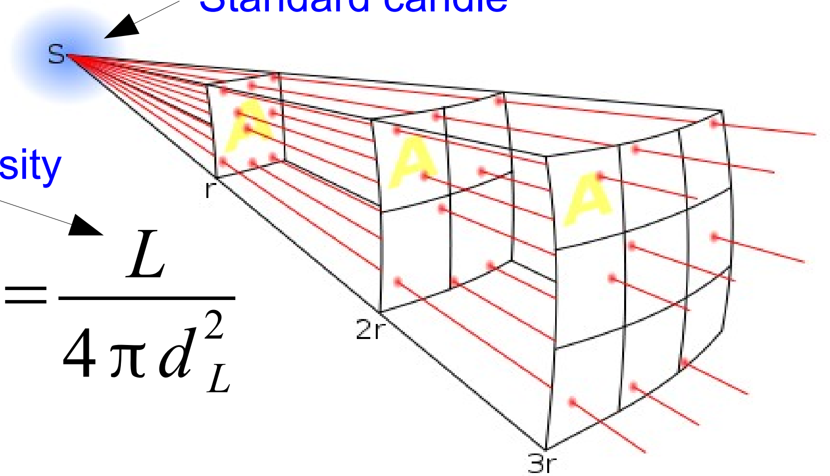
$$d_L(a(z)) \equiv \frac{1}{a} \begin{pmatrix} \sinh\chi(a) \\ \chi(a) \\ \sin\chi(a) \end{pmatrix}$$

Standard candle

Luminosity

$$F = \frac{L}{4\pi d_L^2}$$

Flux



# 1.4 Distances: Hubble length/horizon...

- The (comoving) Hubble length denotes the time-scale over which the scale factor changes appreciably:

$$\text{comoving Hubble length} \equiv c\,\mathscr{H}^{-1}$$

- **Significance**: “genuine” general relativistic effects are important on scales close to or larger than the Hubble length.

- Sometimes also called the Hubble horizon, but it is not a real horizon.

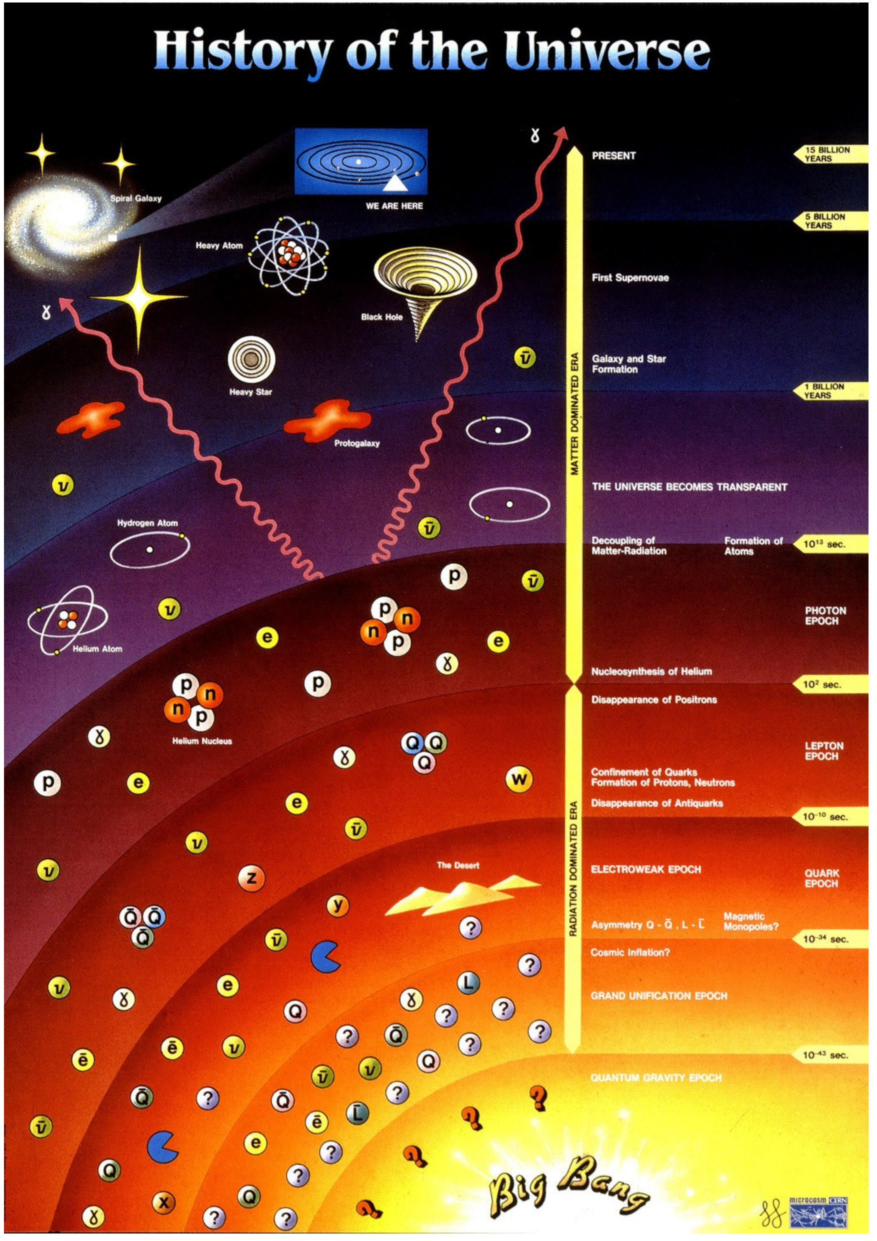
- Today, the Hubble length corresponds roughly to the size of the observable universe.

# 1.5 Particles in cosmology...

- The early universe is very dense and hot.

  → Frequent particle interactions.

- At sufficiently high temperatures, even weak interactions (or weaker) can be maintained in a state of equilibrium.



# 1.5 Particles in cosmology: Neutrino decoupling

- At temperatures > O(1) MeV, weak interactions keep neutrinos coupled to other particles in the thermal bath:

$$\nu + e \leftrightarrow \nu + e\,, \quad \nu\bar{\nu} \leftrightarrow e^{+} e^{-}$$

  - **Weak interaction rate**: $\Gamma = \sigma_{\nu e} n_e \sim G_F^2 T^5$
  - **Hubble expansion rate**: $H = \sqrt{\frac{8\pi G}{3} \sum_i \bar{\rho}_i} \sim \frac{T^2}{m_{\text{pl}}}$

- When the weak rate drops below the Hubble expansion rate, the neutrinos lose thermal contact with other particles → neutrinos decouple.

$$T_{\nu\,\text{dec}} \sim 1 \text{ MeV}$$

Neutrino decoupling temperature

# 1.5 Particles in cosmology: Neutrino decoupling

- After neutrino decoupling, electrons & positrons annihilate at T ~ 0.2 MeV:

$$e^{+}e^{-} \rightarrow \gamma\gamma$$

  - Annihilation reheats the photons, but **not** the neutrinos.

    → The neutrinos emerge a little **colder** than the photons.

  - **Neutrino temperature** after electron-positron annihilation:

$$T_{\nu} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma}$$

# 1.5 Particles in cosmology: Photon decoupling...

- At temperatures > O(1) eV, Thomson scattering keeps **photons** and **free (unbound) electrons** in equilibrium:

$$e^- + \gamma \leftrightarrow e^- + \gamma$$

Free electron density

  - Thomson scattering rate: $\Gamma_T \sim n_e \sigma_T$ $\quad \sigma_T = 0.665 \times 10^{-24}\ \text{cm}^2$

- But the free electron density is governed predominantly by

$$e^- + p^+ \leftrightarrow H + \gamma$$

Hydrogen recombination

- When $n_e$ drops to a point so that $\Gamma_T$ < Hubble rate, photons decouple from electrons:

$$T_{\gamma\,\text{dec}} \sim 0.25\ \text{eV}$$

Photon decoupling temperature

# 2. Inhomogeneities I: cosmological perturbation theory...

# 2.1 Overview...

- Our current understanding of the inhomogeneous universe:

Neutrino decoupling

Matter-radiation equality

Photon decoupling

time

**Inflation**: Quantum fluctuations on a scalar field
→ initial perturbations on spacetime metric
→ initial conditions for CMB & LSS

**Gravity** competes with pressure to amplify primordial inhomogeneities

**CMB anisotropies**: Perturbations in photon energy density and polarisation

**LSS formation**: Perturbations in nonrelativistic matter

Fluid dynamics + GR

# 2.1 Overview...

- We study inhomogeneities by perturbing around the FLRW spacetime geometry and stress-energy tensor:

$\bar{g}_{\mu\nu}$ = Unperturbed FLRW metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

$\bar{T}_{\mu\nu}$ = Homogeneous and isotropic

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}, \qquad |\delta T_{\mu\nu}| \ll \bar{\rho}$$

- **Our goal**:
  - To derive relations between $h_{\mu\nu}$ and $\delta T_{\mu\nu}$ → **Einstein equation**.
  - To find evolution equations for $\delta T_{\mu\nu}$ → **Boltzmann equation**.

- We assume a flat background spatial geometry (K = 0):

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)[-d\eta^2 + \delta_{ij} dx^i dx^j]$$

# 2.2 Metric perturbations...

- The metric perturbations can be parameterised as

$\bar{g}_{\mu\nu}$ = Unperturbed FLRW metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

$$h_{\mu\nu} dx^{\mu} dx^{\nu} = -2 A \, d\eta^2 - 2 B_i \, d\eta \, dx^i + 2 H_{ij} dx^i dx^j$$

- $A \rightarrow$ 1 d.o.f.
- $B_i \rightarrow$ 3 d.o.f.
- $H_{ij} \rightarrow$ Symmetric in $i$ and $j$; 6 d.o.f.

} 10 d.o.f.

# 2.2 Metric perturbations: decomposition...

- Because the FLRW background has 3 rotational symmetries, we can decompose the perturbations $A$, $B_i$ and $H_{ij}$ into irreducible scalar, vector and tensor representations under rotation.

$$B_i = \partial_i B^{(S)} + B_i^{(V)}$$

Longitudinal; 1 d.o.f.

Transverse; 2 d.o.f. $\partial^i B_i^{(V)} = 0$

Traceless; solenoidal; 2 d.o.f. $\partial^i H_i^{(V)} = 0$

$$H_{ij} = H_L \delta_{ij} + [\partial_i \partial_j - \frac{1}{3}\delta_{ij}\partial^2] H_T + \frac{1}{2}[\partial_j H_i^{(V)} + \partial_i H_j^{(V)}] + H_{ij}^{(T)}$$

Related to trace of $H_{ij}$; 1 d.o.f.

Traceless; longitudinal; 1 d.o.f.

Traceless; transverse; 2 d.o.f.

$$\partial^i H_i^{(T)j} = 0, \quad H_i^{(T)i} = 0$$

# 2.2 Metric perturbations: decomposition...

- To summarise:
  - 4 scalars: $A, B^{(S)}, H_L, H_T$ — 4 x 1 d.o.f.
  - 2 vectors: $B^{(V)}, H^{(V)}$ — 2 x 2 d.o.f.
  - 1 tensor: $H^{(T)}$ — 1 x 2 d.o.f.

= 10 d.o.f

# 2.2 Metric perturbations: decomposition...

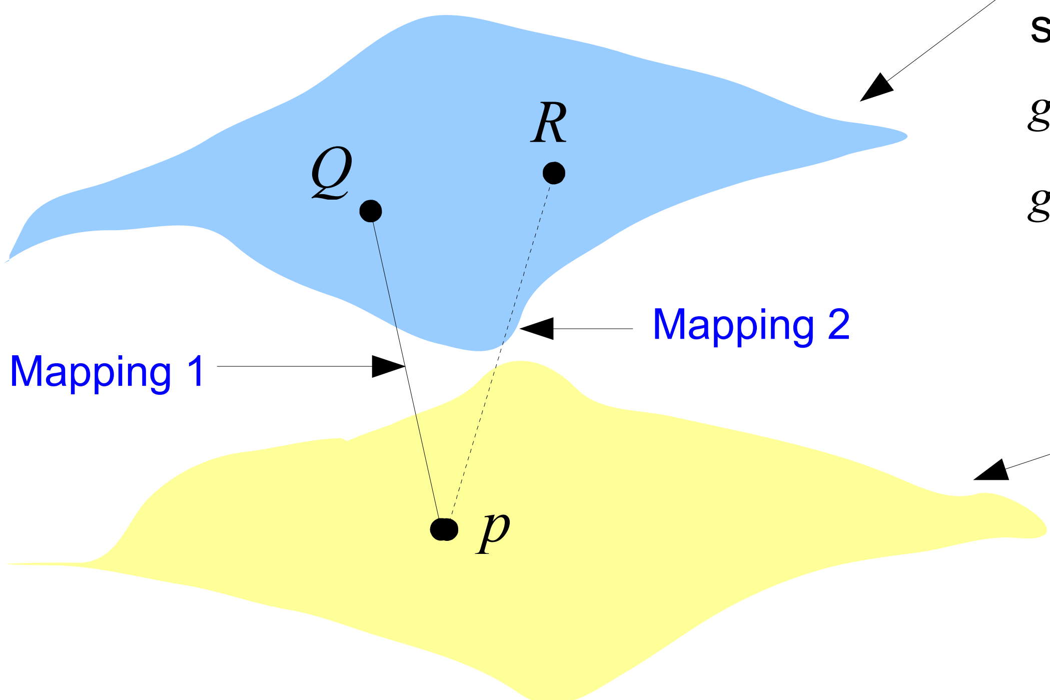
- To summarise:
  - 4 scalars: $A, B^{(S)}, H_L, H_T$ — 4 x 1 d.o.f.
  - 2 vectors: $B^{(V)}, H^{(V)}$ — 2 x 2 d.o.f.
  - 1 tensor: $H^{(T)}$ — 1 x 2 d.o.f.

  = 10 d.o.f

- **Not** all 10 d.o.f. are physical!
  - **Four** are gauge modes.

- Gauge modes arise because we are free to choose the mapping between the **real, inhomogeneous spacetime** to the **fictitious FLRW** spacetime around which we formulate the perturbation theory.

# 2.2 Metric perturbations: gauge modes...




Real, inhomogeneous spacetime

$$g_{\mu\nu}(Q)=\bar{g}_{\mu\nu}(p)+a^2 h_{\mu\nu}(p) \quad \text{Map 1}$$

$$g'_{\mu\nu}(R)=\bar{g}_{\mu\nu}(p)+a^2 h'_{\mu\nu}(p) \quad \text{Map 2}$$

Fictitious FLRW background spacetime

$\bar{g}_{\mu\nu}$

Different mappings (i.e., gauge choices) give different $h_{\mu\nu}(x)$.

# 2.2 Metric perturbations: gauge modes...

- A bit of common sense tells us that:

| | | | |
|---|---|---|---|
| Physical d.o.f. | 1 tensor; 1 x 2 d.o.f. | | |
| | 1 vector; 1 x 2 d.o.f. | 1 vector; 1 x 2 d.o.f. | Gauge modes |
| | 2 scalars; 2 x 1 d.o.f. | 2 scalars; 2 x 1 d.o.f. | |

# 2.2 Metric perturbations: gauge modes...

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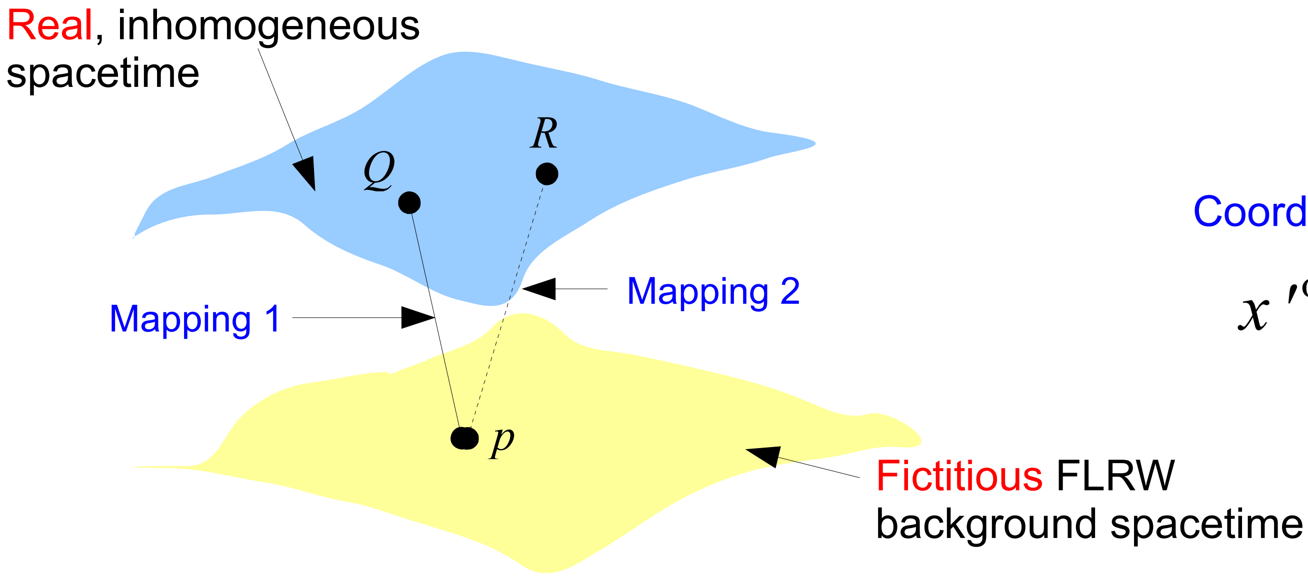
| | | |
|---|---|---|
| Physical d.o.f. | 1 tensor; 1 x 2 d.o.f. | |
| | 1 vector; 1 x 2 d.o.f. | 1 vector; 1 x 2 d.o.f. |
| | 2 scalars; 2 x 1 d.o.f. | 2 scalars; 2 x 1 d.o.f. |
| | | Gauge modes |

- We can remove redundant degrees of freedom by, e.g.,
  - Scalars: setting any two of $A$, $B^{(S)}$, $H_T$ and $H_L$ to zero.
  - Vectors: setting any one of $B^{(V)}$ and $H^{(V)}$ to zero.
- **Warning**! Your results may still contain gauge artifacts!

# 2.3 Gauge-invariant PT...

- Consider a change of mapping as a coordinate transformation in the real, inhomogeneous spacetime, but **not** in the fictitious FLRW background.




Coordinate transformation

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x)$$

Infinitesimal

Gauge transformation law

$$\rightarrow \quad a^2 h'_{\mu\nu}(x) = a^2 h_{\mu\nu}(x) - \bar{g}_{\mu\gamma}(x)\partial_{\nu}\xi^{\gamma} - \bar{g}_{\gamma\nu}(x)\partial_{\mu}\xi^{\gamma} - \xi^{\gamma}\partial_{\gamma}\bar{g}_{\mu\nu}(x)$$

To first order in $h_{\mu\nu}$ and $\xi$

# 2.3 Gauge-invariant PT: gauge transformations...

- Gauge transformation laws for scalar, vector and tensor components:

$$A' = A - \frac{\partial T}{\partial \eta} - \mathscr{H} T$$

$$B^{(S)}{}' = B^{(S)} - T + \frac{\partial L^{(S)}}{\partial \eta}$$

$$B^{(V)}{}' = B^{(V)} + \frac{\partial L^{(V)}}{\partial \eta}$$

$$H_L{}' = H_L - \frac{1}{3}\partial^2 L^{(S)} - \mathscr{H} T$$

$$H_T{}' = H_T - L^{(S)}$$

$$H^{(V)}{}' = H^{(V)} - L^{(V)}$$

$$H^{(T)}{}' = H^{(T)}$$

$$T \equiv \xi^0, \quad \xi_i \equiv \partial_i L^{(S)} + L_i^{(V)}$$

## 2.3 Gauge-invariant PT: Bardeen potentials...

- For scalar perturbations, the Bardeen potentials are invariant under gauge transformation:

$$\Psi \equiv A - \frac{1}{a}\frac{\partial}{\partial \eta}\left[a\left(\frac{\partial H_T}{\partial \eta} + B^{(S)}\right)\right]$$

$$\Phi \equiv -H_L + \frac{1}{3}\partial^2 H_T + \mathscr{H}\left(B^{(S)} + \frac{\partial H_T}{\partial \eta}\right)$$

# 2.3 Gauge-invariant PT: Bardeen potentials...

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$$\Phi \equiv -H_L + \frac{1}{3}\partial^2 H_T + \mathscr{H}\left(B^{(S)} + \frac{\partial H_T}{\partial \eta}\right)$$

- Observe that:

$$\left.\begin{aligned} \Psi &= A\left(H_T = 0, B^{(S)} = 0\right) \\ \Phi &= -H_L\left(H_T = 0, B^{(S)} = 0\right) \end{aligned}\right\} \quad \text{Newtonian gauge}$$

# 2.3 Gauge-invariant PT: Newtonian gauge...

- Spacetime metric with scalar perturbations in the Newtonian gauge:

$$ds^2 = a^2(\eta)[-(1+2\Psi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j]$$

- The Newtonian gauge is a convenient gauge choice because:
  - It is diagonal!
  - The perturbation variables correspond to gauge-invariant quantities.
  - In the nonrelativistic limit, $\Psi = \Phi$ corresponds to the Newtonian gravitational potential.

# 2.3 Gauge-invariant PT: gauge fixing...

- The Newtonian gauge is **not** the only gauge choice with the property that the perturbations degrees of freedom correspond to gauge-invariant variables.

- In general, if the gauge choice completely specifies the gauge-transformation variables $T$, $L^{(S)}$, and $L^{(V)}$, then the gauge is completely fixed.
  - The remaining perturbation degrees of freedom automatically correspond to **gauge-invariant variables**.

# 2.3 Gauge-invariant PT: Newtonian gauge...

- The Newtonian gauge is **completely fixed**:

$$H_T{}' = 0 \quad \Rightarrow \quad L^{(S)} = H_T$$

$$B^{(S)}{}' = 0 \quad \Rightarrow \quad T = B^{(S)} + \frac{\partial L^{(S)}}{\partial \eta} = B^{(S)} + \frac{\partial H_T}{\partial \eta}$$

- Gauge-invariants:

$$A' = A - \left(\frac{\partial}{\partial \eta} + \mathscr{H}\right)\left(B^{(S)} + \frac{\partial H_T}{\partial \eta}\right)$$

$$H_L{}' = H_L - \frac{1}{3}\partial^2 H_T - \mathscr{H}\left(B^{(S)} + \frac{\partial H_T}{\partial \eta}\right)$$

Bardeen potentials

# 2.3 Gauge-invariant PT: spatially flat gauge...

- The spatially flat gauge is **completely fixed**:

$$H_T{}' = 0 \quad \Rightarrow \quad L^{(S)} = H_T$$

$$H_L{}' = 0 \quad \Rightarrow \quad T = \frac{1}{\mathscr{H}}\left(H_L - \frac{1}{3}\partial^2 H_T\right)$$

- Gauge-invariants:

$$A' = A - \frac{\partial}{\partial \eta}\left[\frac{1}{\mathscr{H}}\left(H_L - \frac{1}{3}\partial^2 H_T\right)\right] - H_L + \frac{1}{3}\partial^2 H_T$$

$$B^{(S)} = B^{(S)} - \frac{1}{\mathscr{H}}\left(H_L - \frac{1}{3}\partial^2 H_T\right) + \frac{\partial H_T}{\partial \eta}$$

# 2.3 Gauge-invariant PT: synchronous gauge...

- The synchronous gauge is **NOT completely fixed**.

$$A' = 0 \quad \Rightarrow \quad \frac{\partial T}{\partial \eta} + \mathscr{H} T = \frac{1}{a}\frac{\partial (aT)}{\partial \eta} = A$$

$$\Rightarrow \quad T = a^{-1}\int A\, a\, d\eta + C_1 a^{-1}$$

$$B^{(S)\prime} = 0 \quad \Rightarrow \quad \frac{\partial L^{(S)}}{\partial \eta} = T - B^{(S)}$$

$$\Rightarrow \quad L^{(S)} = \int \left(T - B^{(S)}\right) d\eta + C_2$$

- $C_1$ and $C_2$ are unknown integration constants → The remaining d.o.f.s $H_L$ and $H_T$ may suffer from spurious gauge effects.

# 2.3 Gauge-invariant PT: vector perturbations...

- The vector gauge is **completely fixed**:

$$H(V)'=0 \quad \Rightarrow \quad L^{(V)}=H^{(V)}$$

Gauge-invariant $\Rightarrow$ $$B(V)'=B(V)+\frac{\partial H^{(V)}}{\partial \eta}$$

- Gauge choice 2 is **not completely fixed**:

$$B^{(V)}{}'=0 \quad \Rightarrow \quad \frac{\partial L^{(V)}}{\partial \eta}=B^{(V)}$$

$$\Rightarrow \quad L^{(V)}=\int B^{(V)} d\eta+C_1$$

# 2.4 Perturbations in $T_{\mu\nu}$...

- Perturbed stress-energy tensor:

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}, \qquad |\delta T_{\mu\nu}| \ll \bar{\rho}$$

- For a perfect fluid:

$$T^{\mu}{}_{\nu(\alpha)} = (\rho_\alpha + P_\alpha) u^\mu u_\nu + g^\mu{}_\nu P_\alpha$$

  - $\rho_\alpha = \bar{\rho}_\alpha + \delta\rho_\alpha$ = $\bar{\rho}_\alpha (1 + \delta_\alpha)$ = energy density in the fluid's rest frame
  - $P_\alpha = \bar{P}_\alpha + \delta P_\alpha$ = pressure in the fluids's rest frame
  - $u^\mu$ = 4-velocity of fluid; $|u^i| << |u^0|$

Small quantities in our picture

# 2.4 Perturbations in $T_{\mu\nu}$: fluid perturbations...

- To first order in small quantities:

$$T^0{}_{0(\alpha)} = -\bar{\rho}_\alpha - \delta\rho_\alpha \qquad T^i{}_{0(\alpha)} = -(\bar{\rho}_\alpha + \bar{P}_\alpha) v^i_\alpha$$

$$T^0{}_{i(\alpha)} = (\bar{\rho}_\alpha + \bar{P}_\alpha)(v_{\alpha\, i} - B_i) \qquad T^i{}_{j(\alpha)} = \bar{P}_\alpha \delta^i_j + \delta P_\alpha \delta^i_j$$

  - $\delta\rho_\alpha$ = energy density perturbation
  - $\delta P_\alpha$ = pressure perturbation
  - $v^i_\alpha \equiv dx^i/d\eta$ = peculiar velocity

As measured by a comoving observer

# 2.4 Perturbations in $T_{\mu\nu}$: fluid perturbations...

- To first order in small quantities:

$$T^0{}_{0(\alpha)} = -\bar{\rho}_\alpha - \delta\rho_\alpha \qquad T^i{}_{0(\alpha)} = -(\bar{\rho}_\alpha + \bar{P}_\alpha) v^i_\alpha$$

$$T^0{}_{i(\alpha)} = (\bar{\rho}_\alpha + \bar{P}_\alpha)(v_{\alpha\, i} - B_i) \qquad T^i{}_{j(\alpha)} = \bar{P}_\alpha \delta^i_j + \delta P_\alpha \delta^i_j + \Pi^i{}_{j(\alpha)}$$

$$\Pi^i{}_i = 0$$

Traceless 3-tensor

- $\delta\rho_\alpha$ = energy density perturbation
- $\delta P_\alpha$ = pressure perturbation
- $v^i_\alpha \equiv dx^i/d\eta$ = peculiar velocity   As measured by a comoving observer
- $\Pi^i{}_{j(\alpha)}$ = anisotropic stress in fluid's rest frame (imperfect fluid)

# 2.4 Perturbations in $T_{\mu\nu}$: gauge transformation laws...

- Fluid perturbations also change under gauge transformation, via

$$\delta T^{\mu}{}_{\nu}{}'(x) = \delta T^{\mu}{}_{\nu}(x) - \bar{T}^{\mu}{}_{\gamma}\partial_{\nu}\xi^{\gamma} + \bar{T}^{\gamma}{}_{\nu}\partial_{\gamma}\xi^{\mu} - \xi^{\gamma}\partial_{\gamma}\bar{T}^{\mu}{}_{\nu}$$

- Transformation laws for scalar, vector and tensor fluid perturbations:

$$\delta\rho' = \delta\rho - T\frac{\partial\bar{\rho}}{\partial\eta} \qquad v^{(S,V)}{}' = v^{(S,V)} + \frac{\partial L}{\partial\eta}^{(S,V)}$$

$$\delta P' = \delta P - T\frac{\partial\bar{P}}{\partial\eta} \qquad \Pi^{(S,V,T)}{}' = \Pi^{(S,V,T)}$$

# 2.5 Einstein equation...

- We derive equations of motion for the perturbations from:

$$\bar{G}_{\mu\nu} + \delta G_{\mu\nu} = 8\pi G \left( \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \right)$$

$$\delta G_{\mu\nu} = \delta G^{(S)}_{\mu\nu} + \delta G^{(V)}_{\mu\nu} + \delta G^{(T)}_{\mu\nu} \qquad \delta T_{\mu\nu} = \delta T^{(S)}_{\mu\nu} + \delta T^{(V)}_{\mu\nu} + \delta T^{(T)}_{\mu\nu}$$

- To **linear order** in small quantities:

Linear functions

$$\begin{cases} \delta G^{(S)}_{\mu\nu}(A, B^{(S)}, H_L, H_T) &= 8\pi G\, \delta T^{(S)}_{\mu\nu}(\rho, P, v^{(S)}, \Pi^{(S)}) \\ \delta G^{(V)}_{\mu\nu}(B^{(V)}, H^{(V)}) &= 8\pi G\, \delta T^{(V)}_{\mu\nu}(v^{(V)}, \Pi^{(V)}) \\ \delta G^{(T)}_{\mu\nu}(H^{(T)}) &= 8\pi G\, \delta T^{(T)}_{\mu\nu}(\Pi^{(T)}) \end{cases}$$

Linear functions

- Scalar, vector and tensor perturbations evolve independently.

# 2.5 Einstein equation: scalar perturbations...

- Scalar equations of motion:

$$\partial^2\Phi - 3\,\mathcal{H}\left(\dot{\Phi} + \mathcal{H}\,\Psi\right) = 4\pi G a^2 \sum_\alpha \bar{\rho}_\alpha \delta_\alpha$$

$$\partial^2\left(\dot{\Phi} + \mathcal{H}\,\Psi\right) = -4\pi G a^2 \sum_\alpha \left(\bar{\rho}_\alpha + \bar{P}_\alpha\right)\partial^2 v_\alpha^{(S)}$$

$$\ddot{\Phi} + \mathcal{H}\left(\dot{\Psi} + 2\,\dot{\Phi}\right) + \left(2\,\dot{\mathcal{H}} + \mathcal{H}^2\right)\Psi + \frac{1}{3}\partial^2\left(\Phi - \Psi\right) = 4\pi G a^2 \sum_\alpha \delta P_\alpha$$

$$\partial^2\left(\Phi - \Psi\right) = 8\pi G a^2 \sum_\alpha \partial^2 \Pi_\alpha^{(S)}$$

- Of these equations, there are only two linearly independent combinations

# 2.5 Einstein equation: Fourier decomposition...

- The equations of motion are formally **2nd order linear PDEs** → Can be solved by applying a Fourier transform:

$$f(\eta, x^i) = \frac{1}{(2\pi)^3} \int d^3k \, \tilde{f}(\eta, k^i) \, e^{i k_i x^i}$$

  - $k^i$ = comoving wavevector of perturbation

- For the EoMs: $\partial^2 \rightarrow -k_i k^i = -\delta_{ij} k^i k^j \equiv k^2$

- Fourier modes evolve **independently** of one another
- Evolution depends on absolute value of $k$, **not** the direction.

# 2.5 Einstein equation: scalar perturbations...

- Equations of motion for scalar perturbations in Fourier space:

$$-k^2\Phi - 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Psi) = 4\pi G a^2 \sum_\alpha \bar{\rho}_\alpha \delta_\alpha \qquad (1)$$

$$(\dot{\Phi} + \mathcal{H}\Psi) = -4\pi G a^2 \sum_\alpha (\bar{\rho}_\alpha + \bar{P}_\alpha) v_\alpha^{(S)} \qquad (2)$$

$$\ddot{\Phi} + \mathcal{H}(\dot{\Psi} + 2\dot{\Phi}) + (2\dot{\mathcal{H}} + \mathcal{H}^2)\Psi - \frac{1}{3}k^2(\Phi - \Psi) = 4\pi G a^2 \sum_\alpha \delta P_\alpha$$

$$\Phi - \Psi = 8\pi G a^2 \sum_\alpha \Pi_\alpha^{(S)}$$

# 2.5 Einstein equation: Newtonian limit...

- Combine (1) and (2) to give:

Dimensionless

$$-k^2 \Phi = 4\pi G a^2 \sum_\alpha \bar{\rho}_\alpha \left[ \delta_\alpha - 3 \frac{\mathscr{H}}{k} (1 + w_\alpha)(k v_\alpha^{(S)}) \right]$$

- Important quantity:

$$\frac{\mathscr{H}}{k} \sim \frac{\text{Hubble length}^{-1}}{\text{Comoving wavelength}^{-1}}$$

>> 1 → "Super-horizon"
<< 1 → "Sub-horizon"

- In the **subhorizon limit**:

$$-k^2 \Phi \simeq 4\pi G a^2 \sum_\alpha \bar{\rho}_\alpha \delta_\alpha$$

→ $\mathscr{H}$ quantifies the length scale at which we expect "genuine" GR effects to arise.

Poisson equation in Newtonian gravity
(if nonrelativistic matter)

# 2.5 Einstein equation: scalar perturbations...

- Conservation of energy-momentum: $\nabla_\mu T^{\mu\nu}_{(\alpha)} = 0$

$$\left(\frac{\partial}{\partial\eta} + 3\mathscr{H}\right)\delta\rho_\alpha + 3\mathscr{H}\,\delta P_\alpha = (\bar{\rho}_\alpha + \bar{P}_\alpha)\left(k^2 v^{(s)}_\alpha + 3\dot{\Phi}\right)$$

$$\left(\frac{\partial}{\partial\eta} + 4\mathscr{H}\right)\left[(\bar{\rho}_\alpha + \bar{P}_\alpha)\, v^{(s)}_\alpha\right] = -\delta P_\alpha + \frac{2}{3}k^2\Pi^{(s)}_\alpha - (\bar{\rho}_\alpha + \bar{P}_\alpha)\Psi$$

# 2.5 Einstein equation: scalar perturbations...

- Conservation of energy-momentum: $\nabla_\mu T^{\mu\nu}_{(\alpha)} = 0$

$$\left(\frac{\partial}{\partial\eta} + 3\mathscr{H}\right)\delta\rho_\alpha + 3\mathscr{H}\,\delta P_\alpha = (\bar{\rho}_\alpha + \bar{P}_\alpha)\left(k^2 v_\alpha^{(s)} + 3\dot{\Phi}\right)$$

$$\left(\frac{\partial}{\partial\eta} + 4\mathscr{H}\right)\left[(\bar{\rho}_\alpha + \bar{P}_\alpha)\, v_\alpha^{(s)}\right] = -\delta P_\alpha + \frac{2}{3}k^2\Pi_\alpha^{(s)} - (\bar{\rho}_\alpha + \bar{P}_\alpha)\Psi$$

- Not enough equations?
  - **Perfect fluid** ($\Pi_\alpha = 0$): $\delta P_\alpha$ & $\delta\rho_\alpha$ linked by the fluid sound speed:

$$c_s^2 \equiv \frac{\delta P_\alpha}{\delta\rho_\alpha}$$

Property of fluid $\alpha$

  - **Imperfect fluid** ($\Pi_\alpha \neq 0$): need other supplementary equations (e.g., Boltzmann equation).

# 2.5 Einstein equation: vector perturbations...

- Equations of motion for vector perturbations in Fourier space:

$$k^2\left(B^{(V)}+\dot{H}^{(V)}\right)=16\pi G a^2\sum_\alpha\left(\bar{\rho}_\alpha+\bar{P}_\alpha\right)\left(v_\alpha^{(V)}-B_\alpha^{(V)}\right)$$

$$\left(\frac{\partial}{\partial\eta}+2\mathscr{H}\right)\left(B^{(V)}+\dot{H}^{(V)}\right)=-8\pi G a^2\sum_\alpha\Pi_\alpha^{(V)}$$

Energy-momentum conservation

$$\left(\frac{\partial}{\partial\eta}+4\mathscr{H}\right)\left[\left(\bar{\rho}+\bar{P}\right)\left(v^{(V)}-B^{(V)}\right)\right]=\frac{1}{2}\Pi_\alpha^{(V)}$$

# 2.5 Einstein equation: tensor perturbations...

- Equations of motion for tensor perturbations in Fourier space:

$$\ddot{H}^{(T)} + 2\mathcal{H}\dot{H}^{(T)} + k^2 H^{(T)} = 4\pi G a^2 \sum_{\alpha} \Pi_{\alpha}^{(T)}$$

Gravitational waves!

$$g_{ij} = a^2 \begin{pmatrix} 1 + H_+^{(T)} & H_\times^{(T)} & 0 \\ H_\times^{(T)} & 1 - H_+^{(T)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Traceless;
transverse
(divergenceless)

2 d.o.f = 2 polarisations

# 2.6 Section summary...

- Perturbations in the spacetime metric can be decomposed into 4 scalar, 2 vector, and 1 tensor components.
  - **Caution**: 2 scalars and 1 vector are gauge modes!

- Fluid perturbations are expressed in terms of a perturbed stress-energy tensor, which can likewise be decomposed in terms of rotation invariants.

- Metric and fluid perturbations are related via the Einstein equation.
  - At 1$^{st}$ order: one independent set of equations for each of the scalar, vector, and tensor components.
  - Fourier decomposition: $k$ modes evolve independently of one another.